Social Choice and Welfare
https://doi.org/10.1007/s00355-024-01541-2

ORIGINAL PAPER

Natural interviewing equilibria in matching settings

Allan Borodin[1] · Joanna Drummond[1] · Kate Larson[2] · Omer Lev[3]

Received: 6 April 2023 / Accepted: 27 June 2024

Abstract

A common assumption in matching markets is that both sides fully know their preferences. However, when there are many participants this may be neither realistic nor feasible. Instead, agents may have some partial (perhaps stochastic) information about alternatives and will invest time and resources to better understand the inherent benefits and tradeoffs of different choices. Using the framework of matching medical residents with hospital programs, we study strategic behaviour by residents in a setting where hospitals maintain a publicly known master list of residents (i.e., all hospitals have an identical ranking of all the residents, for example, based on grades) and residents have to decide with which hospitals to interview, before submitting their preferences to the matching mechanism. We first show the existence of pure strategy equilibrium under very general conditions. We then study the setting when residents' preferences are drawn from a known Mallows distribution. We prove that assortative equilibrium (k top residents interview with k top hospitals, etc.) arises only when residents interview with a small number of programs. Surprisingly, such equilibria (or even weaker notions of assortative interviewing) do not exist when residents can interview with many hospital programs, even when residents' preferences are very similar. Simulations on possible outcome equilibrium indicate that some residents will pursue a reach/safety strategy.

✉ Omer Lev
omerlev@bgu.ac.il

Allan Borodin
bor@cs.toronto.edu

Joanna Drummond
jdrummond@cs.toronto.edu

Kate Larson
kate.larson@uwaterloo.ca

1 Department of Computer Science, University of Toronto, 10 King's College Road, Toronto, ON M5S 3G4, Canada

2 Cheriton School of Computer Science, University of Waterloo, Waterloo N2L 3G1, ON, Canada

3 Department of Industrial Engineering and Management, Ben-Gurion University of the Negev, 1 Ben-Gurion Blvd., Beersheba 8410501, Israel

Published online: 30 August 2024

1 Introduction

Since Gale and Shapley's groundbreaking work (Gale and Shapley 1962), the use of stable matching mechanisms has proliferated across numerous domains. Applications range from matching children to schools to matching refugees to countries (Andersson 2019). A central goal of these mechanisms is to ensure that participants (or agents) have no incentive to try to manipulate the final outcome of the matching process by being strategic about the choices they make and actions they take. However, when deployed in practice, many of the assumptions behind the Gale-Shapley (deferred-acceptance) algorithm no longer hold. For example, agents may have partial preferences or ties (Drummond and Boutilier 2014; Rastegari et al 2013; Irving et al 2009), quotas imposed on matching outcomes (Goto et al 2016), distributional constraints (Kurata et al 2017), and computational constraints, for which compact representations of preferences are useful (e.g., Gelain et al 2009; Pini et al 2014).

One real-world domain where matching mechanisms are implemented is medical residencies (Roth 2002). In many countries, such as Canada and the United States, medical students are assigned to hospital residencies through matching mechanisms. For example, the National Residency Matching Program (NRMP), an American program for matching medical residents to hospitals, in 2023 offered 37,425 positions for first-year residents for 5487 hospital programs (National Resident Matching Program 2023). In this paper, we focus on a problem that arises in medical residency matching settings, where the number of options for residents is large. As noted above, in the NRMP residents need to choose from over 5000 positions, yet they apply to only eleven on average (Anderson et al 2000). Furthermore, residents are often faced with significant uncertainty regarding which hospital may be the best match for themselves. While there are publicly available rankings for hospitals, an individual's preferences will also be influenced by specific, personal considerations (e.g., the personal chemistry with the people in the hospital). One way to address this uncertainty is to *interview* with a set of hospitals; this allows the resident to understand and refine their personal ranking between the possible hospitals. However, this requires the resident to choose the set of hospitals they will interview, based on the limited information they have.

The problem of selecting an appropriate interviewing set gives rise to new strategic concerns and widely used mechanisms—which are strategyproof when assuming every resident initially knows their full ranking of the hospitals—are no longer strategyproof (Haeringer and Klijn 2009; Calsamiglia et al 2010). This observation, that interview-set selection is a strategic decision, motivates our work. Inspired by the medical residencies matching problem, we analyze the Nash equilibrium strategies that arise when residents must select interview sets, knowing that a widely used matching mechanism, Resident-Proposing Deferred Acceptance (RP- DA), will be used. In particular, we examine the possibility of *assortative* equilibria, in which residents are divided into groups, with each group interviewing in the same sets of hospitals.

We provide an example to help illustrate some of the strategic reasoning which arises when residents must select interview sets.

Example 1 Suppose we have 4 hospitals—h_1, h_2, h_3, h_4, and 4 residents—r_1, r_2, r_3, r_4. All hospitals know the residents' quality (r_1 being the best, followed by r_2, then r_3, and r_4 is the worst), and every resident knows their position in the hospitals' ranking. Suppose residents have two possible rankings of hospitals: with probability 0.5 a resident's ranking of hospitals is $h_1 \succ h_2 \succ h_3 \succ h_4$, and with probability 0.5, it is $h_2 \succ h_1 \succ h_4 \succ h_3$. Assume that residents can only interview with at most 2 hospitals.

Residents r_1 and r_2 can choose to interview at h_1 and h_2, while residents r_3 and r_4 can choose to interview at hospitals h_3 and h_4. Such a choice is both assortative and stable—both r_1 and r_2 know they will never prefer h_3 and h_4 over the hospitals they interview with; and because of this, both r_3 and r_4 know hospitals h_1 and h_2 will surely be taken already by the time it is their turn to interview, so no point in interviewing there. In this case, there is no other equilibrium.

If the probability of any ordering is as likely as any other, then many other interviewing strategies are stable, including non-assortative ones. For example, r_1 and r_3 interviewing at h_2 and h_4 while r_2 and r_4 interview at h_1 and h_3.

1.1 Our contributions

Under the assumption that hospitals maintain a common master list over residents (e.g. GPAs, exam results, etc. (Hafalir 2008; Zhu 2014; Chen and Pereyra 2015; Ajayi 2011)), we explore the structure of Nash equilibrium when residents are required to select k hospitals with which to interview (and, thus, rank for the matching mechanism). We show the following:

- A pure strategy Nash equilibrium exists for this game.
- Using the Mallow's model for sampling resident preferences, an *assortative* equilibrium exists for very small interviewing sets. This equilibrium is "natural" in that hospitals and residents are stratified: highly ranked residents interview at well-regarded hospitals; medium residents interview at medium hospitals; and low-ranked residents interview at low-ranked hospitals. Indeed, we initially believed this would be the natural equilibria in most cases (as Ajayi 2011 seemed to indicate).
- However, in the Mallow's model, if residents use larger interview sets, assortative strategies no longer form an equilibrium. That said, new equilibria appear, where residents select interview sets that contain both "reach" and "safety" alternatives.

We note that throughout the paper we use the terminology of hospitals and residents, but emphasize that this is merely for clarity and consistency. Our results hold for any setting in which one side cannot provide a full ranking of the other, and must decide how to focus its attention so as to learn more about particular alternatives. Such scenarios could include students interviewing at schools, universities, and recruiting faculty candidates, among others.

2 Related research

While there is a rich literature on matching markets and stability (e.g. Gusfield and Irving 1989), the importance and impact of interviews in these markets is not as well understood. In recent work (Echenique et al 2022), through analysis of the NRMP, observed that most doctors match with one of their most preferred internship programs, despite having very similar preferences. They argued that this apparent contradiction is an artifact of the interview process that precedes the match. These findings highlight the importance of better understanding market interactions, including interviews, that happen before (and after) the actual market. A similar lesson can be taken from Harless and Manjunath (2018) work that studied the impact of the allocation rule (e.g. matching mechanism) on the interviewing process, illustrating how these two steps influence each other. Indeed, several works on interviewing—using various models with varying similarity to ours—have examined the equilibria that arises when assuming the existence of interviews.

One thread of research has studied interviewing policies that aim to minimize the total number of interviews conducted while also ensuring stability in the final match. For example, Rastegari et al (2013) showed that while finding the minimal interviewing policy is NP-hard in general, there are special cases where a polynomial-time algorithm exists, while Drummond and Boutilier (2014) approached the problem using the framework of minimax regret and proposed heuristic approaches for interviewing policies. These papers assume that interview policies are implemented centrally, ignoring the situation where agents may choose with whom to interview, whereas our work explicitly studies strategic issues arising from situations where agents choose their interview strategies.

There is a body of literature that addresses strategic interviewing in matching markets, but many of the papers ask different questions or make different modelling assumptions than we do in our work. Manjunath and Morill (2023) studied the problem of "interview hoarding" where one side of the market (e.g. residents) can interview as many candidates as they wish, while the other side (e.g. hospitals) are limited in the number of interviews they may conduct. The authors show that this leads to problematic outcomes compared to settings where interviews are limited on both sides. In particular, no resident that would have been matched in the setting with limited interviews is better off in the unlimited setting and many residents are worse off. This is consistent with Kadam (2015) findings that relaxing residents' interview constraints can adversely impact lower-ranked residents. These results support our modelling choice of restricting the size of agents' interview sets. We note, however, that research has also shown that limiting the size of agents' interview sets may have strategic implications. For example, in two related papers, Haeringer and Klijn (2009) and Calsamiglia et al (2010) show that limiting the number of interviews an agent may partake in can lead to less stability in the market and encourage agents to misreport their preferences, while He and Magnac (2017) showed, empirically, that imposing an interviewing cost may lead to decreased match quality. Because of these findings, we make no claim that we are using "optimal" interview set sizes, and we consider the question of the size of interview sets to be outside the scope of this paper.

Several other authors have also studied matching markets with limited/fixed interview sets (Immorlica and Mahdian 2005; Beyhaghi et al 2017; Beyhaghi and Tardos 2019). Unlike our work, these papers typically assume uncorrelated preferences (i.e. every hospital independently ranks residents idiosyncratically), allow for a fixed probability of selection (Immorlica and Mahdian 2005), or uniform preferences within subgroups of residents/hospitals (Beyhaghi et al 2017). We believe that these modelling assumptions are overly strong and empirical evidence indicates that preferences are correlated (e.g. Echenique et al 2022), which we try to capture in our preference models.

We are particularly interested in what we call "natural" interviewing equilibria. These equilibria are assortative in that top residents interview with (and are matched with) top hospitals while bottom-ranked residents interview with (and are matched with) bottom-ranked hospitals etc. Lee (2017) showed the existence of such equilibria in matching markets. Their results, however, relied on several strong assumptions including a large market assumption and strong restrictions on the preference models. We are interested in understanding whether it is possible to support assortative equilibria under broader assumptions. Other papers (e.g. Chade and Smith 2006; Chade et al 2014; Ali and Shorrer 2023), motivated by the college-selection problem, have also studied the structure of the resulting equilibria. While Chade and Smith (2006) showed that students would greedily select which colleges to interview with under the assumption that admissions prospects across colleges were stochastically independent, Ali and Shorrer (2023) argued that changes in the underlying model, namely allowing for correlations, results in equilibrium outcomes where students apply for both "reach" and "safety" colleges. While the problem we study is very different since we assume a centralized matching market while these papers study a decentralized process, we find these papers relevant and informative as they hint at the importance of correlated preferences for students/residents in whether assortative or "reach and safety" strategies form equilibria.

Finally, we mention the work of Lee and Schwarz (2017). They studied a multi-stage worker-firm game where one side of the market (e.g. firms) had to first, at some fixed cost, select workers with which to interview before proceeding to a centralized matching market running (firm-proposing) deferred acceptance. Their key finding was if there was no coordination, then all firms were best off each picking k random workers to interview. However, if firms could coordinate then it was best for them to each select k workers so that there was perfect overlap (forming a set of disconnected complete bipartite interviewing subgraphs), i.e., an assortative equilibirum. This finding, while very elegant, relies heavily on the assumption that all firms and workers are *ex-ante* homogeneous, with agents' revealed preferences being idiosyncratic and independent. In particular, for the results to hold either agents have effectively no information about their preferences before they interview, or the market must be perfectly decomposable into homogeneous sub-markets that are known before the interviewing process starts In this paper, we study a similar, multi-stage game, but we relax these assumptions. Instead, we assume that agents' preferences are correlated and make no assumptions about the decomposability of the market into sub-markets.[1] We are interested in understanding and characterizing the resulting equilibrium outcomes, providing insights into how sensitive so called "natural" equilibria are to the underlying preference structures of the agents.

[1] Lee and Schwarz (2017) model includes interview costs and variable interviewing sizes, which our model avoids for simplicity. Adding them is discussed in the conclusion.

3 An interviewing game with limited interviews

Using the resident-hospital matching problem as our basic framework, we assume there is a set, $R = \{r_1, \ldots, r_n\}$ of residents and a set, $H = \{h_1, \ldots, h_n\}$ of hospitals.[2] Every $h_i \in H$ has (strict) preferences over R, and every $r_i \in R$ has (strict) preferences over H. These are represented by $H_\succ = \{\succ_{h_1}, \ldots, \succ_{h_n}\}$ and $R_\succ = \{\succ_{r_1}, \ldots, \succ_{r_n}\}$ respectively.

We are interested in *one-to-one matchings*; residents can only do their residency at a single hospital, and hospital programs can accept at most one resident. A *matching* is a 1–1 function $\mu : R \cup H \rightarrow R \cup H$, such that $\forall r \in R$, $\mu(r) \in H \cup \{r\}$, and $\forall h \in H$, $\mu(h) \in R \cup \{h\}$. If $\mu(r) = r$ or $\mu(h) = h$ then we say that r or h is unmatched. We assume that residents prefer to be assigned to any hospital over not being matched, and hospitals prefer to have any resident over not filling the position. A matching μ is *stable* if there does not exist some $(r, h) \in R \times H$, such that $h \succ_r \mu(r)$ and $r \succ_h \mu(h)$.

Critically, we assume the existence of a *master list*, $\succ_{ML}$, over residents, which is shared by all $h \in H$, such that $\succ_h = \succ_{ML}$. This assumption implies that all hospitals share identical preferences over residents. This captures scenarios such as when grades or GPAs are used to rank residents, or when residents are required to write standardized exams (Irving et al 2008; Chen and Pereyra 2015; Zhu 2014)). Without loss of generality, $r_i \succ_{ML} r_{i+1}, \forall i < n$. We further assume that every $r \in R$ is aware of their ranking on this master list.

Residents, on the other hand, have idiosyncratic preferences over hospitals. This may be based on, for example, location, potential colleagues, career opportunities for partners, *etc*. In particular, we assume there is some underlying, commonly known, preference distribution, D, from which each $r \in R$ draws $\succ_r$ independently. If resident r draws preference ranking η from D, then $h_i \succ_\eta h_j$ means that h_i is preferred to h_j by r under η.

Critical to our model is the assumption that residents do not initially know their true preferences, but *refine* their information by conducting *interviews* with hospitals. If a resident was able to interview every hospital in H then they would know their true preferences. However, this is infeasible and instead, each resident has an *interviewing budget* of $k < n$ hospitals. Let $I(r) \subseteq H$, $|I(r)| \leq k$ be resident r's interview set, consisting of the set of hospitals r has selected to interview. Once the interviews are completed, r knows its preferences over $h \in I(r)$, though does not necessarily have any additional information over $h' \in H \setminus I(r)$.

Once all residents have interviewed with their selected hospitals, they enter into a matching process, using the preference information obtained through their interviews. In this paper, we use the standard resident-proposing deferred acceptance (RP- DA). The resulting matching, μ, is guaranteed to be stable, resident-optimal, and hospital-pessimal (Gale and Shapley 1962). This stable matching is also guaranteed to be unique, as stable matching problems with master lists have unique stable solutions (Irving et al 2008). Thus our results directly hold for any mechanism that returns a stable

[2] The assumption that there are an equal number of residents and hospitals is without loss of generality. If there are more residents than hospitals, then the lowest ranked residents will not obtain any interviews and can therefore be ignored. If there are more hospitals than residents, we can add "dummy" residents having the lowest ranks and the matching mechanism can ignore the match of any dummy resident.

matching, including hospital-proposing deferred acceptance and the greedy linear-time algorithm (Irving et al 2008).

We summarize our model and assumptions for the interviewing game in which residents engage. We call this game the **Interviewing Game with Limited Quotas**, or **ILQ** .

- Each $r \in R$ and $h \in H$ is informed of the master list $\succ_{ML}$.
- Each resident $r \in R$ simultaneously selects an interviewing set $I(r) \subset H, |I(r)| \leq k$.
- Each resident $r \in R$ interviews with hospitals in $I(r)$ and learn their own preference, $\succ_{r|I(r)}$ over members of $I(r)$.
- A central matching system runs resident-proposing deferred acceptance (RP- DA) using $\succ_{ML}$ as the preference for all $h \in H$, and $\succ_{r|I(r)}$ for all $r \in R$. Any hospital $h \notin I(r)$ is reported to be unacceptable by r.

3.1 Utility functions for the interviewing game

We require a clear specification of the residents' utility functions, as this supports the choices they make when deciding which hospitals to interview. Thus, in this subsection, we describe how we derive principled utility functions for the residents, based on their preferences and the expected match.

We first assume that residents share some common scoring function, $v : H \times H_{\succ} \mapsto \mathbb{R}$ such that for any ranking over hospitals, $\eta \in H_{\succ}$, $h_i \succ_{\eta} h_j$ if and only if $v(h_i, \eta) > v(h_j, \eta)$. The existence of such a scoring function is used in other literature (e.g. Coles and Shorrer 2014) and allows for flexibility in the modelling of the problem. For now, we merely assume the existence of such a scoring function and will explore different instantiations later.

