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Majority voting solution concepts and redistributive taxation

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Abstract. Strong assumptions are usually needed to guarantee the existence of a Condorcet winner in majority voting games. The theoretical literature has developed various solution concepts to accommodate the general absence of Condorcet winner, but very little is known on their economic implications. In this paper, I select three such concepts (the uncovered set, the bipartisan set and the minmax set), defined as game-theoretical solution concepts applied to a Downsian electoral competition game. These concepts are then computed by means of simulations in a simple model of purely redistributive taxation, where factor supply varies with net factor rewards. All three concepts give rather sharp predictions and are not too sensitive to small variations of the preference profiles.

1 Introduction

In democratic societies, most social decisions are the outcome – either directly or indirectly – of a majority voting process. Contemporary economists have developed a keen interest in majority voting processes, particularly when eco-

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nomic decisions are at stake. This query dates back to the $XVIII^{th}$ century, with the seminal work of Condorcet. He proposed a rigorous framework to analyze voting procedures and also defined an equilibrium concept. When a polity has to choose an option among a given set, a likely candidate is the option which receives a majority of votes when faced with any other possible option. This option is called the Condorcet winner of the vote.

However, Condorcet himself was well aware that such a winner does not necessarily exist. Indeed, even if individual preferences are well behaved, the social relation obtained by majority voting may contain a cycle, where for example option *a* is majority preferred to option *b*, *b* is majority preferred to option *c* and *c* is majority preferred to option *a*.

Confronted with this problem, one research program has made progress by narrowing the model domain. By suitably restricting the set of admissible preferences and the structure of the options set, one is able to ensure the existence of a Condorcet winner. The most famous result of this research program is the median voter theorem: by requiring that available options be ranked on a single dimension and that individual preferences be single-peaked on this dimension, one obtains a Condorcet winner, which corresponds to the median peak on this single dimension. The median voter theorem is often used in economic models when voting takes place on a single dimension.

However, this line of research has its limits. First, economists cannot assume that individual preferences are single-peaked. Preferences are usually not expressed directly on the space of options, but indirect preferences about these options must be constructed from direct preferences. And many usual forms of direct preferences do not lead to single-peaked indirect preferences. Second, if options are organized on a multi-dimensional space (because for example the polity has to vote simultaneously on various instruments), Plott (1967) has shown that the restrictions to be imposed on individual preferences to obtain a Condorcet winner are far more drastic than simply being single-peaked on this multi-dimensional space. His theorem indicates the general absence of a Condorcet winner in multi-dimensional votes.

To circumvent these problems, another strand of research has developed which studies alternative equilibrium concepts in voting games. This strand of literature, developed both in cooperative game theory and in social choice theory, studies Condorcet-consistent concepts (i.e. concepts that exclusively select the Condorcet winner when it exists) in abstract environments and mainly from a normative/axiomatic perspective.

The objective of this paper is to analyze the implications of these equilibrium concepts in an economic environment. Very little is known about the economic implications of these equilibrium concepts. In a recent paper, Epstein (1997) shows that in games characterized by purely distributive politics (so-called Pareto constant games), the uncovered set is equal to the entire Pareto set, minus some boundary points. Having shown that restricting outcomes to the uncovered set is not very powerful, Epstein then claims that, "in these settings, social choice theory is unable to place meaningful restrictions on equilibrium outcomes", arguing for a corresponding emphasis on "the rules which dictate the play of the game (...) in distributive politics" (the so-called structure-induced equilibria).

While I certainly agree that the precise rules of the games are of paramount importance in any voting game, I nevertheless think that social choice theory can help in reducing the set of interesting outcomes. Laslier and Picard (1999) show that, in the divide-the-dollar game with *n* players (which belongs to the Pareto constant class studied by Epstein), the minimal covering set (or at least an "almost covering" set which they call Hex) gives a much sharper prediction than the uncovered set. They also compute (and henceforth show the existence of) the equilibria in mixed strategies and show that the equilibrium allocations are always in Hex and that divisions close to the equal one are proposed more often. Hence, even in the class of games studied by Epstein, solution concepts other than the uncovered set can prove very useful in restricting the set of interesting options.

In this paper, I compute by means of simulations the uncovered set, the bipartisan set, and the minmax set in a specific economic environment. These three social choice correspondences were chosen in part because they also correspond to game-theoretical solution concepts applied to a particular voting game, the two-party electoral competition game pioneered by Downs (1957). Hence, besides being already axiomatically characterized, they also possess positive foundations.

These concepts are applied to a simple model of purely redistributive taxation with positive factor supply elasticity. This model[1], where vote takes place on a two-dimensional policy space, exemplifies how individual interests may be in conflict when voting on economic parameters such as taxation rates (even if individual preferences are not as maximally opposed as in Epstein's Pareto-constant class of games). Our analysis shows that even the uncovered set is usually very selective with an average size of 5% of the feasible options. The other two concepts are even more focused (2% for the bipartisan set and usually a singleton for the minmax). Also, the choice sets are shown not to be too sensitive to small variations of the preference profiles. The main message of this paper is then that such solution concepts as the uncovered set, the minmax or the bipartisan set give interesting results even in redistributive situations, at least with a two-dimensional policy space.

The plan of the paper is as follows. Section 2 presents the pure redistribution model. Section 3 defines the uncovered set, the bipartisan set, the minmax set and the Pareto set in the context of a two-party electoral competition game. Section 4 gives the results of the simulations of these solution concepts in the pure redistribution model. Section 5 concludes the paper.

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[1] This model was already studied by Gevers and Jacquemin (1987), who investigated the minmax set. They also looked at the uncovered set in an unpublished paper (Gevers and Jacquemin 1988). Our goal is to extend their analysis to other solution concepts. The reader will find more simulations of majority voting solutions in this model in De Donder (1998).

2 The pure redistribution model

The economy described in Gevers and Jacquemin (1987) is composed of a finite odd number n of artisans and of three goods, one output and two inputs (say capital and labor). Both inputs are measured in efficiency units, and their marginal product is supposed constant. Artisans differ by their efficiency in providing the two inputs. Without government, each artisan would consume his/her own input.

The fiscal authority taxes proportionally capital and labor supply at rates s and t respectively. The proceeds are then redistributed to all agents in the form of a uniform lump-sum subsidy g.

In this economy, two types of decisions must be taken: a collective one (the choice of a triplet (s, t, g)) and n decentralized ones (the choice by each agent of how much he/she will contribute to production, given his/her skills, tastes and the fiscal environment).

The interactions between these $n + 1$ choices can be seen as a two-stage optimization process. In the lower stage, each artisan, considering the fiscal variables (s, t, g) as fixed, chooses his/her utility maximizing levels of labor and capital supply. In the upper stage, each individual votes over the fiscal variables (s, t, g) taking into account the effects of these fiscal variables on aggregate consumer behavior. In other words, the collective choice of the fiscal policy is made under the assumption of perfect forecast of aggregate consumer behavior.

More precisely, the government budget constraint describes an implicit relation between s, t and g. Under favorable circumstances, a unique value of g is associated with any pair of values (s, t). This relation, known as the Dupuit-Laffer curve, reduces by one the number of degrees of freedom enjoyed by the polity in its choice of the fiscal variables. I suppose that individuals vote on pairs (s, t).

Individual voters' preferences differ about (s, t) choices. I consider that artisans share the same utility function and only differ by their ability to provide efficient factors. Furthermore, factor supplies are linear in net factor reward. The reader will find in the Appendix the complete formulation of this model. As pointed out by Gevers and Jacquemin (1987) and shown in the Appendix, the interest for this formulation does not come from its empirical plausibility, but from the fact that, after a slight change of scale, it is consistent with the abstract Euclidean preferences (i.e. indifference curves as concentric circles) often used in the voting literature.

In this very simple model, each agent's preferred tax rate is proportional to the difference between the mean efficiency of the artisans' supply of this factor and the artisan's own efficiency (see the Appendix). This means that only agents whose ability is less than the mean ability favor taxation of the factor (the other agents favor income subsidization), and that the preferred tax rate is linearly decreasing with individual efficiency. Figure 1 plots indifference curves for various individuals.

