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Patent package structures and sharing rules for royalty revenue

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Abstract
The role of patent pools—one-stop systems that gather patents from multiple patent holders and offer them to users as a package—is gaining research attention. To bolster the scarce stream of the literature that has addressed how a patent pool agent distributes royalty revenues among patent holders, we conduct an axiomatic analysis of sharing rules for royalty revenue derived from patents managed by a patent pool agent. In our framework, the patent pool agent organizes the patents into some packages, which we call a package structure. By using the hypergraph formulation developed by van den Nouweland et al. (Int J Game Theory 20:255–268, 1992), we analyze sharing rules that consider the package structure. In our study, we propose a sharing rule and show that it is the unique rule that satisfies efficiency, fairness, and independence requirements. In addition, we analyze sharing rules that enable a patent pool agent to organize a revenue-maximizing and objection-free profile.

1 Introduction

Against the background of the rapid increase in the number of patents, the role of patent pools—one-stop systems that gather patents from multiple patent holders and offer them to users as a package—is gaining research attention, as the fragmentation of patent ownership increases the costs of negotiation between patent owners and users.

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Shapiro (2001) and Lerner and Tirole (2004) showed that by forming patent pools and collectively determining license fees, patent holders can achieve lower prices compared with setting competitive license fees individually.

Most economics research on patent pools has focused on their impact on economic efficiency and relationship with competition policy (Layne-Farrar and Lerner 2011). By contrast, few studies have addressed how a patent pool agent distributes royalty revenues among patent holders. Among the studies conducted on sharing rules,

Aoki and Nagaoka (2004) analyzed patent pool participation by assuming that patent holders share pool revenues equally.

Kim (2004) examined the impact of vertically integrated firms that act as licensors upstream and licensees downstream. The author assumed that pool members aim to maximize the pool's revenue and agree on an (unspecified) sharing rule.[1] Specifically, the research focused on the economic effects such as the final product price resulting from the presence of integrated firms in patent pools.

Layne-Farrar and Lerner (2011) empirically investigated the relationship between sharing rules and participation in nine large patent pools such as MPEG and Bluetooth. They found that these nine pools adopted one of the following three sharing rules: the numeric proportional sharing rule, the value-based sharing rule, and royalty-free. They showed that pools with numeric proportional sharing rules attract fewer joiners and that patent holders with higher "values" are less likely to accept numeric proportional sharing rules. They also found that none of the nine pools adopted the equal sharing rule assumed by Aoki and Nagaoka (2004).

Tesoriere (2019) theoretically studied whether a sharing rule is stable against arbitrage, meaning that pool members have no incentive to trade their patents. The study showed that the numeric proportional sharing rule is the only such rule that achieves the stability concept and that it induces a firm with few patents to stay outside the patent pool.

In our study, we conduct an axiomatic analysis of sharing rules, with a focus on the common practice of patent pool agents offering their patents in the form of multiple packages. One example from Japan is ULDAGE, a patent pool agent that manages over 800 patents necessary for television broadcasting. These 800 patents are organized into five patent packages: ARIB (for receivers of 2 K digital broadcasting), CATV (for 2 K digital cable television), Satellite-UHDTV (for 4 K/8 K), CATV-UHDTV (for cable 4 K/8 K), and IPTV (for optical fiber cable IP broadcasting). Since certain core patents have multiple technological purposes, some of these 800 patents belong to two or more packages. In addition to this agent, other prominent patent pool agents organize their patents into multiple packages. For example, Via LA (a merger of Via Licensing and MPEG LA) offers packages necessary for electronic and information technologies such as AAC, MPEG-4, and Display Port, while SISVEL provides packages such as Wi-Fi 6 and Cellular IoT. In our study, we call such a profile of packages offered by a patent pool agent a *package structure* and examine sharing rules that address the complexities arising from such package structures.

[1] See Subsection 4.2 of Kim (2004) for the details. In the conclusion (Kim 2004), the author also mentioned the importance of exogenously specified sharing rules.

Table 1 An example of a package structure

	Package A	Package B	Package C	Package D
Patent 1	+		+	
Patent 2	+			
Patent 3		+	+	
Patent 4		+		
Patent 5			+	
Patent 6				+
Patent 7				+

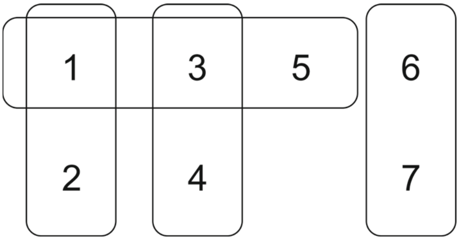

Fig. 1 A hypergraph $\{\{1, 2\}, \{3, 4\}, \{1, 3, 5\}, \{6, 7\}\}$

A package structure defines the assignment of patents to specific packages. Table 1 provides an example of a package structure, demonstrating how seven patents are organized into four packages by a (hypothetical) patent pool agent. For instance, patent 1 is contained in packages A and C. A package structure can also be formulated as a *hypergraph*, which is an extension of the concept of network structures. Figure 1 illustrates the package structure in Table 1 as a hypergraph. By formulating a package structure as a hypergraph, we can leverage existing research on allocation rules involving hypergraphs in the field of cooperative game theory. This allows us to apply these studies to analyze sharing rules that consider package structures.

van den Nouweland et al. (1992) introduced hypergraphs to model communication structures among individuals and studied sharing rules for the profits generated by coalitions of individuals. They generalized the Myerson value (Myerson 1977), a sharing rule for cooperative games with network structures, to the class of games with hypergraph structures.[2]

Our objective is to provide axiomatic foundations for the numerical proportional rule among the sharing rules empirically investigated by Layne-Farrar and Lerner (2011). While Tesoriere (2019) characterized the numerical proportional rule based on its robustness against patent exchanges among patent holders, we are motivated to demonstrate that the numerical proportional rule can be characterized by changes in the structure of patent packages and their corresponding worth through the framework of cooperative games. Specifically, we adapt the framework developed by van den

[2] The Myerson value is an extension of the Shapley value (Shapley 1953). Myerson (1980) extended the Myerson value (for games with network structures) to the class of *NTU*-games with hypergraph structures. van den Nouweland et al. (1992) generalized the Myerson value to the class of *TU*-games with hypergraph structures.

Nouweland et al. (1992) by replacing individuals with patents and communication structures with package structures. However, directly applying the approach of van den Nouweland et al. (1992) to our framework poses certain challenges, as the interpretation of a “pie” to be distributed among members differs between their framework and ours. To illustrate this gap, consider the following distributions of payoffs among elements 1–7 in the example shown in Fig. 1.

In the framework of van den Nouweland et al. (1992), the seven elements represent individuals, and the hypergraph structure specifies the communication relationships between them. To understand their discussion, we denote the profit generated by the members of S as $v(S)$ and the share assigned to an individual i as ψ_i. As shown in Fig. 1, individuals 1 to 5 collectively generate their profit $v(\{1, 2, 3, 4, 5\})$, and individuals 6 and 7 yield $v(\{6, 7\})$. In this sense, a sharing rule, ψ, is assumed to satisfy

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 = v(\{1, 2, 3, 4, 5\}), \psi_6 + \psi_7 = v(\{6, 7\}).$$

However, in our framework, the hypergraph structure represents a set of patent packages, indicating that patents 1 to 5 belong to three packages: $\{1, 2\}$, $\{3, 4\}$, and $\{1, 3, 5\}$. These packages generate royalty revenue $v(\{1, 2\}) + v(\{3, 4\}) + v(\{1, 3, 5\})$. Similarly, patents 6 and 7 yield $v(\{6, 7\})$. In this context, ψ_i represents the share assigned to a patent i. Therefore, in our framework, a sharing rule ψ is required to satisfy

$$\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7 = v(\{1, 2\})+v(\{3, 4\})+v(\{1, 3, 5\})+v(\{6, 7\}).$$

We propose sharing rules that satisfy this requirement and proceed to axiomatically characterize them.

Furthermore, we address substitutability among packages. In the aforementioned example, the value, or royalty revenue, generated from a package depends only on the patents included in that package. However, when a patent pool agent introduces a new package that implements a technology related to an existing package, the introduction of the new package may impact the revenue of the existing package. Therefore, by extending the aforementioned model, we also introduce a model in which the value of each package is not only determined by its constituent patents but also influenced by the structure of the other packages offered by the agent.

As shown in the subsequent sections, when substitutability is considered, selecting a package profile that maximizes overall revenue may result in reduced earnings for some packages. This could lead to objections from patent holders whose patents are included in the affected packages. In this paper, we show that the sharing rule that equally divides the total revenue may serve as an option for patent pool agents to organize a package profile that maximizes the total revenue and prevents patent holders from deviating from the pool.