Second, we make the critical observation that a resident, $r \in R$, need only be concerned about other residents that are higher ranked in the master list, $\succ_{ML}$. If a lower ranked resident, r', is matching to some $h \in I(r)$, then it must be the case the r is matched to some h' such that $h' \succ_r h$ since otherwise the matching would be unstable.

This greatly simplifies the formulation of the utility function for a resident as we need only consider the interview sets of higher-ranked residents, its own choice with whom to interview, and the probability with which it has a particular preference ranking over hospitals.

We introduce notation to help support the development of the utility function for a resident. Consider some resident, r_j, and interview sets, $I(r_1), \ldots, I(r_{j-1})$, for all $r_i \succ_{ML} r_j$. Furthermore, define $m = \mu_{|r_1,\ldots,r_{j-1}}$ to be the partial matching that arises when RP- DC is run. The set of preferences that are consistent with this partial match is

$$\begin{aligned} T(r_j, m) =& \{\xi \in H_{\succ} | \exists I(r_j) \text{ s.t. } \forall h' \in I(r_j) \text{ s.t. } h' \succ_{\xi} m(r_j), \\ & \exists r_a \text{ s.t. } r_a \succ_{ML} r_j \wedge m(r_a) = h'\}. \end{aligned}$$

Observe that $T(r, m) \neq \emptyset$ for all $r \in R$, and that $T(r_1, m) = H_{\succ}$.

Given some preference distribution D, the probability that some particular partial match, m', arises, given interview sets $I(r_1), \ldots, I(r_{j-1})$ is simply the probability that the residents had preferences consistent with $T(r_j, m')$:

$$P(m'|(I(r_i))_{i=1}^{j-1}) = \prod_{i=1}^{j-1} \sum_{\xi \in T(r_i, m)} P(\xi|D).$$

Now resident r_j must determine the probability with which it will be matched to a particular hospital, h, since its utility is determined by how it perceives the program quality. We define

$$M^*(I(r_j), (I(r_i))_{i=1}^{j-1}, \eta, h) = \{m|m(r_j) = h; \forall r_i \in \{r_1, \ldots, r_{j-1}\}, m(r_i) \in I(r_i); \\ \forall x \in I(r_j),\ \text{if } x \succ_\eta h, \exists r_i \text{ s.t. } x \in I(r_i) \text{ and } m(r_i) = x\}$$

to be the set of (partial) matches where r_j is matched to hospital h, given interview sets for residents $r_1, \ldots, r_{j-1}$, interview set $I(r_j)$ for resident r_j with preference ranking η. Since the preference rankings of residents r_l such that $r_j \succ_{ML} r_l$ do not change what hospital r_j is matched to, for any complete matching, μ, we have

$$P(\mu(r_j) = h|\eta, I(r_j), (I(r_i))_{i=1}^{j-1}) = \sum_{m \in M^*(I(r_j), (I(r_i))_{i=1}^{j-1}, \eta, h)} P(m'|(I(r_i))_{i=1}^{j-1}).$$

Bringing everything together, the utility function for resident r_j, given its interview set, $I(r_j)$ is

$$u_{r_j}(I(r_j)) = \sum_{h \in I(r_j)} \sum_{\eta \in H_\succ} v(h, \eta) P(\eta|D) P(\mu(r_j) = h|\eta, I(r_j), (I(r_i))_{i=1}^{j-1}). \quad (1)$$

This utility function has an intuitive interpretation: it weights the value of a hospital by how likely the resident will be matched to it, given the interview-set choices of "more desirable" residents.

4 Equilibria analysis: general results

We start our analysis by studying the most general form of the **ILQ** game possible. Recall that an **ILQ** game is defined as $\Psi = \langle n, k, D, v \rangle$ where $n = |R| = |H|$, k is the number of interviews any resident can conduct (also known as the quota), D is the underlying distribution from which residents' preferences are being drawn, and v is the scoring function over hospitals that residents use. We start by placing no restrictions on the structure of the underlying preference rankings of the residents, nor do we place any constraints on their utility functions. Furthermore, to simplify notation we drop n from the **ILQ** notation unless it influences the results. We start by presenting an existence result, namely the existence of a pure strategy equilibrium for this game.[3] We follow this by outlining general conditions under which this equilibrium might take a particularly appealing form, namely assortative interviewing. We then instantiate the residents' preference models using a common probabilistic model for preferences (the ϕ-Mallows model) and explore how this class of preference rankings support assortative interviewing.

Theorem 1 *Given any **ILQ** game $\Psi = \langle k, D, v\rangle$ with $k > 0$, there exists a pure strategy equilibrium.*

Proof We wish to show that if every resident chooses their expected utility-maximizing interviewing set, this results in an equilibrium. Given any resident r_j who is jth in the hospitals' rank- ordered list, r_j's expected payoff function only depends on residents $r_1, \ldots, r_{j-1}$. As r_j knows that each other resident r_i is drawing from distribution D *i.i.d.*, they can calculate $r_1, \ldots, r_{j-1}$'s expected utility maximizing interview set, using Eq. 1. Their payoff function depends only on D and $I(r_1), \ldots, I(r_{j-1})$, all of which they now have. They then calculate the expected payoff for each $\binom{n}{k}$ potential interviewing sets, and interview with the one that maximizes their expected utility. Of course, when there are ties between the expected payoff of different strategies, multiple equilibria may arise. □

Theorem 1 is an existence theorem. It does not provide any additional insight into the equilibrium behavior, nor does it provide any insight as to how this equilibria may be computed beyond a brute-force approach. This leads us to our next set of questions, namely under what conditions will a particular class of natural interviewing strategies form an equilibrium. We are particularly interested in *assortative interviewing strategies*.

Definition 1 Given **ILQ** game $\Psi = \langle k, D, v\rangle$ with $k > 0$, an interviewing strategy profile is *assortative* if and only if for $j = 0, 1, 2, \ldots, \frac{n}{k} - 1$, each resident $r \in \{r_{jk+1}, \ldots, r_{jk+k}\}$ chooses to interview with the set of k hospitals $\{h_{jk+1}, \ldots, h_{jk+k}\}$.[4]

We view assortative strategies as being "natural" in that hospitals and residents are stratified: highly-ranked residents on the master list interview at well-regarded hospitals; mid-ranked residents interview at what they expect to be mid-ranked hospitals; and low-ranked residents interview at low-ranked hospitals. We start by deriving conditions that ensure assortative interviewing. We show that there exist scenarios in which one need only focus on the behavior of a single agent, namely r_k where k is the interviewing budget. If assortative interviewing is a best response for resident r_k when all other residents $i < k$ interview assortatively, then assortative interviewing is a best response for *every* resident r_i $(i < k)$ when all other residents interview assortatively. In other words, determining if assortative interviewing is a best response

[3] Independently, Kadam (2015) provides a proof of a pure equilibrium similar to ours, despite the differing models, as both proofs rely on the serial structure of the deferred-acceptance algorithm.

[4] We are assuming for convenience that k divides n. When k does not divide n, there will be some remaining $k' < k$ residents that will interview with the remaining k' hospitals: $h_{\lfloor \frac{n}{k} \rfloor k+1}, \ldots, h_n$.

for r_k is sufficient to show that assortative interviewing is a best response for the first k residents (and is thus an equilibrium for them in this game).

Theorem 2 *Let $\Psi = \langle k, D, v\rangle$ be an **ILQ** game with quota k, preference distribution D, and resident scoring function, v. Assume residents $r_1, \ldots, r_{k-1}$ all interview assortatively. Then, if resident r_k's best response is to interview assortatively under this setting, it is a best response for any resident $r_1, \ldots, r_k$ to interview assortatively. Moreover, this forms a unique best-response for $r_1, \ldots, r_k$.*

Proof We introduce an indicator function to simplify notation for when a hospital is a resident's top available choice. For any hospital h and agent i, let $b^j(h, \eta) = 1$ if and only if h is available when r_j makes their choice (i.e., $r_1, \ldots, r_{j-1}$ have not been allocated h), and is their most-desirable available alternative (i.e., $h \succ_\eta h'$ for all other h' available); and 0 otherwise. Directly following from Eq. 1 the utility of resident r_j when interviewing with hospitals $S \subset H$ is:

$$u_{r_j}(S) = \sum_{h\in S}\sum_{\eta\in H_\succ} v(h,\eta)P(\eta, D)b^j(h,\eta)$$

Since for r_1, it is always true that $b^1(h, \eta) = 1$ for any desired h (since r_1 goes first, no $h \in H$ has been allocated by another $r \in R$), suppose it will interview in a set of k hospitals $\{h_1, \ldots, h_k\}$ (the numbering according to r_1's choices as determined by the distribution D). We are concerned with the best response strategy of r_k which only depends on the strategies of r_i for $i < k$. Suppose there is no assortative equilibrium, and let $r_i, i < k$, be the resident with the lowest index for which it is better off interviewing in set $S' \neq \{h_1, \ldots, h_k\}$. Then $b^i(h, \eta) \geq b^k(h, \eta)$, with the inequality being strict for some $h \in \{h_1, \ldots, h_k\}$. Note that for any $h \notin \{h_1, \ldots, h_k\}$, $b^i(h, \eta) = 1$.

Hence, if $u_{r_i}(\{h_1, \ldots, h_k\}) < u_{r_i}(S')$, this means if all agents $r_1, \ldots, r_{k-1}$ are being assortative (so $b^k(h, \eta) = 1 = b^i(h, \eta)$ for $h \in S' \backslash \{h_1, \ldots, h_k\}$), $u_{r_k}(\{h_1, \ldots, h_k\}) < u_{r_k}(S')$. That is, if it is not beneficial for r_i to be assortative, it would not be beneficial for r_k to be assortative if $r_1, \ldots, r_{k-1}$ are assortative.

Note that, as all these players have a strictly dominant strategy, this is a unique equilibrium for this game. □

4.1 Interviewing equilibria when preferences are drawn from the Mallows model

While Theorems 1 and 2 hold for general preferences, we are interested in understanding the impact that the underlying preference model has on the strategic choices of the residents. To this end, we investigate the strategic behaviour that arises when residents' preferences are drawn from the ϕ-Mallows model (Mallows 1957), a probabilistic ranking model that is standardly used for modelling preferences and has been used in previous investigations of preference elicitation schemes for stable matching problems (Drummond and Boutilier 2013, 2014; Brilliantova and Hosseini 2022; Freeman et al 2021). Its particular relevance in our setting is that residents often have a vague ranking of hospitals, based on a common list (e.g., US News Ranking of hospitals)—the Mallows reference ranking—but in practice, their personal preferences may be a noisy variant of it.

4.1.1 The Mallows model

The ϕ-Mallows model (or just Mallows model Mallows (1957)), $D^{\phi,\sigma}$, is a distance-based probabilistic ranking model, characterized by a reference ranking σ, and a dispersion parameter $\phi \in (0, 1]$. Given the parameters, σ and ϕ, the probability of any given ranking η is:

$$P(\eta|D^{\phi,\sigma}) = \frac{\phi^{d(\eta,\sigma)}}{Z}$$

where $d(\eta, \sigma)$ is the Kendall-τ distance metric that counts the number of pairwise disagreements between the two rankings, η and ϕ, and Z is a normalization factor; $Z = \sum_{\eta \in A_{\succ}} \phi^{d(\eta,\sigma)} = (1)(1 + \phi)(1 + \phi + \phi^2) \dots (1 + \cdots + \phi^{|A|-1})$ (Lu and Boutilier 2011). The parameter ϕ controls the likelihood of drawing a ranking that is significantly different from the reference ranking, σ. As $\phi \to 0$, the probability of drawing the reference ranking approaches 1.0, while as $\phi \to 1$, the Mallows distribution is equivalent to drawing a ranking from the uniform distribution.

One interpretation of the Mallows model has rankings being generated by inserting alternatives into a ranking, where the insertion point is a function of ϕ. Because of this, when comparing only a small subset of alternatives in the ranking, the probability that any two of the alternatives of interest are in a specific order may not depend on the total number of alternatives. Furthermore, it is possible to determine the probability that any given alternative will be inserted in a particular position in a ranking simply by computing the probability it will be inserted in that position after all other alternatives have been ranked. We will use these properties in our analysis and so state them here and include the proofs in Appendix A for completeness.

Lemma 1 *Given some Mallows model $D^{\phi,\sigma}$ with a fixed dispersion parameter ϕ and reference ranking σ ordering n agents, in which $a_i \succ a_j$ ($1 \leq i, j \leq n$), the probability that a ranking η is drawn from $D^{\phi,\sigma}$ such that $a_i \succ_\eta a_j$ is equal to drawing from some distribution $D^{\phi,\sigma'}$ where σ is a suffix or prefix of σ' (that is, there is σ, an ordering of n agents, and σ', an ordering of n' agents ($n' > n$), and σ' can be divided into σ, an ordering of the first/last n agents, and an ordering of the last/first $n' - n$ agents).*

Corollary 1 *Given any reference ranking σ and two adjacent alternatives in σ: a_i, a_{i+1},*

$$P(a_i \succ a_{i+1}|D^{\phi,\sigma}) = \frac{1}{1 + \phi}.$$

We extend Corollary 1 to include three consecutive items.

Corollary 2 *Given any reference ranking σ and alternatives a_i, a_{i+1}, a_{i+2} and some $\eta \in \{a_i, a_{i+1}, a_{i+2}\}_{\succ}$, the probability that some ranking β is drawn from $D^{\phi,\sigma}$ that is consistent with η is:*

$$P(\beta|D^{\phi,\sigma}) = \frac{\phi^{d(\eta, a_i \succ a_{i+1} \succ a_{i+2})}}{(1+\phi)(1+\phi+\phi^2)}$$

It is useful to know the probability that any one alternative will be in any particular position in a rank ordered list. We show that this is effectively equivalent to ordering all other alternatives, and then calculating the probability that we can put the alternative in question in its desired slot.

Lemma 2 *The probability that a_1 will be ranked in place j is $\frac{\phi^{j-1}}{1+\phi+\cdots+\phi^{n-1}}$. Furthermore, the probability that a_n will be ranked in place j is $\frac{\phi^{n-j}}{1+\phi+\cdots+\phi^{n-1}}$. Similarly, the probability a_j will be ranked in first place is $\frac{\phi^{j-1}}{1+\phi+\cdots+\phi^{n-1}}$.*

It is possible to bound the probability that any two alternatives will be "out of order" in any given ranking;

Lemma 3 *Let $\eta \in D^{\phi,\sigma}$ be such that $a_j \succ_\eta a_i$ for some $i < j$, then $P(\eta) < \frac{\phi^{j-i}}{Z}$.*

Finally, we include an observation that follows from the definition of the Mallows' model:

Observation 1 *If $|j-i| > |j-i'|$, probability a_i is in place j is smaller than probability $a_{i'}$ is in place j. Similarly, probability a_j is in place i is smaller than probability a_j is in place i'.*

4.1.2 Equilibrium analysis

We now study the equilibria that arise in the interviewing game when residents' preferences are drawn from some underlying ϕ-Mallows model. This allows us to control and, thus, better understand, how diversity of residents' preferences influences the structure of the underlying interviewing equilibrium. For ease of notation, let $\Psi = \langle k, \phi, v \rangle$ be an instance of an **ILQ** game with interview quota k, a Mallows model with dispersion parameter ϕ, and a scoring function v.

We start by considering the class of games where $\phi = 0.0$, namely $\Psi = \langle k, 0.0, v \rangle$. Recall that as $\phi \to 0$, the probability of drawing the reference ranking σ goes to 1. This means that all residents have common preferences, namely the reference ranking which we define as σ such that $h_i \succ h_{i+1}$ for all $1 \leq i < n$. It is straightforward to see that any strategy profile such that each resident r_i interviews with hospital h_i forms an equilibrium. Thus, trivially, assortative interviewing is an equilibrium as well.

We now consider the general case, $\Psi = \langle k, \phi, v \rangle$, where no restrictions are placed on any of the three parameters. We observe that if resident r_k can not improve its expected utility by interviewing with hospital h_{k+1} instead of any of the hospitals in $\{h_1, \ldots, h_k\}$, then in general the best thing resident r_k can do is set its interview set to be $I(r_k) = \{h_1, \ldots, h_k\}$. We formalize this in Lemma 4 and defer the proof to Appendix B. Note that this result greatly simplifies the equilibrium analysis going forward: we need only consider k possible interviewing sets, instead of $\binom{n}{k}$ to determine if assortative interviewing is the best strategy for r_k.

Lemma 4 *Given* **ILQ** *game* $\Psi = \langle k, \phi, v\rangle$*, if resident* r_k*'s expected payoff from interviewing with hospitals* $\{h_1, \ldots, h_k\}$ *(when residents* $r_1, \ldots, r_{k-1}$ *have interviewed with them as well) is higher than their expected payoff from interviewing with hospitals* $\{h_1, \ldots, h_{k+1}\}\backslash\{h_j\}$ *for all* $j \in \{h_1, \ldots, h_k\}$*, then resident* r_k*'s best response is to interview with* $\{h_1, \ldots, h_k\}$ *(i.e., assortatively).*

We now provide a necessary and sufficient condition for assortative interviewing to hold for **ILQ** game $\Psi = \langle k, \phi, v\rangle$. Let $P(h_i)$ denote the probability that hospital h_i is available for resident r_k (i.e., residents $r_1, \ldots, r_{k-1}$ are all matched to different alternatives).