The upper left panel of Fig. 1 shows indifference curves for an individual

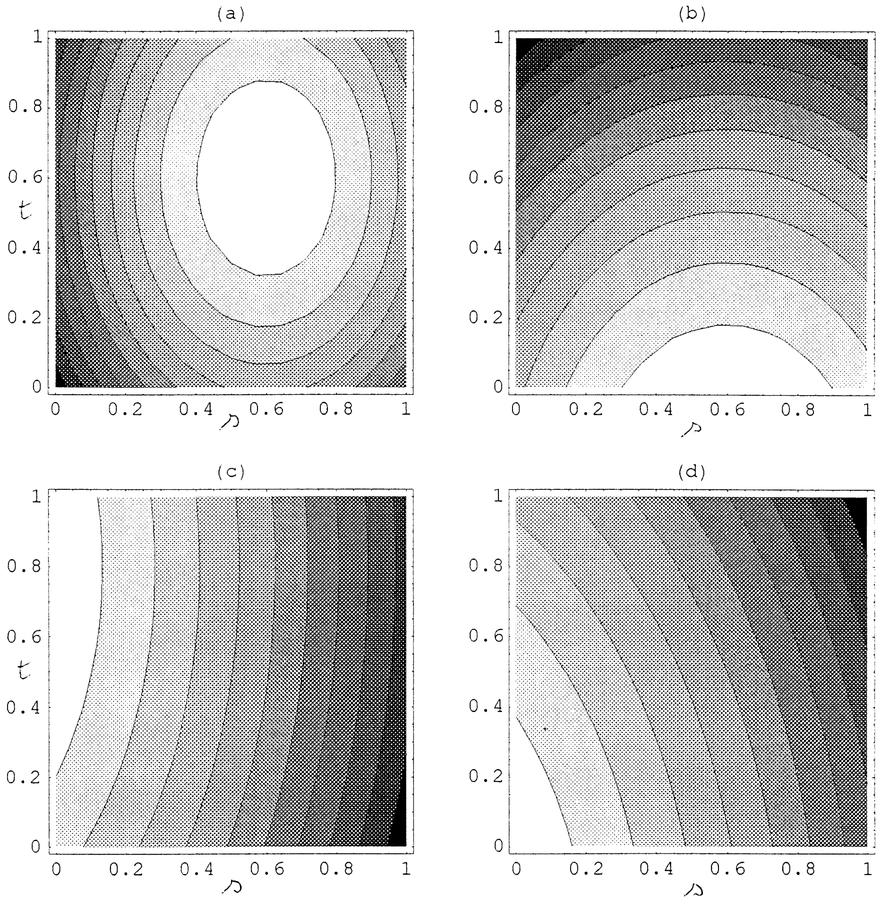

Fig. 1. Indifference curves of four individuals in the (s, t) space. Mean labor productivity in the economy: 2.2; Mean capital productivity in the economy: 1.4.
Individual (**a**): labor productivity: 1.9 capital productivity: 0.8; Individual (**b**): labor productivity: 2.4 capital productivity: 0.8; Individual (**c**): labor productivity: 1.8 capital productivity: 2; Individual (**d**): labor productivity: 2.4 capital productivity: 2

who is less able than the society mean in supplying both capital and labor, while the lower right panel shows the preferences of an individual who is more able than the economy mean in supplying both factors. The other two panels show preferences for the intermediate cases.

Since I am interested in majority voting, the first question that I may address is the existence of a Condorcet winner (a pair (s, t) preferred by a majority of voters over any other possible pair) in this bidimensional choice space. Davis et al. (1972) first stated the conditions under which a Condorcet winner can occur in two-dimension open sets in the space of Euclidean preferences, if the number of voters is even: a necessary and sufficient condition is that there exists a median in all directions (i.e. a point such that *every* line through it divides ideal points equally). This median point is the Condorcet winner. This condition is so demanding that, starting from a situation where such a point exists, the slightest modification of a single individual's preference is almost surely fatal to the equilibrium[2].

The general absence of the Condorcet winner may indicate that the Condorcet requirements are too strong. Indeed, social choice theory has developed many alternative solution concepts corresponding to various weakenings of the Condorcet criterion. These solution concepts are said to be Condorcet consistent whenever they select the Condorcet winner exclusively when it exists. On the other hand, when a Condorcet winner does not exist, different Condorcet consistent concepts may give different results.

These Condorcet consistent solution concepts have been studied mainly in abstract environments. The objective of this paper is to apply some of them to the pure redistribution model outlined above. Moreover, these solution concepts have been studied mainly from a normative perspective. Since this paper has a more positive viewpoint, I have selected three Condorcet-consistent solution concepts having an interpretation in terms of a particular voting mechanism: the two-party electoral competition proposed by Downs (1957). These concepts are presented in the next section and applied to the redistribution model in Sect. 4.

3 Solution concepts

The final objective of this paper is to compute, by means of simulations, the solution concepts defined in this section in the case of the pure redistribution model introduced above. To simulate this economy, I thus have to provide a discrete approximation of the set of feasible tax pairs. Henceforth, I define the solution concepts in the framework of a finite set of possible options.

I first describe the abstract framework of analysis and the electoral competition that takes place in this framework.

3.1 Political setting, notation and definitions

The particular voting mechanism I use is based on the one pioneered by Hotelling (1929) and Downs (1957). I suppose that two political parties compete for the votes of n individuals. Each party simultaneously chooses an option as its proposed platform, which it claims it will impose if elected. Each individual then votes for the party whose platform is better according to his/

[2] Plott (1967) has derived similar very demanding symmetry conditions for a Condorcet winner to exist when indifference curves are simply differentiable. See McKelvey and Schofield (1986), McKelvey and Schofield (1987) and Saari (1997) for more recent references on the non-existence of a Condorcet point. Also, when the number of individuals is even, a Condorcet point can be found that is structurally stable (i.e. exists for an open set in the space of individual parameters): see Schofield (1986).

her preference ordering. The party receiving the most votes wins the election and imposes its platform as the choice of the polity. In case of ties, a fair coin decides which party wins the election.

I introduce the following notation:
X is a finite set of strategies (identical for both parties);
I a set of individuals, indexed by $i = 1 \ldots n$, with n odd;
R_i a complete binary relation on X representing individual i's preference ordering. I do not rule out individual indifferences between options at the outset, but specify later which simplifications obtain if these indifferences are excluded[3]. From this relation, I can define two binary relations, I_i and P_i, constituting respectively the symmetric and asymmetric part of R_i:
$\forall x, y \in X, xI_i y$ if and only if $xR_i y$ and $yR_i x$
$\forall x, y \in X, xP_i y$ if and only if $xR_i y$ and not $yR_i x$

Let n_{xy} be the number of individuals strictly preferring option x to option y, i.e.

$$n_{xy} = |\{i \in I \text{ such that } xP_i y\}|, \quad \forall x, y \in X$$

By definition of the electoral competition, n_{xy} also represents the number of votes received by the party whose platform is x when opposed in elections to other party's platform y if I suppose that indifferent individuals abstain.

Next, I define the binary relation B on X.

Definition 1. *xBy means that strategy x beats strategy y under plurality voting:*

$$xBy \Leftrightarrow n_{xy} > n_{yx}$$

Definition 2. *$a \in X$ is a Condorcet winner if and only if $aBb, \forall b \in X \backslash \{a\}$*

Definition 3. *A tournament is composed of a finite set of alternatives X and of an asymmetric and complete binary relation B over this set. It is denoted by (X, B).*

Thus, if individual preference orderings are strict and if the number of individuals is odd, the asymmetric binary relation B is complete and the pair (X, B) that it defines is a tournament.

The objective assigned to the parties is crucial. I follow Downs in assuming that "the fundamental hypothesis of our model (is that) parties formulate policies in order to win elections, rather than win the elections in order to formulate policies" (1957, p. 28). More precisely, I suppose that parties are *only* interested in winning the elections since they do not derive any intrinsic utility from the particular option chosen, and have a zero marginal utility for votes once a bare majority is achieved. Hence, B can be used to describe parties' payoffs in the following way. The payoff of party 1 when it plays x

[3] These simplifications will be used in the next section, since individual indifference never show up in the simulations.

while party 2 plays y (denoted by m_{xy}) is represented by:

$$m_{xy} = \begin{cases} 1 & \text{if} \quad xBy \\ -1 & \text{if} \quad yBx \\ 0 & \text{if} \quad \text{not } xBy \text{ and not } yBx \end{cases}$$

Player 2's payoff in this case being defined as $-m_{xy}$, this game is zero-sum. Furthermore, since both players have the same set X of pure strategies and since $m_{xy} = -m_{yx}$, this game is also symmetric[4].

