Social Choice and Welfare (2024) 62:367–396
https://doi.org/10.1007/s00355-023-01492-0

ORIGINAL PAPER

Inequality measurement with coarse data

Xiangyu Qu[1]

Received: 13 September 2021 / Accepted: 22 September 2023 / Published online: 16 November 2023

Abstract
Measuring inequality is a challenging task, particularly when data is collected in a coarse manner. This paper proposes a new approach to measuring inequality indices that considers all possible income values and avoids arbitrary statistical assumptions. Specifically, the paper suggests that two sets of income distributions should be considered when measuring inequality, one including the highest income per individual and the other including the lowest possible income per individual. These distributions are subjected to inequality index measures, and a weighted average of these two indices is taken to obtain the final inequality index. This approach provides more accurate measures of inequality while avoiding arbitrary statistical assumptions. The paper focuses on two special cases of social welfare functions, the Atkinson index and the Gini index, which are widely used in the literature on inequality.

1 Introduction

In recent years, inequality has become a major focus in economics, particularly in the aftermath of financial crises, social conflicts, and the pandemic (Atkinson et al. 2011). However, measuring inequality can be a challenging task due to a coarseness of data. While there are various indices such as Atkinson (1970), Gini (1921) and Theil (1967) indices.[1] that can be used to measure inequality if an economic distribution within a population can be precisely and correctly summarized in a single statistic, the reality is that datasets are often not presented in such a straightforward manner.

One reason for this is that data is often collected in a way that is not precise, such as income data being collected in the form of income bands rather than precise

[1] We refer to Cowell (2011) for a survey of various inequality indices.

✉ Xiangyu Qu
xiangyuqu@gmail.com

[1] CNRS, Centre d'Economie de la Sorbonne and School of Management, Wuhan University of Technology, Wuhan, China

individual income data.[2] As an example, the Current Population Survey, which serves as the primary source for labor force statistics in the United States, includes a question in its questionnaire: “Which category best represents the total combined income of all members of your FAMILY over the past 12 months?”.[3] Respondents are provided with 16 possible intervals to choose from, such as ($5,000 to $7,499) or ($7,500 to $9,999). Another reason is that while the data collected may be precise, it may not be processed in a unique way. For example, equivalised income is a measure of household income that is adjusted for differences in household size and composition. However, there are a wide range of equivalence scales that exist, making it difficult to estimate inequality when the dataset is coarse.[4] Therefore, although inequality is an important subject in economics, the challenge of dealing with coarse data makes measuring it a daunting task.

One common approach is to use a statistical model to impute individual incomes based on the reported income band data. This can be done by assuming that individual incomes are the group means or median (Heitjan 1989; Henson 1967). The imputed individual income data can then be used to calculate inequality measures such as the Gini index. These methods involve making strong assumptions about the shape of the income distribution and introduce errors and biases into the estimates of income inequality, which can affect the accuracy of policy decisions based on the data.

To accurately measure inequality parameters when data is coarse, it is important to avoid arbitrary statistical model assumptions. Ideally, the inequality measure should be independent of these assumptions, ensuring that estimates of inequality are consistent across different statistical models. This paper proposes new methods for measuring inequality indices that address this issue. We argue that all possible income values should be considered when measuring inequality, without ignoring any possible distribution. Specifically, we consider two sets of income distributions, one including the highest income per individual and the other including the lowest possible income per individual. We subject these distributions to inequality index measures, and then take a weighted average of these two indices to obtain the final inequality index. By using this approach, we can obtain more accurate measures of inequality while avoiding arbitrary statistical assumptions. More precisely, consider a multi-valued distribution F. Let $\overline{F}$ ($\underline{F}$) define the upper (lower) limit distribution. We suggest that a society could measure the inequality of F in the following way:

$$I(F) = \lambda_F \cdot \phi(\overline{F}) + (1 - \lambda_F) \cdot \phi(\underline{F}),$$

where ϕ is a measure of single-valued distribution. The parameter $0 \leq \lambda_F \leq 1$ can be interpreted as a measure of the social attitude toward inequality of upper limit distribution of F.

[2] Income bands refer to grouping individuals or households into broad categories based on their reported income levels. I highly appreciate that one referee suggests this example.

[3] Consult the website of *United States Census Bureau* for further details.

[4] I highly appreciate that an editor suggests this example.

