



Group strategy-proof rules in multidimensional binary domains

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Abstract

We consider a setting in which the alternatives are binary vectors and the preferences of the agents are determined by the Hamming distance from their most preferred alternatives. We consider only rules that are unanimous, anonymous, and component-neutral, and focus on strategy-proofness, weak group strategy-proofness, and strong group strategy-proofness. We show that component-wise majority rules are strategy-proof, and for three agents or two components also weakly group strategy-proof, but not otherwise. These rules are even strongly group strategy-proof if there are two or three agents. Our main result is an impossibility result: if there are at least four agents and at least three components, then no rule is strongly group strategy-proof.

1 Introduction

We consider social decision-making problems with n agents, each of whom has an opinion (yes or no, one or zero) on each of m issues, to be called components. Any m -vector of zeros and ones is called an alternative, and the goal is to find a social, compromise alternative.

As an application, think of the agents as the members of a committee that has to establish the suitability of a number of candidates (the components) for a position or job: each committee member reports for each candidate whether it finds that candidate

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suiting or not, and based on these reports the committee decides on the suitability of each candidate. Clearly, it is desirable that the committee members report their true opinions. As will be mentioned later on, related applications can be found in judgement aggregation or in classification problems.

The preference of an agent for an alternative is determined by the (Hamming) distance of that alternative to the agent's top alternative: the smaller that distance, the better. Thus, an agent's preference is completely determined by that agent's top alternative, and each agent is asked to report this alternative. A rule assigns to each profile of reported alternatives a (social) alternative. We assume, almost throughout, that a rule is unanimous (if all reported alternatives are equal then that common alternative should be chosen), anonymous (only the reported alternatives matter and not who reports what), and component-neutral (components are treated equally: renaming the components in the preference profile results in renaming the components in the outcome in the same way). We denote the set of all rules with these three properties by \mathcal{F} .

We will be interested in strategy-proofness (no agent can be better off by not reporting the top alternative truthfully), and in particular in weak group strategy-proofness (no group of agents can make each of its members better off by not reporting truthfully), and strong group strategy-proofness (no group of agents can make each of its members at least as well and at least one member better off by not reporting truthfully). This model fits within a large literature. We first describe our main results, and next relate our model and results to this literature.

Unsurprisingly, due to the strong restriction on the domain of preferences, strategy-proofness in this context is a fairly weak requirement. For instance, component-wise majority rules are strategy-proof. A component-wise majority rule simply assigns to each component a zero if the majority of agents assigns a zero to that component, and a one if the majority of agents assigns a one, with a tie-breaker in the case that the number of agents is even. In the appendix of the paper, for the case of two agents and two components, we describe all (six) strategy-proof rules in \mathcal{F} (Proposition A.1); this is a rather elaborate task, and we learn from it that, without additional conditions, characterizing all such rules for the general case will be practically infeasible. For this reason, we limit our study of strategy-proofness to giving some partial results (Sect. 3.1).

The central question of the paper is how far we can go in strengthening strategy-proofness to group strategy-proofness. We first consider weak group strategy-proofness (Sect. 3.2). Somewhat surprisingly, it turns out that for at least four agents and at least three components, component-wise majority rules are not weakly group strategy-proof (Proposition 3.10). This, however, does not mean that there are no weakly group strategy-proof rules in \mathcal{F} , and for three components and two or more agents we exhibit an example of such a rule (Example 3.11). The main result of the paper is an impossibility result on strong group strategy-proofness: for at least four agents and at least three components, there exists no strongly group strategy-proof rule in \mathcal{F} (Theorem 4.6). This result is based on Proposition 4.5, which says that in the case of at least four agents a strongly group strategy-proof rule in \mathcal{F} has a so-called dominant alternative: this is an alternative, necessarily with all components zero or

all components one due to our conditions on rules in \mathcal{F} , which is always chosen if reported by at least one agent.

Clearly, in the context of this paper the consequences of individual strategy-proofness, weak group strategy-proofness, and strong group strategy-proofness are rather different; indeed, the conditions identified by Barberà et al. (2010) for equivalence of these three strategy-proofness conditions are not satisfied here.

Our model is a special case of the classical social choice model. If all preferences were allowed then the well-known result of Gibbard (1973) and Satterthwaite (1975) would imply that there is no strategy-proof rule in \mathcal{F} . Of course, our domain of preferences is very small, and for this reason it is not a surprise that there exist strategy-proof rules in \mathcal{F} . Nevertheless, similar models in the literature do exhibit non-existence of such rules. Observe that in our model, alternatives can be seen as the vertices in a hypercube (a square if there are two components, a cube if there are three, etc.). Preference is then simply determined by the Euclidian distance between the vertex corresponding to the top alternative and any other vertex, measured along the edges of the hypercube. This model closely resembles the model of Schummer and Vohra (2002) for more general graphs, but there every point of each edge is an alternative (thus, the number of alternatives is infinite). This difference appears to be essential: they find that there exists no strategy-proof rule in \mathcal{F} , in contrast to our findings. The model is also a special case of the model of Peters et al. (2021), but they consider a different class of preferences, namely all preferences that are single-peaked with respect to a spanning tree of the graph: for instance, in the case of two components, a preference $11 \succ 10 \succ 00 \succ 01$ would be allowed, but in our model, if the top alternative is 11, then 01 is (always) preferred to 00. Again, this difference appears to be essential, since also Peters et al. (2021) find that there exists no strategy-proof rule in \mathcal{F} .

Our model of a multidimensional binary domain with preferences determined by Hamming distance has been studied before in the literature. Amanatidis et al. (2015), within the same model, consider so-called ordered weighted averaging operators, ranging from minimizing the maximal Hamming distance to minimizing the sum of Hamming distances (minisum), but focus on computational complexity issues. They do show that many of the resulting rules are manipulable (not strategy-proof), but that the minisum rule is one exception—this was already shown by Brams et al. (2007) and Legrand et al. (2007). Barrot et al. (2017) provide a more detailed study of the manipulability of rules in this class. Also Xia and Conitzer (2010) consider multidimensional but not per se binary domains and derive an impossibility result concerning strategy-proofness under weaker restrictions on preferences.

Preferences based on Hamming distance, as in our paper, are in particular also separable: if we fix all alternatives on a subset of the components, for instance to $x_1 \in \{0, 1\}$ on the first component and $x_2 \in \{0, 1\}$ on the second component, then a preference on this restricted set does not depend on how exactly these alternatives are fixed, in our example on x_1 and x_2 . Therefore, the fact that component-wise majority rules are strategy-proof also follows from one of the results (Theorem 4.1) of Le Breton and Sen (1999).

Another strand of literature to which our model is related, is that of judgment aggregation: a rule aggregates the judgments of the agents over a number of different

issues to a common judgment over these issues (without there being a conclusion, as opposed to the classical judgment aggregation problem). Dietrich and List (2007) introduce strategy-proofness (non-manipulability) in the judgment aggregation model, consider a larger class of preferences, possibly incomplete, and show (their Theorem 1) that a rule is strategy-proof if and only if it is independent (the value assigned to each component depends only on the agents' values assigned to that component and not on the values the agents assign to any other component) and monotonic (in the values per component). Terzopoulou and Endriss (2019) show that this result still holds if preferences are, additionally, complete (their Theorem 1). A typical example of such a rule is component-wise majority. Our class of preferences is smaller and, indeed, we show that other strategy-proof rules in \mathcal{F} exist: see our Example 3.6, which exhibits such a rule which is not independent. Botan et al. (2016) consider the same preferences as we do, and show that an unbiased rule (i.e., flipping zeros and ones in a component leads to a corresponding flip (only) in the same component of the social alternative) is strategy-proof if and only if it is independent and monotonic (their Theorem 4). Our rule in Example 3.6 is, indeed, not unbiased.¹ In another result, Dietrich and List (2007) impose additional conditions (compared to Terzopoulou and Endriss (2019), Theorem 1) on a rule, resulting in a dictatorship (their Theorem 2). Botan et al. (2016) also consider manipulation by groups, but mainly focus on unbiased rules, as explained above. Other related work on binary aggregation, but not focusing on manipulation issues, includes Dokow and Holzman (2009) and Endriss and Grandi (2014).

Formally, our model also resembles approval voting (Brams and Fishburn 2007). Components can be interpreted as candidates, and reported alternatives as approval votes: one means approval, zero means disapproval. The alternative assigned by a rule then gives the approved candidates. In models considering approval voting sometimes more detailed preferences (linear orders) of agents over candidates are considered, which are absent in our model.