Lemma 5 *Given* **ILQ** *game* $\Psi = \langle k, \phi, v\rangle$*, if residents* $r_1, \ldots, r_{k-1}$ *all interview assortatively (i.e., with hospital set* $S = \{h_1, \ldots, h_k\}$*), then assortative interviewing is a best response for resident* r_k *if and only if the following inequality is satisfied for all* $h_j \in \{h_1, \ldots, h_k\}$ *when* $S' = S\backslash\{h_j\} \cup \{h_{k+1}\}$*:*

$$\begin{aligned} P(h_j)\mathbb{E}(v(h_j)|D^{\phi,\sigma}) \geq \quad & P(h_j)\mathbb{E}(v(h_{k+1})|D^{\phi,\sigma}) \\ & + \sum_{\eta \in H_{\succ}} P(\eta|D^{\phi,\sigma}) \cdot \Big[\sum_{h_i \in S'} P(h_i)\mathbb{1}_{h_{k+1} \succ_\eta h_i} v(h_{k+1}, \eta)\Big] \end{aligned}$$

where

$$\mathbb{1}_{h_i \succ_\eta h_j} = \begin{cases} 1, & \text{if } h_i \succ_\eta h_j \\ 0, & \text{otherwise} \end{cases}$$

We now present our key result for this section. We provide a necessary and sufficient condition for assortative interviewing to form an equilibrium for a given **ILQ** game, $\Psi = \langle k, \phi, v\rangle$. Furthermore, this condition can be checked efficiently since it only involves checking k possible interview sets for a single resident, r_k.

Theorem 3 *Given* **ILQ** *game* $\Psi = \langle k, \phi, v\rangle$*, then satisfying the inequality found in Lemma 5 for all* $h_j \in \{h_1, \ldots, h_k\}$ *is both sufficient and necessary to show that all residents interviewing assortatively form an equilibrium for this game.*

Proof For the first k residents, this follows directly from combining Theorem 2 and Lemma 5. The theorem would be correct if we could apply this proposition and lemma iteratively, one group of k hospitals and residents at a time. Thanks to the Mallows distribution's properties, we can: If r_k's best response was assortative, we know that all the residents $r_1, \ldots, r_k$ interviewed assortatively, thus all hospitals $h_1, \ldots, h_k$ are taken. This means that the same equations that told us that r_k's best response (to $r_1, \ldots, r_{k-1}$) was assortative tell us that r_{2k}'s best response (to $r_{k+1}, \ldots, r_{2k-1}$) is assortative: Since a switch between h_1 and h_2 has the same probability as switching between h_{k+1} and h_{k+2}, if Theorem 2 and Lemma 5 can be applied once on hospitals and residents $1, \ldots, k$, they can be applied again for $k+1, \ldots, 2k$, as all equations remain the same, due to the practical "disappearance" of the hospitals $h_1, \ldots, h_k$ for agents $r_{k+1}, \ldots, r_{2k}$ (thus their order can be ignored). Now that we have shown that the first two groups of k residents interview assortatively, we can use the same argument iteratively for the next k residents, and so on. □

To conclude this section we note that while we focussed on assortative equilibria since they are elegant and simplifies the problem of determining equilibrium strategies for the residents, other equilibria may also exist. For example, consider the special case where $\Psi = \langle k, 1.0, v \rangle$. When $\phi = 1.0$ the resulting Mallows distribution is uniform. As first noted by Lee and Schwarz (2017) under a different model, when residents and hospitals are divided into n/k subsets and matched inside these subsets, this also forms an equilibrium.

5 Assortativity, utility function structure, and quotas

We now focus our attention on understanding the interplay between the number of interviews residents may conduct and the structure of the underlying utility functions. We continue to be interested in characterizing the conditions in which "natural" or assortative interviewing equilibria exist.

To ground the work we continue to assume that residents' preferences are drawn from some underlying ranking distribution generated by the ϕ-Mallows model, and then we instantiate the residents' utility functions in three different ways, drawing inspiration from both the social choice literature (Brandt et al 2016; Loewenstein et al 1989; Messick and Sentis 1985) and the matching literature (e.g. Coles and Shorrer 2014; Calsamiglia et al 2020). Let h_i be the i^{th} ranked hospital in a resident's ranking η.

Plurality-based: A utility function is *plurality-based* if

$$v(h_i) = \begin{cases} 1, & \text{if } i = 1 \\ 0, & i > 1 \end{cases}$$

Borda-based: A utility function is *Borda-based* if for any h_i,

$$v(h_i) = n - i + 1$$

where n is the number of alternatives (hospitals) in the market. This is equivalent, in a sense, to the expected rank, though the values are inverted—the most preferred choice has a maximal Borda score, but the expected rank value is minimal (1).

Exponential: A utility function is exponential if

$$v(h_i) = \left(\frac{\epsilon}{2}\right)^{i-1}, \text{ for } 0 < \epsilon < 1.$$

These three functions capture a wide range of residents' preferences. If best modelled using plurality-based utility functions, residents care only about being matched to their top choice. Borda-based, on the other hand, provides a linear utility function that decreases as a resident is matched to a less preferred hospital. The class of exponential utility functions forms a bridge between plurality and Borda.

Our first result identifies a condition under which residents with plurality-based utility functions will interview assortatively in equilibrium. The proofs are provided in Appendix D.

Lemma 6 *Given **ILQ** game $\Psi = \langle k, \phi, v \rangle$ where v is the plurality-based utility function, a necessary and sufficient condition for assortative interviewing to be an equilibrium is*

$$P(h_j) \geq \phi^{k-j+1}$$

where $P(h_j)$ is the probability that hospital h_j is available for resident r_k.

There is a strong relationship between the conditions under which assortative interviewing is an equilibrium when residents have plurality-based utility functions and when they have exponential utility functions.

Lemma 7 *Given **ILQ** game $\Psi = \langle k, \phi, v \rangle$, if*

$$P(h_j) \geq \phi^{k-j+1}$$

when v are plurality-based utility functions, then there exists exponential utility functions that also result in assortative interviewing being in equilibria.

One can immediately develop some intuition from these Lemmas by considering the extreme values for the ϕ-parameter. For example, if $\phi = 1.0$, then the distribution from which residents' preferences are drawn is uniform.[5] In this case, assortative interviewing will only be supported in equilibrium if resident r_k is certain to be matched with h_1. This is clearly very strong and unlikely to hold in many real-world settings. On the other hand, if ϕ is close to zero, meaning that residents' true rankings over hospitals are likely to be similar to each other, then assortative interviewing is supported as long as there is some (possibly fairly small) chance that h_1 will be available to be matched to r_k. We will leverage Lemmas 6 and 7 in the rest of this section to gain a clearer picture of the characteristics of assortative equilibria.

5.1 Assortative equilibria when $k = 2$

If residents are only allowed to interview with two hospitals, then assortative interviewing forms an equilibrium under certain conditions. In particular, the existence of assortative interviewing depends on both the structure of the residents' utility functions and the dispersion parameter, ϕ, of the underlying Mallows model.

[5] This is also known as Impartial Culture, a term introduced in Garman and Kamien (1968).

Theorem 4 *Given **ILQ** game $\Psi = \langle k, \phi, v\rangle$ with $k = 2$ and v being plurality-based utility functions, assortative interviewing forms an equilibrium when* $0 < \phi \leq 0.6180$.

A direct consequence of Theorem 4 and Lemma 7 is that for exponential scoring functions, when $0 < \phi < 0.6180$, there exists an ε such that if residents' scoring function is an exponential function dominated by $(\frac{\varepsilon}{2})^{(i-1)}$ with $\varepsilon > 0$, assortative interviewing is an equilibrium for that ϕ.

We are also able to show a similar result when the utility functions of residents are Borda-based, though assortative interviewing is in equilibrium for a significantly smaller range of ϕ, meaning that the preferences of the residents are much less diverse. This illustrates the strong connection and interplay between the structure of the utility functions of the residents, the underlying preference distribution, and the number of interviews residents may participate in.

Theorem 5 *Given **ILQ** game $\Psi = \langle k, \phi, v\rangle$ with $k = 2$ and v being Borda-based utility functions, assortative interviewing forms an equilibrium when* $0 < \phi \leq 0.265074$.

5.2 Assortative equilibria when $k = 3$

Interestingly, when residents can interview with up to three hospitals, assortative interviewing continues to be an equilibrium for plurality-based and exponential utility functions but is no longer an equilibrium if residents have Borda-based utility functions. We begin with the negative result for Borda-based utility functions.

Theorem 6 *Given **ILQ** game $\Psi = \langle k, \phi, v\rangle$ with $k = 3$ and v being Borda-based utility functions, then assortative interviewing may not form an equilibrium or any* $0 < \phi \leq 1$.

Alternatively, for plurality and exponential-based utility functions, assortative interviewing still forms an equilibrium for certain ranges of ϕ in the Mallows model. We observe, however, that the range of ϕ is smaller than in the case where $k = 2$, indicating again the sensitivity of residents' strategic decisions on all aspects of the **ILQ** game.

Theorem 7 *Given **ILQ** game $\Psi = \langle k, \phi, v\rangle$ with $k = 3$ and v being plurality-based utility functions, assortative interviewing forms an equilibrium when* $0 < \phi \leq 0.4655$.

The existence of assortative interviewing, when v are exponential-based, is an immediate consequence of Theorem 7 and Lemma 7.

5.3 Assortative equilibria when $k \geq 4$

We finally consider the setting where residents can interview with more than three hospitals. Unfortunately, our results are negative; we show there are settings, characterized by k and ϕ, such that assortative interviewing is not an equilibrium, irrespective of the underlying utility function. We begin by showing that when there is a setting for which there is no assortative equilibria for plurality, then there is no scoring function with assortative equilibria. We use this result to show that, for sufficiently small dispersion parameter ϕ and for $k > 3$ interviews, assortative interviewing cannot be an equilibrium under any scoring function. We then provide a specific counterexample for *all* ϕ when $k = 4$ for plurality, implying there is no assortative equilibrium for any scoring function. This suggests that, for a wide category of resident valuation functions under a Mallows distribution, contrary to some real-world behaviour, assortative interviewing is not an equilibrium.

Theorem 8 *Given* ***ILQ*** *game* $\Psi = \langle k, \phi, v\rangle$ *with* $k \geq 4$ *and* v *being plurality-based, if hospital* h_1 *causes the condition in Lemma 5 to be falsified (i.e.,* $\{h_2, \ldots, h_{k+1}\}$ *has a better expected payoff than* $\{h_1, \ldots, h_k\}$*), then for* $k \geq 4$ *and* ϕ*, assortative interviewing is not an equilibrium for any valuation function.*

Intuitively, there is a tradeoff between the likelihood that a hospital will be available for resident r_k by the time it is their turn to be matched and the expected value of that hospital. Both are strongly tied to the dispersion parameter ϕ of the Mallows model: as the dispersion parameter approaches 1.0, the difference in the expected value of any given hospital goes to 0. As the dispersion parameter approaches 0.0, the expected value of any hospital h_i goes to the value of its slot in expectation, $v(s_i)$. However, the likelihood it is taken by some higher ranked r_j (i.e., with $j < i$) also approaches 1. The following theorem addresses the latter case: for sufficiently small dispersion, even though the expected value of a hospital is high, the likelihood it will be available is so low that residents are disincentivized from choosing to interview with it.

Theorem 9 *Given* ***ILQ*** *game* $\Psi = \langle k, \phi, v\rangle$ *with* $k > 4$*, there exists* $0 < \varepsilon < 1$ *such that for any scoring function* v *no assortative interviewing forms an equilibrium for dispersion parameter* $0 < \phi < \varepsilon$*.*

We now show that for $k = 4$, assortative interviewing is not an equilibrium for *any* $\phi < 1$ and any scoring rule. We then continue to show that for $k > 4$ and ϕ sufficiently small, assortative interviewing is not an equilibrium.

Theorem 10 *Given* ***ILQ*** *game* $\Psi = \langle k, \phi, v\rangle$ *with* $k = 4$ *and any scoring function* v*, assortative interviewing is not an equilibrium for any dispersion parameter* $0 < \phi < 1$*.*

It seems unlikely that for $k > 4$, assortative interviewing is an equilibrium. Intuitively, if it is an equilibrium it should be for low ϕ: this is when the expected value of hospital h_i is very close to $v(s_i)$. However, this is also when residents $r_1, \ldots, r_{k-1}$ are all most likely to be matched with hospitals $h_1, \ldots, h_{k-1}$. We leave open the possibility that there may exist some δ such that when $0 < \varepsilon < \phi < \delta \leq 1$, assortative interviewing is an equilibrium for plurality.

6 Beyond assortative interviewing

The results in the previous sections are mixed. While we believe that assortative interviewing is an interesting phenomenon and is "natural" as it provides an intuitive strategy for residents, we have also shown that such equilibria are only guaranteed to exist when residents have a limited number of interviewing options.

This inspires us to do two things. First, we observe that our definition of assortative interviewing is strong. Thus, we explore the ramifications of weakening the definition. As we will show, interestingly, our weaker definition does not help and instead can add further complications to the problem. This motivates us to expand the class of what we consider "natural" outcomes and initiate an investigation into the class of *reach and safety* strategies.

6.1 The weakness of weak assortative strategies

Our definition of assortative interviewing was very strong. Definition 1 imposed two key restrictions. First, it assumed that each resident, with an interview budget of size k, interviews with k consecutive hospitals (according to the reference ranking). Second, the definition assumed that the top k residents interviewed with the top k hospitals, the following k residents (ranked from $k+1$ to $2k$) interviewed with the next k hospitals, etc. We consider two relaxations of this definition: *pseudo-strong assortative* and *weak assortative* interviewing.

Definition 2 Given **ILQ** game with quota $k > 0$, an interviewing strategy profile is *pseudo-strong assortative* if given $j = 0, 1, 2, \ldots, \frac{n}{k} - 1$, for each group of k hospitals such that $H_j = \{h_{1+kj}, h_{2+kj}, \ldots, h_{k(j+1)}\}$, there exists a subset of residents, $R_j \subset R$, $|R_j| = k$ such that all residents in R_j interview with H_j.[6]

Note that this definition relaxes the assumption that consecutive residents interview with the same hospitals. For example, if there was an outcome such that resident r_1, r_3 and r_5 interview at h_1, h_2 and h_3, while residents r_2, r_4 and r_6 interview at hospitals h_4, h_5 and h_6, this would be pseudo-strong but not strongly assortative.

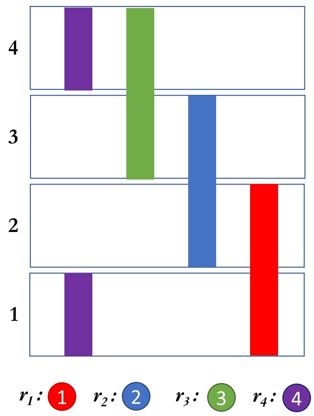
Definition 3 Given **ILQ** game with quota $k > 0$, we say that an interviewing strategy profile is *weakly assortative* iff for all $r_i \in R$, $I(r_i) = \{h_j, h_{j+1}, \ldots, h_{j+k-1}\}$ for some j.

Weak assortative interviewing has that each resident selects k consecutive hospitals to interview and places no other restrictions on the residents' strategies. That is, it is conceivable that almost every resident is targeting a different part of the hospital list.

Strong and pseudo-strong strategies result in a similar structure of resident and hospital interviews: hospitals are divided into consecutive sets (the top-k hospitals, the 2nd-k hospitals, etc.), and each resident interviews in one of those sets. Since a resident only interviews with one of those sets (e.g., a resident cannot interview in some of the top-k and some of the 2nd-k), such a structure may happen when the difference between the expected value of each hospital is so small that the mere interview of a resident in a hospital (which decreases the probability of another resident getting it) reduces the expected value in a way that another agent prefers to avoid it completely.

[6] The ordering of the hospitals is the residents' expected ordering. This would, for example, coincide with the reference ranking in the Mallows model.

Fig. 1 Interviewing sets when $|R| = 4$, $k = 2$, and we require weak assortative strategies.



This can happen when ϕ is very close to 1, in which case all hospitals are almost equivalent to resident preferences. It can also occur when the valuation function is such that the value of each hospital is very close, regardless of their ranking.

Since residents do not interview with different sets of hospitals (top-k, 2nd-k, etc.), this also means that if r_1 and r_2 interview in the same hospitals in equilibrium, this indicates r_2 sees some value in these hospitals, even if r_1 interviews there, which means that r_3 would consider these hospitals as well (assuming $k > 2$, of course), so while r_3 may gradually change the interview set, they will not completely avoid r_1 and r_2's hospitals (resulting in an equilibrium that is not strong or pseudo-strong).

The above argument, however, does not answer what happens when we consider weakly assortative interviewing that is neither strong nor pseudo-strong. Surprisingly, weak assortative interviewing turns out to be more problematic than strong (or pseudo-strong) assortative interviewing.

Theorem 11 *Given **ILQ** game $\Psi = \langle k, \phi, v \rangle$ where all residents employ a weakly assortative interviewing strategy (that is, at least one resident is not strong or pseudo-strong). Then there is a non-zero probability that some resident will be unmatched.*

Note that such a situation cannot happen with strongly (or pseudo-strong) assortative strategies—sets of k residents all interview at the same k hospitals, meaning they are guaranteed to be matched.

While Theorem 11 is an inherently negative result, it is not the only negative aspect that arises when one considers weakening strong assortativity.