The remainder of this paper is organized as follows. Section 2 introduces the model and proposes our sharing rule. Section 3 introduces and analyzes the sharing rule that reconciles revenue maximization and objection-freeness. Section 4 provides a summary and additional remarks. Section 5 provides all the proofs and counterexamples for the independence of the axioms.

2 Model and sharing rule

2.1 Preliminary

Let $N = \{1, \ldots, n\}$ be the set of all patents gathered by a patent pool agent, where the agent may represent an organization, manager, or administrator handling a set of patents. We call each $S \subseteq N$ a *package*. For every $S \subseteq N$, let $v(S)$ denote the worth of package S, where $v(\emptyset) = 0$. We suppose that the worth of a package represents the royalty revenue earned by the package. We assume that $v(S) \geq 0$ for every $S \subseteq N$.

The patent pool agent selects which packages to offer for sale from the set of all possible packages $2^N \backslash \{\emptyset\}$. We use $\mathcal{H} \subseteq 2^N \backslash \{\emptyset\}$ to denote the set of packages S selected by the patent pool agent. We call $\mathcal{H}$ a package structure or a *profile* of packages. For a given N, we call $(v, \mathcal{H})$ a patent package game, and let $\mathrm{PP}(N)$ denote the set of all patent package games for patent set N.

Moreover, for every $S \subseteq N$ and $\mathcal{H} \subseteq 2^N \backslash \{\emptyset\}$, let $\mathcal{H}(S) = \{T \in \mathcal{H} | T \subseteq S\}$. Let $N^*_{\mathcal{H}} \subseteq N$ denote the set of patents that belong to at least one package in $\mathcal{H}$, i.e., $N^*_{\mathcal{H}} = \cup_{T \in \mathcal{H}} T$.

Let $\psi : \mathrm{PP}(N) \rightarrow \mathbb{R}^N_+$ denote a *sharing rule*, where $\psi_i(v, \mathcal{H})$ represents the share assigned to patent i,[3] Therefore, a sharing rule is required to divide the total revenue $\sum_{T \in \mathcal{H}} v(T)$ earned by profile $\mathcal{H}$.

Axiom 1 A sharing rule ψ satisfies pool efficiency (PE) if for every $(v, \mathcal{H}) \in \mathrm{PP}(N)$,

$$\sum_{j \in N} \psi_j(v, \mathcal{H}) = \sum_{T \in \mathcal{H}} v(T).$$

Next, we introduce the fairness requirement, which was initially proposed by Myerson (1980) for games with network structures and later extended by van den Nouweland et al. (1992) to games with communication structures. Although the term "fairness" generally carries a wide range of meanings, we use this term for the sake of terminological consistency.

Axiom 2 A sharing rule ψ satisfies fairness (F) if for every $(v, \mathcal{H}) \in \mathrm{PP}(N)$, $T \in \mathcal{H}$, and $i, j \in T$,

$$\psi_i(v, \mathcal{H}) - \psi_i(v, \mathcal{H} \backslash \{T\}) = \psi_j(v, \mathcal{H}) - \psi_j(v, \mathcal{H} \backslash \{T\}).$$

The fairness requirement states that all patents in a package T gain or lose the same amount if T is removed from profile $\mathcal{H}$. More specifically, this axiom stipulates that patents forming a package, especially when indispensable to its functionality, be treated with equal weight. This is because each patent holds veto power over the formation of the package. Moreover, even in cases where the patents may not be essential, the axiom requires a sharing rule to treat them equally within the package if

[3] A sharing rule assigns a share to each *patent* and a patent *holder* who contributes some patents to the agent receives the sum of the shares assigned to the patents. Moreover, an international agent usually calculates royalty revenue by country. Therefore, in this study, we focus on revenue sharing within one country.

exogenous values, such as the number of citations and remaining years of validity, are not considered. Note that the absence of external values is also assumed by Tesoriere (2019) and Aoki and Nagaoka (2004). Layne-Farrar and Lerner (2011) discovered that two DVD pools implemented value-based sharing rules, while six pools utilized sharing rules unaffected by external value weights.

The following axiom requires the share of patent i to be independent of the packages that do not involve i.

Axiom 3 A sharing rule ψ satisfies package independence (PI) if for every $(v, \mathcal{H}) \in \text{PP}(N)$, $T \in \mathcal{H}$, and $i \in N \setminus T$,

$$\psi_i(v, \mathcal{H}) = \psi_i(v, \mathcal{H} \setminus \{T\}).$$

This axiom requires that the share allocated to patent i depends only on the packages in which i is included as an element. This stipulation asserts that if the contribution of i to the profile $\mathcal{H}$ remains unchanged (with v remaining constant), then the share received by i should not change. In other words, it ensures that if neither v nor the collection of packages containing i is affected by the change in profile $\mathcal{H}$, the agent does not have to recalculate the allocation to each patent i. The package independence requirement can be seen as a complementary counterpart of the fairness requirement. A sharing rule that incorporates exogenous value weights, such as those employed by the DVD pools, may not necessarily adhere to this requirement because of the exogenous weights.

The following result shows that our sharing rule is the unique rule that satisfies PI along with PE and F.

Proposition 2.1 *A sharing rule ψ satisfies PE, PI, and F if and only if $\psi_i(v, \mathcal{H}) = \sum_{T:i \in T \in \mathcal{H}} \frac{v(T)}{|T|}$ for every $i \in N$.*

We call this sharing rule the *package-wise equal sharing rule*. In addition to this characterization, by applying Theorem 2.3 of van den Nouweland et al. (1992), which characterizes the Myerson value, we can associate the package-wise equal sharing rule with the *Shapley value*[4] For every $(v, \mathcal{H}) \in \text{PP}(N)$ and $i \in N$, $\text{Sh}_i(v^{\mathcal{H}}) = \sum_{T:i \in T \in \mathcal{H}} \frac{v(T)}{|T|}$, where for every $(v, \mathcal{H}) \in \text{PP}(N)$ and $S \subseteq N$, let $v^{\mathcal{H}}(S) = \sum_{T \in \mathcal{H}(S)} v(T)$ denote the total revenue obtained from the packages in $\mathcal{H}$ that consist only of patents in S. This equivalence with the Shapley value suggests that the package-wise equal sharing rule assigns to patent i the expected value of the marginal contributions of patent i.

[4] Specifically, van den Nouweland et al. (1992) used *component efficiency*: for communication situations. The component efficiency requires an allocation to be efficient for every component (see van den Nouweland et al. (1992) for details). Moreover, the Shapley value (Shapley 1953) is defined as follows. For every $i \in N$, $\text{Sh}_i(v) = \sum_{S:i \in S \subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} (v(S) - v(S \setminus \{i\}))$. For every S with $i \in S \subseteq N$, $v(S) - v(S \setminus \{i\})$ represents the *marginal contribution* of i to S. Therefore, the Shapley value can be seen as the expected value of the marginal contributions of i.

Table 2 ψ violates F and PI

	$\mathcal{H}$	$v(\{1\}, \mathcal{H})$	$v(\{2\}, \mathcal{H})$	$v(\{1, 2\}, \mathcal{H})$	(ψ_1, ψ_2)
$\mathcal{H}_1$	$\{\{1\}, \{2\}\}$	1	2	–	(1, 2)
$\mathcal{H}_2$	$\{\{1\}, \{1, 2\}\}$	2	–	4	(4, 2)
$\mathcal{H}_3$	$\{\{1\}, \{2\}, \{1, 2\}\}$	1	1	4	(3, 3)

2.2 Sharing rule and substitutability among packages

We introduce substitutability among packages. When the agent opts to introduce a new package T along with the current packages S in $\mathcal{H}$, the addition of the new package may influence the revenue of the existing ones in the presence of substitutability among packages. In other words, the revenue of package S in $\mathcal{H}$ can be different from that of S in $\mathcal{H} \cup \{T\}$.

Therefore, we now use $v(S, \mathcal{H})$, instead of $v(S)$, to denote the worth of package $S \in \mathcal{H}$. This notation states that the revenue of package S may depend not only on package S but also on entire package structure $\mathcal{H}$. For every $\mathcal{H} \subseteq 2^N \backslash \{\emptyset\}$ and $S \in \mathcal{H}$, we call $(S, \mathcal{H})$ an *embedded package*.[5] Let EM_N be the set of all embedded packages and $v : \mathrm{EM}_N \to \mathbb{R}_+$. Moreover, let $\mathrm{PP}^*(N)$ denote the set of all patent package games $(v, \mathcal{H})$ with substitutability.[6] Let $\psi : \mathrm{PP}^*(N) \to \mathbb{R}^N_+$ denote a sharing rule.