Definition 4. *The pair (X, M) is a two-player, zero-sum, symmetric game where $M = [m_{xy}]_{x,y \in X}$ represents the payoff matrix of party 1. (X, M) is called the majority game.*

If individual preferences are strict, the majority game defined by these preferences has a unique Nash equilibrium in pure strategies if and only if there exists a Condorcet winner. In this case, both parties choose the Condorcet winner as a strategy. If indifferences are allowed, the set of Nash equilibria in pure strategies consists of all unbeaten alternatives. In this case, the non-emptiness of this set does not require the existence of a Condorcet winner. However, this set reduces to the Condorcet winner when the latter exists.

I now present three solution concepts which are, in the terminology of Dutta and Laslier (1999), strong Condorcet consistent, in the sense that they *exclusively* select the Condorcet winner whenever it exists.

3.2 The uncovered set and the set of weakly undominated strategies

I first introduce Fishburn (1977)–Miller (1980)'s *covering relation*.

Definition 5. *x covers y (denoted by xCy) if x beats y in pairwise comparison and x beats whatever y beats, i.e.*

$$\forall x, y \in X, \quad xCy \textit{ if and only if} \begin{cases} & xBy \\ \forall w \in X, & yBw \Rightarrow xBw \end{cases}$$

Definition 6. *The set of maximal elements of the covering relation*

$$UC(X, B) = \{a \in X \textit{ such that for no } b \in X \textit{ does } bCa\}$$

is called the uncovered set.

If individual preference orderings are strict, the covering relation has a straightforward counterpart defined on the corresponding majority game. This counterpart is called the weak dominance relation.

Definition 7. *Strategy x weakly dominates strategy y in the majority game (X, M) if x performs as well as y when faced with any strategy z, and fares strictly better when faced with at least one strategy, i.e.*

[4] i.e., exchanging strategies between the two players amounts to exchanging their payoffs.

$$m_{xz} \geq m_{yz} \quad \forall z \in X$$

with a strict inequality for at least one z.

Since the covering relation is equivalent to the weak dominance relation if individual preference orderings are strict, it follows that the uncovered set is equivalent to the set of weakly undominated strategies in this case. It will prove easier to work with this latter set in the section devoted to the simulations.

Remark 1. If individual preference orderings are not strict, the equivalence between weak dominance and covering is no longer true. Indeed, in this case, x weakly dominates y does not request that x beats y, and thus does not imply that x covers y. One could easily reexpress the covering relation definition to keep the equivalence with weak dominance, for example by replacing the requirement xBy by not yBx. However, Dutta and Laslier (1999) argue that this redefinition is not relevant, because it loses the main property (called Expansion – see Moulin 1986) of the uncovered set. Hence, Dutta and Laslier (1999) and Peris and Subiza (1995) propose to extend the usual covering definition in the following way: x covers* y if $m_{xy} > 0$ and $m_{xz} \geq m_{yz}$, $\forall z \in X \backslash \{y\}$. This covering relation is thus more demanding than the weak dominance.

3.3 The Nash equilibrium in mixed strategies

Since the majority game is finite, there exists equilibria in mixed strategies (see Ordeshook 1986). Following Dutta and Laslier (1999), I adopt the following definition

Definition 8. *The essential set is the set of options played with a strictly positive probability in some mixed Nash equilibrium of the majority game.*

If individual preference orderings are strict, all the off-diagonal entries of the payoff matrix of the majority game are odd integers. It then follows from Laffond et al. (1997) that there exists a unique Nash equilibrium in mixed strategies.

Definition 9. *The bipartisan set, denoted by $BP(X, B)$, is the support of the unique Nash equilibrium in mixed strategies of a majority game derived from an odd number of strict preference orderings.*

3.4 The minmax set and the Pareto set

The two following concepts need more information than the relation B, namely the size of the majorities for every pairwise comparison. Consequently, I represent the payoff matrix of player 1 by the matrix of net pluralities, i.e. $NP = [n_{xy} - n_{yx}]_{x,y \in X}$ and call (X, NP) the *plurality game*.

The following concept was introduced by Kramer (1977) and Simpson (1969):

Definition 10. *The minmax set selects as solutions of the plurality game* (X, NP) *the strategies*

$$SK(X, NP) = argmin_{x \in X}(\max_{y \in X \setminus \{x\}} n_{yx} - n_{xy})$$

i.e. those strategies whose maximal opposition is the weakest.

The interest devoted to the minmax in the voting literature partly comes from the fact that it has attraction properties in a dynamic voting game introduced by Kramer (1977). In this voting game, two myopic political parties repeatedly compete for the votes of a set of individuals having stationary preferences. The peculiarity of this game is that when a party is elected, it is committed to keep the same political platform for the next election. In absence of unbeaten alternative, this assumption implies that both parties alternate in office. Kramer further assumes that parties are interested in maximizing the size of their majority (or net plurality), and thus that the opposing party will always select an option which maximizes its voting share when opposed to the incumbent policy. This dynamic voting process can be represented by the sequence of winning platforms, which Kramer calls a *vote-maximizing trajectory*. Kramer demonstrates that for Euclidean preferences the minmax set behaves like a basin of attraction in the sense that any vote-maximizing trajectory converges to the minmax set.

I also use the Pareto set:

Definition 11. *The Pareto set is composed of options which are not strictly Pareto dominated, i.e.*

$$PA(X, NP) = \{x \in X \text{ such that } \nexists y \in X \setminus \{x\} : n_{yx} = n\}$$

Unless agents are completely informed and strategic commitments cannot be violated, the Pareto set is not to be considered as a positive solution concept, but as a normative benchmark against which positive solution concepts will be tested in the voting game.

3.5 Set-theoretical comparisons

To obtain clues as to the maximal amount of disagreement between the concepts defined above, I compare their solution sets from a set-theoretical viewpoint.

As originally argued by McKelvey (1986)[5] and proved by Banks, Duggan and Le Breton (1998), $BP(X, B) \subseteq UC(X, B)\ \forall (X, B)$. Furthermore, one can see that any Pareto-dominated option is also weakly dominated[6] and thus that

[5] McKelvey (1986, theorem 3b)'s proof was incorrect, since it relied on eliminating weakly dominated strategies. As Remark 1 showed, the set of weakly undominated strategies is larger than the uncovered set when indifferences are allowed.

[6] If x is Pareto-dominated by y, y has a better position than x in *every* individual preference ordering. Hence, using majority voting, y beats all the options that x beats, which means that y covers x.

$UC(X, B) \subseteq PA(X, NP)$. One can also see that $SK(X, NP) \subseteq PA(X, NP)$, since any Pareto-dominated option has the worst maximum opposition score of n, which prevents this option from belonging to the minmax set.

De Donder et al. (2000) show that the minmax set can have an empty intersection with many other solution concepts, including UC (and thus BP). In fact, the minmax set can reduce to such an undesirable option as the Condorcet loser, i.e. the option which is beaten by all other feasible options.

The solution concepts presented in this section (and many others) have been widely studied in abstract environment but very little is known on their economic implications. My objective in the next section is to apply these concepts to the model of purely redistributive taxation described in Sect. 2.

4 Simulations

In this section, I present simulations results on the model of Sect. 2. As in all simulations, I need to discretize the feasible set of options. I do this by restricting both tax rates s and t to take values ranging from 0 to 1 by steps of 0.05. Since any combination of (s, t) in this grid is allowed, the feasible set is composed of 441 different tax pairs[7]. This number is sufficiently large to prevent existence of a Condorcet winner in most cases, but is still tractable by modern personal computers.

I performed 80 different samplings of individual production abilities. In each sampling, these abilities are drawn from the same bivariate log-normal distribution, with a correlation coefficient of the underlying normal bivariate distribution of 0.4. Values for the other parameters were chosen to get a ratio of capital income to labor income of roughly one half, as an approximation of the situation that prevails in developed economies. Dispersion parameters were chosen so that the ratio of the median to the average supply be 0.8 for labor and 0.2 for capital. These simulation parameters are those used in Gevers and Jacquemin (1987). The size of the samples was fixed at 999.

As shown in the Appendix, preferred tax rates are decreasing with individual abilities, with mean preferred tax rate being zero. The choice of a positively skewed distribution of abilities, for which the median ability is less than the mean, insures that most individuals prefer taxation to income subsidization. This is important since, by restricting voting to positive values of t and s, I rule out income subsidization schemes.

Figure 2 shows the Dupuit-Laffer surface for a typical sample, where the maximum fiscal revenues roughly correspond to $t = 0.8$ and $s = 0.6$.