The model in this paper is also studied in computer science under the name of (and with application to) classification: the alternatives correspond to examples described by binary labels, and a rule corresponds to a classifier, aimed at finding an alternative that best matches the reported examples. See, e.g., Meir et al. (2010).

The paper is organized as follows. Section 2 introduces the model and states a few preliminary observations. Sections 3 consider strategy-proofness and weak group strategy-proofness, and Sect. 4 considers strong group strategy-proofness. Section 5 concludes. A few results are delegated to the Appendix.

2 Model and preliminary results

The set of *agents* is $N = \{1, \dots, n\}$, where $n \in \mathbb{N}$ with $n \geq 2$. A subset S of N is also called a *group*.

The set of *components* is $M = \{1, \dots, m\}$, where $m \in \mathbb{N}$ with $m \geq 2$, and the set of *alternatives* is $A = \{0, 1\}^M$. For $a = (a_1, a_2, \dots, a_m) \in A$, $a_j \in \{0, 1\}$ is the *value*

¹ As the reader can check later, for this rule f and $n = 2$, we have $f(100, 111) = 100$ but $f(011, 000) = 000 \neq 011$.

of the j^{th} component of a . Instead of (a_1, \dots, a_m) we also use the notation $a_1 \dots a_m$. We denote by $0_{x_1} 1_{y_1} 0_{x_2} 1_{y_2} \dots 0_{x_k} 1_{y_k}$ the alternative containing x_1 zeros followed by y_1 ones followed by x_2 zeros, and so on. If a subscript x_j or y_j is equal to 1, we leave it out: for example, 010_{m-2} starts with a zero, followed by a one, which is followed by $m - 2$ zeros.

For all $a, b \in A$ let $r(a, b) = \sum_{j=1}^m |a_j - b_j|$, i.e., $r(a, b)$ is the Hamming distance between a and b . A *preference* of an agent i on A is characterized by its unique top alternative $a^i \in A$: then an alternative b is (weakly) preferred by agent i to alternative c if and only if $r(b, a^i) \leq r(c, a^i)$. The *disutility* of agent i with top alternative a^i for alternative b is $r(b, a^i)$. The set of (*preference*) *profiles* is denoted by A^N . We use the notation a^N for the profile (a^1, \dots, a^n) , and for $S \subseteq N$ we denote by $a^S b^{N-S}$ the profile $(c^1, \dots, c^n) \in A^N$ with $c^i = a^i$ for all $i \in S$ and $c^i = b^i$ for all $i \in N - S$. Similar notations will be clear from the context. In general, we use superscripts for agents and subscripts for components.

A *rule* is a map $f : A^N \rightarrow A$. The alternative assigned by a rule f to a preference profile will also be called an *outcome*.

A rule f is *unanimous* if $f(a, \dots, a) = a$ for all $a \in A$. Two profiles a^N and b^N are *similar* if there is a permutation π of N such that $a^i = b^{\pi(i)}$ for every $i \in N$. A rule f is *anonymous* if $f(a^N) = f(b^N)$ for all similar profiles a^N and b^N . For $a \in A$ and a permutation π of M we denote by $\pi a \in A$ the alternative with $(\pi a)_j = a_{\pi(j)}$ for every $j \in M$. A rule f is *component-neutral* if $f(\pi a^1, \dots, \pi a^n) = \pi f(a^N)$ for every $a^N \in A^N$ and every permutation π of M . Throughout the rest of this paper we consider only rules that are unanimous, anonymous, and component-neutral. The set of all such rules is denoted by \mathcal{F} .

We say that a profile b^N is obtained from a^N by *swapping* the components j and k if for all $i \in N$ we have $b^i = \pi a^i$, where π is the permutation of M with $\pi(j) = k$, $\pi(k) = j$, and $\pi(\ell) = \ell$ otherwise. A profile a^N is *symmetric* in components j and k if a^N is similar to the profile b^N obtained by swapping the components j and k . The following observation will be used regularly.

Lemma 2.1 *Let $f \in \mathcal{F}$ and let profile a^N be symmetric in components j and k . Then $f(a^N)_j = f(a^N)_k$.*

Proof Profile a^N is similar to the profile b^N obtained from a^N by swapping the components j and k . By component-neutrality of f , we have $f(a^N)_j = f(b^N)_k$. Anonymity of f implies $f(a^N) = f(b^N)$. Hence, $f(a^N)_j = f(b^N)_k = f(a^N)_k$. □

Another consequence of anonymity is that we can use the notation $(a^{[n_1]}, b^{[n_2]}, \dots)$ for the profile in which n_1 agents have preference a , n_2 agents have preference b , etc., without confusion.

In this paper, the conditions of central interest for a rule f are the following.

Definition 2.2 An agent i can *manipulate* f at profile a^N if there is a $b^i \in A$ such that $r(f(b^i, a^{N-\{i\}}), a^i) < r(f(a^N), a^i)$; f is *strategy-proof* if at every profile there is no agent who can manipulate f .

Definition 2.3 A group S can *strongly manipulate* f at profile a^N if there are $b^i \in A$, $i \in S$, such that $r(f(b^S, a^{N-S}), a^i) < r(f(a^N), a^i)$ for every $i \in S$; f is *weakly*

group strategy-proof if at every profile there is no group which can strongly manipulate f .

Definition 2.4 A group S can weakly manipulate f at profile a^N if there are $b^i \in A$, $i \in S$, such that $r(f(b^S, a^{N-S}), a^i) \leq r(f(a^N), a^i)$ for every $i \in S$, with at least one inequality strict; f is strongly group strategy-proof if at every profile there is no group which can weakly manipulate f .

Observe that f is weakly group strategy-proof if no group can strongly manipulate and strongly group strategy-proof if no group can weakly manipulate: the latter condition is more demanding than the former, therefore resulting in a stronger property.

Other properties of a rule which are of interest in this paper, are the following. A rule f is weakly Pareto optimal if for every profile $a^N \in A^N$, there does not exist an alternative $b \in A$ such that $r(b, a^i) < r(f(a^N), a^i)$ for all $i \in N$. A rule f is strongly Pareto optimal if for every profile $a^N \in A^N$, there does not exist an alternative $b \in A$ such that $r(b, a^i) \leq r(f(a^N), a^i)$ for all $i \in N$, with at least one inequality strict. One can easily see that in the case of two agents, strategy-proofness and weak (strong) Pareto optimality imply weak (strong) group strategy-proofness. A rule f is component-wise unanimous if for each $j \in M$ and $x \in \{0, 1\}$, $a_j^i = x$ for each agent i implies $f(a^N)_j = x$. Clearly, if f is weakly Pareto optimal, then it is also component-wise unanimous.

3 Strategy-proofness and weak group-strategy proofness

As already mentioned in the Introduction, in the present model with a quite restricted domain of preferences, strategy-proofness is not very demanding. In Sect. 3.1 we will collect some results on strategy-proofness. We focus on weak group strategy-proofness in Sect. 3.2 and strong group strategy-proofness in Sect. 4.

3.1 Strategy-proofness

We just present a few eclectic results—even within \mathcal{F} the condition of strategy-proofness is too weak to obtain very general results.

We first consider the natural concept of a component-wise majority rule: such a rule assigns to each profile an alternative of which each component is the majority of the respective components in the preferences of the agents.² The formal definition is as follows.

Definition 3.1 A rule f is a component-wise majority rule with tie breaker $t \in \{0_m, 1_m\}$ if for every profile $a^N \in A^N$ and every $j \in M$, if $|\{i \in N \mid a_j^i = 1\}| > \frac{n}{2}$ then $f(a^N)_j = 1$, if $|\{i \in N \mid a_j^i = 0\}| > \frac{n}{2}$ then $f(a^N)_j = 0$, and if $|\{i \in N \mid a_j^i = 0\}| = |\{i \in N \mid a_j^i = 1\}|$, then $f(a^N)_j = t_j$.

² Nevertheless, the component-wise majority rule can also be criticized. Suppose there is a minimal majority of 1's for each component, but some agent happens to have a 0 at each component. Then that agent, under component-wise majority, gets his/her worst outcome, which does not seem quite fair. See also Amanatidis et al. (2015) for a similar example.