We begin by adopting the approach of Atkinson (1970), Kolm (1969), and Sen (1973) to characterize a class of social welfare functions that induce the desired measurement. Specifically, we focus on two special cases: the Atkinson index and the Gini index. These indices are widely used in the literature on inequality and their associated social welfare functions are theoretically meaningful. The robust Atkinson social welfare function is additively separable, which is normatively appealing. In contrast, the robust Gini social welfare function is non-additive, but maintains a property known as comonotonic additivity à la Schmeidler (1989). Both indices have potential connections to political economy models, as demonstrated in Salas and Rodríguez (2013) and Rodríguez and Salas (2014). We also seek to establish a set of ethical axioms that characterize these robust social welfare functions, reflecting both inequality and imprecision considerations.

This paper provides a complementary approach to studying epistemic uncertainty in addition to the existing research on subjective uncertainty by scholars such as Ben-Porath et al. (1997), Gajdos and Tallon (2002), Gajdos and Maurin (2004), Chew and Sagi (2012). Epistemic uncertainty refers to the uncertainty arising from limitations of the data or our knowledge of the world, while subjective uncertainty pertains to uncertainty due to chance or randomness. Our study focuses on set-valued problems, which fall under the category of epistemic uncertainty. However, it is not yet clear what the state space is within our framework. Although studies in psychology suggest that people can intuitively distinguish between these two types of uncertainty, research in the field of inequality, particularly theoretical studies, has not explicitly focused on epistemic uncertainty. Thus, our paper proposes a new inequality index with a theoretical foundation under conditions of uncertainty when the state space cannot be naturally constructed.

Since Atkinson (1970), the inequality literature has had close connections with decision theory. Our robust social welfare functions are no exception and are related to concepts such as maxmin expected utility, α-maxmin expected utility of Ghirardato et al. (2004), and Hurwicz expected utility of Gul and Pesendorfer (2015). However, our focus is on the environment where no state space is present, which makes the objective ambiguity model of Olszewski (2007) a closer fit to our approach. Although the concepts are similar, our motivation and application are significantly different. At a technical level, our main distinction is that we allow for a non-additive measure with respect to single-valued distribution.

In the next section, we explore a social welfare approach to construct a robust measure of inequality. We discuss how to extend two widely used inequality indices, the Atkinson and Gini indices, to robust indices. Section 3 focuses on the robust Atkinson index and the robust Gini index. We axiomatize the robust Atkinson and Gini social welfare functions, which induce the corresponding robust indices. Finally, in Sect. 4, we conclude and provide further discussion. All proofs are included in the appendix.

2 Inequality measurement

2.1 Setup

Consider a society $\mathcal{N}$ consists of $n > 2$ *individuals*. Let $X = \mathbb{R}_+$ be the set of possible individual allocations. We denote by $\mathcal{X}$ the collection of all non-empty compact subsets of X. An allocation profile is denoted by $F = (F_1, \dots, F_n)$, where each $F_i \in \mathcal{X}$ contains all possible allocations of individual i. An allocation profile is *deterministic* (also known as a *distribution*) and is written as $f = F$, if each F_i is a singleton, i.e. $F_i \in X$. Let $\mathcal{F}$ be the collection of all possible allocation profiles and let X^n denote the set of all deterministic allocation profiles. We denote $\mathbb{1} \in X^n$ as the deterministic profile f where $f_i = 1$ for all i. If there is no confusion, we write deterministic profile $f \in F$ if $f_i \in F_i$ for each i.

For $Y, Z \in \mathcal{X}$, we write $Y \geq Z$ if $y \geq z$ for all $y \in Y$ and $z \in Z$. For $F \in \mathcal{F}$, we denote $\overline{F}$ as the upper limit distribution in F if $\overline{F} \in F$ and $\overline{F}_i \geq F_i$ for all i. Similarly we denote $\underline{F}$ as the lower limit distribution in F if $\underline{F} \in F$ and $\underline{F}_i \leq F_i$ for all i. Also, for $F, G \in \mathcal{F}$, we write $F \geq G$ if $F_i \geq G_i$ for all i.

For $f \in X^n$, we write $\mu(f) = \frac{1}{n}\sum_{i=1}^{n} f_i$ for the *mean* of f. Also, let $\tilde{f}$ be the deterministic allocation profile obtained from f by rearranging the allocation in increasing order, i.e. there exist a permutation $\pi : \mathcal{N} \to \mathcal{N}$ such that $f_{\pi(i)} = \tilde{f}_i$ and $\tilde{f}_1 \leq \ldots \leq \tilde{f}_n$.