By replacing $v(T)$ with $v(T, \mathcal{H})$, we can immediately generalize the package-wise equal sharing rule:

$$\psi_i(v, \mathcal{H}) = \sum_{T:i \in T \in \mathcal{H}} \frac{v(T, \mathcal{H})}{|T|}.$$

Moreover, by setting $v_*^{\mathcal{H}}$ as follows, the equivalence with the Shapley value also holds: For every $\mathcal{H} \subseteq 2^N \backslash \{\emptyset\}$ and $S \subseteq N$, $v_*^{\mathcal{H}}(S) = \sum_{T \in \mathcal{H}(S)} v(T, \mathcal{H})$; For every $(v, \mathcal{H}) \in \mathrm{PP}^*(N)$ and $i \in N$, $\mathrm{Sh}_i(v_*^{\mathcal{H}}) = \sum_{T:i \in T \in \mathcal{H}} \frac{v(T, \mathcal{H})}{|T|}$.

In view of these generalizations, one may conjecture that the preceding preliminary result could also be smoothly generalized to the class with substitutability. However, the following example demonstrates that the sharing rule no longer satisfies F and PI (redefined by replacing PP(N) with PP*(N)) in the presence of substitutability. Let $N = \{1, 2\}$ and consider v satisfying Table 2, where for simplicity let $v(S, \mathcal{H}') = 0$ for every $\mathcal{H}' \subseteq 2^N \backslash \{\emptyset\}$ with $\mathcal{H}' \notin \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ and every $S \in \mathcal{H}'$.[7]

[5] The original form of this concept was introduced by Thrall (1962) and Thrall and Lucas (1963) to describe *externalities* among coalitions and later called an *embedded coalition* in the class of partition function form games. We extend this concept to our framework.

[6] This framework can be generated from the demand side for patent packages à la the Shapiro–Cournot model (Shapiro (2001)). Let $\mathcal{H} = \{S_1, \ldots, S_m\}$ be a package structure. For every $S \in \mathcal{H}$, consider the demand $D_S^{\mathcal{H}}(r_{S_1}, \ldots, r_{S_m})$ for package S, where $r = (r_S)_{S \in \mathcal{H}}$ represents a profile of the license fees r_S for packages $S \in \mathcal{H}$. Assuming the patent pool agent maximizes the total revenue obtained from $\mathcal{H}$, let $r^* = (r_S^*)_{S \in \mathcal{H}} = \arg\max_r \sum_{S \in \mathcal{H}} r_S D_S^{\mathcal{H}}(r)$ subject to $0 \leq r_S \leq \bar{r}_S$ for every $S \in \mathcal{H}$. We set $v(S, \mathcal{H}) = r_S^* D_S^{\mathcal{H}}(r^*)$ for every $S \in \mathcal{H}$.

[7] No problem arises by assigning an arbitrary value to $v(S, \mathcal{H}')$ for every $\mathcal{H}' \subseteq 2^N \backslash \{\emptyset\}$ with $\mathcal{H}' \notin \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ and every $S \in \mathcal{H}'$

The rightmost column of the table describes the shares $\psi_i = \sum_{T:i\in T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|T|}$. If ψ satisfied F, then we would obtain $\psi_1(v, \mathcal{H}_3) - \psi_1(v, \mathcal{H}_1) = \psi_2(v, \mathcal{H}_3) - \psi_2(v, \mathcal{H}_1)$. However, we have $3 - 1 \neq 3 - 2$. Moreover, if ψ satisfied PI, then we would have $\psi_1(v, \mathcal{H}_3) = \psi_1(v, \mathcal{H}_2)$. However, we have $3 \neq 4$.

Therefore, Proposition 2.1 does not hold in the presence of substitutability. This gap arises because once $\mathcal{H}$ changes to $\mathcal{H}'$, the worth of each package $S \in \mathcal{H} \cap \mathcal{H}'$ may also change. However, by extending F and PI in a way that they restrict a sharing rule for changes of the *worth* v of packages, instead of the *structures* $\mathcal{H}$ of packages, we can overcome this gap.

Axiom 4 A sharing rule ψ satisfies fairness for worth (FW) if for every $\mathcal{H} \subseteq 2^N \backslash \{\emptyset\}$, $T \in \mathcal{H}$, and v, v' with $v'(T, \mathcal{H}) > v(T, \mathcal{H})$ and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N \backslash \{(T, \mathcal{H})\}$, we have

$$\psi_i(v', \mathcal{H}) - \psi_i(v, \mathcal{H}) = \psi_j(v', \mathcal{H}) - \psi_j(v, \mathcal{H}).$$

for every $i, j \in T$.

FW requires that if the worth of package T in $\mathcal{H}$ changes with keeping the worth of the other packages unchanged, then all the patents in T should obtain or lose the same amount. In this version of the fairness requirement, it is similarly stated that if all patents in a package are essential to the functionality provided by the package, these patents should be considered equally weighted when there are no exogenous value indices.

Axiom 5 A sharing rule ψ satisfies package independence for worth (PIW) if for every $\mathcal{H} \subseteq 2^N \backslash \{\emptyset\}$, $T \in \mathcal{H}$, and v, v' with $v'(T, \mathcal{H}) > v(T, \mathcal{H})$ and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N \backslash \{(T, \mathcal{H})\}$, we have

$$\psi_i(v', \mathcal{H}) = \psi_i(v, \mathcal{H})$$

for every $i \in N \backslash T$.

PIW states that if the worth of package T in $\mathcal{H}$ changes with keeping the worth of the other packages unchanged, then the change in package T does not affect the shares allocated to the patents outside of T. In other words, it ensures that if the change in v does not impact the worth of packages containing i, the allocation to i remains unchanged.

While F and PI constrain a sharing rule applied to the domain of v without considering substitutability, FW and PIW constrain a sharing rule applied to the domain of v with substitutability. In this sense, FW and PIW serve as generalizations of F and PI. Due to the different domains, there is no direct strong or weak relationship between F and FW (and, PI and PIW).[8]

[8] As demonstrated in Table 2, the package-wise equal sharing rule does not straightforwardly obey F and PI in the presence of substitutability. This is because substitutability allows S to change its worth depending on $\mathcal{H}$ and, hence, the requirements of F and PI become very demanding. Nevertheless, in the next section, we will propose another sharing rule that satisfies the generalized fairness requirement even in the presence of substitutability.

For the sake of completeness, by replacing $v(T)$ with $v(T, \mathcal{H})$, we redefine PE in the presence of substitutability.

Axiom 6 A sharing rule ψ satisfies PE if for every $(v, \mathcal{H}) \in \mathrm{PP}^*(N)$,

$$\sum_{j \in N} \psi_j(v, \mathcal{H}) = \sum_{T \in \mathcal{H}} v(T, \mathcal{H}).$$

The following proposition shows that these three requirements axiomatically characterize the package-wise equal sharing rule in the presence of substitutability.

Proposition 2.2 *A sharing rule ψ satisfies PE, FW, and PIW if and only if $\psi_i(v, \mathcal{H}) = \sum_{T:i \in T \in \mathcal{H}} \frac{v(T,\mathcal{H})}{|T|}$ for every $i \in N$.*

Proposition 2.2 can be seen as an extension of Proposition 2.1, where the axioms F and PI are replaced by FW and PIW. However, the proofs of Propositions 2.2 and 2.1 are significantly different because FW and PIW impose constraints on changes in the worth v of packages, while F and PI focus on changes in package structure $\mathcal{H}$.

These results provide an axiomatic rationale for using the package-wise equal sharing rule and, equivalently, (an extension of) the numeric proportional sharing rule empirically analyzed by Layne-Farrar and Lerner (2011).

3 Objection-freeness and revenue maximization

In addition to ensuring a fair distribution of royalty revenue, the patent pool agent faces the task of selecting profile $\mathcal{H}$ that maximizes royalty revenue $\sum_{T \in \mathcal{H}} v(T, \mathcal{H})$. However, even if the agent chooses a revenue-maximizing profile, it may not be accepted by all patent holders who contribute to the profile. This is because some patent holders may object to the chosen profile if another profile offers them higher profits under the sharing rule.

Table 3 provides a numerical example that demonstrates the conflict between revenue-maximizing profiles and objection-free profiles.[9] In this example, we consider two patents: patent 1 owned by patent holder 1 and patent 2 owned by patent holder 2. We test the sharing rule $\psi_i(v, \mathcal{H}) = \sum_{T:i \in T \in \mathcal{H}} \frac{v(T,\mathcal{H})}{|T|}$ in this example. The rightmost column of the table shows the shares based on the package-wise equal sharing rule.