As proved in the Appendix, voters preferences on the underlying two-dimension policy space are Euclidean after a slight change of scale, i.e. indif-

[7] Even if this assumption is made essentially for technical reasons, it could be defended more basically, on the empirical ground that political parties platforms rarely state their proposed tax rates up to the third decimal …

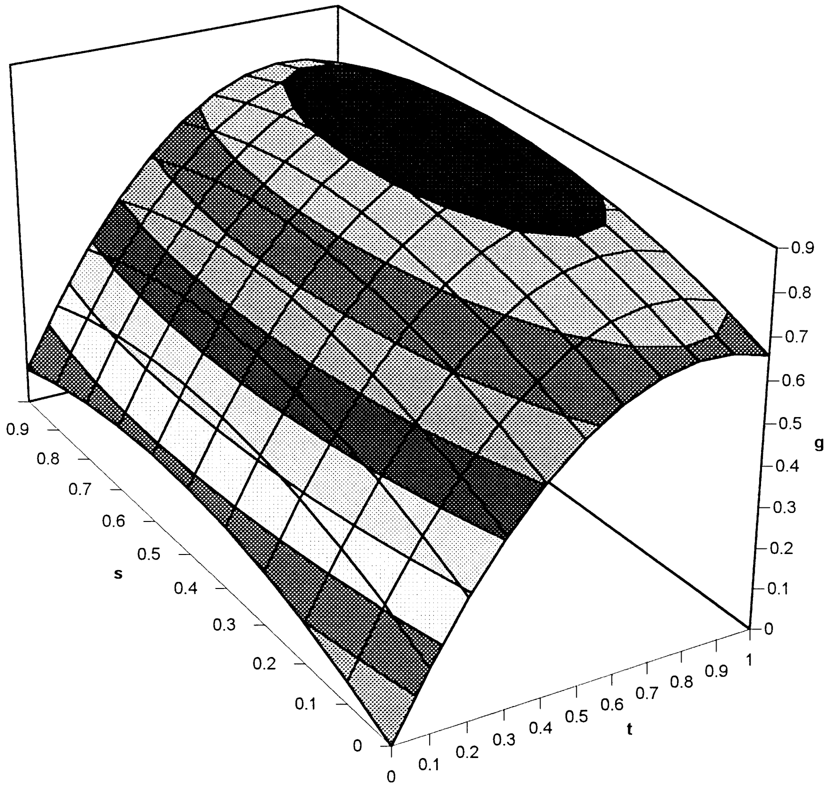

Fig. 2. Dupuit-Laffer surface showing lump-sum subsidy g for tax rates (s, t)

ference curves are concentric circle in the $\left(t, \frac{s}{\sqrt{2}}\right)$ plane. We know from Davis et al. (1972) and Plott (1967) that, if n voter ideal points are located in an Euclidean space of dimensionality d, there is no Condorcet winner except on a set of configurations of measure 0 when $d \geq 3$ or when $d = 2$ and n is odd, which is the case here[8]. My conjecture is that the probability of obtaining a Condorcet winner when feasible options are restricted to a grid imposed on the underlying Euclidean space is very low. This conjecture is corroborated by the simulations, since 79 out of our 80 samplings give no Condorcet winner (for the remaining sample, making the grid finer makes the Condorcet winner disappear).

[8] Tovey (1992) further shows that, for n even, $d = 2$ and for any voter v, v's probability to be the Condorcet winner is $1/2^{n-2}$ when $n - 1$ voters are scattered independently at random according to any non-singular sign-invariant distribution. Thus the probability of obtaining a Condorcet winner in this case is of order $n/2^{n-2}$. See also Schofield and Tovey (1992).

Table 1. Size of the choice sets

	Average	Minimum	Maximum
PA	59.1%	40%	76.2%
UC	4.8%	2.9%	7%
BP	1.8%	1.1%	2.9%
SK	#1	#1	#3

Moreover, the probability for any individual to be indifferent between any two points belonging to the grid is very low, since it corresponds to the probability that the individual's blisspoint be situated exactly at the same distance from these two points. This was confirmed by the simulations where indifferences never occurred. Hence, individuals' preferences on the restricted set of feasible options are strict. This greatly simplifies the computation of the solution sets since it guarantees the equivalence between the weak dominance and covering relations, and also the unicity of the Nash equilibrium in mixed strategies.

In the sequel, I impose the two-party electoral competition that has been described in Sect. 3. I then compute, for each of the 80 samplings, the choice sets of the solution concepts defined in this section[9,10].

In all samples, the uncovered set, the Pareto set, the minmax set and the bipartisan set were connected. The Pareto set and the minmax set were always convex in terms of the underlying Euclidean space (but not the bipartisan set). Table 1 shows, on the 79 samples without a Condorcet winner in the standard grid, the average, the minimum and the maximum size of each solution set (as a proportion of the 441 feasible options, except for the minmax that is so small that the absolute number of options is reported).

The options that do not belong to the Pareto set are on the downward sloping side of the Dupuit-Laffer surface. Starting from such high values of

[9] Here is how I proceeded. For each sampling of individual abilities, I first computed the matrix of net pluralities $NP = [n_{xy} - n_{yx}]_{x,y \in X}$ and then the majority payoff matrix $M = [m_{xy}]_{x,y \in X}$. The uncovered set was computed by eliminating the weakly dominated lines in this matrix M. The bipartisan set was obtained using a linear optimization procedure which consisted of solving the set of inequalities that any mixed strategies equilibrium must obey in two-person constant-sum games (see Laffond et al. 1997). To spare computation time, I computed $BP(UC(X,B),B)$ which, by the Strong Superset Property of the bipartisan set (Laffond et al. (1993), Proposition 4) is equivalent to $BP(X,B)$. Both the minmax set and the Pareto set were easily obtained from the matrix NP.

[10] Gevers and Jacquemin (1987) have studied both the minmax and the vote-maximizing trajectories in this model. Their simulations confirm the minmax stability properties even when preferences are not Euclidean (i.e. in the cases where the supply functions [Eqs. (3) and (4) in the Appendix] are log-linear or quadratic). They also study the comparative statics of the size of the minmax and of the vote-maximizing trajectories when allowing for changes in various parameters of the model.

t and/or s (see Fig. 2), reducing taxation increases both pre-tax incomes and the transfer level, making everybody better off. Pareto dominated options typically represent 40% of the feasible options.

The other three solution concepts are much more selective, the most selective one being the minmax (SK), which on average reduces to a singleton, while the other two concepts select between 1% and 7% of the feasible options.

Comparing the locations of the solution sets, Sect. 3.5 already showed that $BP(X,B) \subseteq UC(X,B) \subseteq PA(X,NP)\ \forall(X,B)$ and that, although always included in PA, SK might have an empty intersection with both BP and (a fortiori) UC. I never obtain such a situation in the simulations, since SK is always included in UC and often (73%) in BP (in 6% of the cases it simply intersects with BP and in 21% of them they have an empty intersection[11]).

To summarize these simulation results, one could say that, although the uncovered set, the minmax set and the bipartisan set are all quite selective, the bipartisan set seems to strike the best balance between size and location. Looking more thoroughly at the bipartisan set, one can not detect a regular pattern between the probability to be played in the equilibrium of any one point in this set and the location of this point inside the set. More precisely, the probability of playing a given point in the "center" of the bipartisan set is not always higher than the probability of playing a point at the boundary of this set. Furthermore, although always connected, the bipartisan set may include "holes", i.e. points with zero probability (not belonging to BP) may be surrounded by points played with strictly positive probabilities (and thus belonging to BP).

Another point of interest resides in the location and sensitivity of the various choice sets with respect to the precise realization of individual productivities. Are the choice sets always localized in the same region of the policy set or are their location very sensitive to small changes in the preference profiles? Figures 3, 4 and 5 attempt to answer these questions for, respectively, UC, SK and BP. They show, for any value of (t,s), the proportion of the 80 samples in which this pair[12] belongs respectively to the uncovered set, the minmax set, and the bipartisan set. For example, Fig. 3 shows that the pair $(t = 15\%, s = 55\%)$, which corresponds to the maximum of the density function, belongs to the uncovered set for 60 out of the 80 samplings. Another way to interpret this Fig. is to say that, when individual characteristics are randomly drawn from a lognormal bivariate distribution, this pair has a 75% probability of belonging to the uncovered set. If the uncovered set was very sensitive to the precise realization of the individual abilities, the density shown in Fig. 3 would be quite flat, with many points sharing the same (quite low)

[11] Even in this last case, the minmax is usually not very distant, the difference being at most 10%, for both tax rates, with the nearest point in the bipartisan set.

[12] Figures 3, 4 and 5 show smooth extrapolations of the 441 feasible tax pairs, because such extrapolations are much easier to read than histograms showing only the proportions at these 441 points.