Theorem 12 *Given **ILQ** game $\Phi = \langle k, \psi, v \rangle$ with $k = 2$, $\phi < 1$, and $|R| = |H| = 4$, then there does not exist a Nash equilibrium in which residents have weak-assortative strategies.*

The proof of Theorem 12 (found in the appendix) illustrates a cascading effect where residents (particularly the bottom-ranked resident) have incentives to break the sequential structure of hospitals when selecting their interview sets. This is illustrated in Fig. 1. This is further exacerbated as the number of residents ($|R|$) increases. With a larger k, a similar process would occur, and as also intuited in Theorem 11, creating more than k residents interviewing at the same hospital means the probability of a resident being left without a hospital grows. While we hypothesize that for $k > 2$ and $n > k$, there is no Nash equilibrium at all with only weakly assortative strategies (in cases without tie-breaking), it is clear that if there is a sufficiently negative cost to being left without a hospital (as there is in the real world), weakly assortative interviewing cannot happen, as weakly assortative strategies result in hospitals with over k interviewees (and thus residents without hospitals). These residents would instead seek to reduce this probability by interviewing at the hospitals with highest availability probability, which means they would not interview in a hospital with more than k interviewees.

6.2 Reach and safety strategies for a small interviewing quota

While the example shown in Fig. 1 lacked the assortativity structure we have been interested in, it still illustrated an interesting phenomenon, *reach-and-safety* strategic behaviour. We investigate such behaviour empirically and relate the emergence of such behaviour to the underlying preference models of the residents. We focus on small **ILQ** games as we concentrate on exact computation of the underlying equilibrium, but we hypothesize that our findings generalize to larger settings.

Consider the case for $k = 2$ interviews where (for the Borda scoring rule) we only guarantee assortative interviewing for some sufficiently small dispersion parameter ϕ. To gain better insight into the strategic behaviour of the residents as a function of ϕ, we calculated the exact values of ϕ where the interviewing equilibria changes in small markets. In doing so, we see that the structure of the interviewing equilibria contain both "reach" and "safety" schools, where participants diversify their interviewing portfolio to get both the benefit of a desirable, unlikely option, and a likely, but less desirable option.

Figure 2 depicts a market with 4 hospitals, 4 residents, and 2 interviews ($n = 4$, $k = 2$). The figure shows what sets are being chosen by the different residents for any dispersion ϕ. As ϕ increases, we explicitly see the trade-off between a safer choice, and a better expected-payoff value for individual alternatives. For small ϕ, as the theoretical results showed, assortative interviewing is optimal, and r_2 chooses $\{h_1, h_2\}$, while r_3 and r_4 choose $\{h_3, h_4\}$. Interestingly, for $\phi \in [0.5, 0.62]$, r_2's best option is to split the difference, and interview with one hospital (h_3) they are guaranteed to get and one hospital (h_2) that will be available with sufficiently high probability, and has a higher expected value. This choice available to r_2 further results in some of the "reach" vs. safe behaviour we see in college admissions markets; namely, r_3's best response now is to interview with h_1, h_4 (i.e., a "reach" choice, and a "safe" bet), while r_4, being left without any truly "safe" option, aims slightly higher than its rank. As ϕ grows and approaches 1, any ordering of hospitals is as likely as another, making r_2's choice $\{h_3, h_4\}$, which are as likely as any to be highly ranked, and are available. The desire to avoid interviewing hospitals that are already chosen by many other residents also drives r_3 and r_4 to $\{h_2, h_3\}$ and $\{h_1, h_4\}$, respectively; that is, they both want to avoid competing with r_1 and r_2.

We expand on these results and now consider the case of $n = 6$ residents with $k = 2, 3$ and 4 interviews per resident. Here we see in Figs. 3, 4, and 5, similar

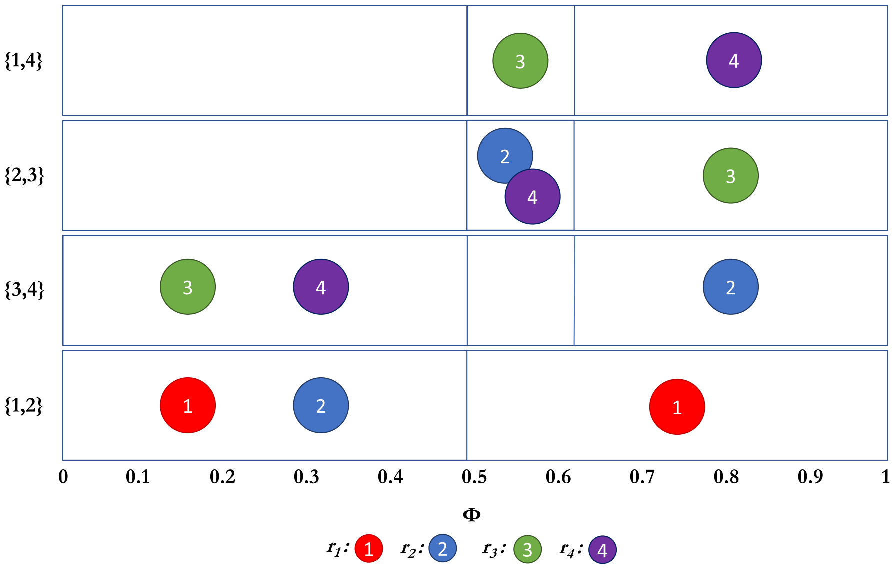


Fig. 2 Interviewing sets of residents as a function of ϕ when using the Borda scoring function, with 4 participants, and interview set size of 2.

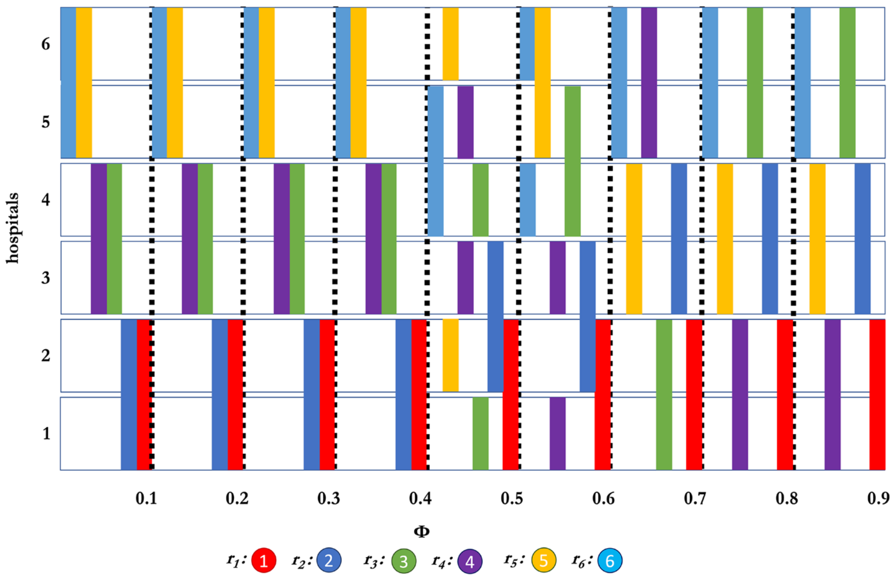


Fig. 3 Interviewing sets of residents as a function of ϕ when using the Borda scoring function, with 6 participants, and interview set size of 2.

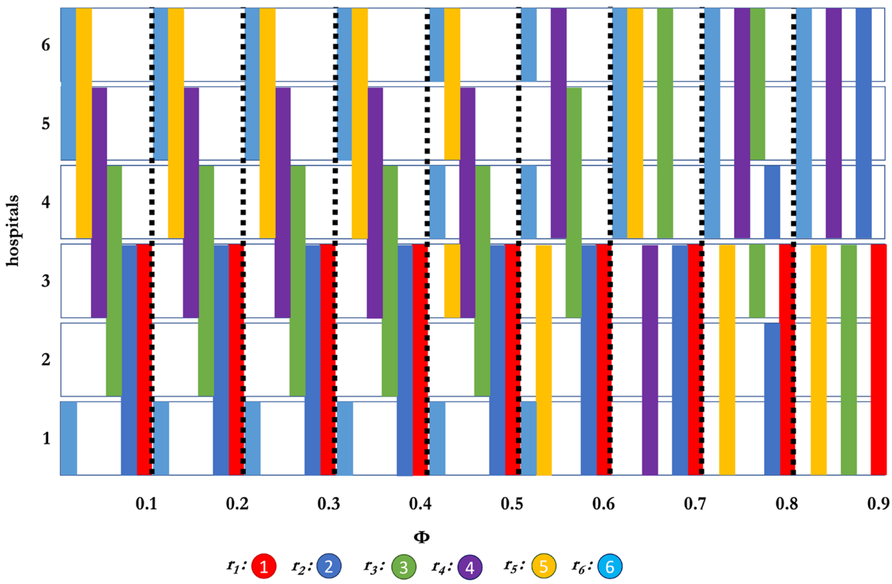


Fig. 4 Interviewing sets of residents as a function of ϕ when using the Borda scoring function, with 6 participants, and interview set size of 3.

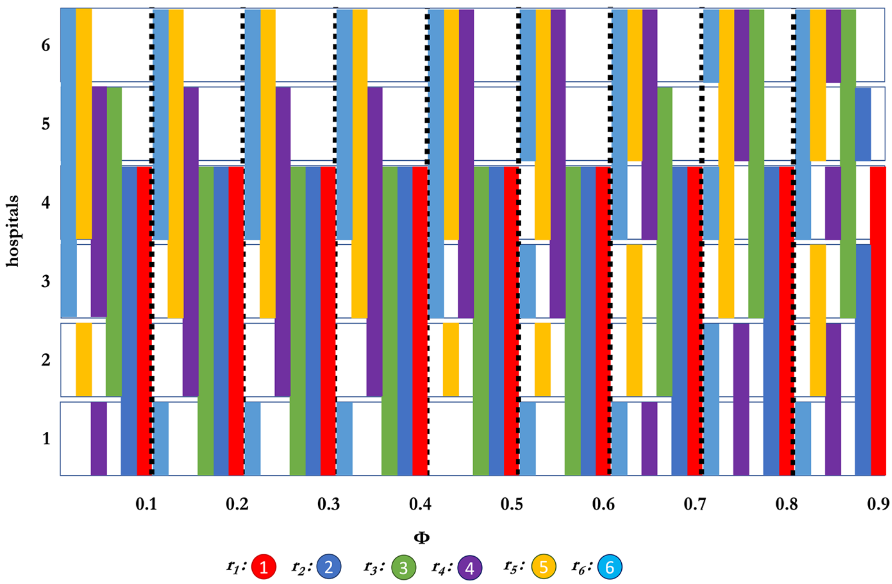


Fig. 5 Interviewing sets of residents as a function of ϕ when using the Borda scoring function, with 6 participants, and interview set size of 4.

equilibrium strategies as for the $n = 4$, $k = 2$ case. For $k = 2$, and $\phi \leq 0.4$, we again see that assortative interviewing is an equilibrium. When $\phi = 0.5$, we observe that the second resident departs from strict assortative interviewing in favour of a weak version of assortative interviewing and this, in turn, affects the other players, as, for example, the third resident applies what is a safety move (hospital 4, which is theirs if they want it) with a reach move (hospital 1, the top choice). Of some interest, for $\phi \geq 0.7$, all residents except r_1 use a weak assortative strategy, that is, they interview in sets of hospitals which are adjacent in rank, rather than splitting their interviews between radically different ranked hospitals.

Turning to $k = 3$ interviews per resident, we see as Theorem 6 claimed, that the third resident does not interview assortatively. While residents 3, 4, and 5 are mostly weakly assortative (except the third resident and $\phi = 0.8$, where it tries a small reach choice), the sixth resident goes consistently for a reach and safety strategy, as it interviews in the top hospital as well. The resident's behaviour only changes when ϕ is large enough ($\phi > 0.6$) when the chance of the true ranking being different from the ground truth is much higher. Of interest, when $\phi = 0.9$, the second resident chooses hospitals 4, 5, 6 (even knowing that at least two of the hospitals in $\{1,2,3\}$ will be available. But when ϕ is sufficiently close to 1, the distribution is approaching the uniform distribution so that this resident might as well choose hospital 4, 5, 6 as they might very well be as desirable as 1, 2, 3 where the residents' top choices might be taken.

Finally, for $k = 4$, we see that resident 1 (as we know must happen) interviews assortatively for all settings of ϕ while other residents are much more willing to experiment. Not included in Fig. 5 are further results, showing that even for some very small ϕ ($0.1 > \phi \geq 10^{-20}$), there are residents which are not even weakly assortative. We hypothesize that this "reach" and "safety" behaviour is present in markets with larger interviewing quotas.

7 Conclusions and future directions

We investigated equilibria in **ILQ** games, inspired by the matching of medical residents to hospital programs. A key feature of this game is that residents must interview with hospitals to discover their true preferences, but are limited in the number of interviews they may conduct. This introduces a new level of complication as residents need to carefully consider how to optimize their interviewing strategies given the interviews choices of other residents. While we showed the existence of a pure-strategy Nash equilibria for this game, we where particularly interested in understanding under what circumstances assortative interviewing forms an equilibrium, as such strategies are "easy" for residents to execute, result in stable outcomes where everyone is matched, and which earlier work had suggested might exist (Lee 2017).

We summarize our findings into a few key take-aways that may provide useful guidelines for market designers:

- Assortative interviewing is supported in equilibrium, but its existence depends critically on how correlated residents' preferences are, the limit on the number of interviews, and the structure on the underlying value functions for alternatives.

- If the underlying value function is Borda-based, then assortative interviewing only forms an equilibrium when preferences of residents are closely correlated (as measured by the ϕ parameter in the underlying $\phi-$Mallows model). If the underlying value function is plurality-based or exponential then assortative interviewing is more broadly supported.
- Limiting the number of interviews residents can undertake is critical if assortative interviewing it to be supported. If residents are allowed to interview with 4 or more hospitals then assortativity might not be supported in equilibrium.
- Relaxing the definition of assortativity (to weak-assortativity) does not help.

There are many research questions raised by our results, to which at least some of out technical results and techniques may also contribute. Most concretely, we hypothesize Theorem 9 could be replaced by extending Theorem 10 for all $k \geq 4$. Second, while we believe that the space of scoring functions used in this paper was broad in its scope, we always assumed that residents' underlying ranked preferences were drawn from a distribution generated by the ϕ-Mallows model. While the ϕ-Mallows model is standard in the literature, it is possible that other preference distributions (e.g., Plackett-Luce) may better support assortative interviewing. Second, the analysis relies on the assumption that one side of the market maintained a master list. While master-lists do occur in real-world matching markets, lifting this assumption will obviously generalize the setting, and may invalidate our results. More specifically, the removal of the master-list assumption would complicate the analysis significantly, increasing the complexity of the payoff function formulation.

Furthermore, we could consider modifying our definition of an interview set. Currently we assume that residents could interview up to k hospitals for free, but an alternative model to consider would be to allow each resident r to have a "budget" b_r, and incur a cost, $c_r(h)$, when interviewing hospital h, with the constraint that if S is the set of hospitals interviewed by resident r, then $\sum_{h \in S} c_r(h) \leq b_r$. Such a budget, even if the cost is equal for all hospitals, will change the equilibrium in a variety of ways, including by making it no longer always a dominating strategy to interview at all k hospitals, as sometime—particularly for very high/low ranked residents—interviewing at some hospitals might not offer enough expected utility. It may also give rise to a setting equivalent to the hospitals having a limited number of potential interview slots, which would make the hospitals strategic players as well, as they wish to find the candidates which are both highly ranked and that will also choose them.[7] A similar form of either assortative or "reach"/"safety" may happen, this time by the hospitals, though that is outside the scope of this paper.

A long-term research goal is to better understand the extent to which "natural equilibria" exist in matching games, and how such equilibria correspond with observed behaviour in actual markets. One such possibility is for interviewing to be assortative for "safety" programs while allowing for one or a few "reach" programs. (See for example the strategy of resident 6 for small values of the Mallows' parameter in Fig. 4.) Furthermore, we are interested in techniques that could reduce the cognitive burden placed on participants in matching markets, while also reducing inefficiencies.

[7] Though note that in an equilibrium, the number of interviews in each hospital will still be k, for the reasons outlined following Theorem 12 when the cost for residents of being left without a hospital is high.

For example, there may be ways to leverage research on preference elicitation for matching markets (e.g., Drummond and Boutilier 2014) with matching market design so as to guide participants to interview with the appropriate programs so as to improve the overall quality of the match.

Funding Open access funding provided by Ben-Gurion University.

Data availability Not applicable.

Appendix A Proofs from Sect. 4.1.1

In this appendix we provide the proofs of the results that were presented in Sect. 4.1.1.