[9] This example is also generated from the demand side with the following setups. For every $\mathcal{H} = \mathcal{H}_2, \ldots, \mathcal{H}_8$ and $S \in \mathcal{H}$, let $D_S^{\mathcal{H}}(r) = a_S^{\mathcal{H}} - b_S^{\mathcal{H}} \cdot (\sum_{S' \in \mathcal{H}} \theta_{SS'} r_{S'})$, where $|\theta_{SS'}|$ represents the degree of substitutability between S and S': specifically, $\theta_{SS'} = \theta_{S'S}$ for every $S, S' \subseteq N$; $\theta_{SS'} = 1$ if $S = S'$; and $\theta_{SS'} \leq 0$ if $S \neq S'$. Note that $\theta_{SS'} = 0$ states that there is no substitutability between the two packages. The numerical example Table 3 is obtained from, for example, the following parameters: $(\theta_{\{1\}\{1\}}, \theta_{\{1\}\{2\}}, \theta_{\{1\}\{1,2\}}) = (1, 0, -1)$, $(\theta_{\{2\}\{1\}}, \theta_{\{2\}\{2\}}, \theta_{\{2\}\{1,2\}}) = (0, 1, -1)$, and $(\theta_{\{1,2\}\{1\}}, \theta_{\{1,2\}\{2\}}, \theta_{\{1,2\}\{1,2\}}) = (-1, -1, 1)$; $a_{\{1\}}^{\mathcal{H}_2} = a_{\{2\}}^{\mathcal{H}_3} = 2\sqrt{2}$, $a_{\{1,2\}}^{\mathcal{H}_4} = 4\sqrt{2}$, $(a_{\{1\}}^{\mathcal{H}_5}, a_{\{2\}}^{\mathcal{H}_5}) = (2\sqrt{2}, 2\sqrt{2})$, $(a_{\{1\}}^{\mathcal{H}_6}, a_{\{1,2\}}^{\mathcal{H}_6}) = (a_{\{2\}}^{\mathcal{H}_7}, a_{\{1,2\}}^{\mathcal{H}_7}) = (2\sqrt{2}, \sqrt{2})$, $(a_{\{1\}}^{\mathcal{H}_8}, a_{\{2\}}^{\mathcal{H}_8}, a_{\{1,2\}}^{\mathcal{H}_8}) = (\sqrt{2}, \sqrt{2}, \sqrt{2}/2)$; $b_S^{\mathcal{H}} = 1/2$ for $(S, \mathcal{H}) = (\{1, 2\}, \mathcal{H}_8)$ and 1 otherwise; and $\bar{r}_S = \sqrt{2}$ for every $S \subseteq N$.

Table 3 All the profiles allow some patent holders to object

	$\mathcal{H}$	$v(\{1\}, \mathcal{H})$	$v(\{2\}, \mathcal{H})$	$v(\{1,2\}, \mathcal{H})$	(ψ_1, ψ_2)
$\mathcal{H}_1$	$\emptyset$	–	–	–	(0, 0)
$\mathcal{H}_2$	$\{\{1\}\}$	2	–	–	(2, 0)
$\mathcal{H}_3$	$\{\{2\}\}$	–	2	–	(0, 2)
$\mathcal{H}_4$	$\{\{1, 2\}\}$	–	–	8	(4, 4)
$\mathcal{H}_5$	$\{\{1\}, \{2\}\}$	2	2	–	(2, 2)
$\mathcal{H}_6$	$\{\{1\}, \{1, 2\}\}$	4	–	2	(5, 1)
$\mathcal{H}_7$	$\{\{2\}, \{1, 2\}\}$	–	4	2	(1, 5)
$\mathcal{H}_8$	$\{\{1\}, \{2\}, \{1, 2\}\}$	2	2	2	(3, 3)

The revenue-maximizing profile is $\mathcal{H}_4$ as the total revenue is 8. However, patent holder 1 has an incentive to object to $\mathcal{H}_4$ by demanding that the agent includes the singleton package of patent 1 in $\mathcal{H}_4$, as $\psi_1(\mathcal{H}_6) = 5 > 4 = \psi_1(\mathcal{H}_4)$. Patent holder 1 can propose the singleton package {1} without requiring the support of patent holder 2. In this sense, the revenue-maximizing profile allows some patent holders to object.

To further analyze the relationship between revenue-maximizing profiles and objection-free profiles, we formally define the concept of an objection. Let $\mathcal{H} \subseteq N\backslash\{\emptyset\}$ and ψ be an arbitrary sharing rule. An *objection* to profile $\mathcal{H}$ for ψ is $(S, \mathcal{H}')$ that satisfies the following three conditions:

(a) $\psi_j(v, \mathcal{H}') > \psi_j(v, \mathcal{H})$ for every $j \in S$,
(b1) $\mathcal{H}'\backslash\mathcal{H} \subseteq 2^S\backslash\{\emptyset\}$,
(b2) $\mathcal{H}\backslash\mathcal{H}' \subseteq \{T \in \mathcal{H} | S \cap T \neq \emptyset\}$.

Condition (a) is an incentive requirement, which states that the proposed profile $\mathcal{H}'$ provides higher payoffs to the owners of the patents in S. Conditions (b1) and (b2) are feasibility requirements: (b1) states that the newly proposed packages, $\mathcal{H}'\backslash\mathcal{H}$, consist only of patents in S and can be formed without involving patents in $N\backslash S$; and (b2) states that the packages proposed for deletion, $\mathcal{H}\backslash\mathcal{H}'$, contain a patent in S. In other words, (b1) states that for a new package to be formed, it requires the agreement of *all* patent holders of S, and (b2) states that a package is canceled if *at least one* member disagrees. Profile $\mathcal{H}$ is *objection-free* for ψ if no objections are raised against $\mathcal{H}$ for ψ.

The concept of objection is close to that of *deviation* used to define the core for TU-games and that of an objection proposed by Aumann and Maschler (1964) for the bargaining set. Moreover, the feasibility conditions, (b1) and (b2), are consistent with that of the formation and elimination of links in a network. For a link between two nodes (or individuals) to be formed, the agreement from both nodes is needed, whereas the elimination of a link only requires disagreement from either of the nodes.

Given the definition of an objection, we reexamine the example in Table 3. The example also shows that the set of objection-free profiles can be empty. Patent holder 2 has an incentive to withdraw its patent from package {1, 2} in $\mathcal{H}_6$ and instead propose package {2}, which leads to $\mathcal{H}_5$. In $\mathcal{H}_5$, both patent holders have an incentive to jointly offer package {1, 2} instead of the two separate packages, which results in

$\mathcal{H}_4$. Therefore, we have a cycle: $\mathcal{H}_4 \to (\mathcal{H}_6$ or $\mathcal{H}_7) \to \mathcal{H}_5 \to \mathcal{H}_4$. For $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{H}_8$, both patent holders have a common incentive to cancel the singleton packages and jointly offer the two-patent package.

This analysis suggests that the package-wise equal sharing rule (i) does not necessarily make a revenue-maximizing profile objection-free in the presence of substitutability[10] and (ii) does not guarantee the existence of objection-free profiles. If (i) is resolved, then the established revenue-maximizing profile is objection-free and resolves (ii). Therefore, we explore a sharing rule that resolves (i), namely, a sharing rule that makes revenue-maximizing profiles objection-free.

The set of objection-free profiles can vary depending on v and the choice of a sharing rule, while that of revenue-maximizing profiles depends only on v. Therefore, for every $v : \mathrm{EM}_N \to \mathbb{R}_+$, let $RM(v)$ denote the set of revenue-maximizing profiles: $RM(v) = \{\mathcal{H} \subseteq 2^N\backslash\{\emptyset\} | \sum_{T\in\mathcal{H}} v(T, \mathcal{H}) \geq \sum_{T\in\mathcal{H}'} v(T, \mathcal{H}')$ for every $\mathcal{H}' \subseteq 2^N\backslash\{\emptyset\}\}$. Moreover, let $OF^{\psi}(v)$ denote the set of objection-free profiles for the sharing rule ψ.

This notation allows us to formulate our objective as follows: What sharing rule ψ achieves $RM(v) \subseteq OF^{\psi}(v)$? The following axiom plays a key role in answering this question. It states that if the total revenue either increases or is unchanged, then the shares do not decrease.

Axiom 7 A sharing rule ψ satisfies pool monotonicity (PM) if for every $i \in N$ and $(v, \mathcal{H}), (v', \mathcal{H}') \in \mathrm{PP}^*(N)$, $\sum_{T\in\mathcal{H}'} v'(T, \mathcal{H}') \geq \sum_{T\in\mathcal{H}} v(T, \mathcal{H}) \Rightarrow \psi_i(v', \mathcal{H}') \geq \psi_i(v, \mathcal{H})$.