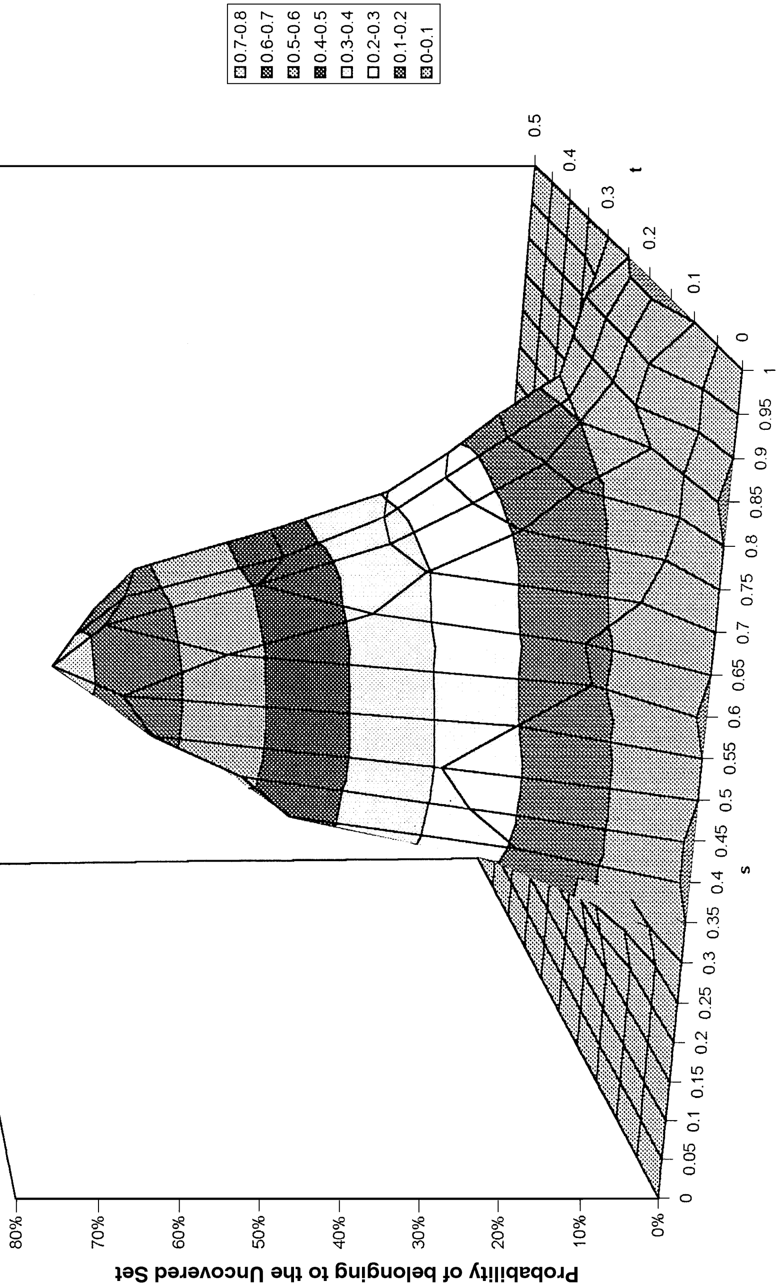

Fig. 3. Empirically inferred probability of belonging to the uncovered set

probability of belonging to the uncovered set. On the contrary, Fig. 3 shows a roughly bell-shaped surface whose summit corresponds to three quarters of the total number of samplings. Thus, the uncovered set does not seem to be very sensitive to the precise samplings and pairs around $(t = 15\%, s = 55\%)$ share a high probability of belonging to the uncovered set when individual characteristics are drawn from a lognormal bivariate distribution.

The same conclusion can be drawn from Figs. 4 and 5 for, respectively, *SK* and *BP*, although the maximum probability reached by any pair is smaller for *BP* (51% for the pair $(0.15, 0.55)$) and especially for *SK* (24% of occurrences for the pair $(0.2, 0.55)$). This smaller probability may be attributed to the smaller size of these two choice sets: indeed, the same shift will have a much greater impact on the composition of a small set than on a big set, leading to lower probabilities of occurrence in the small set. This is especially clear in the case of the minmax set, which usually consists of a singleton: although the probability of occurrence of single points is quite low, the minmax point is included in three quarters of the simulations in the subset $\{0.15 \leq t \leq 0.2, 0.5 \leq s \leq 0.7\}$.

Figures 6 to 9 examine the behavior of the uncovered set with respect to variations in the correlation coefficient between both individual abilities in the sampling. When the correlation between both abilities is perfect, all individuals blisspoints are distributed on a line. Hence, Plott (1967)'s symmetry conditions are respected and, as is shown in Fig. 6, the uncovered set reduces to the Condorcet winner, i.e. the tax pair most preferred by the median-endowed individual. Starting from this situation, the correlation coefficient is slowly decreased to introduce more heterogeneity in individuals' preferences. To isolate the effect of the correlation between abilities, values of the abilities in producing labor are kept unchanged and only the sampling of the other ability parameter is modified according to the correlation coefficients. Figures 7, 8 and 9 show most individuals' blisspoints and the uncovered set for correlation coefficient equal to respectively 0.9, 0.7 and 0.4. While the exact values of the blisspoints are drawn and approximate a continuum[13], the uncovered set is finite in size since it is constrained by the grid[14]. As the figures show, the uncovered set progressively expands around its initial position. This result is clearly in line with Banks, Duggan and Le Breton (1998), who show that the

[13] Figures 6 to 9 only show blisspoints whose two components are greater to -0.2. Since the distribution of abilities is positively skewed, a minority of very well endowed individuals have preferences very much in favor of income subsidization. Since we concentrate on the positive orthant, we have not depicted these blisspoints. All solution concepts have of course been computed taking into account all individuals preferences.

[14] It would be interesting to zoom in on the region containing the uncovered set, to try to fill it out and see if, for example, it is convex or bounded by a convex figure. Unfortunately, the uncovered set does not satisfy the strong superset property, meaning that it is not invariant to the presence of losers. Thus, we can not reduce the mesh of the grid and concentrate on the uncovered set neighborhood: deleting some or the totality of the losers (i.e. points not belonging to the uncovered set) may affect this solution set.