Clearly, component-wise majority rules are unanimous, anonymous, and component-neutral and, thus, in \mathcal{F} . The tie breaker t is only relevant when the number of agents is even. The following proposition shows that, indeed, component-wise majority rules are examples of strategy-proof rules.³

Proposition 3.2 *Each component-wise majority rule is strategy-proof.*

Proof Let f be a component-wise majority rule with tie breaker t . Suppose that f is not strategy-proof. Then there exists a profile a^N , an agent i , and a preference b^i for agent i such that $r(f(b^i, a^{N-i}), a^i) < r(f(a^N), a^i)$. Writing $a = f(a^N)$ and $b = f(b^i, a^{N-i})$, this implies that there is a component $j \in M$ such that $b_j = a_j^i \neq a_j$. Without loss of generality we may assume that $a_j^i = b_j = 1$ and $a_j = 0$. Since $a_j = f(a^N)_j = 0$, either strictly more than $\frac{n}{2}$ preferences in the profile a^N have 0 at component j or exactly $\frac{n}{2}$ preferences in the profile a^N have 0 at component j and $t_j = 0$. Since $a_j^i = 1$ and each agent other than i has the same preference in the profiles a^N and (b^i, a^{N-i}) , we have $b_j = 0$, which is a contradiction. \square

Proposition 3.2 would still hold for general tie breakers $t \in A$, different from 0_m or 1_m , but then component-neutrality is violated. It does not necessarily hold if ties are broken in different ways. We illustrate this by the following example.

Example 3.3 Let $m = 3$ and $n = 4$. For any preference profile and any component let the rule f assign 0 [1] if there is a strict majority for 0 [1]. In case of a tie at some component, let f assign 1 if the total number of 1s in the profile strictly exceeds the total number of 0s; otherwise, let f assign 0. Then f is unanimous, anonymous, and component-neutral. By definition it respects component-wise strict majority, without being a component-wise majority rule in the sense of Definition 3.1. Observe that $f(010, 010, 110, 111) = 110$ and $f(000, 010, 110, 111) = 010$. Since agent 1 with preference 010 prefers 010 over 110, f is not strategy-proof. \triangleleft

Another observation is that component-wise majority rules are strongly Pareto optimal.

Proposition 3.4 *Let f be a component-wise majority rule. Then f is strongly Pareto optimal.*

Proof Note that f minimizes total Hamming distance, that is, for every profile a^N and every $b \in A$ we have

$$\sum_{i \in N} r(f(a^N), a^i) \leq \sum_{i \in N} r(b, a^i).$$

Hence, there is no $b \in A$ with $r(b, a^i) \leq r(f(a^N), a^i)$ for all $i \in N$ such that at least one of these inequalities is strict. Thus, f is strongly Pareto optimal. \square

³ As already mentioned in the Introduction, this proposition can also be derived from Theorem 4.1 of Le Breton and Sen (1999).

In fact, Proposition 3.4 would still hold if ties are broken in arbitrary ways.

A simple consequence of Propositions 3.2 and 3.4 is that for two agents any component-wise majority rule is strongly (and hence also weakly) group strategy-proof.

In Proposition A.1 in the Appendix we describe all strategy-proof rules in \mathcal{F} for the simple case when there are only two agents and only two components. It turns out that there are six such rules, including the two component-wise majority rules, and that all these rules have a so-called dominant alternative. In general, rule f has a *dominant alternative* $d \in A$ if $f(a^N) = d$ whenever $a^i = d$ for some $i \in N$. Clearly, if such a dominant alternative exists, it is unique. Moreover, there are only two possibilities.

Lemma 3.5 *Let $f \in \mathcal{F}$ have a dominant alternative a . Then $a = 0_m$ or $a = 1_m$.*

Proof Suppose, contrary to what we wish to prove, that $a_j \neq a_k$ for some $j, k \in M$. Let $a' \in A$ with $a'_j = a_k, a'_k = a_j$, and $a'_l = a_l$ otherwise. Consider the profiles $V = (a, a', 0_m^{N-\{1,2\}})$ and $V' = (a', a, 0_m^{N-\{1,2\}})$. Then $f(V) = f(V') = a$ by anonymity. On the other hand, component-neutrality of f and $f(V) = a$ imply $f(V') = a'$. Since $a \neq a'$, this is a contradiction. \square

Dominant alternatives play an important role in the next section, where we consider strong group-strategy-proofness. As to component-wise majority rules, if there are three or more agents then these do not have a dominant alternative, as is easy to see: if there were a dominant alternative d , then it should be selected even at the profile (d, d', \dots, d') where $d'_j = 1 - d_j$ for every component j , which is not the case. It is also easy to see that in the case of two agents a component-wise majority rule has dominant alternative 0_m if $t = 0_m$ and 1_m if $t = 1_m$.

We conclude our specific consideration of strategy-proofness with the following example of a rule in \mathcal{F} , for three alternatives and an arbitrary number of agents, which is strategy-proof and has a dominant alternative.

Example 3.6 The rule f is defined as follows. Let $m = 3$ and let a^N be a preference profile.

- (i) If $a^i = b$ for every $i \in N$ and some $b \in A$, then $f(a^N) = b$.
- (ii) Otherwise, if $a^i \in \{011, 101, 110, 111\}$ for every $i \in N$, then $f(a^N) = 111$.
- (iii) Otherwise, if there are $i \in N$ and $j \in M$ such that $a^i_j = 1, a^i_k = 0$ for all $k \neq j$, and $a^h_j = 1$ for all $h \neq i$, then $f(a^N) = a^i$.
- (iv) In all remaining cases, $f(a^N) = 000$.

By (i), f is unanimous, and it is straightforward to verify that f is also anonymous and component-neutral, so that $f \in \mathcal{F}$. By going over the cases it can be checked that f is strategy-proof. Clearly, 000 is a dominant alternative. Also, e.g. $f(011, \dots, 011, 101) = 111$, hence if $n \geq 3$ then at component 1 there is a strict majority for 0, whereas f assigns 1. If $n = 2$, then $f(011, 101) = 111$ and $f(001, 101) = 001$, so that f is not a component-wise majority rule either: this would imply both $t = 111$ and $t = 000$, an impossibility. \triangleleft

3.2 Weak group strategy-proofness

We now strengthen strategy-proofness to weak group strategy-proofness. We first establish some useful results concerning weakly group strategy-proof rules in \mathcal{F} . Next, we consider again component-wise majority rules.

We first observe that weak group strategy-proofness implies weak Pareto optimality.

Lemma 3.7 *Let $f \in \mathcal{F}$ be a weakly group strategy-proof rule. Then f is weakly Pareto optimal.*

Proof By way of contradiction suppose there exist a profile a^N and an alternative b such that for each agent $i \in N$, $r(b, a^i) < r(f(a^N), a^i)$. By unanimity, $f(b, \dots, b) = b$, hence N can strongly manipulate at a^N . This contradicts weak group strategy-proofness of f . \square

Now we turn again to component-wise majority rules.⁴ We have already seen that for the case of two agents, both component-wise majority rules are weakly group strategy-proof—this follows from Propositions 3.2 and 3.4. We next consider the case of three agents.

Proposition 3.8 *Let $n = 3$. Then the component-wise majority rule is weakly group strategy-proof.*

Proof Let f be the component-wise majority rule. Suppose for contradiction that $S \subseteq N$ can strongly manipulate f at profile a^N . By Proposition 3.2, f is strategy-proof, so S can not be a singleton set. By Proposition 3.4, f is strongly Pareto optimal. Hence $S \neq N$. Thus, $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. Since f is the component-wise majority rule, component j of $f(a^N)$ depends only on component j of the preferences of the agents. If $a_j^1 = a_j^2$, then $f(a^N)_j = a_j^1 = a_j^2$, so agents 1 and 2 do not misreport at component j . If $a_j^1 \neq a_j^2$, and if the outcome is changed by misreporting, then the Hamming distance for one agent is increased by 1 and for the other agent it is decreased by 1. So, it is not possible to decrease the Hamming distance of both agents 1 and 2 simultaneously. Hence S cannot strongly manipulate f at profile a^N . This contradiction completes the proof. \square

We also obtain a positive result if there are two components.

Proposition 3.9 *Let $m = 2$. Then each component-wise majority rule is weakly group strategy-proof.*

Proof Let a^N be a preference profile and without loss of generality assume that $f(a^N) = 00$. Let $S \subseteq N$ and $b^S \in A^S$. If $f(b^S, a^{N-S}) = 10$, then S strongly manipulates if $a^i \in \{10, 11\}$ for all $i \in S$, but then $|\{i \in S \mid b_1^i = 1\}| \leq |\{i \in S \mid a_1^i = 1\}|$, hence

$$|\{i \in N \mid a_1^i = 1\}| \geq |\{i \in S \mid b_1^i = 1\}| + |\{i \in N - S \mid a_1^i = 1\}|$$

⁴ Recall that, by definition, there are two component-wise majority rules if n is even, and only one if n is odd.

so that $f_1(b^S, a^{N-S}) = 0$, a contradiction. The case where $f(b^S, a^{N-S}) = 01$ is similar. If $f(b^S, a^{N-S}) = 11$, then S strongly manipulates if $a^i = 11$ for all $i \in S$, and then

$$|\{i \in N \mid a^i = 11\}| \geq |\{i \in S \mid b^i = 11\}| + |\{i \in N - S \mid a^i = 11\}|$$

so that $f(b^S, a^{N-S}) = 00$, a contradiction. Hence f is weakly group strategy-proof. \square

In all other cases component-wise majority rules fail to be weakly group strategy-proof.