Moreover, we redefine F in the presence of substitutability by replacing PP(N) in Axiom 2 with PP*(N).

Axiom 8 A sharing rule ψ satisfies F if for every $(v, \mathcal{H}) \in \mathrm{PP}^*(N)$, $T \in \mathcal{H}$, and $i, j \in T$,

$$\psi_i(v, \mathcal{H}) - \psi_i(v, \mathcal{H}\backslash\{T\}) = \psi_j(v, \mathcal{H}) - \psi_j(v, \mathcal{H}\backslash\{T\}).$$

The following proposition shows that a single sharing rule, which is different from the rule $\psi_i(v, \mathcal{H}) = \sum_{T:i\in T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|T|}$, satisfies F and PM along with PE.

Proposition 3.1 *A sharing rule ψ satisfies PE, F, and PM if and only if* $\psi_i(v, \mathcal{H}) = \sum_{T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|N|}$ *for every* $i \in N$.

Proposition 3.1 suggests that PE, F, and PM yield the sharing rule that distributes the total revenue equally among all patents. This rule directly divides the total revenue among all patents, while the rule $\psi_i(v, \mathcal{H}) = \sum_{T:i\in T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|T|}$ divides the revenue of each package among the constituent patents of the package. The following proposition shows that the rule characterized in Proposition 3.1 makes revenue-maximizing profiles objection-free for an arbitrary v.

Proposition 3.2 *If a sharing rule ψ satisfies:*

[10] In the absence of substitutability, the sharing rule readily achieves this requirement because the full profile $\mathcal{H} = 2^N\backslash\{\emptyset\}$ maximizes $\sum_{T\in\mathcal{H}} v(T)$, and no one has an incentive to withdraw its patents.

Table 4 The combination of PM and PE does not imply $RM(v) = OF^{\psi}(v)$

	$\mathcal{H}$	$v(\{1\}, \mathcal{H})$	$v(\{2\}, \mathcal{H})$	$v(\{1,2\}, \mathcal{H})$	$(\tilde{\psi}_1, \tilde{\psi}_2)$
$\mathcal{H}_1$	$\{\{1\}, \{1,2\}\}$	0	–	1	(1, 0)
$\mathcal{H}_2$	$\{\{1\}, \{2\}, \{1,2\}\}$	0	0	2	(2, 0)

- *PM, then* $RM(v) \subseteq OF^{\psi}(v)$ *for every* $v : EM_N \to \mathbb{R}_+$.
- *PM, F, and PE, then* $RM(v) = OF^{\psi}(v)$ *for every* $v : EM_N \to \mathbb{R}_+$.

Proposition 3.2 highlights the central role of PM in reconciling revenue maximization and objection-freeness. The first statement suggests that a pool monotonic sharing rule ensures the existence of a revenue-maximizing and objection-free profile since $RM(v)$ is always nonempty for an arbitrary v. Therefore, regardless of the situation v the pool agent faces, pool monotonicity enables the agent to choose a revenue-maximizing and objection-free profile. In other words, assuming that the pool agent chooses a revenue maximizing profile, pool monotonicity is a sufficient condition of a sharing rule for the selected profile to be stable in the sense of the objection-freeness. However, it also suggests that PM may allow for the possibility of $OF^{\psi}(v)\setminus RM(v)$ to be nonempty. We can consider profile $\mathcal{H} \in OF^{\psi}(v)\setminus RM(v)$ to be an inefficient "stuck" profile, as it fails to maximize the total revenue, while all potential objections are not feasible.

The second statement shows that F and PE eliminate these stuck profiles and that a profile becomes revenue-maximizing if and only if it is objection-free. Moreover, F and PE are "tight" requirements because removing either could allow for the possibility of a sharing rule that makes $OF^{\psi}(v)\setminus RM(v)$ nonempty, as described below.

Consider the following sharing rule: $\tilde{\psi}_i(v, \mathcal{H}) = \omega_i \sum_{T\in\mathcal{H}} v(T, \mathcal{H})$ with $\omega_1 = 1$ and $\omega_j = 0$ for $j \in 2, ..., n$. This rule satisfies PM and PE but not F. We apply this sharing rule to the example in Table 4, where let $v(S, \mathcal{H}) = 0$ for every $\mathcal{H} \subseteq 2^N\setminus\{\emptyset\}$ with $\mathcal{H} \notin \{\mathcal{H}_1, \mathcal{H}_2\}$ and $S \in \mathcal{H}$.

In this example, $\mathcal{H}_2$ is the unique revenue-maximizing profile, while both $\mathcal{H}_1$ and $\mathcal{H}_2$ are objection-free profiles because without the involvement of patent 2, the holder of patent 1 cannot change $\mathcal{H}_1$ to $\mathcal{H}_2$.[11]

Now, consider the sharing rule $\hat{\psi}_i(v, \mathcal{H}) = c$ for every $i \in N$, where c is an arbitrary constant. This rule readily violates PE but satisfies PM and F since it always assigns c to all patents. As a result, no one has any incentive to object to any profile, and $OF^{\hat{\psi}}$ includes all profiles. However, the revenue maximization requirement still selects $\mathcal{H}_2$.

The combination of Propositions 3.1 and 3.2 implies that the sharing rule $\psi_i(v, \mathcal{H}) = \sum_{T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|N|}$ guarantees the existence of objection-free profiles and that each objection-free profile maximizes the total revenue. In this sense, our result suggests that this sharing rule serves as a mediator between the patent pool agent, which aims to maximize revenue, and patent holders, which may have an incentive to object.

[11] If condition (a) requires "weak inequalities $\geq$ for all $j \in S$ and strict one $>$ for some $i \in S$," then the combination of PM and PE implies $RM(v) = OF^{\psi}(v)$.

4 Conclusion

In this study, we conducted an axiomatic analysis of sharing rules for royalty revenue by incorporating package structures. The main findings can be summarized as follows.

- The package-wise equal sharing rule, which assigns to each patent i the summation of the average revenues of the packages containing patent i, is the unique sharing rule that satisfies the pool efficiency, fairness, and package independence requirements. Moreover, this sharing rule incorporates the contributions of each patent to the packages and can be seen as an extension of the numeric proportional sharing rule empirically analyzed by Layne-Farrar and Lerner (2011) (Sect. 2).
- The package-wise equal sharing rule does not guarantee that revenue-maximizing profiles are objection-free. The sharing rule that equally distributes the total revenue among all patents ensures that revenue-maximizing profiles are objection-free. This rule is characterized by the pool efficiency, fairness, and pool monotonicity requirements (Sect. 3).

In this study, a sharing rule is formulated as a function of v and $\mathcal{H}$ (i.e., royalty revenues and a patent package structure). However, other factors can contribute to the evaluation of patents, such as total citation counts and the remaining years until expiration, as highlighted by Layne-Farrar and Lerner (2011). To incorporate citation counts into a sharing rule, it would be necessary to consider a directed graph that represents the citation network among patents. This graph would capture the relationships between patents based on citations, allowing us to incorporate citation scores as an additional criterion in the patent evaluation process. The remaining years until patent expiration can be represented as an n-dimensional vector. However, further research is needed to propose a sharing rule that integrates all these factors and remains practical and explainable.

Moreover, we conjecture that the entire set of $v \in \text{PP}^*(N)$ forms the class of "hypergraph function form games," akin to partition function form games. To analyze this class of games, it is needed to establish key properties such as superadditivity and convexity, similar to what Hafalir (2007) has done for partition function form games. Investigating these properties is part of our ongoing and future research.

5 Proofs and independence of axioms

Proof of Proposition 2.1

A sharing rule ψ satisfies PE, PI and F if and only if $\psi_i(v, \mathcal{H}) = \sum_{T: i \in T \in \mathcal{H}} \frac{v(T)}{|T|}$ for every $i \in N$.