Proof (Lemma 1) Suppose σ is a prefix of σ'. Then, let σ be some ranking with p elements, including elements a_i and a_j. Let σ' be a ranking of $p+1$ elements with σ as its prefix, and an additional element a_p added at the end. We prove this by starting from the definition of $P(a_i \succ a_j | D^{\phi,\sigma'})$, and using algebraic manipulations to show this is equivalent to the definition of $P(a_i \succ a_j | D^{\phi,\sigma})$.

$$P(a_i \succ a_j | D^{\phi,\sigma'}) = \frac{\sum_{\eta' \in \{a_0,\dots,a_{p-1},a_p\}_{\succ}^{a_i \succ a_j}} \phi^{d(\eta',\sigma')}}{1(1+\phi)\dots(1+\dots+\phi^{p-1}+\phi^p)} \tag{2}$$

However, because a_i, a_j are in ranking σ, the only difference between summing over the set of all rankings in $\{a_0,\dots,a_p\}_{\succ}^{a_i \succ a_j}$ and $\{a_0,\dots,a_{p-1}\}_{\succ}^{a_i \succ a_j}$ is that there for each permutation generated by $\{a_0,\dots,a_{p-1}\}_{\succ}$, there are p permutations in $\{a_0,\dots,a_p\}_{\succ}$, each one with a_p in a different place (and thus a different Kendall-τ distance). Fixing some $\eta \in \{a_0,\dots,a_{p-1}\}_{\succ}$, if a_p is in the last rank position (as it is in σ'), the distance is simply $d(\eta,\sigma)$. If a_p is in the second-to-last position, we have now added in an additional discordant pair, so the distance is $d(\eta,\sigma)+1$. Using this, we generate the following:

$$\begin{aligned}
P(a_i \succ a_j | D^{\phi,\sigma'}) &= \frac{\sum_{\eta \in \{a_0,\ldots,a_{p-1}\}_{\succ}^{a_i \succ a_j}} \sum_{l=0}^{p} \phi^{d(\eta,\sigma)+l}}{1(1+\phi)\ldots(1+\ldots+\phi^p)} \\
&= \frac{\left[\sum_{\eta \in \{a_0,\ldots,a_{p-1}\}_{\succ}^{a_i \succ a_j}} \phi^{d(\eta,\sigma)}\right]\left[\sum_{l=0}^{p} \phi^l\right]}{1(1+\phi)\ldots(1+\cdots+\phi^p)} \\
&= \frac{\left[\sum_{\eta \in \{a_0,\ldots,a_{p-1}\}_{\succ}^{a_i \succ a_j}} \phi^{d(\eta,\sigma)}\right](1+\ldots+\phi^p)}{1(1+\phi)\ldots(1+\cdots+\phi^{p-1})(1+\cdots+\phi^p)} \\
&= \frac{\sum_{\eta \in \{a_0,\ldots,a_{p-1}\}_{\succ}^{a_i \succ a_j}} \phi^{d(\eta,\sigma)}}{1(1+\phi)\ldots(1+\cdots+\phi^{p-1})} \\
&= P(a_i \succ a_j | D^{\phi,\sigma})
\end{aligned}$$

By symmetry, this also holds when σ is a suffix of σ'. □

Proof (Corollary 1) Consider $\sigma = a_i \succ a_{i+1}$, a reference ranking with only our two elements in it. Then, the set of all potential rankings such that $a_i \succ a_{i+1}$ under $D^{\phi,\sigma}$ is solely the ranking $a_i \succ a_{i+1}$. By the definition of the Mallows model, this ranking has probability $\frac{1}{1+\phi}$. We add some arbitrary prefix σ' to σ and some arbitrary suffix σ'' to σ to create a new reference ranking γ. By Lemma 1, the probability that some η is drawn from $D^{\phi,\gamma}$ such that $a_i \succ_\eta a_{i+1}$ is $\frac{1}{1+\phi}$ as required. □

Proof (Corollary 2) Consider $\sigma^* = a_i \succ a_{i+1} \succ a_{i+2}$, a reference ranking with three elements in it. The set of all potential rankings under D^{ϕ,σ^*} such that $a_i \succ a_{i+1} \succ a_{i+2}$ is solely that ranking. Using the same argument as in Lemma 1, we note that creating some new reference ranking $\gamma = \sigma' \succ \sigma^* \succ \sigma''$ and drawing from $D^{\phi,\gamma}$ does not change the likelihood that we draw a ranking consistent with $a_i \succ a_{i+1} \succ a_{i+2}$.

Therefore, the probability that we draw a ranking β consistent with some permutation η of a_i, a_{i+1}, a_{i+2} under the distribution $D^{\phi,\gamma}$ is simply the probability that we drew η under the distribution D^{ϕ,σ^*}, which is $\frac{\phi^{d(\eta,\sigma^*)}}{(1+\phi)(1+\phi+\phi^2)}$. □

Proof (**Lemma 2**) This is equivalent to generating the set of all $(n-1)!$ possible rankings excluding alternative a_1 (a_n), and then adding a_1 (a_n) in place j. Whatever the ranking, adding a_1 (a_n) in place j adds $j-1$ ($n-j$) to each possible ranking's Kendall's τ distance from $\sigma \setminus \{a_1\}$ ($\sigma \setminus \{a_n\}$), making the distance from σ grow by exactly $j-1$ ($n-j$). Similarly, adding a_j in first place adds $j-1$ to the distance from $\sigma \setminus \{a_j\}$, increasing the distance from σ by $j-1$.

However, we also added in an additional element to the ranking (growing from $n-1$ to n), and must include that in the normalization factor Z. The normalization factor for $n-1$ alternatives is $(1+\phi)(1+\phi^2)\ldots(1+\cdots+\phi^{n-2})$. The normalization factor for n elements is identical, but multiplied by $1+\cdots+\phi^{n-1}$. □

Proof (Lemma 3) For any a_ℓ, $i > \ell > j$: if $a_\ell \succ_\eta a_i$, this adds at least 1 to the Kendall-τ distance of η from σ (due to $a_i \succ_\sigma a_\ell$). But if $a_i \succ_\eta a_\ell$, this means that

$a_j \succ_\eta a_\ell$, again adding 1 to the Kendall-τ distance of η from σ. So the Kendall-τ distance of η from σ is at least $\sum_{\ell=i}^{j-1} 1 = j - i$, and therefore, $P(\eta) < \frac{\phi^{j-i}}{Z}$. □

Appendix B Proofs from Sect. 4.1.2

Proof (Lemma 4) The idea behind the proof is that if there is a set of hospitals that are better than interviewing assortatively, since no other resident prior to r_k interviews there, the hospitals in this set that are outside of $\{h_1, \ldots, h_k\}$ have an ordering. That is, the expected utility from adding h_{k+1} is larger than that of adding h_{k+2}, since, in expectation h_{k+1} is likely to be ranked higher by the resident than h_{k+2}. Therefore, taking out the hospital with the least expected utility from $\{h_1, \ldots, h_k\}$ and adding h_{k+1} in its stead should already be beneficial, since any other hospital added to the interviewing set will remove a hospital with a higher utility (than the one removed for h_{k+1}), and replace it with lesser utility hospital (since an hospital from $h_{k+2}, \ldots, h_n$ has smaller expected utility). Therefore, if there is a set that is better than assortative, it should show up already when replacing some hospital in $\{h_1, \ldots, h_k\}$ by h_{k+1}.

For any hospital h, let $b(h, \eta) = 1$ iff h is available for r_k, and $h \succ_\eta h_j$ for all other h_j available; and 0 otherwise. Directly following from the utility function, the utility of resident r_k when interviewing with some set of hospitals $S = \{h_1, \ldots, h_k\}$ can thus be written as:

$$u_{r_k}(S) = \sum_{h \in S} \sum_{\eta \in H_\succ} v(h, \eta) P(\eta, D^{\phi,\sigma}) b(h, \eta)$$

As we assume knowledge of the strategies for residents $r_1, \ldots, r_{k-1}$, we can calculate the probability that any given hospital is available. We thus can calculate the contribution of each hospital interview to the total utility, as $P(\eta, D^{\phi,\sigma})$ and $v(h, \eta)$ are known a priori. Moreover, when $r_1, \ldots, r_k$ all interview with the same k hospitals, $b(h, \eta)$ is equivalent to the probability that hospital h is available for r_k (which we denote by $P(h)$): resident r_k gets whatever hospital $r_1, \ldots, r_{k-1}$ do not take.

Now, assume there is in equilibrium some set S' of hospitals such that $u_{r_k}(S') > u_{r_k}(S)$. Define $\bar{S} = S \setminus S'$; denote the members of $\bar{S}$ as $h'_1, \ldots, h'_l$. Also, note that h_{k+1} must be in $S' \setminus S$, as $\bar{S} \neq S$ and h_{k+1} dominates all alternatives in $\{h_{k+1}, \ldots, h_n\}$: h_{k+1} is available for r_k with probability 1 (as are all other alternatives not in S), and has higher expected value than any other h_j s.t. $h_{k+1} \succ_\sigma h_j$. Without loss of generality, let h'_1 be the hospital in $\bar{S}$ that minimizes the benefit gained from swapping some element in $\bar{S}$ with one of the more "desirable" elements in S'. More formally, h'_1 is the hospital in $\bar{S}$ that minimizes

$$y_1 = \sum_{\eta \in H_\succ} P(\eta | D^{\phi,\sigma}) b(h'_1, \eta) \left[v(h'_1, \eta) - v(h_{k+1}, \eta) \right]$$

y_1 is the value that is lost when h'_1 is the only available hospital from $h_1, \ldots, h_k$, and h_{k+1} must be chosen instead. The value added by interviewing in h_{k+1} instead of h'_1 is formally: $z_1 = \sum_{\eta \in H_\succ} P(\eta | D^{\phi,\sigma}) b(h_{k+1}, \eta) v(h_{k+1}, \eta)$. Then, $u_{r_k}(S \cup$

$\{h_{k+1}\}\backslash\{h'_1\}) = u_{r_k}(S) - y_1 + z_1$. If $y_1 \leq z_1$, the lemma is proven; Otherwise, we assume $z_1 - y_1 < 0$ and establish a contradiction.

Without loss of generality, let h'_2 be the hospital in $\bar{S} \setminus \{h'_1\}$ that minimizes

$$y_2 = \sum_{\eta \in H_\succ} P(\eta|D^{\phi,\sigma})b(h'_2,\eta)\big[v(h'_2,\eta) - \max(v(h_{k+1},\eta), v(h_{k+2},\eta))\big]$$
$$= \sum_{\eta \in H_{\succ|h_{k+1}\succ h_{k+2}}} P(\eta|D^{\phi,\sigma})b(h'_2,\eta)\big[v(h'_2,\eta) - v(h_{k+1},\eta)\big]$$
$$+ \sum_{\eta \in H_{\succ|h_{k+2}\succ h_{k+1}}} P(\eta|D^{\phi,\sigma})b(h'_2,\eta)\big[v(h'_2,\eta) - v(h_{k+2},\eta)\big]$$

Again, y_2 is the benefit we get from h'_2, the alternative we are swapping out for h_{k+2}. The value added from h_{k+2} is $z_2 = \sum_{\eta \in H_\succ} P(\eta|D^{\phi,\sigma})b(h_{k+2},\eta)v(h_{k+2})$. Since h_{k+1} and h_{k+2} have the same probability of being available, but the expected value of $v(h_{k+1})$ is more than that of $v(h_{k+2})$, we know $z_2 < z_1$. Thanks to Corollary 1:

$$\sum_{\eta \in H_{\succ|h_{k+1}\succ h_{k+2}}} P(\eta|D^{\phi,\sigma})b(h'_2,\eta)\big[v(h'_2,\eta) - v(h_{k+1},\eta)\big] = \frac{1}{1+\phi}y_2$$

Looking at the equivalent section of y_1:

$$\sum_{\eta \in H_{\succ|h_{k+1}\succ h_{k+2}}} P(\eta|D^{\phi,\sigma})b(h'_1,\eta)\big[v(h'_1,\eta) - v(h_{k+1},\eta)\big] > \frac{1}{1+\phi}y_1$$

but thanks to y_1 minimality:

$$\sum_{\eta \in H_{\succ|h_{k+1}\succ h_{k+2}}} P(\eta|D^{\phi,\sigma})b(h'_2,\eta)\big[v(h'_2,\eta) - v(h_{k+1},\eta)\big]$$
$$> \sum_{\eta \in H_{\succ|h_{k+1}\succ h_{k+2}}} P(\eta|D^{\phi,\sigma})b(h'_1,\eta)\big[v(h'_1,\eta) - v(h_{k+1},\eta)\big]$$

and therefore $y_2 > y_1$. Thus:

$$\begin{aligned} u_{r_k}(S \setminus \{h'_1, h'_2\} \cup \{h_{k+1}, h_{k+2}\}) &= u_{r_k}(S) - y_1 + z_1 - y_2 + z_2 \\ &< u_{r_k}(S) - 2y_1 + 2z_1 \\ &< u_{r_k}(S) \end{aligned}$$

Note that due to similar considerations, all other alternatives in $S \setminus S'$ must also have y_i such that $y_i > y_1$ and $z_i < z_1$, by the construction of y_1 and z_1. Let $l = |\bar{S}|$.

Thus:

$$u_{r_k}(S') = u_{r_k}(S \setminus \bar{S}) + \sum_{i=1}^{l} z_i - y_i < u_{r_k}(S) - ly_1 + lz_1 < u_{r_k}(S)$$

This contradicts our assumption that $u_{r_k}(S') > u_{r_k}(S)$; thus, if such an S' exists, $y_1 \geq z_1$, and showing that S dominates $S \backslash \{h_j\} \cup \{h_{k+1}\}$ is sufficient for all $h_j \in S$. □

Proof (Lemma 5) By Lemma 4, showing that the marginal contribution from h_j is bigger than the marginal contribution from h_{k+1} is sufficient to show that S dominates any other interviewing set. Using the payoff function in Sect. 3.1, this means that we want to find conditions such that the utility to r_k provided by h_j is larger than that of h_{k+1}:

$$\sum_{\eta \in H_{\succ}} v(h_j, \eta) P(\mu(h_j) = r_k | S, \eta, D^{\phi,\sigma}) P(\eta | D^{\phi,\sigma}) \geq \sum_{\eta \in H_{\succ}} v(h_{k+1}, \eta) P(\mu(h_{k+1}) = r_k | S', \eta, D^{\phi,\sigma}) P(\eta | D^{\phi,\sigma}) \tag{3}$$

Note that, when interviewing with set S, the probability $\mu(h_j) = r_k$ is simply the probability that no resident in $r_1, \ldots, r_{k-1}$ chooses h_j. Thus, the left hand side of Eq. 3 simplifies to:

$$\begin{aligned} \sum_{\eta \in H_{\succ}} v(h_j, \eta) P(\mu(h_j) = r_k | S, \eta, D^{\phi,\sigma}) P(\eta | D^{\phi,\sigma}) \\ = P(h_j) \sum_{\eta \in H_{\succ}} v(h_j, \eta) P(\eta | D^{\phi,\sigma}) \\ = P(h_j) \mathbb{E}(v(h_j) | D^{\phi,\sigma}) \end{aligned} \tag{4}$$

We now also wish to simplify the right hand side. Note that there are two cases in which resident r_k is matched with h_{k+1} when interviewing with set S': either h_j is the only hospital available (i.e., $r_1, \ldots, r_{k-1}$ have all been matched with $\{h_1, \ldots, h_k\} \setminus \{h_j\}$), or for some $h_i \in \{h_1, \ldots, h_k\} \backslash \{h_j\}$, h_i is available and under the ranking η in consideration, $h_{k+1} \succ_\eta h_i$. Again, $\mathbb{1}(y)$ denote an indicator function, where $\mathbb{1}(y) = 1$ iff y is true, and 0 otherwise. More formally, we express the RHS of the condition in

Eq. 3 using the indicator function, and simplify:

$$\sum_{\eta \in H_{\succ}} P(\eta | D^{\phi,\sigma}) \cdot \big[v(h_{k+1}, \eta) P(h_j) + \sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_{\eta} h_i) v(h_{k+1}, \eta) \big] =$$
$$= P(h_j) \mathbb{E}(v(h_{k+1}) | D^{\phi,\sigma}) + \sum_{\eta \in H_{\succ}} P(\eta | D^{\phi,\sigma}) \cdot \big[\sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_{\eta} h_i) v(h_{k+1}, \eta) \big] \tag{5}$$

Combining the simplifications provided in Eqs. 4 and 5 completes the proof. □

Appendix C Sufficient inequality for checking assortativity

We provide a simplified condition for assortative interviewing that is sufficient though not necessary. This condition is easier to compute than the condition in Lemma 5, and thus will be valuable later on, when verifying whether specific valuation functions admit assortative interviewing equilibria.

Lemma 8 *Given an interviewing quota of k interviews, a dispersion parameter ϕ, and a scoring function v, if residents $r_1, \ldots, r_{k-1}$ all interview assortatively (i.e., with hospital set $S = \{h_1, \ldots, h_k\}$), then satisfying the following inequality for all $h_j \in \{h_1, \ldots, h_k\}$ when $S' = S \setminus \{h_j\} \cup \{h_{k+1}\}$ is sufficient to show that assortative interviewing is a best response for resident r_k:*

$$P(h_j) \mathbb{E}(v(h_j) | D^{\phi,\sigma}) \geq P(h_j) \mathbb{E}(v(h_{k+1}) | D^{\phi,\sigma}) + \sum_{h_i \in S'} P(h_i) \mathbb{E}(v(h'_k) | D^{\phi,\sigma'}) \frac{\phi}{Z(1-\phi)}$$

where σ' is equivalent to the reference ranking σ with one element h_i s.t. $h_j \succ_{\sigma} h_i$ removed, and h'_k is the kth item in σ'.

Proof We begin from the sufficient and necessary condition stated in Lemma 5. Note that we can generate any ranking such that $h_{k+1} \succ h_i$ (for some given i) by iterating over all permutations of $H \setminus \{h_i\}$, and for each permutation, placing h_i in every slot below h_{k+1}. There are at most $n-1$ slots that h_i could be placed in.