Proposition 3.10 *Let $n \geq 4$ and $m \geq 3$. Then no component-wise majority rule is weakly group strategy-proof.*

Proof First, we consider the cases with n odd. Let f be the component-wise majority rule f . Consider the profile $V = \left(0_{m-3}001, 0_{m-3}010, 0_{m-3}100, 0_m^{\lfloor \frac{n-5}{2} \rfloor}, 1_m^{\lfloor \frac{n-1}{2} \rfloor}\right)$. Then $f(V) = 0_{m-3}13$. The disutility for the first three agents is 2. If these three agents form a group and all report 0_m , the new profile is $V' = \left(0_m^{\lfloor \frac{n+1}{2} \rfloor}, 1_m^{\lfloor \frac{n-1}{2} \rfloor}\right)$, and $f(V') = 0_m$. Hence the disutility for the first three agents of the new alternative is 1, which is strictly less.⁵ Hence group $\{1, 2, 3\}$ can strongly manipulate f , and so f is not weakly group strategy-proof.

Second, let $n = 4$ and let f be the component-wise majority rule with $t = 0_m$ (the case $t = 1_m$ is analogous). Then $f(1100_{m-3}, 1010_{m-3}, 0110_{m-3}, 0_m) = 0_m$, whereas $f(1110_{m-3}^{[3]}, 0_m) = 1110_{m-3}$. Hence, the disutility of agents 1, 2, and 3 decreases from 2 to 1, so that group $\{1, 2, 3\}$ strongly manipulates. Hence also in this case f is not weakly group strategy-proof.

Finally, we consider the cases where n is even, $n \geq 6$. Let f be the component-wise majority rule with tie breaker $t = 0_m$ (again, the case $t = 1_m$ is analogous). Consider the profile

$$V = \left(1000_{m-3}, 0100_{m-3}, 0010_{m-3}, 0110_{m-3}, 1010_{m-3}, 1100_{m-3}, 0_m^{\lfloor \frac{n-6}{2} \rfloor}, 1_m^{\lfloor \frac{n-6}{2} \rfloor}\right).$$

By tie-breaking, $f(V) = 0_m$. This alternative has disutility 2 for agents 4, 5, and 6; if these agents all report 1110_{m-3} , the new outcome is 1110_{m-3} , which has disutility 1 for these agents. Hence group $\{4, 5, 6\}$ can strongly manipulate f , and so f is not weakly group strategy-proof. \square

We conclude our consideration of weak group strategy-proofness with an example showing that there do exist weakly group strategy-proof rules in \mathcal{F} also if the component-wise majority rule(s) is (are) not weakly group-strategy-proof.⁶

⁵ This manipulation by the first three agents can be seen as an example of vote trading, cf. Riker and Brams (1973).

⁶ The example is for three components, but we conjecture that an example can also be constructed for more than three components.

Example 3.11 Let $m = 3$. We define a weakly group strategy-proof rule in \mathcal{F} .

Say an alternative (or an agent) is of type 0 if it (or its preference) is in the set $\{000, 001, 010, 100\}$ and type 1 if it (or its preference) is in the set $\{011, 101, 110, 111\}$. A preference profile is of type 0 if it contains at least $\frac{n}{2}$ preferences of type 0, and of type 1 otherwise. We define a rule f as follows. Consider a profile a^N of type 0. If there is a component $j \in \{1, 2, 3\}$ such that $a_j^i = 1$ for each $i \in N$, then $f(a^N) = a$ where $a_j = 1$ and $a_k = 0$ for $k \neq j$. If there is no such j , then $f(a^N) = 000$. Now consider a profile a^N of type 1. If there is a component $j \in \{1, 2, 3\}$ such that $a_j^i = 0$ for each $i \in N$, then $f(a^N) = a$ where $a_j = 0$ and $a_k = 1$ for $k \neq j$. If there is no such j , then $f(a^N) = 111$. Then $f \in \mathcal{F}$.

We show that f is weakly group strategy-proof. Assume the contrary, i.e., there exists a profile a^N , a group S and a profile b^N where $b^i = a^i$ for each agent $i \notin S$, such that each $i \in S$ strictly prefers $f(b^N)$ over $f(a^N)$, i.e., $r(f(b^N), a^i) < r(f(a^N), a^i)$ for all $i \in S$.

First consider the case where $f(a^N) = 001$. Then, a^N is of type 0 and for each $i \in N$, $a_3^i = 1$. So every agent i of type 0 has the preference 001 and hence $i \notin S$. This implies that at the new profile b^N at least $\frac{n}{2}$ agents have preference 001. Hence also b^N is of type 0 and $f(b^N) = 000$ or $f(b^N) = 001$, thus $f(b^N) = 000$. Since for each $i \in N$, $a_3^i = 1$, they all prefer 001 over 000. This is a contradiction. The cases where $f(a^N) \in \{010, 100, 011, 101, 110\}$ are analogous.

Now consider the case where $f(a^N) = 000$. Then, a^N is of type 0 and for each $j \in \{1, 2, 3\}$ there is $i \in N$ such that $a_j^i = 0$. First assume that $f(b^N) = 001$. Fix an $i \in N$ satisfying $a_3^i = 0$. As $f(b^N) = 001$, we must have $b_3^i = 1$, and hence $i \in S$. This is a contradiction since i strictly prefers $000 = f(a^N)$ over $001 = f(b^N)$. Thus, $f(b^N) \neq 001$. Similarly one shows that $f(b^N) \neq 010$ and $f(b^N) \neq 100$. Since the disutility of any agent of type 0 at $f(a^N) = 000$ is either 0 or 1, and $f(b^N) \neq 010$ and $f(b^N) \neq 100$, it follows that none of the agents of type 0 is in S , so profile b^N is also of type 0. Hence, $f(b^N) = 000$, but this is a contradiction. An analogous argument works for the case where $f(a^N) = 111$. ◁

4 Strong group strategy-proofness

We now further strengthen weak to strong group-strategy-proofness. Thereby, the following concept will play an important role. A *switching point* of a rule $f \in \mathcal{F}$ is an integer n_0 with $1 \leq n_0 \leq n$ such that

$$f(0_m^{[n']}, 1_m^{[n-n']}) = \begin{cases} 0_m & \text{if } n' \geq n_0 \\ 1_m & \text{if } n' < n_0. \end{cases}$$

Clearly, if a switching point exists then it is unique.

The following lemma shows that strategy-proofness alone already implies the existence of a switching point. The argument is straightforward: since a rule in \mathcal{F} assigns 0_m if every agent reports 0_m , and 1_m if every agent reports 1_m , there must be a point where this outcome switches, and strategy-proofness implies that after the switch the

outcome stays 1_m . This is also a familiar phenomenon in the literature: for instance, it is analogous to the monotonicity requirement of Dietrich and List (2007), and closely related to the concept of a phantom voter in Moulin (1980). Besides, the lemma shows that a strategy-proof rule has a dominant alternative if this switching point is 1 or n .

Lemma 4.1 *Let $f \in \mathcal{F}$ be strategy-proof. Then f has a switching point n_0 . Further, if f does not have a dominant alternative, then $2 \leq n_0 \leq n - 1$.*

Proof For $l \in \{0, 1, \dots, n\}$, denote by V^l the profile $(0_m^{[l]}, 1_m^{[n-l]})$. Observe that V^l is symmetric in any two components. Thus, Lemma 2.1 implies that $f(V^l)_j = f(V^l)_{j'}$ for any $j, j' \in M$. Hence, for every $l \in \{0, 1, \dots, n\}$, $f(V^l) = 0_m$ or $f(V^l) = 1_m$.

Let $n_0 = \min\{l \in \{0, 1, \dots, n\} \mid f(V^l) = 0_m\}$. By unanimity, $f(V^0) = 1_m$ and $f(V^n) = 0_m$. Thus, $1 \leq n_0 \leq n$.