Proof The sharing rule ψ straightforwardly satisfies the three axioms. Hence, we focus on the uniqueness. Suppose that there are two sharing rules ψ and ψ' such that both satisfy the three axioms. In the same manner as Myerson (1980), van den Nouweland et al. (1992), and Casajus (2009), there is $(v, \mathcal{H}) \in \text{PP}(N)$ satisfying the following

(1) and (2)[12]:

$$\psi(v, \mathcal{H}) \neq \psi'(v, \mathcal{H}), \tag{1}$$

$$\psi(v, \mathcal{M}) = \psi'(v, \mathcal{M}) \text{ for every } \mathcal{M} \subseteq \mathcal{H}. \tag{2}$$

Moreover, by (1), we have

$$\text{there is } k \in N \text{ such that } \psi_k(v, \mathcal{H}) > \psi'_k(v, \mathcal{H}). \tag{3}$$

If the patent k belongs to exactly one package that is a singleton, i.e., $\{T \in \mathcal{H} | k \in T\} = \{\{k\}\}$, or belongs to no package, then PI and PE result in a contradiction by removing all packages from $\mathcal{H}$ except for $\{k\}$. Therefore, we consider $T \in \mathcal{H}$ with $|T| \geq 2$ and $k \in T$. For every $i \in T \in \mathcal{H}$ with $i \neq k$, we have

$$\begin{aligned}
\psi_i(v, \mathcal{H}) - \psi_k(v, \mathcal{H}) &\overset{\text{F}}{=} \psi_i(v, \mathcal{H}\backslash\{T\}) - \psi_k(v, \mathcal{H}\backslash\{T\}) \\
&\overset{(2)}{=} \psi'_i(v, \mathcal{H}\backslash\{T\}) - \psi'_k(v, \mathcal{H}\backslash\{T\}) \\
&\overset{\text{F}}{=} \psi'_i(v, \mathcal{H}) - \psi'_k(v, \mathcal{H}).
\end{aligned}$$

By (3), we obtain $\psi_i(v, \mathcal{H}) > \psi'_i(v, \mathcal{H})$. Letting $C_k \in \mathcal{C}(\mathcal{H})$ be the component containing patent k,[13] we repeat this for every $i \in C_k$ and then obtain

$$\sum_{i \in C_k} \psi_i(v, \mathcal{H}) > \sum_{i \in C_k} \psi'_i(v, \mathcal{H}).$$

For the component $C_k \in \mathcal{C}(\mathcal{H})$, PE implies that

$$\sum_{i \in N\backslash C_k} \psi_i(v, \mathcal{H}) < \sum_{i \in N\backslash C_k} \psi'_i(v, \mathcal{H}).$$

However, for an arbitrary $T \in \mathcal{H}(C_k)$, we have

$$\begin{aligned}
\sum_{i \in N\backslash C_k} \psi_i(v, \mathcal{H}) &\overset{\text{PI}}{=} \sum_{i \in N\backslash C_k} \psi_i(v, \mathcal{H}\backslash\{T\}) \\
&\overset{(2)}{=} \sum_{i \in N\backslash C_k} \psi'_i(v, \mathcal{H}\backslash\{T\}) \\
&\overset{\text{PI}}{=} \sum_{i \in N\backslash C_k} \psi'_i(v, \mathcal{H}),
\end{aligned}$$

which is a contradiction. □

[12] Note that for $\mathcal{H}_0 = \emptyset$, we have $\psi_i(v, \mathcal{H}_0) = \psi'_i(v, \mathcal{H}_0)$ for every $i \in N$ by PE and the non-negativity of ψ.

[13] For the details of components of a hypergraph, see van den Nouweland et al. (1992). The collection of the components of $\mathcal{H}$, i.e., $\mathcal{C}(\mathcal{H})$, partitions N, and every pair of the components has an empty intersection.

Independence of axioms for Proposition 2.1

[A sharing rule that satisfies PI and F but not PE]

Sharing rule ψ^1

$$\psi_i^1(v, \mathcal{H}) = \sum_{T: i \in T \in \mathcal{H}} v(T).$$

satisfies PI and F but not PE.

[A sharing rule that satisfies PE and PI but not F]

Let $\psi_i^2(v, \mathcal{H}) = \sum_{T: i \in T \in \mathcal{H}} Z(i, T, v)$, where

$$Z(i, T, v) = \begin{cases} v(T) & \text{if index } i \text{ is the minimum in } T, \\ 0 & \text{otherwise.} \end{cases}$$

This rule satisfies PE and PI but not F.

[A sharing rule that satisfies PE and F but not PI]

Let ψ^3 be

$$\psi_i^3(v, \mathcal{H}) = \frac{\sum_{T \in \mathcal{H}} v(T)}{|N|}.$$

This sharing rule straightforwardly satisfies PE and F. However, it violates PI: Let $N = \{1, 2\}$, $\mathcal{H} = \{\{1\}, \{2\}\}$, and $v(\{1\}) = v(\{2\}) = 1$. We have $\psi_1^3(v, \mathcal{H}) = 1 \neq 0.5 = \psi_1^3(v, \mathcal{H} \setminus \{\{2\}\})$.

Proof of Proposition 2.2

A sharing rule ψ satisfies PE, FW, and PIW if and only if $\psi_i(v, \mathcal{H}) = \sum_{T: i \in T \in \mathcal{H}} \frac{v(T, \mathcal{H})}{|T|}$ for every $i \in N$.

Proof Sufficiency: We show that $\psi_i(v, \mathcal{H}) = \sum_{T: i \in T \in \mathcal{H}} \frac{v(T, \mathcal{H})}{|T|}$ satisfies PE, FW, and PIW. For every $(v, \mathcal{H}) \in \mathrm{PP}^*(N)$, we have

$$\sum_{j \in N} \psi_j(v, \mathcal{H}) = \sum_{j \in N} \sum_{\substack{T \in \mathcal{H} \\ j \in T}} \frac{v(T, \mathcal{H})}{|T|} = \sum_{T \in \mathcal{H}} \sum_{j \in T} \frac{v(T, \mathcal{H})}{|T|} = \sum_{T \in \mathcal{H}} v(T, \mathcal{H}).$$

Therefore, ψ satisfies PE.

Next, let $\mathcal{H} \subseteq 2^N \setminus \{\emptyset\}$, $T^* \in \mathcal{H}$, and v, v' with $v'(T^*, \mathcal{H}) > v(T^*, \mathcal{H})$ and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N \setminus \{(T^*, \mathcal{H})\}$. For every $i, j \in T^*$, we have

$$\begin{aligned}\psi_i(v',\mathcal{H})-\psi_i(v,\mathcal{H}) &= \sum_{T:i\in T\in\mathcal{H}}\frac{v'(T,\mathcal{H})}{|T|}-\sum_{T:i\in T\in\mathcal{H}}\frac{v(T,\mathcal{H})}{|T|}\\ &=\frac{v'(T^*,\mathcal{H})}{|T^*|}-\frac{v(T^*,\mathcal{H})}{|T^*|}\\ &=\sum_{T:j\in T\in\mathcal{H}}\frac{v'(T,\mathcal{H})}{|T|}-\sum_{T:j\in T\in\mathcal{H}}\frac{v(T,\mathcal{H})}{|T|}\\ &=\psi_j(v',\mathcal{H})-\psi_j(v,\mathcal{H}).\end{aligned}$$

Therefore, ψ satisfies FW.

Moreover, for every $i\in N\backslash T^*$,

$$\psi_i(v,\mathcal{H})=\sum_{T:i\in T\in\mathcal{H}}\frac{v(T,\mathcal{H})}{|T|}=\sum_{T:i\in T\in\mathcal{H}}\frac{v'(T,\mathcal{H})}{|T|}=\psi_i(v',\mathcal{H}).$$

Hence, ψ satisfies PIW.

Uniqueness: Let ψ be a sharing rule that satisfies PE, FW, and PIW. Let $(v,\mathcal{H})\in \mathrm{PP}^*(N)$. For every $i\in N\backslash N^*_{\mathcal{H}}$, we have $\psi_i(v,\mathcal{H})\overset{\text{PE,PIW}}{=}0=\sum_{T:i\in T\in\mathcal{H}}\frac{v(T,\mathcal{H})}{|T|}$ as $\{T:i\in T\in\mathcal{H}\}=\emptyset$. Henceforth, we consider $i\in N^*_{\mathcal{H}}$. Let $\tau_i(\mathcal{H})$ denote the set of packages in $\mathcal{H}$ that contain i:

$$\tau_i(\mathcal{H})=\{T\in\mathcal{H}|i\in T\}=\{T_1,\ldots,T_k,\ldots,T_m\}.$$

First, consider v_0 with $v_0(S,\mathcal{H})=0$ for every $(S,\mathcal{H})\in \mathrm{EM}_N$. By PE and the nonnegativity of ψ, $\psi_i(v_0,\mathcal{H})=0$.