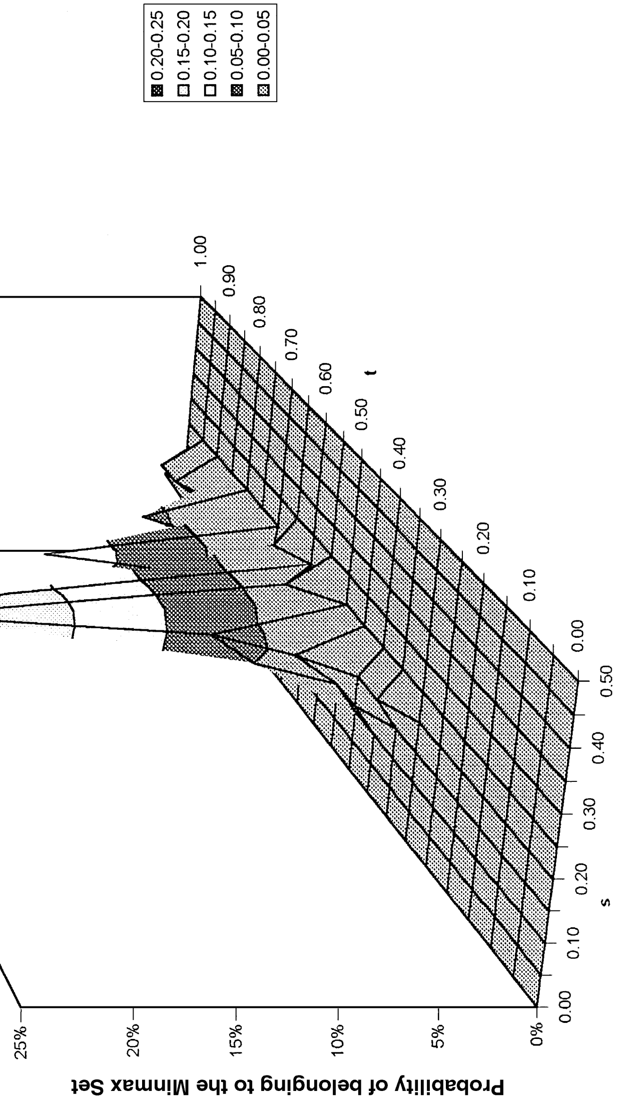

Fig. 4. Empirically inferred probability of belonging to the minmax set

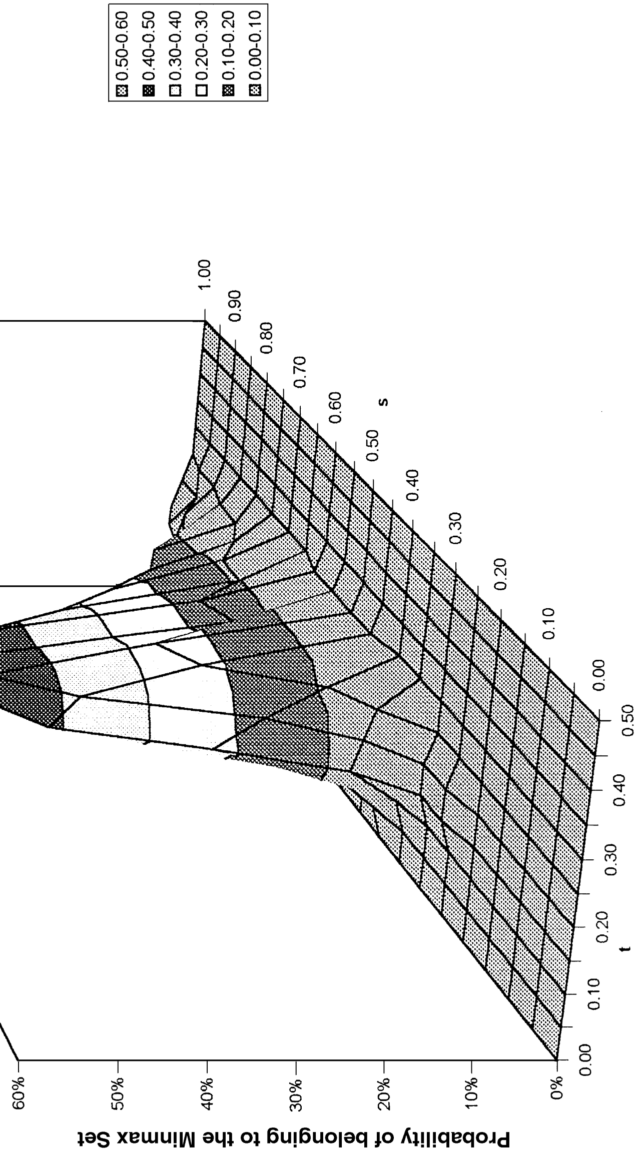

Fig. 5. Empirically inferred probability of belonging to the bipartisan set

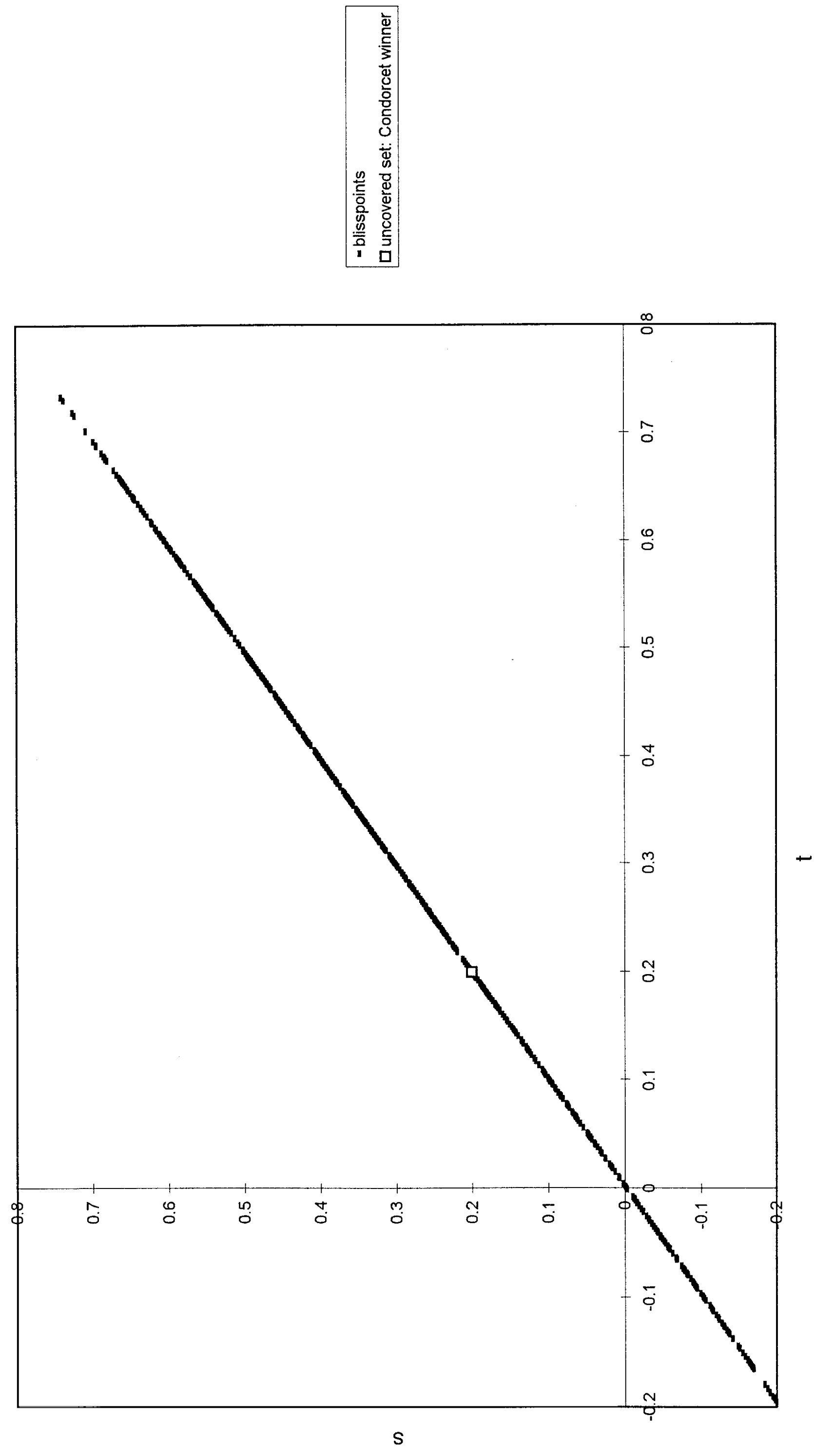

Fig. 6. Blisspoints and uncovered set for perfect correlation of abilities

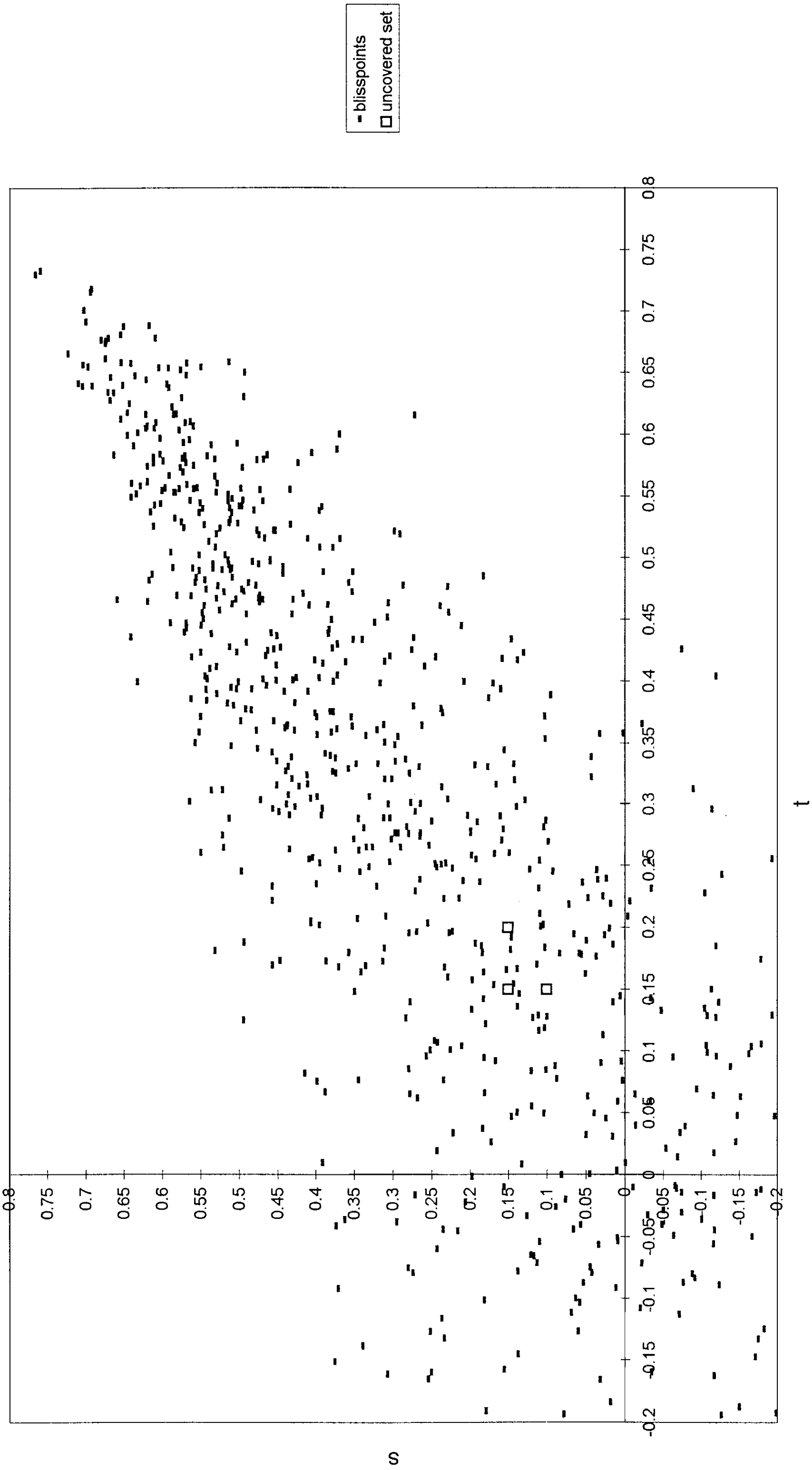

Fig. 7. Blisspoints and uncovered set for correlation coefficient of 0.9 (grid size: 0.05 × 0.05)

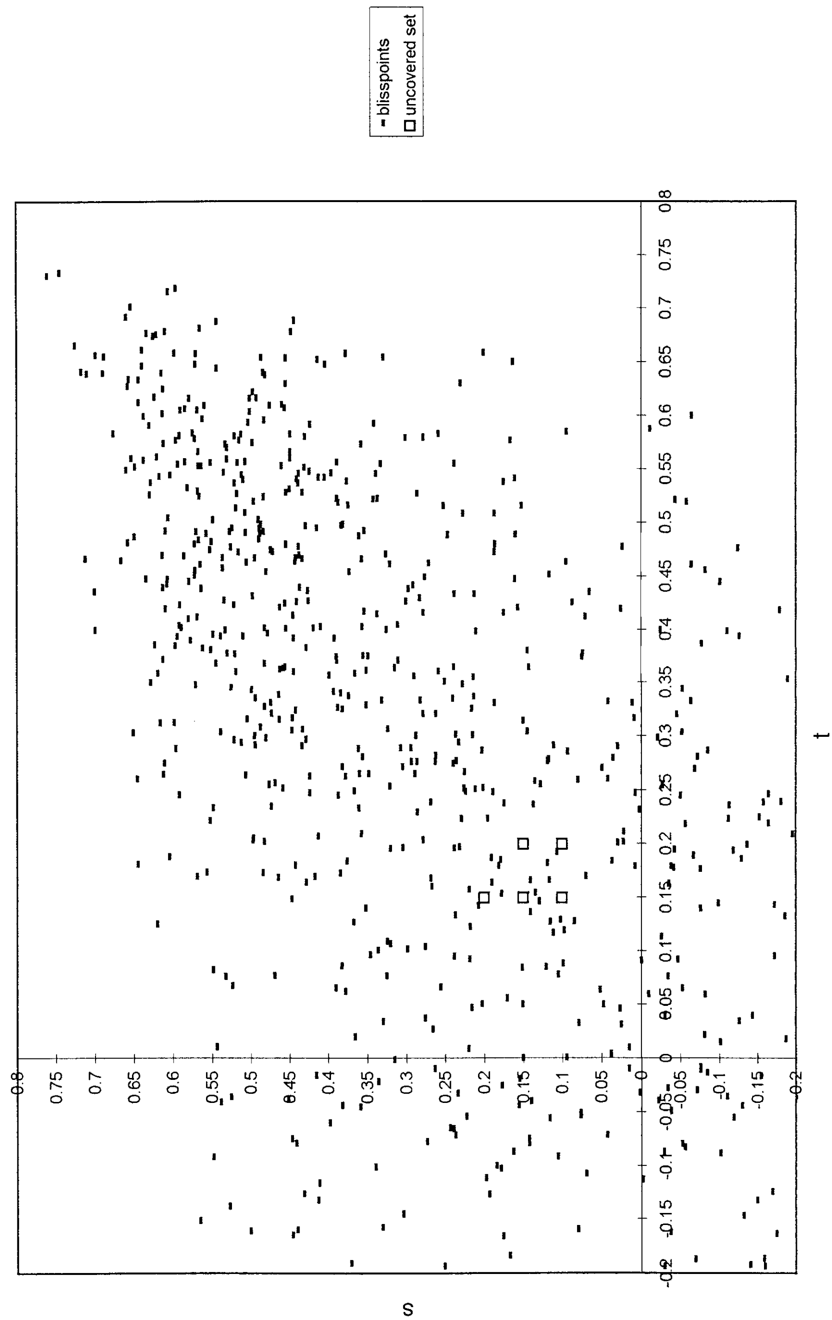

Fig. 8. Blisspoints and uncovered set for correlation coefficient of 0.7 (grid size: 0.05 × 0.05)