If $f(V^{n_0+1}) = 1_m$, an agent with preference 0_m in V^{n_0+1} can report 1_m so that the alternative $f(V^{n_0}) = 0_m$ results, contradicting strategy-proofness of f . Hence, $f(V^{n_0+1}) = 0_m$. Iterating this argument, we obtain $f(V^l) = 0_m$ for each $n_0 \leq l \leq n$. Hence, $f(V^l) = 0_m$ if $l \geq n_0$ and $f(V^l) = 1_m$ if $l < n_0$. Therefore, n_0 is a switching point.

Suppose, additionally, that f has no dominant alternative. We show that $2 \leq n_0 \leq n - 1$. By way of contradiction, suppose $n_0 = 1$. Hence, $f(V^1) = 0_m$. Since 0_m is not a dominant alternative of f , there exists a profile $(0_m, a^{[n-1]})$ such that $f(0_m, a^{[n-1]}) \neq 0_m$. Consider a sequence of profiles $R^1 = (0_m, 1_m^{[n-1]})$ and $R^i = (0_m, a^2, \dots, a^i, 1_m^{[n-i]})$ for $i = 2, \dots, n$. Since $f(R^1) = f(V^1) = 0_m$ and $f(R^n) = f(0_m, a^{[n-1]}) \neq 0_m$, there exists i such that $f(R^i) = 0_m$ and $f(R^{i+1}) \neq 0_m$. Then agent $i + 1$ in profile R^i , who has top alternative 1_m , can strictly benefit from reporting a^{i+1} , thus obtaining $f(R^{i+1}) \neq 0_m$. This contradicts strategy-proofness of f . Therefore, $2 \leq n_0$. By a similar argument it follows that $n_0 \leq n - 1$. This completes the proof of the lemma. \square

Our main result will be that, if there are at least four agents and at least three alternatives, then no rule in \mathcal{F} is strongly group strategy-proof. First, we consider the other cases and show that then we do have existence of a strongly group strategy-proof rule. The following result strengthens Proposition 3.8.

Proposition 4.2 *Let $n \in \{2, 3\}$. Then each component-wise majority rule is strongly group strategy-proof.*

Proof If $n = 2$, then this follows from Propositions 3.2 and 3.4. For the case $n = 3$, again by Propositions 3.2 and 3.4, we only need to consider two-agent groups, and by the same argument as used in the proof of Proposition 3.8 it follows that no such group can deviate so that both members are at least as well off, with one strictly better off. This concludes the proof. \square

For $m = 2$ and $n \geq 4$ we present the following example.

Example 4.3 Let $m = 2$ and $n \geq 4$. We define f as follows. Let a^N be a preference profile. If $a^i = a \in A$ for all $i \in N$, then $f(a^N) = a$. Otherwise, if $a^i = 00$ for some $i \in N$, then $f(a^N) = 00$. Otherwise, if $a^i = 11$ for some $i \in N$, then $f(a^N) = 11$. In

all other cases, $f(a^N) = 00$. Then it is not hard to verify that $f \in \mathcal{F}$ and f is strongly group strategy-proof. \triangleleft

Summarizing, we have established that strongly group strategy-proof rules in \mathcal{F} exist if $n = 2$ or $n = 3$, or if $m = 2$. It turns out that this does not hold in all other cases. To show this, we use Proposition 4.5 below. In order to prove this proposition, we use the following lemma, which in fact holds for weakly group strategy-proof rules in \mathcal{F} . Note that by Lemma 4.1, every weakly group strategy-proof rule in \mathcal{F} has a switching point.

Lemma 4.4 *Let $f \in \mathcal{F}$ be weakly group strategy-proof. Let n_0 be the switching point of f . Then:*

- [1] if $l < n_0$, then $f(0_m^{[l]}, (1_k 0_{m-k})^{[n-l]}) = 1_k 0_{m-k}$ for all $k \geq 2$;
- [2] if $l \geq n_0$, then $f(1_m^{[n-l]}, (0_k 1_{m-k})^{[l]}) = 0_k 1_{m-k}$ for all $k \geq 2$.

Proof We only prove [1], the proof of [2] is analogous. Let $l < n_0$.

For $m = 2$, the statement reduces to $f(0_2^{[l]}, 1_2^{[n-l]}) = 1_2$, which is true since $l < n_0$. Now assume that $m \geq 3$. Denote by V^k the profile $(0_m^{[l]}, (1_k 0_{m-k})^{[n-l]})$. We prove that $f(V^k) = 1_k 0_{m-k}$ for all $k = 2, \dots, m$, by using backward induction on k .

For $k = m$, the statement follows from the fact that $l < n_0$. Assume it to be true for $k + 1$ where $k \geq 2$. We prove that it is also true for k . Since V^k is symmetric in each pair of components $j, j' \in \{1, \dots, k\}$, Lemma 2.1 implies $f(V^k)_j = f(V^k)_{j'}$ for all such j and j' . Since f is weakly group strategy-proof, by Lemma 3.7 it is also weakly Pareto optimal and therefore component-wise unanimous. Hence, $f(V^k)_j = 0$ for each $j \in \{k + 1, \dots, m\}$. Thus, either $f(V^k) = 1_k 0_{m-k}$ or $f(V^k) = 0_m$.

If $f(V^k) = 0_m$, then the $n - l$ agents with preference $1_k 0_{m-k}$ can report $1_{k+1} 0_{m-(k+1)}$ and achieve $1_{k+1} 0_{m-(k+1)}$ by the induction hypothesis. The disutility of $1_{k+1} 0_{m-(k+1)}$ is equal to 1 for each of these agents, which is strictly better than the disutility k of 0_m . This contradicts weak group strategy-proofness of f . Therefore, $f(V^k) = 1_k 0_{m-k}$, which concludes the proof of the induction step and of the lemma. \square

Proposition 4.5 *Let $n \geq 4$, and let $f \in \mathcal{F}$ be strongly group strategy-proof. Then f has a dominant alternative.*

Proof Suppose, to the contrary, that f has no dominant alternative. Then by Lemma 4.1 f has a switching point n_0 with $2 \leq n_0 \leq n - 1$. For $k = 1, \dots, n - 1$ denote by V^k the profile $(0_m^{[k]}, 1_m^{[n-k]})$. We have $f(V^1) = 1_m$ and $f(V^{n-1}) = 0_m$.

Case 1: $m = 2$.

If n is even then consider the profile $V = (00, 01^{[\frac{n-2}{2}]}, 10^{[\frac{n-2}{2}]}, 11)$. By Lemma 2.1, $f(V) = 00$ or $f(V) = 11$. If $f(V) = 00$, then group $\{2, \dots, n\}$ weakly manipulates by each member reporting 11, resulting in $f(V^1) = 11$. If $f(V) = 11$, then group $\{1, \dots, n - 1\}$ weakly manipulates by each member reporting 00, resulting in $f(V^{n-1}) = 00$.

If n is odd then consider the profile $V = (00^{[2]}, 01^{[\frac{n-3}{2}]}, 10^{[\frac{n-3}{2}]}, 11)$. Again by Lemma 2.1, $f(V) = 00$ or $f(V) = 11$. If $f(V) = 11$, then group $\{1, \dots, n - 1\}$

weakly manipulates by each member reporting 00, resulting in $f(V^{n-1}) = 00$, thus contradicting strong strategy-proofness of f . Suppose $f(V) = 00$. If $n_0 > 2$ then, if each member of group $\{3, \dots, n\}$ reports 11, the alternative $f(V^2) = 11$ results, so that $\{3, \dots, n\}$ weakly manipulates. Therefore, $n_0 = 2$. Now consider the profile $V' = (00, 01^{\lfloor \frac{n-3}{2} \rfloor}, 10^{\lfloor \frac{n-3}{2} \rfloor}, 11^{\lfloor 2 \rfloor})$ then by a similar argument $n_0 = n - 1$, hence $n = 3$, contradicting our assumption that $n \geq 4$.

Case 2: $m \geq 3$ and n is even.

Consider the profile $V = (0_m, 010^{\lfloor \frac{n}{2} - 1 \rfloor}, 100^{\lfloor \frac{n}{2} - 1 \rfloor}, 1_m)$. Since V is symmetric in components 1 and 2, Lemma 2.1 implies $f(V)_1 = f(V)_2$. By a similar argument, $f(V)_j = f(V)_{j'}$ for all $j, j' \in \{3, \dots, m\}$. Thus, $f(V)$ is equal to $0_m, 0_2 1_{m-2}, 1_2 0_{m-2}$, or 1_m .