Next, for every $k=1,\ldots,m$, define v_k as follows:

$$v_k(T_k,\mathcal{H})=v(T_k,\mathcal{H}) \tag{4}$$

and

$$v_k(S,\mathcal{M})=v_{k-1}(S,\mathcal{M}) \text{ for every } (S,\mathcal{M})\in \mathrm{EM}_N\backslash\{(T_k,\mathcal{H})\}. \tag{5}$$

By PIW, we have

$$\psi_j(v_k,\mathcal{H})=\psi_j(v_{k-1},\mathcal{H}) \text{ for every } j\in N\backslash T_k. \tag{6}$$

Moreover, FW enables us to define

$$a_k:=\psi_j(v_k,\mathcal{H})-\psi_j(v_{k-1},\mathcal{H}) \text{ for every } j\in T_k. \tag{7}$$

By PE, for every $k = 1, \ldots, m$, we obtain

$$\sum_{j \in N} \psi_j(v_k, \mathcal{H}) - \sum_{j \in N} \psi_j(v_{k-1}, \mathcal{H}) = \sum_{h=1}^{k} v_k(T_h, \mathcal{H}) - \sum_{h=1}^{k-1} v_{k-1}(T_h, \mathcal{H}). \quad (8)$$

The left-hand side of (8) is

$$\sum_{j \in N} \psi_j(v_k, \mathcal{H}) - \sum_{j \in N} \psi_j(v_{k-1}, \mathcal{H}) \overset{(6)}{=} \sum_{j \in T_k} \psi_j(v_k, \mathcal{H}) - \sum_{j \in T_k} \psi_j(v_{k-1}, \mathcal{H})$$
$$\overset{(7)}{=} |T_k| \cdot a_k.$$

The right-hand side of (8) is

$$\sum_{h=1}^{k} v_k(T_h, \mathcal{H}) - \sum_{h=1}^{k-1} v_{k-1}(T_h, \mathcal{H}) \overset{(5)}{=} v_k(T_k, \mathcal{H}) \overset{(4)}{=} v(T_k, \mathcal{H}).$$

Hence, we obtain $a_k = \frac{v(T_k, \mathcal{H})}{|T_k|}$.

From $i \in T_k$ and (7), it follows that

$$\begin{aligned} \psi_i(v_k, \mathcal{H}) &= a_k + \psi_i(v_{k-1}, \mathcal{H}) \\ &= a_k + a_{k-1} + \psi_i(v_{k-2}, \mathcal{H}) \\ &= a_k + a_{k-1} + \cdots + a_1. \end{aligned}$$

Therefore, we have $\psi_i(v, \mathcal{H}) \overset{\text{PIW}}{=} \psi_i(v_m, \mathcal{H}) = a_m + a_{k-1} + \cdots + a_1 = \sum_{T : i \in T \in \mathcal{H}} \frac{v(T, \mathcal{H})}{|T|}$. □

Independence of axioms for Proposition 2.2

[A sharing rule that satisfies PIW and FW but not PE]

Let ψ^4 be

$$\psi_i^4(v, \mathcal{H}) = \sum_{T : i \in T \in \mathcal{H}} v(T, \mathcal{H}).$$

This sharing rule satisfies FW and PIW by replacing $\frac{v(T, \mathcal{H})}{|T|}$ with $v(T, \mathcal{H})$ in the proof of the sufficiency-part of Proposition 2.2. However, ψ^4 violates PE as it is not divided by $|T|$.

[A sharing rule that satisfies PE and FW but not PIW]

Let ψ^5 be

$$\psi_i^5(v, \mathcal{H}) = \frac{1}{|C|} \sum_{T \in \mathcal{H}(C)} v(T, \mathcal{H}),$$

where C is the component of $\mathcal{H}$ that contains i. This sharing rule straightforwardly satisfies PE. It also satisfies FW as follows: Let $\mathcal{H} \subseteq 2^N\backslash\{\emptyset\}$, $T^* \in \mathcal{H}$, and v, v' with $v'(T^*, \mathcal{H}) > v(T^*, \mathcal{H})$ and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N\backslash\{(T^*, \mathcal{H})\}$. For every $i, j \in T^*$, we have

$$\psi_i^5(v', \mathcal{H}) - \psi_i^5(v, \mathcal{H}) = \frac{1}{|C|} v'(T^*, \mathcal{H}) - \frac{1}{|C|} v(T^*, \mathcal{H}) = \psi_j^5(v', \mathcal{H}) - \psi_j^5(v, \mathcal{H}).$$

However, it violates PIW: Consider, for example, $N = \{1, 2\}$, $\mathcal{H} = \{\{1\}, \{1, 2\}\}$, $v(\{1, 2\}, \mathcal{H}) = 1$, and $v(S, \mathcal{M}) = 0$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N\backslash\{(\{1, 2\}, \mathcal{H})\}$. Let $v'(\{1\}, \mathcal{H}) = 5$, and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N\backslash\{(\{1\}, \mathcal{H})\}$. Then, we have $\psi_2^4(v', \mathcal{H}) = \frac{1+5}{2} \neq \frac{1}{2} = \psi_2^4(v, \mathcal{H})$.

[A sharing rule that satisfies PE and PIW but not FW]

Let $\psi_i^6(v, \mathcal{H}) = \sum_{T:i\in T\in\mathcal{H}} Z(i, T, v)$, where

$$Z(i, T, v) = \begin{cases} v(T, \mathcal{H}) & \text{if index } i \text{ is the minimum in } T, \\ 0 & \text{otherwise.} \end{cases}$$

This sharing rule satisfies PE because

$$\sum_{j\in N} \psi_j^6(v, \mathcal{H}) = \sum_{j\in N} \sum_{\substack{T\in\mathcal{H} \\ j\in T}} Z(j, T, v) = \sum_{T\in\mathcal{H}} \sum_{j\in T} Z(j, T, v) = \sum_{T\in\mathcal{H}} v(T, \mathcal{H}).$$

Moreover, it satisfies PIW as follows: Let $\mathcal{H} \subseteq 2^N\backslash\{\emptyset\}$, $T^* \in \mathcal{H}$, and v, v' with $v'(T^*, \mathcal{H}) > v(T^*, \mathcal{H})$ and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N\backslash\{(T^*, \mathcal{H})\}$. For every $i \in N\backslash T^*$, we have $\psi_i^6(v', \mathcal{H}) = \sum_{T:i\in T\in\mathcal{H}} Z(i, T, v')$. Since $i \notin T^*$, we have $Z(i, T, v') = Z(i, T, v)$ for every $T : i \in T \in \mathcal{H}$. Therefore, $\sum_{T:i\in T\in\mathcal{H}} Z(i, T, v') = \sum_{T:i\in T\in\mathcal{H}} Z(i, T, v) = \psi_i^6(v, \mathcal{H})$. However, ψ^6 violates FW: Consider $N = \{1, 2\}$, $\mathcal{H} = \{\{1, 2\}\}$, $v(\{1, 2\}, \mathcal{H}) = 1$, and $v(S, \mathcal{M}) = 0$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N\backslash\{(\{1, 2\}, \mathcal{H})\}$. Let $v'(\{1, 2\}, \mathcal{H}) = 5$, and $v'(S, \mathcal{M}) = v(S, \mathcal{M})$ for every $(S, \mathcal{M}) \in \mathrm{EM}_N\backslash\{(\{1, 2\}, \mathcal{H})\}$. Then, we have $\psi_1^6(v', \mathcal{H}) - \psi_1^6(v, \mathcal{H}) = 5 - 1 \neq 0 - 0 = \psi_2^6(v', \mathcal{H}) - \psi_2^6(v, \mathcal{H})$.

Proof of Proposition 3.1

A sharing rule ψ satisfies PE, F, and PM if and only if $\psi_i(v, \mathcal{H}) = \sum_{T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|N|}$ for every $i \in N$.

Proof The sharing rule $\sum_{T\in\mathcal{H}} \frac{v(T,\mathcal{H})}{|N|}$ immediately satisfies the three axioms. We show that the function is the unique sharing rule satisfying PE, F, and PM.

Let $(v, \mathcal{H}) \in \mathrm{PP}^*(N)$ and arbitrarily fix $i \in N$. Let $\mathcal{H}' = \mathcal{H} \cup \{N\}$. Moreover, define v' as follows:

$$v'(N, \mathcal{H}') := \sum_{T \in \mathcal{H}} v(T, \mathcal{H}), \tag{9}$$
$$v'(S, \mathcal{M}) := 0 \text{ for every } (S, \mathcal{M}) \in \mathrm{EM}(N) \backslash \{(N, \mathcal{H}')\}.$$

We have $\sum_{T \in \mathcal{H}' \backslash \{N\}} v'(T, \mathcal{H}') = 0$. From F, it follows that $\psi_i(v', \mathcal{H}') - \psi_i(v', \mathcal{H}' \backslash \{N\}) = \psi_j(v', \mathcal{H}') - \psi_j(v', \mathcal{H}' \backslash \{N\})$ for every $j \in N$. PE and the non-negativity of ψ imply that

$$\psi_j(v', \mathcal{H}' \backslash \{N\}) = 0 \text{ for every } j \in N.$$

Hence, $\psi_i(v', \mathcal{H}') = \psi_j(v', \mathcal{H}')$ for every $j \in N$. By PE, $\psi_i(v', \mathcal{H}') = \frac{1}{|N|} \sum_{T \in \mathcal{H}'} v'(T, \mathcal{H}')$. Moreover, by PM, we have $\psi_i(v, \mathcal{H}) = \psi_i(v', \mathcal{H}')$. Hence, we obtain

$$\psi_i(v, \mathcal{H}) = \psi_i(v', \mathcal{H}') = \frac{1}{|N|} \sum_{T \in \mathcal{H}'} v'(T, \mathcal{H}') \overset{(9)}{=} \frac{1}{|N|} \sum_{T \in \mathcal{H}} v(T, \mathcal{H}).$$