Let σ' be identical to the reference ranking σ, except with h_i removed. Rename every element after h_i such that it corresponds to its current index: in other words, $h'_j = h_{j+1}$ for all $j \geq i$. Let η' be some arbitrary ranking drawn from $D^{\phi,\sigma'}$. Let

$H' = H \setminus \{h_i\}$. Remember, $S' = \{h_1, \ldots, h_{k+1}\} \backslash \{h_j\}$. Thus, we note that:

$$\sum_{\eta \in H_{\succ}} \sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_\eta h_i) v(h_{k+1}, \eta) P(\eta | D^{\phi,\sigma})$$
$$\leq \sum_{h_i \in S'} \left[P(h_i) \left(\sum_{\eta' \in H'_{\succ}} v(h'_k, \eta') P\left(\eta' | D^{\phi,\sigma'}\right) \left(\sum_{l=1}^{n} \frac{\phi^l}{Z} \right) \right) \right]$$

However, note that ϕ^l is a geometric series. We let $n \to \infty$, giving us:

$$\sum_{h_i \in S'} \left[P(h_i) \mathbb{E}\left(v(h'_k)\, | D^{\phi,\sigma'}\right) \sum_{l=1}^{n} \frac{\phi^l}{Z} \right] \leq \sum_{h_i \in S'} P(h_i) \mathbb{E}(v(h'_k) | D^{\phi,\sigma'}) \frac{\phi}{Z(1-\phi)} \quad (6)$$

Thus, because Eq. 6 is an upper bound, it is sufficient to show the following, as required:

$$P(h_j) \mathbb{E}\left(v(h_j) | D^{\phi,\sigma}\right) \geq P(h_j) \mathbb{E}\left(v(h_{k+1}) | D^{\phi,\sigma}\right)$$
$$+ \sum_{h_i \in S'} P(h_i) \mathbb{E}\left(v\left(h'_k\right) | D^{\phi,\sigma'}\right) \frac{\phi}{Z(1-\phi)}$$

□

Appendix D Proofs from Sect. 5

Before proving the results in this section we introduce some notation. Given some resident with ranking η over hospitals, define s_i to be the i'th ranked hospital in this list. While in general we hesitate to introduce new notation, it is important to distinguish between some hospital in H and the i'th ranked one from a resident's perspective.

Proof (Lemma 6) We begin with the condition in Lemma 5:

$$P(h_j) \mathbb{E}\left(v(h_j) | D^{\phi,\sigma}\right) > P(h_j) \mathbb{E}\left(v(h_{k+1}) | D^{\phi,\sigma}\right) +$$
$$\sum_{\eta \in H_{\succ}} P(\eta | D^{\phi,\sigma}) \cdot \left[\sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_\eta h_i) v(h_{k+1}, \eta) \right]$$

We instantiate this condition for the plurality function, noting that $v(h, \eta) > 0$ iff h is top-ranked in η.

$$P(h_j)\mathbb{E}\left(v(h_{k+1})|D^{\phi,\sigma}\right) + \sum_{i=1}^{j-1} P(h_i)\mathbb{E}\left(v(h_{k+1})|D^{\phi,\sigma}\right) + \sum_{i=j+1}^{k} P(h_i)\mathbb{E}\left(v(h_{k+1})|D^{\phi,\sigma}\right) = \sum_{i=1}^{k} P(h_i)\mathbb{E}\left(v(h_{k+1})|D^{\phi,\sigma}\right)$$

But, again, as the expected value for any hospital h is simply the probability that h is s_1 this further simplifies to:

$$P(h_{k+1} = s_1)\sum_{i=1}^{k} P(h_i) = P(h_{k+1} = s_1)$$

Note that $\sum_{i=1}^{k} P(h_i) = 1$ as all residents $r_1, \ldots, r_{k-1}$ have been matched with exactly $k-1$ hospitals in $h_1, \ldots, h_k$, leaving exactly one hospital left with probability 1.

Applying Lemma 2 to both sides of the inequality (recall that $\mathbb{E}(v(h_j)|D^{\phi,\sigma})$ is simply $P(h_j = s_1)$):

$$P(h_j)\frac{\phi^{j-1}}{1 + \cdots + \phi^{n-1}} \geq \frac{\phi^k}{1 + \cdots + \phi^{n-1}}$$

$$P(h_j) \geq \phi^{k-j+1}$$

□

Proof (Lemma 7) Looking at the condition of Lemma 5

$$P(h_j)\mathbb{E}(v(h_j)|D^{\phi,\sigma}) \geq P(h_j)\mathbb{E}\left(v(h_{k+1})|D^{\phi,\sigma}\right) + \sum_{\eta \in H_{\succ}} P\left(\eta|D^{\phi,\sigma}\right)\left[\sum_{h_i \in S'} P(h_i)\mathbb{1}(h_{k+1} \succ_\eta h_i)v(h_{k+1}, \eta)\right]$$

We will first expand the value expectation ($\mathbb{E}$):

$$P(h_j)\sum_{i=1}^{n}P(h_j=s_i)v(s_i)\geq P(h_j)\sum_{i=1}^{n}P(h_{k+1}=s_i)v(s_i)$$
$$+\sum_{\substack{\eta\in H_{\succ}|\\ h_{k+1}=s_1}}P(\eta|D^{\phi,\sigma})\sum_{h_i\in S'}P(h_i)\mathbb{1}(h_{k+1}\succ_\eta h_i)v(s_1)$$
$$+\cdots+\sum_{\substack{\eta\in H_{\succ}|\\ h_{k+1}=s_{n-1}}}P(\eta|D^{\phi,\sigma})\sum_{h_i\in S'}P(h_i)\mathbb{1}(h_{k+1}\succ_\eta h_i)v(s_{n-1})$$

Note that for any $1\leq\ell\leq n$,

$$v(s_\ell)>P(h_j)P(h_j=s_\ell)v(s_\ell)+$$
$$\sum_{\substack{\eta\in H_{\succ}|\\ h_{k+1}=s_\ell}}P(\eta|D^{\phi,\sigma})\sum_{h_i\in S'}P(h_i)\mathbb{1}(h_{k+1}\succ_\eta h_i)v(s_\ell)$$

Thus, combining these and Lemma 5, it is sufficient to show the following holds whenever plurality admits an assortative interviewing equilibrium:

$$P(h_j)P(h_j=s_1)v(s_1)\geq P(h_j)P(h_{k+1}=s_1)v(s_1)+\sum_{\ell=2}^{n}v(s_\ell)\tag{7}$$

We assume that for plurality valuation, the condition has a strict inequality. In other words:

$$P(h_j)P(h_j=s_1)>P(h_j)P(h_{k+1}=s_1)$$

Hence, there is an $\bar{\epsilon}\leq 1$ such that for all $1\leq j\leq k$,

$$P(h_j)P(h_j\text{ in }s_1)-\bar{\epsilon}>P(h_j)P(h_{k+1}\text{ in }s_1)$$

Now, for $\epsilon<\frac{\bar{\epsilon}}{2}$, examine the valuation function $v(s_\ell)=\epsilon^{\ell-1}$. Note that $\sum_{\ell=2}^{n}\epsilon^{\ell-1}\leq\sum_{\ell=1}^{\infty}\epsilon^{\ell}=\frac{\epsilon}{1-\epsilon}\leq 2\epsilon$. This simplifies such that it satisfies Eq. 7, as required:

$$P(h_j)P(h_j=s_1)>P(h_j)P(h_{k+1}=s_1)+2\epsilon$$
$$\geq P(h_j)P(h_{k+1}=s_1)+\sum_{\ell=2}^{n}v(s_\ell)$$

□

Proof (Theorem 4) We begin by using the condition from Lemma 6 for h_1. We thus wish to show conditions on ϕ s.t. $P(h_1)\geq\phi^2$, when resident r_2 chooses their interview set. For r_2, h_1 is available iff r_1 happened to draw a ranking over their preferences s.t. $h_2\succ h_1$. Then, by Corollary 1, $P(h_1)=\frac{\phi}{1+\phi}$, implying we need to satisfy the equation $\frac{\phi}{1+\phi} \geq \phi^2$, which is true whenever $0 < \phi \leq 0.6180$. Doing the same for h_2 provides a bound of $0 < \phi \leq 0.7549$, so we take the tighter bound of 0.618. □

Proof (Theorem 5) We begin by noting that, because of Lemma 8, we only need to show that assortative interviewing is an equilibrium when $0 < \phi \leq 0.265074$ for resident r_2, and it will hold for all r_i. Furthermore, by Lemma 4, we only need to prove that $\{h_1, h_2\}$ dominates both $\{h_1, h_3\}$ and $\{h_2, h_3\}$ to show that it dominates all other possible interviewing sets of size 2.

We prove that choosing $\{h_1, h_2\}$ is better than choosing $\{h_2, h_3\}$, for all values of ϕ such that $0 < \phi \leq 0.265074$. We prove this by summing over all possible preference rankings that induce a specific permutation of the alternatives h_1, h_2, h_3. We then pair these summed permutations in such a manner that makes it easy to find a lower bound for $u_{r_2}(\{h_1, h_2\}) - u_{r_2}(\{h_2, h_3\})$. This lower bound is entirely in terms of ϕ, meaning that for any ϕ such that this bound is above 0, it will be above 0 for any market size n.

We look at three cases, pairing all possible permutations of h_1, h_2, h_3 as follows:
Case 1: all rankings η consistent with $h_2 \succ h_1 \succ h_3$ or η' consistent with $h_2 \succ h_3 \succ h_1$;
Case 2: all rankings η consistent with $h_1 \succ h_2 \succ h_3$ or η' consistent with $h_3 \succ h_2 \succ h_1$;
Case 3: all rankings η consistent with $h_1 \succ h_3 \succ h_2$ or η' consistent with $h_3 \succ h_1 \succ h_2$.

Note that as we have enumerated all possible permutations of h_1, h_2, h_3, these three cases generate every ranking in $H_{\succ}$. Furthermore, for any one of the three cases, we can iterate only over all possible rankings η that are consistent with the first member of the pair, and generate the ranking η' consistent with the second member of the pair by simply swapping two alternatives in the rank. Moreover, given some η, the number of discordant pairs in η' is simply the number of discordant pairs in η, plus the number of additional discordant pairs between h_1, h_2, h_3 caused by swapping the two alternatives.

For clarity, let $u_{r_2}(\{h_1, h_2\}) - u_{r_2}(\{h_2, h_3\}) = U_1 + U_2 + U_3$, where U_1, U_2, U_3 correspond to our three cases. We also introduce the notation $P_{\mu(r_i)}(h)$ to denote the probability that r_i is matched to hospital h under matching μ. That is, $P_{\mu(r_i)}(h) = P(\mu(r_i) = h)$.

Case 1. Because we have fixed $h_2 \succ h_1 \succ h_3$ or $h_2 \succ h_3 \succ h_1$, we know exactly what r_2's match will be. As we know r_1's interviewing set ($\{h_1, h_2\}$), and the distribution r_1's preferences are drawn *i.i.d.*, we know the likelihood that either h_1 or h_2 is available; by Lemma 1, $P(\mu(r_1) = h_1) = \frac{1}{1+\phi}$. Using this information, the payoff function, and the definition of η, η', we find a lower bound:

$$U_1 = \sum_{\eta \in P(H)^{h_2 \succ h_1 \succ h_3}} P_{\mu(r_1)}(h_2) \Big[(v(h_1, \eta) - v(h_3, \eta)) P(\eta | D^{\phi,\sigma}) + (v(h_1, \eta') - v(h_3, \eta')) P(\eta' | D^{\phi,\sigma}) \Big]$$

$$U_1 \geq P_{\mu(r_1)}(h_2)(1)(1-\phi) P(h_2 \succ h_1 \succ h_3) = \left(\frac{\phi}{1+\phi}\right)\left(\frac{\phi}{(1+\phi)(1+\phi+\phi^2)}\right)(1-\phi) \quad (8)$$

Case 2. We fix $h_1 \succ h_2 \succ h_3$ or $h_3 \succ h_2 \succ h_1$. This case is analogous to Case 1:

$$U_2 = \sum_{\eta \in P(H)^{h_1 \succ h_2 \succ h_3}} P_{\mu(r_1)}(h_1)\Big[(0)P(\eta|D^{\phi,\sigma}) + (v(h_2,\eta') - v(h_3,\eta'))P(\eta'|D^{\phi,\sigma})\Big]$$
$$+ P_{\mu(r_1)}(h_2)\Big[(v(h_1,\eta) - v(h_3,\eta))P(\eta|D^{\phi,\sigma}) + (v(h_1,\eta') - v(h_3,\eta'))P(\eta'|D^{\phi,\sigma})\Big]$$
$$U_2 \geq P(h_1 \succ h_2 \succ h_3)\frac{2}{1+\phi}(\phi - \phi^3 - \phi^4) \tag{9}$$

Case 3. We fix $h_1 \succ h_3 \succ h_2$ or $h_3 \succ h_1 \succ h_2$. Again, we look at pairs of rankings η, η', where η is consistent with $h_1 \succ h_3 \succ h_2$, and η' is identical to η, except $\text{rank}(h_1, \eta) = \text{rank}(h_3, \eta')$, and $\text{rank}(h_3, \eta) = \text{rank}(h_1, \eta')$.

Then, as before, we sum over all possible rankings consistent with $h_1 \succ h_3 \succ h_2$, but we break this into two subcases, so that $U_3 = U_{3a} + U_{3b}$:

$$U_{3a} = \sum_{\eta \in H_{\succ|h_1 \succ h_3 \succ h_2}} P_{\mu(r_1)}(h_1)[(v(h_2,\eta) - v(h_3,\eta))P(\eta|D^{\phi,\sigma}) + (v(h_2,\eta') - v(h_3,\eta'))P(\eta'|D^{\phi,\sigma})]$$
$$U_{3b} = \sum_{\eta \in H_{\succ|h_1 \succ h_3 \succ h_2}} P_{\mu(r_1)}(h_2)[(v(h_1,\eta) - v(h_3,\eta))P(\eta|D^{\phi,\sigma}) + (v(h_1,\eta') - v(h_3,\eta'))P(\eta'|D^{\phi,\sigma})]$$

Case U_{3b} is similar to Cases 1 and 2:

$$U_{3b} = \sum_{\eta \in P(H)^{h_1 \succ h_3 \succ h_2}} P_{\mu(r_1)}(h_2)[(v(h_1,\eta) - v(h_3,\eta))\frac{\phi^{d(\eta,\sigma)}}{Z} + (v(h_3,\eta) - v(h_1,\eta))\frac{\phi^{d(\eta,\sigma)+1}}{Z}$$
$$U_{3b} \geq \frac{\phi}{\phi+1}(1-\phi)P(h_1 \succ h_3 \succ h_2) \tag{10}$$

Case U_{3a}, however, is different from all other cases, in that *all* terms are negative. We note that U_{3a} as above is a monotonically decreasing function in terms of n. Thus, if U_{3a} converges as $n \to \infty$, we have found a lower bound for all n. Using this technique, we show the following bound holds:

$$U_{3a} \geq P_{\mu(r_1)}(h_1)\frac{-\phi}{(1+\phi)(1+\phi+\phi^2)}\Big(\frac{\phi}{(1-\phi)^4} + \frac{1}{3(1-\phi)^3} + \frac{2}{3}\Big)(1+\phi) \tag{11}$$

We have considered all cases, and can now combine them together. We add the bounds for U_1 (Eq. 8), U_2 (Eq. 9), U_{3a} (Eq. 11), and U_{3b} (Eq. 10). We simplify using Corollaries 1 and 2, giving us:

$$u_{r_2}(\{h_1, h_2\}) - u_{r_2}(\{h_2, h_3\})$$
$$\geq \frac{\phi^2}{(1+\phi)(1+\phi)(1+\phi+\phi^2)}(1-\phi) + \frac{2(\phi - \phi^3 - \phi^4)}{(1+\phi)(1+\phi)(1+\phi+\phi^2)}$$
$$- \frac{\phi}{(1+\phi)(1+\phi)(1+\phi+\phi^2)}\Big(\frac{\phi}{(1-\phi)^4} + \frac{1}{3(1-\phi)^3} + \frac{2}{3}\Big)(1+\phi)$$
$$+ \frac{\phi^2}{(1+\phi)(1+\phi)(1+\phi+\phi^2)}(1-\phi) \tag{12}$$

Thus, Eq. 12 gives us a lower bound for the difference in expected utility between $\{h_1, h_2\}$ and $\{h_2, h_3\}$ for resident r_2, for all n. Using numerical methods to approximate the roots of Eq. 12, we get that there is a root at 0, and a root at $\phi \approx 0.265074$.

As the calculations are analogous, we omit the discussion of their derivation, but it can be shown that:

$$u_{r_2}(\{h_1, h_2\}) - u_{r_2}(\{h_1, h_3\}) \geq \frac{1}{(1+\phi)(1+\phi+\phi^2)} \times \left[1 + \phi - 2\phi^2 - 2\phi^3 - 2\phi^3\left(\frac{\phi}{(1-\phi)^4} + \frac{1}{3(1-\phi)^3} + \frac{2}{3}\right)\right] \tag{13}$$

Using numerical methods, it can be shown that this is positive for $0 < \phi < 0.413633$.

Thus, for the interval $0 < \phi \leq 0.265074$, we have shown that r_2's best move in this interval is to interview with $\{h_1, h_2\}$. Then, by Lemma 8, this is an equilibrium for all r_i as required. □

Proof (Theorem 6) We provide a counterexample for $n = 4, k = 3$. Suppose residents r_1 and r_2 interview assortatively, both interviewing with $S = \{h_1, h_2, h_3\}$. We show that for resident r_3, interviewing with interviewing set $S' = \{h_2, h_3, h_4\}$ dominates interviewing with $S = \{h_1, h_2, h_3\}$ for all ϕ.