If $f(V) = 0_2 1_{m-2}$ or $f(V) = 1_m$, agents $2, \dots, n - 1$ in the profile V have disutility $m - 1$. Then $f(0_m, 0_m^{\lfloor n-2 \rfloor}, 1_m) = f(V^{n-1}) = 0_m$, and this alternative has disutility 1 for agents $2, \dots, n - 1$. Thus, group $\{2, \dots, n - 1\}$ can weakly (even strongly) manipulate at V , contradicting strong (even weak) group strategy-proofness of f .

If $f(V) = 1_2 0_{m-2}$ then, since $f(V^{n-1}) = 0_m$, group $\{1, \dots, n - 1\}$ can weakly manipulate at V by each member reporting 0_m : alternative 0_m is strictly preferred by agent 1, and agents $2, \dots, n - 1$ are indifferent. Again, this contradicts that f is strongly group strategy-proof.

If $f(V) = 0_m$ then, since $f(0_m, 1_2 0_{m-2}^{\lfloor n-1 \rfloor}) = 1_2 0_{m-2}$ by Lemma 4.4, group $\{2, \dots, n\}$ can weakly manipulate at V : $1_2 0_{m-2}$ is strictly preferred by agent n and agents $2, \dots, n - 1$ are indifferent. This again contradicts that f is strongly group strategy-proof, and completes the proof for the case that n is even and $m \geq 3$.

Case 3: $m \geq 3$ and n is odd.

Consider the profile $V = (0_m^{\lfloor 2 \rfloor}, 010^{\lfloor \frac{n-3}{2} \rfloor}, 100^{\lfloor \frac{n-3}{2} \rfloor}, 1_m)$. By Lemma 2.1, $f(V)_1 = f(V)_2$ and $f(V)_j = f(V)_{j'}$ for all $j, j' \in \{3, \dots, m\}$. Thus, $f(V)$ is equal to $0_m, 0_2 1_{m-2}, 1_2 0_{m-2}$, or 1_m .

If $f(V) = 0_2 1_{m-2}$, $f(V) = 1_2 0_{m-2}$, or $f(V) = 1_m$, then group $\{1, \dots, n - 1\}$ can weakly manipulate by each member reporting 0_m , resulting in $f(V^{n-1}) = 0_m$, which is strictly better for agents 1 and 2 and weakly better for agents $\{3, \dots, n - 1\}$. This contradicts strong group strategy-proofness of f .

If $f(V) = 0_m$, then consider the group $\{3, \dots, n\}$. If every member of this group reports $1_2 0_{m-2}$, the new profile is $(0_m^{\lfloor 2 \rfloor}, 1_2 0_{m-2}^{\lfloor n-2 \rfloor})$. If $n_0 > 2$, then by Lemma 4.4, $f(0_m^{\lfloor 2 \rfloor}, 1_2 0_{m-2}^{\lfloor n-2 \rfloor}) = 1_2 0_{m-2}$, implying that $\{3, \dots, n\}$ can weakly manipulate at V (in particular, agent n is strictly better off). Since f is strongly group strategy-proof, this implies $n_0 = 2$. Now consider the profile $V' = (0_m, 011^{\lfloor \frac{n-3}{2} \rfloor}, 101^{\lfloor \frac{n-3}{2} \rfloor}, 1_m^{\lfloor 2 \rfloor})$. By a similar same argument as for profile V , we obtain $n_0 = n - 1$, hence $n = 3$, contradicting the assumption that $n \geq 4$. This completes the proof of the proposition. □

Proposition A.2 in the Appendix states that, if $n = m = 3$, then the component-wise majority rule is the unique strongly group strategy-proof in \mathcal{F} that does not have

a dominant alternative. In particular, this implies that the statement in Proposition 4.5 does not hold for $n = m = 3$.

Theorem 4.6 *Let $n \geq 4$ and $m \geq 3$. Then no rule in \mathcal{F} is strongly group strategy-proof.*

Proof Let $f \in \mathcal{F}$ and for contradiction assume that f is strongly group strategy-proof. By Proposition 4.5, f has a dominant alternative. By Lemma 3.5, without loss of generality, this dominant alternative is 0_m .

Consider the profile $V = (100_{m-2}, 011_{m-2}, 110_{m-2}^{[n-2]})$. Since V is symmetric in any two components $j, j' \in \{3, \dots, m\}$, Lemma 2.1 implies $f(V)_j = f(V)_{j'}$ for any such j and j' . Therefore, $f(V)$ is equal to one of the eight alternatives xyz_{m-2} with $x, y, z \in \{0, 1\}$.

First we show that $z = 0$ if $m \geq 4$. If $z = 1$, then $f(V)$ has disutility at least $m - 2$ for agent 1. If that agent reports 0_m , then f assigns the dominant alternative 0_m , resulting in disutility 1 for agent 1, which is strictly better since $m \geq 4$. This contradicts the assumption that f is strategy-proof. Hence $z = 0$ if $m \geq 4$.

If $m = 3$ and $z \neq 0$, i.e., $z = 1$, then $f(V) = f(100, 011, 110^{[n-2]}) = 101$ since otherwise agent 1 can strictly improve by reporting the dominant alternative 000 .

Case 1: $m \geq 4$ or $m = 3$ and $z = 0$.

If $f(V) = 010_{m-2}$, then agent 1 can report 0_m and is again better off. If $f(V) = 100_{m-2}$, then agent 2 can report 0_m and is better off. If $f(V) = 0_m$ and all agents report 110_{m-2} , then f assigns 110_{m-2} : this is weakly preferred by every agent and strictly preferred by the agents $3, \dots, n$. Hence, $f(V) = 110_{m-2}$.

Consider the profile $V' = (10_{m-1}, 01_{m-1}, 1_{m-1}^{[n-2]})$. Since V' is symmetric in any two components $j, j' \in \{2, \dots, m\}$, Lemma 2.1 implies that $f(V')_j = f(V')_{j'}$ for any such j and j' . Therefore, $f(V')$ is equal to one of the four alternatives xy_{m-1} with $x, y \in \{0, 1\}$.

If $y = 1$, then the disutility of $f(V')$ for agent 1 is at least $m - 1$. If agent 1 reports the dominant alternative 0_m , then f assigns 0_m , which agent 1 strictly prefers. Thus, $y = 0$. If $f(V') = 100_{m-2}$, then agent 2 can report 0_m and be strictly better off. Hence, $f(V') = 0_m$.

At V' , the disutility for agents $3, \dots, n$ is m . If each of them reports 110_{m-2} , the resulting profile is V and $f(V) = 110_{m-2}$. At V' , agents $3, \dots, n$ strictly prefer 110_{m-2} over 0_m . This contradicts the assumption that f is strongly group strategy-proof and completes the proof for this case.

Case 2: $m = 3$ and $z = 1$.

As in Case 1 we consider the profile $V' = (100, 011, 111^{[n-2]})$, and obtain $f(V') = 000$. Then the disutility of agents $3, \dots, n$ at $f(V')$ is equal to 3. If, instead, these agents report 110 , then $f(V) = 101$ results, which has disutility 1 for each agent $3, \dots, n$, violating strong (even weak) group strategy-proofness of f .

This completes the proof of the theorem. \square

5 Conclusion

We have studied rules on a multidimensional binary domain with preferences determined by Hamming distance. Under the natural conditions of unanimity, anonymity

and component-neutrality, we have focused on strategy-proofness conditions: individual, weak group, and strong group strategy-proofness, with results ranging from many possibilities to impossibility. Specifically, we show that component-wise majority rules are strategy-proof, and for three agents or two components also weakly group strategy-proof, but not otherwise. These rules are even strongly group strategy-proof if there are two or three agents. Our main result is an impossibility result: if there are at least four agents and at least three components, then no rule is strongly group strategy-proof.

From our analysis, in particular of special cases, it appears that it is a difficult if not infeasible task to find all strategy-proof or weakly group strategy-proof rules under the conditions of unanimity, anonymity and component-neutrality, unless additional requirements are imposed. For instance, is it possible to extend Example 3.11 to a characterization of a class of weakly strategy-proof rules? This is left for future research.

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A Appendix

A.1 Strategy-proofness and $m = n = 2$

We consider the $m = n = 2$ case, for which we describe all strategy-proof rules in \mathcal{F} .

Proposition A.1 *Let $n = m = 2$. \mathcal{F} contains exactly six strategy-proof rules, two of which are the component-wise majority rules. All these rules have a dominant alternative.*

Proof To completely specify a strategy-proof rule $f \in \mathcal{F}$, because of unanimity, anonymity, and component-neutrality, we only need to fix the alternatives that f assigns to four profiles: (00, 01), (00, 11), (01, 10), and (01, 11). The profile (00, 11) is symmetric in the components 1 and 2, so by Lemma 2.1, $f(00, 11)_1 = f(00, 11)_2$. Thus, $f(00, 11) = 00$ or $f(00, 11) = 11$.