□

Independence of axioms for 3.1

[A sharing rule that satisfies PE and F but not PM]

Let

$$\psi_i^7(v, \mathcal{H}) = a_i(v) + \frac{1}{|N|} \left[\sum_{T \in \mathcal{H}} v(T, \mathcal{H}) - \sum_{j \in N} a_j(v) \right],$$

where $a_i(v) = \omega_i \cdot \min_{\mathcal{H}' : \emptyset \neq \mathcal{H}' \subseteq 2^N \backslash \{\emptyset\}} \sum_{T \in \mathcal{H}'} v(T, \mathcal{H}')$, and $\omega_1 = 1, \omega_j = 0$ for every $j = 2, \ldots, n$. We show that ψ^7 satisfies PE and F. For PE, we have $\sum_{j \in N} \psi_j^7(v, \mathcal{H}) = \sum_{j \in N} a_j(v) + [\sum_{T \in \mathcal{H}} v(T, \mathcal{H}) - \sum_{j \in N} a_j(v)] = \sum_{T \in \mathcal{H}} v(T, \mathcal{H})$. For F, letting $(v, \mathcal{H}) \in \mathrm{PP}^*(N)$, $T^* \in \mathcal{H}$, and $i, j \in T^*$, we have

$$\psi_i^7(v, \mathcal{H}) - \psi_i^7(v, \mathcal{H} \backslash \{T^*\}) = \frac{1}{|N|} \left[\sum_{T \in \mathcal{H}} v(T, \mathcal{H}) - \sum_{T \in \mathcal{H} \backslash \{T^*\}} v(T, \mathcal{H} \backslash \{T^*\}) \right]$$
$$= \psi_j^7(v, \mathcal{H}) - \psi_j^7(v, \mathcal{H} \backslash \{T\}).$$

However, it violates PM as follows. Let $N = \{1, 2\}$, $\mathcal{H}_1 = \{\{1\}\}$, $\mathcal{H}_2 = \{\{2\}\}$, and $\mathcal{H}_3 = \{\{1\}, \{2\}, \{1, 2\}\}$. Moreover, let

$$v(\{2\}, \mathcal{H}_2) = 7,\ v(\{1\}, \mathcal{H}_3) = 2,\ v(\{2\}, \mathcal{H}_3) = 2,\ v(\{1, 2\}, \mathcal{H}_3) = 4,$$

and $v(S, \mathcal{M}) = 99$ for other $(S, \mathcal{M})$. Now, let

$$v'(\{1\},\mathcal{H}_1)=2,\ v'(\{1\},\mathcal{H}_3)=2,\ v'(\{2\},\mathcal{H}_3)=2,\ v'(\{1,2\},\mathcal{H}_3)=6,$$

and $v'(S,\mathcal{M})=99$ for other $(S,\mathcal{M})$. We have $\sum_{T\in\mathcal{H}_3} v'(T,\mathcal{H}_3)=2+2+6=10>8=2+2+4=\sum_{T\in\mathcal{H}_3} v(T,\mathcal{H}_3)$. However, $\psi_1(v',\mathcal{H}_3)=2+\frac{1}{2}[10-2]=6<7.5=7+\frac{1}{2}[8-7]=\psi_1(v,\mathcal{H}_3)$.

[A sharing rule that satisfies PE and PM but not F]

Let $\psi_i^8(v,\mathcal{H})=\omega_i\cdot\sum_{T\in\mathcal{H}} v(T,\mathcal{H})$, where $\sum_{j\in N}\omega_j=1$ and $\omega_j\neq\frac{1}{n}$ for some $j\in N$. This rule straightforwardly satisfies PE and PM but violates F as follows. Let $N=\{1,2\}$ and $\mathcal{H}=\{\{1\},\{2\},\{1,2\}\}$. Moreover, let $v(\{1,2\},\mathcal{H})=10$, $v(\{1\},\mathcal{H})=v(\{1\},\mathcal{H}\backslash\{\{1,2\}\})=0$, and $v(\{2\},\mathcal{H})=v(\{2\},\mathcal{H}\backslash\{\{1,2\}\})=0$. We consider, for example, $(\omega_1,\omega_2)=(0.6,0.4)$. We have $\psi_1^8(v,\mathcal{H})-\psi_1^8(v,\mathcal{H}\backslash\{\{1,2\}\})=6-0>4-0=\psi_2^8(v,\mathcal{H})-\psi_2^8(v,\mathcal{H}\backslash\{\{1,2\}\})$.

[A sharing rule that satisfies PM and F but not PE]

Sharing rule $\psi_i^9(v,\mathcal{H})=\frac{2}{n}\cdot\sum_{T\in\mathcal{H}} v(T,\mathcal{H})$ satisfies F and PM. However, it violates PE as the summation is doubled.

Proof of Proposition 3.2

If a sharing rule ψ satisfies:

- PM, then $RM(v)\subseteq OF^{\psi}(v)$ for every $v:\mathrm{EM}_N\to\mathbb{R}_+$.
- PM, F, and PE, then $RM(v)=OF^{\psi}(v)$ for every $v:\mathrm{EM}_N\to\mathbb{R}_+$.

Proof Let ψ satisfies PM. Fixing an arbitrary $v:\mathrm{EM}_N\to\mathbb{R}_+$, we prove that $RM(v)\subseteq OF^{\psi}(v)$. Assume that there is $\mathcal{H}\in RM(v)$ such that $\mathcal{H}\notin OF^{\psi}(v)$. By $\mathcal{H}\notin OF^{\psi}(v)$, there is $(S,\mathcal{H}')$ such that $\mathcal{H}'$ satisfies (a), (b1), and (b2). It follows from (a) that $\psi_j(v,\mathcal{H}')>\psi_j(v,\mathcal{H})$ for every $j\in S$. Then, we have $\sum_{T\in\mathcal{H}'} v(T,\mathcal{H}')>\sum_{T\in\mathcal{H}} v(T,\mathcal{H})$ because if $\sum_{T\in\mathcal{H}'} v(T,\mathcal{H}')\leq\sum_{T\in\mathcal{H}} v(T,\mathcal{H})$, PM implies that $\psi_j(v,\mathcal{H}')\leq\psi_j(v,\mathcal{H})$ for every $j\in N$, which is a contradiction. Therefore, $\sum_{T\in\mathcal{H}'} v(T,\mathcal{H}')>\sum_{T\in\mathcal{H}} v(T,\mathcal{H})$. However, $\mathcal{H}\in RM(v)$ implies that $\sum_{T\in\mathcal{H}} v(T,\mathcal{H})\geq\sum_{T\in\mathcal{H}'} v(T,\mathcal{H}')$, which contradicts the assumption.

Now, let ψ satisfies PM, F, and PE. By Proposition 3.1, $\psi_i(v,\mathcal{H})=\sum_{T\in\mathcal{H}}\frac{v(T,\mathcal{H})}{|N|}$. Since ψ satisfies PM, it suffices to prove that $OF^{\psi}(v)\subseteq RM(v)$. We assume that there is $\mathcal{H}\in OF^{\psi}(v)$ such that $\mathcal{H}\notin RM(v)$. Since $\mathcal{H}$ is not a revenue-maximizing profile, consider $\mathcal{H}'\in RM(v)$ with $\mathcal{H}'\neq\mathcal{H}$. We have $\psi_j(v,\mathcal{H}')=\sum_{T\in\mathcal{H}'}\frac{v(T,\mathcal{H}')}{|N|}>\sum_{T\in\mathcal{H}}\frac{v(T,\mathcal{H})}{|N|}=\psi_j(v,\mathcal{H})$ for every $j\in N$. Hence, $(N,\mathcal{H}')$ satisfies (a). Moreover, $(N,\mathcal{H}')$ satisfies $\mathcal{H}'\backslash\mathcal{H}\subseteq 2^N\backslash\{\emptyset\}$, i.e., (b1); and every package $T\in\mathcal{H}\backslash\mathcal{H}'$ has a nonempty intersection with N, i.e, (b2). Hence, $(N,\mathcal{H}')$ is an objection to $\mathcal{H}$ for ψ. This contradicts $\mathcal{H}\in OF^{\psi}(v)$. □

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