By Lemma 4, it is sufficient to show that if the marginal value in interviewing with h_4 dominates the marginal value in interviewing with h_1 (as these two sets only differ by these two items), then interviewing with $\{h_2, h_3, h_4\}$ dominates $\{h_1, h_2, h_3\}$. We thus instantiate Eq. 3 for $n = 4$, $k = 3$, S, and S' as above for resident r_3. Note that $Z = (1+\phi)(1+\phi+\phi^2)(1+\phi+\phi^3)$. Let $\mathbb{E}(u(h_i, S))$ denote the expected marginal value in interviewing alternative h_i in set S; remember $v(s_i) = 5 - i$.

$$\mathbb{E}(u(h_1, S)) = \sum_{\eta \in H_{\succ}} v(h_1, \eta) P(\mu(h_1) = r_3 | S, \eta, D^{\phi,\sigma}) P(\eta | D^{\phi,\sigma}) \tag{14}$$

$$\mathbb{E}(u(h_4, S')) = \sum_{\eta \in H_{\succ}} v(h_4, \eta) P(\mu(h_4) = r_3 | S', \eta, D^{\phi,\sigma}) P(\eta | D^{\phi,\sigma}) \tag{15}$$

As before, Eq. 14 is simply the probability that h_1 is available times the expected value of h_1. As noted, $\mathbb{E}(v(h_1)|D^{\phi,\sigma}) = \sum_{i=1}^{4} P(h_1 \text{ in } s_i) \cdot v(s_i) = \sum_{i=1}^{4} P(h_1 \text{ in } s_i) \cdot (5 - i)$. However, using Lemma 2, we know that $P(h_1 \text{ in } s_i) = \frac{\phi^{i-1}}{1+\phi+\phi^2+\phi^3}$, giving:

$$\mathbb{E}(u(h_1, S)) = P(h_1)\mathbb{E}(v(h_1)|D^{\phi,\sigma}) = P(h_1)\frac{4 + 3\phi + 2\phi^2 + \phi^3}{1 + \phi + \phi^2 + \phi^3} \tag{16}$$

Let $P(h_i \text{ taken})$ denote the probability that either r_1 is matched to h_i, or r_2 is matched to h_i (i.e., h_i is taken by the time we get to resident r_3). Also let $P(\mu(r_3) = h_4 | h_4 \text{ in } s_i)$ denote the probability that r_3 is matched to h_4 if h_4 is in slot s_i in r_3's ranking. This is easily calculable by enumerating over the subset of possible rankings such that this occurs, given that r_1 and r_2 have already taken certain alternatives. Then, using Lemma 2 again and an analogous approach as above, we adapt Eq. 15:

$$\begin{aligned}\mathbb{E}(u(h_4, S')) &= \sum_{i=1}^{4} v(s_i) P(h_4 \text{ in } s_i) P(\mu(r_3) = h_4 | h_4 \text{ in } s_i) \\ &= \frac{4\phi^3}{1+\phi+\phi^2+\phi^3} \\ &\quad + \frac{3}{Z}\big(\phi^2+\phi^3+P(h_2 \text{ taken})(\phi^3+\phi^4)+P(h_3 \text{ taken})(\phi^4+\phi^5)\big) \\ &\quad + \frac{2}{Z}\big(P(h_2 \text{ taken})(\phi+\phi^2)+P(h_3 \text{ taken})(\phi^2+\phi^3)+P(h_1)(\phi^3+\phi^4)\big) \\ &\quad + \frac{P(h_1)}{1+\phi+\phi^2+\phi^3} \end{aligned} \tag{17}$$

As we assume that residents r_1 and r_2 both interview with S, the probability that h_1 is available, or h_2 (resp. h_3) is taken is the same across both $\mathbb{E}(u(h_1, S))$ and $\mathbb{E}(u(h_4, S'))$. We instantiate these as follows, by determining the probability that r_1 is matched to some hospital h_j other than h^*, and enumerate the probabilities of all rankings such that r_2 is matched to some hospital $h'_j \neq h^*$ given that r_1 is matched to h_j:

$$\begin{aligned} P(h_1) = &P(\mu(r_1)=h_2|S,D^{\phi,\sigma})\Big(\frac{\phi^2}{1+\phi+\phi^2+\phi^3}+\frac{\phi^2+\phi^3+\phi^4+2\phi^5+\phi^6}{Z}\Big) \\ &+P(\mu(r_1)=h_3|S,D^{\phi,\sigma})\Big(\frac{\phi}{1+\phi+\phi^2+\phi^3}+\frac{\phi^3+2\phi^4+2\phi^5+\phi^6}{Z}\Big) \\ P(h_2 \text{ taken}) = &P(\mu(r_1)=h_2|S,D^{\phi,\sigma})+P(\mu(r_1)=h_3|S,D^{\phi,\sigma})\Big(\frac{\phi}{1+\phi+\phi^2+\phi^3}+\frac{\phi^3+2\phi^4+2\phi^5+\phi^6}{Z}\Big) \\ &+P(\mu(r_1)=h_1|S,D^{\phi,\sigma})\Big(\frac{\phi}{1+\phi+\phi^2+\phi^3}+\frac{1+\phi+\phi^2+\phi^3+\phi^4+\phi^5}{Z}\Big) \\ P(h_3 \text{ taken}) = &P(\mu(r_1)=h_3|S,D^{\phi,\sigma})+P(\mu(r_1)=h_2)\Big(\frac{\phi^2}{1+\phi+\phi^2+\phi^3}+\frac{\phi^2+\phi^3+\phi^4+2\phi^5+\phi^6}{Z}\Big) \\ &+P(\mu(r_1)=h_1|S,D^{\phi,\sigma})\Big(\frac{\phi^2}{1+\phi+\phi^2+\phi^3}+\frac{1+\phi+\phi^2+\phi^3+\phi^4+\phi^5+\phi^6}{Z}\Big) \end{aligned}$$

It is also possible to calculate exact values for the probability that r_1 is matched to h_1, h_2, h_3. We do this by calculating the probability that alternative is first, or the probability that alternative is second, and h_4 is first:

$$\begin{aligned} P(\mu(r_1) = h_1 | S, D^{\phi,\sigma}) &= P(h_1 \text{ in } s_1) + P(h_1 \text{ in } s_2 \text{ and } h_4 \text{ in } s_1) \\ &= \frac{1}{1+\phi+\phi^2+\phi^3} + \frac{\phi^3+\phi^4}{Z} \\ P(\mu(r_1) = h_2 | S, D^{\phi,\sigma}) &= P(h_2 \text{ in } s_1) + P(h_2 \text{ in } s_2 \text{ and } h_4 \text{ in } s_1) \\ &= \frac{\phi}{1+\phi+\phi^2+\phi^3} + \frac{\phi^4+\phi^5}{Z} \\ P(\mu(r_1) = h_3 | S, D^{\phi,\sigma}) &= P(h_3 \text{ in } s_1) + P(h_3 \text{ in } s_2 \text{ and } h_4 \text{ in } s_1) \\ &= \frac{\phi^2}{1+\phi+\phi^2+\phi^3} + \frac{\phi^5+\phi^6}{Z} \end{aligned}$$

By combining the equations for the probabilities we are left with two equations depending only on ϕ. Moreover, after instantiating $\mathbb{E}(u(h_1, S))$ and $\mathbb{E}(u(h_4, S')$ above, we note that both functions are continuous on the interval (0, 1]. Using numerical techniques, it can be shown that there are no zeros for the function $\mathbb{E}(u(h_1, S)) - \mathbb{E}(u(h_4, S'))$ on the interval (0, 1], and the function is negative on the interval (0, 1] providing the counterexample as required. □

Proof (Theorem 7) For $k = 3$, we simply check the constraint

$$P(h_j) \geq \phi^{k-j+1}$$

from Lemma 6 with $h_j = h_1, h_2, h_3$. We find that the marginal contribution from h_1 is less than the marginal contribution of h_2 or h_3, and thus only present the calculation for h_1. We directly compute $P(h_1)$, by multiplying the probability that r_1 did not take h_1, and multiplying it by the probability that r_2 did not take h_1, given that r_1 *also* did not take h_1. To calculate this we enumerate the probabilities of any possible rankings:

$$P(h_1) = P(\mu(r_1) \neq h_1)P(\mu(r_2) \neq h_1|\mu(r_1) \neq h_1)$$
$$P(h_1) = (\frac{\phi + 2\phi^2 + \phi^3}{(1+\phi)(1+\phi+\phi^2)})(\frac{\phi^2 + 2\phi^3}{(1+\phi+\phi^2)})$$

The first parenthesis is using Corollary 2, and the second the probability h_3 is preferred over h_2 using Corollary 1. Using numerical methods to find the roots of $P(h_1) - \phi^3$, we can show that above constraint holds when $0 < \phi \leq 0.5462$. □

D.1 Assortative equilibria when $k \geq 4$

We start by providing an additional lemma regarding a bound on the availability of any given alternative h_i at the time resident r_k is being matched by the mechanism to their favourite remaining hospital. This probability is dependent on ϕ: for any hospital h_i such that $i < k$, as $\phi \to 1$, the probability h_i is available goes to $\frac{1}{k}$; as $\phi \to 0$, this probability goes to 0. Instead of looking at the probability directly, we look at the probability that a preference profile will admit a stable match such that h_i is available, and bound that.

Lemma 9 *Given a Mallows model with dispersion parameter ϕ, assortative interviewing for residents $r_1, \ldots, r_{k-1}$, and a hospital $h_i \in \{h_1, \ldots, h_k\}$ (i.e., the residents' interview set), then any profile $\eta_1, \ldots, \eta_{k-1} \in D^{\phi,\sigma}$ of $k-1$ preferences (for $r_1, \ldots, r_{k-1}$) such that h_i is available for r_k has probability $P(r_1 = \eta_1, r_2 = \eta_2, \ldots, r_{k-1} = \eta_{k-1}) \geq \frac{\phi^\gamma}{Z^{k-1}}$, where $\gamma = \sum_{j=1}^{k-i} j$ and Z is the normalizing factor for a Mallows model.*

Proof In order for h_i to be available, there need to be $r'_{i+1}, \ldots, r'_k$ with preference orders $\eta_{i+1}, \ldots, \eta_k \in D^{\phi,\sigma}$ such that they were assigned hospitals $h_{i+1}, \ldots, h_k$. Hence, at the very least, $h_{i+1} \succ_{\eta_{i+1}} h_i, \ldots, h_k \succ_{\eta_k} h_i$. According to Lemma 3, the probability for each of these events is at most $\frac{\phi}{Z}, \ldots, \frac{\phi^{k-i}}{Z}$ (respectively). Since they are independent of each other, and since the maximum probability for any particular $\eta \in D^{\phi,\sigma}$ is $\frac{1}{Z}$, the probability of a particular preference set occurring in which h_i is available is at least $\frac{\phi^\gamma}{Z^{k-1}}$. □

Proof (Theorem 8) Looking at the condition of Lemma 5 (recall $S' = \{h_1, \ldots, h_k\} \backslash \{h_j\} \cup \{h_{k+1}\}$ for any $h_j \in \{h_1, \ldots, h_k\}$)

$$P(h_j)\mathbb{E}(v(h_j)|D^{\phi,\sigma}) \geq P(h_j)\mathbb{E}(v(h_{k+1})|D^{\phi,\sigma}) + \sum_{\eta \in H_\succ} P(\eta|D^{\phi,\sigma})\left[\sum_{h_i \in S'} P(h_i)\mathbb{1}(h_{k+1} \succ_\eta h_i)v(h_{k+1}, \eta)\right]$$

We again begin by expanding the value expectation ($\mathbb{E}$). This can be divided into n different inequalities:

$$P(h_j)P(h_j \text{ in } s_1)v(s_1) \geq v(s_1)\left[P(h_j)P(h_{k+1} \text{ in } s_1) + \sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_1}} P(\eta|D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i)\mathbb{1}(h_{k+1} \succ_\eta h_i)\right]$$

$$\vdots$$

$$P(h_j)P(h_j \text{ in } s_{n-1})v(s_{n-1}) \geq v(s_{n-1})\left[P(h_j)P(h_{k+1} \text{ in } s_{n-1}) + \sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_{n-1}}} P(\eta|D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i)\mathbb{1}(h_{k+1} \succ_\eta h_i)\right]$$

$$P(h_j)P(h_j \text{ in } s_n)v(s_n) \geq v(s_n)P(h_j)P(h_{k+1} \text{ in } s_n)$$

We shall show that under the theorem's assumptions, none of these inequalities hold for h_1, and therefore the general inequality (Lemma 5) does not hold.

Note that for each inequality we can simply ignore $v(s_\ell)$ $(1 \leq \ell \leq n)$, since they appear on both sides of the inequality. The assumption of the theorem, since we are using plurality, is that the first inequality does not hold, i.e.,

$$P(h_1)P(h_1 \text{ in } s_1) < P(h_1)P(h_{k+1} \text{ in } s_1) + \sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_1}} P(\eta|D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i)\mathbb{1}(h_{k+1} \succ_\eta h_i)$$

As noted in Observation 1 (end of Sect. 3), for any $1 < \ell \leq k$ the probability of h_1 being in any spot s_ℓ is monotonically decreasing with ℓ, while the probability of h_{k+1} being in spot s_ℓ is monotonically increasing with ℓ. Hence, $P(h_1)P(h_1 \text{ in } s_1) > P(h_1)P(h_1 \text{ in } s_\ell)$.

Similarly, $P(h_1)P(h_{k+1} \text{ in } s_1) < P(h_1)P(h_{k+1} \text{ in } s_\ell)$. We analogously see that:

$$\sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_1}} P(\eta | D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_\eta h_i) < \sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_\ell}} P(\eta | D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_\eta h_i)$$

Simply put, the LHS gets smaller, while the RHS increases. Hence, for $1 \leq \ell \leq k$:

$$P(h_1)P(h_1 \text{ in } s_\ell) < P(h_1)P(h_{k+1} \text{ in } s_\ell) + \sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_\ell}} P(\eta | D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_\eta h_i)$$

By Observation 1, for any $\ell > k$, $P(h_1 \text{ in } s_\ell) < P(h_{k+1} \text{ in } s_\ell)$ which gives us:

$$\begin{aligned} &P(h_1)P(h_1 \text{ in } s_\ell) < P(h_1)P(h_{k+1} \text{ in } s_\ell) \implies \\ &P(h_1)P(h_1 \text{ in } s_\ell) < P(h_1)P(h_{k+1} \text{ in } s_\ell) + \\ &\quad + \sum_{\substack{\eta \in H_\succ | \\ h_{k+1} \text{ in } s_\ell}} P(\eta | D^{\phi,\sigma}) \sum_{h_i \in S'} P(h_i) \mathbb{1}(h_{k+1} \succ_\eta h_i) \end{aligned}$$

Starting with the assumption that assortative interviewing does not hold for plurality, we show that none of the inequalities above hold for any slot s_ℓ, and therefore that the condition in Lemma 5 does not hold for $j = h_1$ for any valuation function. □

Proof (Theorem 10) By Theorem 8, if assortative interviewing is not an equilibrium for plurality due to h_1, it is never an equilibrium for any scoring rule. If we compute the marginal contribution from some $h^* \in \{h_1, h_2, h_3, h_4\}$, and the contribution from h^* is strictly less than the contribution from h_5 for any ϕ, assortative interviewing is not an equilibrium for $k = 4$ and plurality. We find that the contribution from h_1 is less than the marginal contribution from h_5.

To calculate $P(h_1)$, we simply iterate over all six possible allocations for r_1, r_2, r_3 such that h_1 is not taken, and directly calculate the probabilities of each ranking profile for r_1, r_2, r_3 that allows that to happen. In the interest of clarity, we only provide a symbolic representation. Let a permutation of h_2, h_3, h_4 be denoted as (a_1, a_2, a_3), and let A be the set of all such permutations (i.e., $(a_1, a_2, a_3) \in A$ is a particular permutation of h_2, h_3, h_4).

$$P(h_1) = \sum_{(a_1,a_2,a_3) \in A} P(\mu(r_1) = a_1) P(\mu(r_2) = a_2 | \mu(r_1) = a_1) P(\mu(r_3) = a_3 | \mu(r_1) = a_1, \mu(r_2) = a_2)$$

We instantiate the above equation using the probabilities of each potential match, and use numerical methods to show the function $P(h_1) - \phi^4$ is negative for any ϕ in $0 < \phi < 1$. □

Proof (Theorem 9)

Due to Theorem 8, it is enough to show there is no assortative equilibrium under plurality (and that h_1 violates Lemma 5's condition). We use the simplification from Lemma 6: $P(h_j) \geq \phi^{k-j+1}$, and we will show it does not hold. Appealing to Lemma 9, we know $P(h_j)$ is of the form:

$$P(h_j) = \frac{X(k)}{Z^{k-1}}\phi^{\sum_{i=1}^{k-j} i} + \frac{X^1(k)}{Z^{k-1}}\phi^{1+\sum_{i=1}^{k-j} i} + \cdots + \frac{X^\ell(k)}{Z^{k-1}}\phi^{(k\sum_{i=1}^{k-j} i)-1} + \frac{1}{Z^{k-1}}\phi^{k\sum_{i=1}^{k-j} i} \tag{18}$$

($X(k), X^1(k), \ldots, X^\ell(k)$ are functions that calculate the number of different sets of possible preference orders for $r_1, \ldots, r_k$, with each set being a particular distance from the ground truth σ, thus having the probability $\phi^{\sum_{i=1}^{k-j} i}$ for $X(k)$, $\phi^{1+\sum_{i=1}^{k-j} i}$ for $X^1(k)$, etc.)