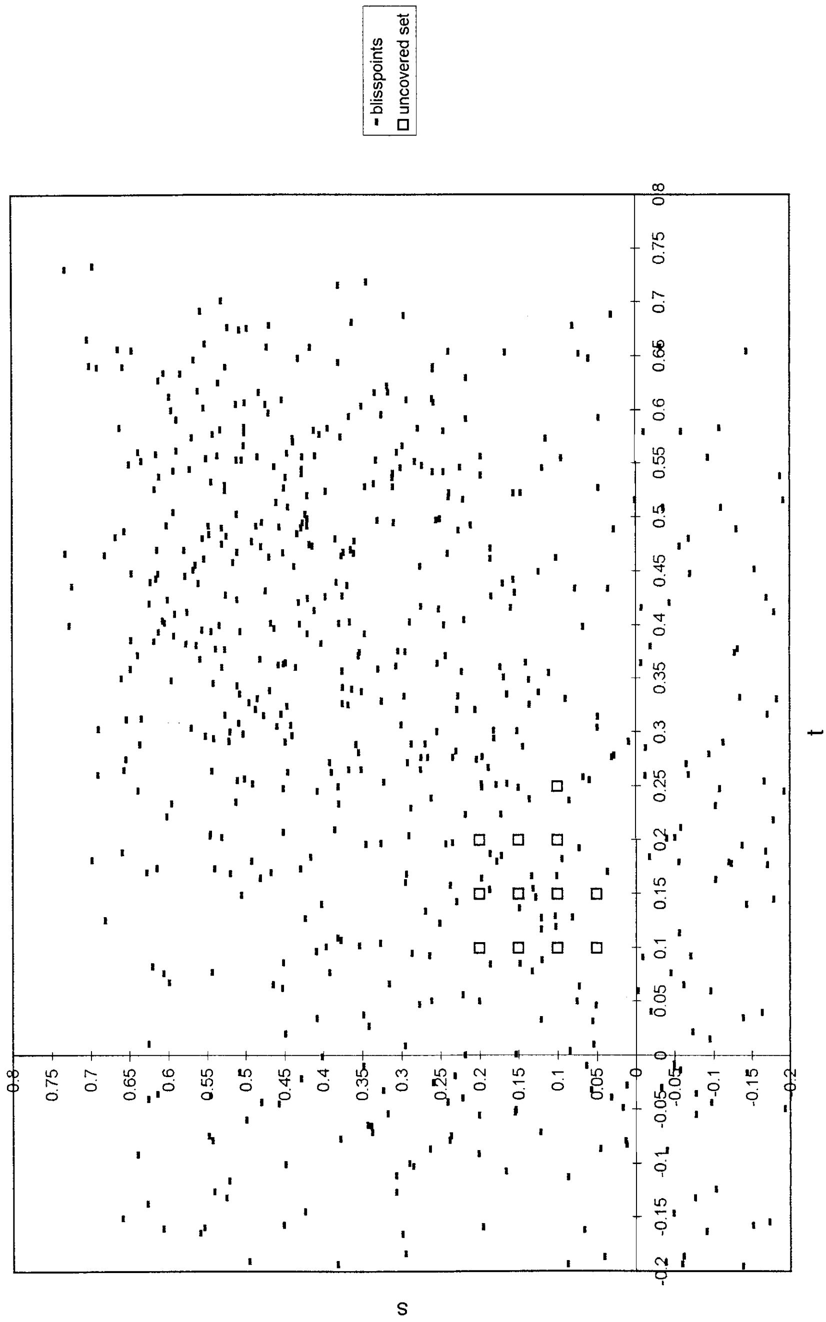

Fig. 9. Blisspoints and uncovered set for correlation coefficient of 0.4 (grid size: 0.05×0.05)

Table 2. Number of elements in the uncovered set and correlation coefficient between abilities

Correlation coefficient	Size of the uncovered set
1 and 0.95	1
0.9	3
0.8 and 0.7	5
0.6	9
0.5	10
0.4	12

UC correspondence is upper-hemicontinuous in a neighborhood of a profile admitting a Condorcet winner. Our simulations do not reject the hypothesis that this property could be verified generally. Table 2 shows the size of the uncovered set for various values of the correlation coefficient.