Consider the case where $f(00, 11) = 00$. Then $f(00, 01) = 00$, otherwise the agent with preference 11 in the profile (00, 11) can report 01 and get a better outcome, contradicting strategy-proofness. Then also $f(00, 10) = 00$ due to component-neutrality and $f(00, 00) = 00$ due to unanimity. Hence, 00 is a dominant alternative. The profile (01, 10) is symmetric in components 1 and 2, so $f(01, 10) = 00$ or

Table 1 The six strategy-proof rules in \mathcal{F} for $m = n = 2$.

Profile	f^1	f^2	f^3	f^4	f^5	f^6
00	00	00	00	00	00	00
00	01	00	00	00	00	01
00	10	00	00	00	00	10
00	11	00	00	00	11	11
01	00	00	00	00	00	01
01	01	01	01	01	01	01
01	10	00	11	00	00	11
01	11	01	11	11	11	11
10	00	00	00	00	00	10
10	01	00	11	00	00	11
10	10	10	10	10	10	10
10	11	10	11	11	11	11
11	00	00	00	00	11	11
11	01	01	11	11	11	11
11	10	10	11	11	11	11
11	11	11	11	11	11	11

Rules f^1 , f^2 , and f^3 have dominant alternative 00, the other rules have dominant alternative 11. Rules f^1 and f^6 are the component-wise majority rules with $t = 00$ and $t = 11$, respectively

$f(01, 10) = 11$. We now consider the profile $(01, 11)$. If $f(01, 11) = 10$, then the agent with preference 01 in the profile $(01, 11)$ can report 11 and get a better outcome. Similarly if $f(01, 11) = 00$, the agent with preference 11 can misreport to 01. Hence, $f(01, 11) = 01$ or $f(01, 11) = 11$. Moreover, if $f(01, 10) = 11$ then also $f(01, 11) = 11$, otherwise the agent with preference 11 in the profile $(01, 11)$ can misreport 10 to get the outcome 11. Hence, there are exactly three strategy-proof rules in \mathcal{F} when $f(00, 11) = 00$. Analogously, there are also three strategy-proof rules in \mathcal{F} when $f(00, 11) = 11$, and these have 11 as dominant alternative.

Thus, there are exactly six strategy-proof rules in \mathcal{F} , and all have a dominant alternative. Two of these rules are the component-wise majority rules. □

The rules derived in Proposition A.1 are displayed in Table 1.

A.2 The component-wise majority rule for $m = n = 3$

As mentioned in Sect. 4, Proposition 4.5 does not hold when $n = 3$ and $m = 3$. In particular, the component-wise majority rule is a strongly group strategy-proof rule in \mathcal{F} that does not have a dominant alternative. We show that there is no other strongly group strategy-proof rule in \mathcal{F} that does not have a dominant alternative.

Proposition A.2 *Let $n = 3$ and $m = 3$. The component-wise majority rule is the unique strongly group strategy-proof rule in \mathcal{F} that does not have a dominant alternative.*

Proof Let $f \in \mathcal{F}$ and assume that f does not have a dominant alternative. Hence, f is also strongly Pareto optimal and therefore component-wise unanimous. The idea is to enumerate all preference profiles and step by step eliminate, for each profile, the outcomes which contradict strong group strategy-proofness of f . In the end, we show that f is the component-wise majority rule.

As f is unanimous, anonymous and component-neutral, we only need to consider the following profiles. Once the outcomes for these profiles are fixed, the outcomes of all other profiles are also determined by unanimity, anonymity and component-neutrality of f .

000,000,001	000,001,001	000,001,100	001,001,100
001,010,100	000,111,111	000,011,111	000,011,011
000,011,110	001,111,111	001,011,111	001,110,111
001,011,011	001,110,110	001,011,110	001,101,011
111,111,110	111,110,110	111,110,011	110,110,011
110,101,011	111,000,000	111,100,000	111,100,100
111,100,001	110,000,000	110,100,000	110,001,000
110,100,100	110,001,001	110,100,001	110,010,100

For every profile, there are 8 possible outcomes from the set $\{000, 001, 010, 100, 011, 101, 110, 111\}$. We start by eliminating the outcomes which contradict either unanimity, anonymity, component-neutrality or the fact that f is strongly group strategy-proof.

- (1) For $f(001, 010, 100)$, we can eliminate all outcomes except 000 and 111 because the other outcomes violate component-neutrality or anonymity. Moreover, we can also eliminate 111 by strong Pareto optimality. Hence, $f(001, 010, 100) = 000$. Analogously, $f(110, 101, 011) = 111$.
- (2) For $f(000, 000, 001)$, we can eliminate all outcomes except 000 because otherwise agents 1 and 2 can misreport to 010 and 100, respectively, and get the outcome 000 by step (1). Hence, $f(000, 000, 001) = 000$. Analogously, $f(111, 111, 110) = 111$.
- (3) For $f(000, 001, 001)$, we can eliminate all outcomes except 000 and 001 by component-wise unanimity of f , at components 1 and 2. Analogously for $f(111, 110, 110)$, we can eliminate all outcomes except 111 and 110.
- (4) We have $f(000, 001, 100) = 000$ by strategy-proofness, since otherwise agent 1 can misreport to 010 and obtain 000 by step (1). Analogously, $f(111, 110, 011) = 111$.
- (5) For $f(001, 001, 100)$, we can eliminate the outcomes 010, 011, 110 and 111 by component-wise unanimity at component 2. We can also eliminate the outcome 100, because otherwise agent 2 can misreport to 010 and obtain the strictly preferred outcome 000 by step (1). Analogously, for $f(110, 110, 011)$, we can eliminate the outcomes 101, 100, 001, 000 and 011.
- (6) For $f(000, 111, 111)$, we can eliminate all outcomes except 000 and 111 by component-neutrality or anonymity. Moreover, we can eliminate 000, other-

- wise, by strong group strategy-proofness, f has a dominant alternative 000. Hence, $f(000, 111, 111) = 111$. Analogously, $f(111, 000, 000) = 000$.
- (7) For $f(000, 011, 111)$, we can eliminate the outcomes 000, 101, 110 and 100 because otherwise agent 2 can misreport to 111 to obtain the strictly preferred outcome 111, by step (6).
Analogously, for $f(111, 100, 000)$, we can eliminate the outcomes 111, 010, 001 and 011.
- (8) $f(000, 011, 011) = 000$ or $f(000, 011, 011) = 011$ by component-wise unanimity and component-neutrality. Hence, by step (6) and strong group strategy-proofness, $f(000, 011, 011) = 011$. Analogously, $f(111, 100, 100) = 100$.
- (9) For $f(000, 011, 110)$, by component-neutrality and unanimity, components 1 and 3 have equal value, so that we can eliminate the outcomes 001, 100, 011, and 110. We can also eliminate 000 and 101 by strong group strategy-proofness and step (6). Finally, $f(000, 011, 110) \neq 111$ by strong Pareto optimality, since agent 1 strictly prefers 010 and agents 2 and 3 are indifferent. Hence, $f(000, 011, 110) = 010$. Analogously, $f(111, 100, 001) = 101$.
- (10) For $f(001, 111, 111)$, we can eliminate the outcomes 010, 011, 100, and 101 by component-neutrality, and we can eliminate 000 and 110 by component-wise unanimity at component 3. Also, $f(001, 111, 111) \neq 001$, otherwise agent 1 can manipulate at $(000, 111, 111)$ by misreporting 001, using step (6). Hence, $f(001, 111, 111) = 111$. Analogously, $f(110, 000, 000) = 000$.
- (11) For $f(001, 011, 111)$, we can eliminate the outcomes 000, 010, 100 and 110 by component-wise unanimity at component 3. We can also eliminate 101, because otherwise agent 2 can misreport to 111 to obtain the strictly preferred outcome 111 by step (10).
Analogously, for $f(110, 100, 000)$, we can eliminate the outcomes 111, 101, 011, 001, and 010.
- (12) For $f(001, 110, 111)$, we can eliminate the outcomes 010, 100, 011 and 101 by component-neutrality at components 1 and 2. We can also eliminate the outcomes 000 and 001 because otherwise agent 2 can misreport to 111 to obtain the strictly preferred outcome 111 by step (10). Finally, we can eliminate the outcome 110, because otherwise agent 1 can misreport to 111 to obtain the strictly preferred outcome 111 by step (2) and anonymity. Hence, $f(001, 110, 111) = 111$. Analogously, $f(110, 001, 000) = 000$.
- (13) For $f(001, 011, 011)$, we can eliminate all outcomes except 001 and 011 by component-wise unanimity at components 1 and 3. Also, $f(001, 011, 011) \neq 001$, since otherwise in the profile $(000, 011, 011)$ agent 1 can misreport to 001 to obtain the outcome 001, which is strictly preferred to 011, and $f(000, 011, 011) = 011$ by step (8). Hence, $f(001, 011, 011) = 011$. Analogously, $f(110, 100, 100) = 100$.
- (14) For $f(001, 110, 110)$, we can eliminate the outcomes 010, 100, 011 and 101 by component-neutrality at components 1 and 2. We can also eliminate the outcomes 000 and 001 because otherwise agent 2 and 3 can misreport to 111 to obtain the strictly preferred outcome 111 by step (10).
Analogously, for $f(110, 001, 001)$, we can eliminate the all the outcomes except 001 and 000.