When $\phi \to 0$, $Z^{k-1} \to 1$, Eq. 18 becomes $P(h_j) \to X(k)\phi^{\sum_{i=1}^{k-j} i}$. In particular, there is ε', such that $P(h_1) < X(k)\phi^{(\sum_{i=1}^{k-j} i)-1}$, and there is $\varepsilon = \min(\varepsilon', \frac{1}{X(k)})$ such that for $\phi < \varepsilon$, for $k > 3$:

$$\phi^{k-j+1} \geq \phi^k \geq \phi^{\left(\sum_{i=1}^{k-j} i\right)-2} > X(k)\phi^{\left(\sum_{i=1}^{k-j} i\right)-1} > P(h_1)$$

This contradicts the condition stated in Lemma 6. □

Appendix E Proofs from Sect. 6

Proof (**Theorem 11**) We look at the top k hospitals—$h_1, \ldots, h_k$. If the same k residents interview in all of them (and none others do), these are strongly/pseudo-strongly assortative, and all the top k hospitals are taken, and we can move on and ignore these k hospitals (and the k residents interviewing there) until we find the first set of k hospitals that do not all interview strongly/pseudo-strongly assortative. For notational simplicity, let us assume that $r_1, \ldots, r_k$ are not strongly assortative with hospitals $h_1, \ldots, h_k$. Hence, at least one of $r_1, \ldots, r_k$ interviews at $\{h_j, h_{j+1}, \ldots, h_{j+k-1}\}$ for some $2 \leq j \leq n-k+1$. Thus, the number of residents from $r_1, \ldots, r_k$ interviewing at h_1 is smaller than k.

- If the number of interviewees at h_1 is $> k$, this means more than k residents are interviewing at the top k hospitals—$h_1, \ldots, h_k$—so at least one of these residents will not get a hospital.
- If the number of interviewees at h_1 is exactly k, recall that for this to be a weak assortative equilibrium there is a resident in $r_1, \ldots, r_k$ that is not interviewing at h_1 but rather interviews at $h_j, \ldots, h_{j+k-1}$ for $2 \leq j \leq k$. Thus, there is a positive probability that this resident will prefer hospitals from the top k hospitals, and will get one of them. However, since k residents are interviewing at h_1, we know that in addition to the resident that does not interview there, there are k residents interviewing at h_1, i.e,. at the top k hospitals (since this an assortative

strategy, anyone interviewing at h_1 interviews at $h_1, \dots, h_k$). Since one of the top k hospitals is taken by the resident not interviewing at h_1, there are not enough hospitals for the k residents interviewing at $h_1, \dots, h_k$, leaving at least one without a hospital.

- If the number of interviewees at h_1 is $< k$ but larger than 1, let us examine the resident with the highest index that interviews at h_1, which we shall denote $\bar{r}$. We examine this resident's choices:
  - If there is no positive probability that any hospital $h_1, h_2, \dots, h_k$ is available, $\bar{r}$ will be left without any hospital.
  - If there is a positive probability that h_1 and some other hospital h_i $(2 \le i \le k)$ are both available to $\bar{r}$, there is a possibility $\bar{r}$ prefers h_i, and thus there is positive probability h_1 will have no resident that choses to go there. Since we have n residents and n hospitals, if h_1 has no resident, there is a resident without a hospital.
  - If there is only a positive probability that h_1 is available and no h_i $(2 \le i \le k)$ has any positive probability of being available, it means that even if the weakly-assortative resident in $r_1, \dots, r_k$ that does not interview at h_1 chooses to go to some h_j, $j > k$, there is still no alternative hospital that will be available for $\bar{r}$ to choose from if h_1 is available. But this means that there is a choice for each resident before the one we analyze that gets a hospital, leaving only h_1 available, i.e., a previous resident interviewing at h_1 chose some hospital $h_{j'} \in \{h_2, \dots, h_k\}$. Therefore, there is a probability that the residents' preferences are such that the previous agent interviewing at h_1 chose h_1, and the weakly-assortative resident that chose h_j can chose $h_{j'}$, and no other residents' choices change, leaving our resident without any available hospital.
- If the number of interviewees at h_1 is exactly 1, if this resident does not choose h_1 but some other hospital, we have $n - 1$ residents (everyone except the one interviewing at h_1) interviewing at $n - 2$ hospitals (all hospitals except h_1, which they didn't interview at, and the one our resident chose), ensuring some will not find a hospital. If we assume the only resident interviewing at h_1 chose h_1, we can simply ignore this resident and the hospital h_1 and repeat this proof with $n - 1$ agents and hospitals.

□

Proof (Theorem 12) Recall that agents wish to maximize their expected utility. Beyond the probability of ranking each hospital over another (which depends on the ground truth σ and ϕ), this utility also depends on each hospital's availability based on the previous residents' choices. For example, when interviewing in a set of hospitals only one of which is free (e.g., when r_1 interviews at h_1 and h_2, if r_2 interviews at h_1 and h_2, r_2 will get the hospital r_1 did not choose), the resident's utility when interviewing at set S of hospitals is $\sum_{h \in S} P(h) E(v(h))$. Therefore, we need to keep in mind the probability of hospitals being available, which changes following each resident's interviews, while the expected value from each hospital is the same for every resident (as they are sampled from the same distribution).

Resident r_1 will always bid on h_1 and h_2. If r_2 does as well, then we have a strong assortative equilibrium. Similarly, if r_2 interviews h_3 and h_4, r_3 would interview at h_1 and h_2 (since $\phi < 1$, h_1 and h_2 are preferable to h_3 and h_4 as well as to h_2 and h_3), hence r_4 would interview at h_3 and h_4 resulting in a pseudo-strong equilibrium. Hence, in order to have at least one resident with a weak assortative strategy, r_2 interviews at h_2 and h_3. Before starting the analysis of r_3 let us note the probability of each hospital being available after r_1 and r_2 have chosen their interviews.

h_1: $\frac{\phi}{1+\phi}$ (i.e., r_1 choosing h_2).
h_2: $\frac{1}{1+\phi}\frac{\phi}{1+\phi} = \frac{\phi}{(1+\phi)^2}$ (i.e., r_1 choosing h_1 and r_2 choosing h_3).
h_3: $\frac{1}{1+\phi}\frac{1}{1+\phi} = \frac{1}{(1+\phi)^2}$ (i.e., r_1 choosing h_1 and r_2 choosing h_2).
h_4: 1 (since neither r_1 or r_2 interview there, h_4 will not be taken by them).

The third resident, r_3 may interview in h_1 and h_2, h_2 and h_3 or h_3 and h_4. We first show that r_3 will not interview in h_1 and h_2. We have:

$$u_{r_3}(h_1, h_2) = \frac{1}{1+\phi}\frac{\phi}{1+\phi}E(v(h_2)) + \frac{\phi}{1+\phi}E(v(h_1))$$

This is because r_3 will get h_2 if r_1 chooses h_1 and r_2 chooses h_3, while r_3 will get h_1 if r_1 goes for h_2.

Interviewing in h_1 and h_3 (which is not weakly assortative) has the utility:

$$u_{r_3}(h_1, h_3) = \frac{1}{1+\phi}\frac{1}{1+\phi}E(v(h_3)) + \frac{\phi}{1+\phi}E(v(h_1))$$

If interviewing in h_1 and h_2 is preferable, these equations imply $E(v(h_3)) < \phi E(v(h_2))$. However, simple analysis shows that this is impossible: the ratio of expected values of consecutive hospitals cannot be less than ϕ (the ratio $\frac{E(v(h_3))}{E(v(h_2))}$ is exactly ϕ when the scoring function is plurality; anything that gives non-zero values to hospitals not in first place will make the ratio higher than ϕ). So interviewing in h_1 and h_2 cannot be a strategy in an equilibrium. Furthermore, as is clear from the availability probabilities, since the probability of h_1 being available is higher than that of h_2, it makes no sense to interview in h_2 and h_3 and not in h_1 and h_3 (the resident increases their chance of getting a better hospital than h_3 by interviewing in h_1).

If all agents are weakly assortative, this leaves the case of r_3 interviewing in h_3 and h_4.[8] Once again, let us look at the probability each hospital is available for r_4:

Before starting the analysis of r_3 let us note the probability of each hospital being available:

h_1: $\frac{\phi}{1+\phi}$ (i.e., r_1 choosing h_2).
h_2: $\frac{1}{1+\phi}\frac{\phi}{1+\phi} = \frac{\phi}{(1+\phi)^2}$ (i.e., r_1 choosing h_1 and r_2 choosing h_3).
h_3: $\frac{1}{1+\phi}\frac{1}{1+\phi}\frac{\phi}{1+\phi} = \frac{\phi}{(1+\phi)^3}$ (i.e., r_1 choosing h_1, r_2 choosing h_2, and r_3 choosing h_4).

[8] Since h_1 and h_4 were not chosen, we know the probability of availability of h_1 must be lower than that of h_3, so $\frac{1}{1+\phi} < \frac{1}{(1+\phi)^2} \Rightarrow \phi < \frac{\sqrt{5}-1}{2}$.

h_4: $\frac{1}{1+\phi}\frac{1}{1+\phi}\frac{1}{1+\phi} = \frac{1}{(1+\phi)^3}$ (i.e., r_1 choosing h_1, r_2 choosing h_2, and r_3 choosing h_3).

This situation can be viewed graphically in Fig. 1. The availability of h_1 is better than that of h_2 and h_3 for any ϕ, and therefore, it will always make sense interviewing at h_1 over h_2 and h_3. So if any interview set is chosen that does not include h_1, it is beneficial to swap one of those for h_1. We now need to show that r_4 interviewing with h_1 and h_2 (which is the only weakly assortative strategy that includes h_1) is not optimal.[9]

It is clear that if $E(v(h_2))$ is close in value to $E(v(h_4))$ then h_1 and h_4 have a higher expected utility than h_1 and h_2, thanks to h_4's greater probability of being available. Notice that r_4's utility from h_1 and h_2 is the same as that of r_3 with those same hospitals, and r_3's strategy indicates that it is preferable to chose h_4 and not interview at h_1 or h_2 at all. If $E(v(h_4))$ was small compared to $E(v(h_2))$ (and thus compared to $E(v(h_1))$ as well), r_3's choice of h_4 would not have happened. Analytical tools can be used to verify that r_4's choice is either h_1 and h_4 (so not weakly assortiative), or r_3 would not have interviewed at h_3 and h_4 (leading to the problems detailed above for r_3, and hence not be weakly assortiative). □

References

Ajayi KF (2011) School choice and educational mobility: Lessons from secondary school applications in Ghana. Tech. rep., University of California-Berkeley Working Paper

Ali SN, Shorrer RI (2023) Hedging in the College Application Problem

Anderson KD, Dorough MM, Stein CR et al (2000) The urology residency matching program in practice. J Urol 163(6):1878–1887

Andersson T (2019) Refugee matching as a market design application. In: Laslier JF, Moulin H, Sanver MR et al (eds) The Future of Economic Design. Springer, New York, pp 445–450

Beyhaghi H, Sabán D, Tardos É (2017) Effect of selfish choices in deferred acceptance with short lists. CoRR arXiv:abs/1701.00849

Beyhaghi H, Tardos É (2019) Effect of limited number of interviews on matching markets. Working Paper

Brandt F, Conitzer V, Endriss U et al (eds) (2016) Handbook of computational social choice. Cambridge University Press, Cambridge, UK

Brilliantova A, Hosseini H (2022) Fair stable matching meets correlated preferences. In: Proceedings of the 21st international conference on autonomous agents and multiagent systems, pp 190–198

Calsamiglia C, Haeringer G, Klijn F (2010) Constrained school choice: an experimental study. Am Econ Rev 100(4):1860–1874

Calsamiglia C, Mora FM, Miralles A (2020) School choice design, risk aversion, and cardinal segregation. Discussion Paper Series 13464, Institute for the Study of Labor (IZA), Bonn, Germany

Chade H, Smith L (2006) Simultaneous search. Econometrica 74(5):1293–1307

Chade H, Lewis G, Smith L (2014) Student portfolios and the college admissions problem. Rev Econ Stud 81(3):971–1002

Chen L, Pereyra JS (2015) Self-selection in school choice: theory and evidence from Mexico City high school match. Working paper

Coles P, Shorrer R (2014) Optimal truncation in matching markets. Games Econ Behav 87:591–615

Drummond J, Boutilier C (2013) Elicitation and approximately stable matching with partial preferences. In: Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence (IJCAI-13), Beijing. AAAI Press, Beijing, pp 97–105

[9] The probability of availability for h_4 is also higher than that of h_2 and h_3, since given r_3's choice to interview h_3 and not h_1 we know h_3's probability of availability ($\frac{1}{(1+\phi)^2}$) for r_3 is higher than that of h_1 for r_3 ($\frac{\phi}{1+\phi}$), and therefore the availability of h_4 for r_4 ($\frac{1}{(1+\phi)^3}$), is higher than that of h_2/h_3 for r_4 (i.e., $\frac{\phi}{(1+\phi)^2} / \frac{\phi}{(1+\phi)^3}$).

Drummond J, Boutilier C (2014) Preference elicitation and interview minimization in stable matchings. In: Proceedings of the twenty-eighth AAAI conference on artificial intelligence (AAAI-14). AAAI Press, Québec City, pp 645–653

Echenique F, González R, Wilson A et al (2022) Top of the batch: interviews and the match. Am Econ Rev Insight 4(2):223–238

Freeman R, Pritchard G, Wilson M (2021) Order symmetry: a new fairness criterion for assignment mechanisms. Tech. rep., Soc. https://doi.org/10.31235/osf.io/xt37c

Gale D, Shapley LS (1962) College admissions and the stability of marriage. Am Math Monthly 69(1):9–15

Garman MB, Kamien MI (1968) The paradox of voting: probability calculations. Behav Sci 13(4):306–316

Gelain M, Pini MS, Rossi F, et al (2009) Male optimal and unique stable marriages with partially ordered preferences. In: Proceedings of the international conference on collaborative agents-research and development (CARE-09). Springer, Berlin, pp 44–55

Goto M, Iwasaki A, Kawasaki Y et al (2016) Strategyproof matching with regional minimum and maximum quotas. Artif Intell 235:40–57

Gusfield D, Irving RW (1989) The stable marriage problem: structure and algorithms. MIT Press, Boston, MA

Haeringer G, Klijn F (2009) Constrained school choice. J Econ Theory 144(5):1921–1947

Hafalir IE (2008) Stability of marriage with externalities. Internat J Game Theory 37(3):353–369

Harless P, Manjunath V (2018) Learning matters: reappraising object allocation rules when agents' strategically investigate. Int Econ Rev 59(2):557–592

He Y, Magnac T (2017) Application costs and congestion in matching markets. TSE Working Paper 17-870, Toulouse School of Economics, Tolouse, France

Immorlica N, Mahdian M (2005) Marriage, honesty, and stability. In: Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, Vancouver, pp 53–62

Irving RW, Manlove DF, Scott S (2008) The stable matching problem with master preference lists. Discret Appl Math 156:2959–2977

Irving RW, Manlove DF, O'Malley G (2009) Stable marriage with ties and bounded length preference lists. J Discrete Algorithm 7(2):213–219

Kadam SV (2015) Interviewing in matching markets. Working Paper

Kurata R, Goto M, Iwasaki A et al (2017) Controlled school choice with soft bounds and overlapping types. J Artif Intell Res 58:153–184

Lee S (2017) Incentive compatibility of large centralized matching markets. Rev Econ Stud 84(1):444–463

Lee RS, Schwarz M (2017) Interviewing in two-sided matching markets. Rand J Econ 48(3):835–855

Loewenstein GF, Thompson L, Bazerman MH (1989) Social utility and decision making in interpersonal contexts. J Pers Soc Psychol 57(3):426–441

Lu T, Boutilier C (2011) Learning Mallows models with pairwise preferences. Proceedings ICML 2011. Omnipress, Bellevue, pp 145–152

Mallows CL (1957) Non-null ranking models. Biometrika 44:114–130

Manjunath V, Morill T (2023) Interview hoarding. Theor Econ 18(2):503–527

Messick DM, Sentis KP (1985) Estimating social and nonsocial utility functions from ordinal data. Eur J Soc Psychol 15:389–399

National Resident Matching Program (2023) National resident matching program, results and data: 2023 main residency match®. Tech. rep., National Resident Matching Program, Washington, DC

Pini MS, Rossi F, Venable KB (2014) Stable matching problems with soft constraints. In: Proceedings of the thirteenth international conference on autonomous agents and multiagent systems (AAMAS 2014). International Foundation for Autonomous Agents and Multiagent Systems, Paris, France, pp 1511–1512

Rastegari B, Condon A, Immorlica N, et al (2013) Two-sided matching with partial information. In: Proceedings of the Fourteenth ACM conference on electronic commerce. ACM, Philadelphia, Pennsylvania, pp 733–750

Roth AE (2002) The economist as engineer: game theory, experimentation, and computation as tools for design economics. Econometrica 70(4):1341–1378

Zhu M (2014) College admissions in China: a mechanism design perspective. Chin Econ Rev 30:618–631