These results show that, starting from a situation where a generalized median (or Condorcet winner) exists, modifying slightly the individuals preferences does not make the uncovered set explode even if it destroys the existence of this generalized median. Indeed, McKelvey (1986) showed that if indifference contours are concentric circles, the uncovered set is precisely contained in a ball around this center, whose bounds he derived. Moreover, there is experimental support for the proposition that points near the center remain attractive, even in the absence of Condorcet winner (see Ordeshook 1986, Sect. 4.9.). Hence, the uncovered set is consistent with the intuition (corroborated by experimental evidence) that "two-candidate majority-rule elections do not evolve into "chaos" in the likely event that a set of restrictive conditions are not met. Rather, the attractiveness of the centre of public opinion is maintained" (Ordeshook 1986, p. 187).

5 Conclusion

The use of voting theory in economic models is often restricted to the application of the well-known median voter theorem. The assumptions imposed by this theorem are nevertheless quite restrictive: the set of options should be ranked on a single dimension while voters' preferences should be single-peaked on that dimension. In this paper, I relax the one-dimension hypothesis, allowing voters to vote simultaneously on two instruments. In this case, the median voter result vanishes and the general rule is the absence of any majority (or Condorcet) winner.

Solution concepts corresponding to different possible generalizations of the Condorcet intuition have been studied in the theoretical literature. This paper selects three such concepts (the uncovered set, the bipartisan set and the minmax set), that have a positive interpretation as game-theoretical solution concepts applied to variants of the Downsian electoral competition game. These solution concepts have been developed in abstract environments and very little is known on their economic implications. Our objective is then to apply them to a model of purely redistributive taxation, where factor supply is sensitive to net factor rewards.

I have shown that the uncovered set, the minmax set and the bipartisan set are all connected. 40% of the feasible options are Pareto-dominated (i.e. on the downward sloping side of the Dupuit-Laffer surface). The three solution concepts are selective (they select between 1% and 7% of the feasible options) with the minmax being the most selective one, often reducing to a singleton.

By definition, the bipartisan set is included in the uncovered set, itself included in the Pareto set. In theory, the minmax set, even if it is always included in the Pareto set, could have an empty intersection with both the uncovered set and the bipartisan set. However, in the simulations, the minmax set is always included in the uncovered set and in most cases is not very different from the bipartisan set.

Although quite selective, the choice sets of the solution concepts are not too sensitive to slight modifications of preference profiles. This "robustness" property is very interesting if one wants to use these concepts either as normative benchmarks or as positive predictions.

Also, this paper checked the behavior of the uncovered set, starting from a situation where a Condorcet winner exists and then progressively moving away from this situation. The uncovered set slowly expands around the location of the previously existing Condorcet point.

In summary, these results indicate that the uncovered set, the minmax set and the bipartisan set may be useful in voting on multidimensional economic issues, and that they deserve more research in such environments. For example, even if, in this paper, computer limitations were such that the problem had only two dimensions, it would surely be very interesting to study these concepts in higher dimensionality problems.

A Appendix: The redistribution model with linear supply curves

The artisans are indexed by $i = 1 \ldots n$, and the triplet (z_i, y_i, r_i) denotes respectively the (nonnegative) amount of private commodity consumption by individual i and the amounts of capital and of labor that he/she supplies (if positive) or buys from other individuals (if negative). The private good z is taken as the numeraire.

Each individual is equipped with the same continuous utility function

$$U(z_i, y_i, r_i; h_i, q_i) = z_i - \frac{(r_i - h_i)^2}{2} - (y_i - q_i)^2 \qquad (1)$$

where h_i and q_i denote respectively the capital and labor productivities[15] of individual i (the only source of heterogeneity between agents in this model).

[15] One could alternatively view these parameters as disutilities of supplying either input.

Each individual maximizes his/her utility function (1) subject to the constraint

$$z_i = (1 - s)y_i + (1 - t)r_i + g \tag{2}$$

If y_i and/or r_i are negative, they are bought at the after tax unit price, which means that the fiscal authority collects no taxes on household services.

From the maximization problem, I obtain the factor supply functions:

$$r_i = h_i + 1 - t \tag{3}$$

$$y_i = q_i + \frac{1 - s}{2} \tag{4}$$

Replacing these two factor supply equations into the utility function (1) and using equation (2) gives the indirect utility function

$$V(s, t, g; h_i, q_i) = (1 - s)q_i + (1 - t)h_i + \frac{(1 - s)^2}{4} + \frac{(1 - t)^2}{2} + g \tag{5}$$

i.e. the maximum utility that each individual can attain given his/her abilities and the fiscal environment, including the lump-sum subsidy.

Aggregate consumer behavior, as represented by the two factor supply equations, reduces by one the number of degrees of freedom enjoyed by the polity in its choice of the fiscal variables. To show this, substitute these two equations in the fiscal budget balance constraint

$$ng = s\sum_{i=1}^{n} y_i + t\sum_{i=1}^{n} r_i \tag{6}$$

to obtain

$$g = sq + \frac{(1 - s)s}{2} + th + t(1 - t) \tag{7}$$

where

$$q = \textstyle\sum_{i=1}^{n} q_i/n$$

$$h = \textstyle\sum_{i=1}^{n} h_i/n.$$

This gives a formulation of the Dupuit-Laffer surface.

Replacing the lump-sum subsidy g by its value (as given by the Dupuit-Laffer equation) in the indirect utility function (5) yields a reduced indirect utility function, i.e. the end product that will allow us to study the conflict of interest between the voters in the collective choice of the taxation rates:

$$W(s, t; h_i, q_i, h, q) = q_i + s(q - q_i) + h_i + t(h - h_i) + \frac{(1 - s)^2}{4} + \frac{(1 - s)s}{2} + \frac{(1 - t)^2}{2} + t(1 - t) \tag{8}$$

Deriving (8) with respect to s and t, one can obtain the co-ordinates of the voter's preferred tax rates, denoted by t_i and s_i:

$$\begin{aligned} t_i &= h - h_i \\ s_i &= 2(q - q_i) \end{aligned} \tag{9}$$

I can use this to parametrize the reduced indirect utility function with respect to h_i, q_i, s_i and t_i. From (9), it is clear that individuals whose ability exceed the mean one want subsidization of the corresponding factor. Also, one can easily see that the mean preferred tax rates are both zero.

I now show that this formulation is, after a slight change of scale, consistent with Euclidean preferences, i.e. that the contour lines are concentric circles.

Fixing the utility level to an arbitrary value of K, equation (8) can be re-expressed as

$$K - q_i - h_i = \frac{3}{4} + \frac{ss_i}{2} + tt_i - \frac{s^2}{4} - \frac{t^2}{2} \tag{10}$$

Rearranging, I obtain

$$\frac{3}{2} + t_i^2 + \frac{s_i^2}{2} - 2(K - q_i - h_i) = (t - t_i)^2 + \left(\frac{s}{\sqrt{2}} - \frac{s_i}{\sqrt{2}}\right)^2 \tag{11}$$

Since the terms on the left hand side do not vary with s and t, Eq. (11) shows that contour lines are in fact concentric circles in the $\left(t, \frac{s}{\sqrt{2}}\right)$ plane, with centre $\left(t_i, \frac{s_i}{\sqrt{2}}\right)$.

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