- (15) For $f(001, 011, 110)$, we can eliminate the outcomes 000 and 101 because otherwise agents 2 and 3 can misreport to 111 to obtain the strictly preferred outcome 111 by step (10). We can eliminate 110, because otherwise agent 1 can misreport to 101 to obtain the strictly preferred outcome 111 by step (1) and anonymity. We can eliminate 100, because otherwise agent 2 can misreport to 111 to obtain the strictly preferred outcome 111 by step (12) and anonymity. We can also eliminate 001, because otherwise agent 3 can misreport to 011 to obtain the strictly preferred outcome 011 by step (13). Analogously, for $f(110, 100, 001)$, we can eliminate the outcomes 111, 010, 001, 011 and 110.
- (16) For $f(001, 101, 011)$, we can eliminate the outcomes 010, 100, 110 and 000 by component-wise unanimity at component 3. The outcomes 101 and 011 are excluded by component-neutrality at components 1 and 2, and anonymity.

Analogously, for $f(110, 010, 100)$, we can eliminate the all outcomes except 110 and 000.

By the above eliminations we are left with following possible outcomes for the given profiles.

000, 000, 001	000	111, 111, 110	111
000, 001, 001	000, 001	111, 110, 110	111, 110
000, 001, 100	000	111, 110, 011	111
001, 001, 100	000, 001, 101	110, 110, 011	111, 110, 010
001, 010, 100	000	110, 101, 011	111
000, 111, 111	111	111, 000, 000	000
000, 011, 111	001, 010, 011, 111	111, 100, 000	110, 101, 100, 000
000, 011, 011	011	111, 100, 100	100
000, 011, 110	010	111, 100, 001	101
001, 111, 111	111	110, 000, 000	000
001, 011, 111	001, 011, 111	110, 100, 000	110, 100, 000
001, 110, 111	111	110, 001, 000	000
001, 011, 011	011	110, 100, 100	100
001, 110, 110	110, 111	110, 001, 001	001, 000
001, 011, 110	010, 011, 111	110, 100, 001	101, 100, 000
001, 101, 011	001, 111	110, 010, 100	110, 000

Further eliminations are as follows:

- (17) By step (9), $f(000, 011, 110) = 010$. By component-neutrality, we have $f(000, 011, 101) = 001$. So for $f(000, 001, 001)$, we can eliminate 000, because otherwise agents 2 and 3 can misreport to 011 and 101, respectively, to obtain 001. Hence, $f(000, 001, 001) = 001$. Analogously, $f(111, 110, 110) = 110$.
- (18) For $f(000, 011, 111)$, we can eliminate the outcome 111 because otherwise agent 1 can misreport to 011 to obtain 011 by step (17), anonymity, and component-neutrality. We can also eliminate the outcomes 001 and 010 because otherwise agent 3 can misreport to 011 to obtain the strictly pre-

- ferred outcome 011 by step (8). Hence, $f(000, 011, 111) = 011$. Analogously, $f(111, 100, 000) = 100$.
- (19) For $f(001, 011, 111)$, we can eliminate the outcome 111 because otherwise agent 1 can misreport to 011 to obtain the strictly preferred outcome 011 by step (17), anonymity, and component-neutrality. We can also eliminate the outcome 001, because otherwise agent 3 can misreport to 011 to obtain the strictly preferred outcome 011 by step (8). Hence, $f(001, 011, 111) = 011$. Analogously, $f(110, 100, 000) = 100$.
- (20) If $f(001, 110, 110) = 111$, then in the profile $(111, 110, 110)$ with $f(111, 110, 110) = 110$ by step (17), agent 1 can misreport to 001 to obtain the strictly better outcome 111. Hence, $f(001, 110, 110) = 110$. Analogously, $f(110, 001, 001) = 001$.
- (21) If $f(001, 001, 100) = 000$, then in the profile $(000, 001, 001)$ with $f(000, 001, 001) = 001$ by step (17), agent 1 can misreport to 100 and obtain 000, which is strictly better than 001. Hence, $f(001, 001, 100) \neq 000$. Suppose that $f(001, 001, 100) = 101$. Then by component-neutrality $f(010, 010, 100) = 110$, and again by component-neutrality $f(100, 100, 010) = 110$. In the profile $(110, 100, 100)$ agent 1 can misreport to 010 and obtain a strictly better outcome by step (13) and anonymity. Hence, $f(001, 001, 100) \neq 101$, and therefore $f(001, 001, 100) = 001$. Analogously, $f(110, 110, 011) = 110$.
- (22) For $f(001, 011, 110)$, we can eliminate the outcomes 010 and 111 because otherwise agent 1 can misreport to 011 to obtain the strictly preferred outcome 011 by step (21), anonymity, and component-neutrality. Hence, $f(001, 011, 110) = 011$. Analogously, $f(110, 100, 001) = 100$.
- (23) For $f(001, 101, 011)$, we can eliminate the outcome 111 because otherwise agent 1 can misreport to 101 to obtain the strictly preferred outcome 101 by step (21), anonymity, and component-neutrality. Hence, $f(001, 101, 011) = 001$. Analogously, $f(110, 010, 100) = 110$.

We conclude that f is the component-wise majority rule. \square

References

- Amanatidis G, Barrot N, Lang J, Markakis E, Ries B (2015) Multiple referenda and multiwinner elections using Hamming distances: complexity and manipulability. In: Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2015)
- Barberà S, Berga D, Moreno B (2010) Individual versus group strategy-proofness: when do they coincide? *J Econ Theory* 145:1648–1674
- Botan S, Novaro A, Endriss U (2016) Group manipulation in judgment aggregation. In: Proceedings of the 15th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2016)
- Barrot N, Lang J, Yokoo M (2017) Manipulation of Hamming-based approval voting for multiple referenda and committee elections. In: Proceedings of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2017)
- Brams SJ, Fishburn PC (2007) Approval voting. Springer, Berlin
- Brams S, Kilgour DM, Sanver MR (2007) A minimax procedure for electing committees. *Public Choice* 132:401–420
- Dietrich F, List C (2007) Strategy-proof judgment aggregation. *Econo Philos* 23(3):269–300
- Dokow E, Holzman R (2009) Aggregation of binary evaluations for truth-functional agendas. *Social Choice Welfare* 32:221–241

- Endriss U, Grandi U (2014) Binary aggregation by selection of the most representative voter. In: Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI 2014)
- Gibbard A (1973) Manipulation of voting schemes: a general result. *Econometrica* 41:587–602
- Le Breton M, Sen A (1999) Separable preferences, strategy-proofness, and decomposability. *Econometrica* 67:605–628
- Legrand R, Markakis E, Mehta A (2007) Some results on approximating the minimax rule in approval voting. In: Proceedings of the 6th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2007)
- Meir R, Procaccia AD, Rosenschein JS (2010) On the limits of dictatorial classification. In: Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2010)
- Moulin H (1980) On strategy-proofness and single peakedness. *Public Choice* 35:437–455
- Peters H, Roy S, Sadhukhan S (2021) Unanimous and strategy-proof probabilistic rules for single-peaked preference profiles on graphs. *Math Oper Res* 46(2):811–833
- Riker WH, Brams SJ (1973) The paradox of vote trading. *Am Polit Sci Rev* 67:1235–1247
- Satterthwaite MA (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. *J Econ Theory* 10(2):187–217
- Schummer J, Vohra RV (2002) Strategy-proof location on a network. *J Econ Theory* 104(2):405–428
- Terzopoulou Z, Endriss U (2019) Strategyproof judgment aggregation under partial information. *Social Choice Welfare* 53(3):415–442
- Xia L, Conitzer V (2010) Strategy-proof voting rules over multi-issue domains with restricted preferences, WINE-10, pp. 402–414

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