2.2 Robust inequality index

To construct a robust inequality index, we adopt Atkinson (1970), Kolm (1969), and Sen (1973) (AKS) approach, which posits that an inequality index should be a transformation of a social welfare function (SWF) that emphasizes the welfare loss due to the inequality in the allocation profile. Formally, a *social welfare function* (SWF) $W : \mathcal{F} \to \mathbb{R}$ maps allocation profiles to real numbers.

To develop a welfare-theoretic approach to the measurement of inequality, we focus on the class of SWFs that display the inequality reduction property. To this end, we assume that the SWF should satisfy the following three assumptions. We say a SWF W is *Schur-concave* on deterministic profiles if for all $f \in X^n$ and all bistochastic matrices M of order n,[5] $W(fM) \geq W(f)$. We say a SWF W is *monotonic* if for all $F, G \in \mathcal{F}$, $W(F) \geq W(G)$ whenever $F \geq G$. We refer to a SWF as *regular* if it is continuous with respect to Hausdorff distance,[6] monotonic and Schur-concavity on deterministic profiles. We assume throughout this section that W is regular.

Given a regular SWF W, for any allocation profile F, we define the *equally distributed equivalent* $\xi(F) \in \mathbb{R}$ as follows:

[5] A $n \times n$ matrix M with nonnegative entries is called a bistochastic matrix order n if each of its rows and columns sums to unity.

[6] For every pair of deterministic allocation profiles f, g, the distance between f and g can be induced by a natural topology, written as $d(f, g)$, on $\mathbb{R}^n$. Therefore, the set of allocation profiles $\mathcal{F}$ can be equipped with Hausdorff distance in the following way: for $F, G \in \mathcal{F}$,

$$\mathrm{dist}(F, G) = \max\left\{ \max_{f \in F} \min_{g \in G} d(f, g), \max_{g \in G} \min_{f \in F} d(f, g) \right\}.$$

$$W(\xi(F)\cdot \mathbb{1}) = W(F).$$

Therefore, $\xi(F)$ is the level of allocation that, if given to each individual, will make the existing profile F socially indifferent. Since W satisfies regularity conditions, this can be used to yield the equally distributed equivalent as a function $\xi : \mathcal{F} \to \mathbb{R}$. In other words, given a profile F, $\xi(F)$ can be uniquely extracted from the above equation. In particular, note ξ is also regular. Further, it is immediate to see $\xi(c\cdot \mathbb{1}) = c$ for all $c > 0$.

Due to monotonicity, for $F \in \mathcal{F}$, we have

$$\xi(\underline{F}) \le \xi(F) \le \xi(\overline{F}).$$

So there exists a unique $\lambda_F \in [0,1]$ such that

$$\lambda_F \xi(\overline{F}) + (1-\lambda_F)\xi(\underline{F}) = \xi(F).$$

Accordingly, we propose a simple transformation of regular SWF as an index of inequality.

Definition 1 A function $I : \mathcal{F} \to \mathbb{R}$ is said to be a *robust index of inequality* if, for all $F \in \mathcal{F}$ with $\underline{F} \neq 0$,

$$I(F) = 1 - \left\{ \lambda_F \frac{\xi(\overline{F})}{\mu(\overline{F})} + (1-\lambda_F)\frac{\xi(\underline{F})}{\mu(\underline{F})} \right\}. \tag{1}$$

The proposed definition of the inequality index coincides with the AKS approach when the profile is deterministic. However, it is important to note that the index is not defined for profiles where each individual has zero allocation, as this is not a feasible allocation. Our proposal is plausible because the index has important properties that the classical index requires.

Proposition 1 *A robust index of inequality I has the following properties*:

(i) *Betweeness*: *Each* $I(F)$ *lies between* $I(\overline{F})$ *and* $I(\underline{F})$.
(ii) *Schur convexity on deterministic profiles*: $I(f) \ge I(fM)$ *for every bistochastic matrix M and deterministic profile f.*
(iii) *Normalization*: *Each* $I(F)$ *lies in* $[0,1]$; *and* $I(F) = 0$ *iff* $\overline{F} = c\cdot \mathbb{1}$ *and* $\underline{F} = c'\cdot \mathbb{1}$ *for some* $c \ge c' > 0$.

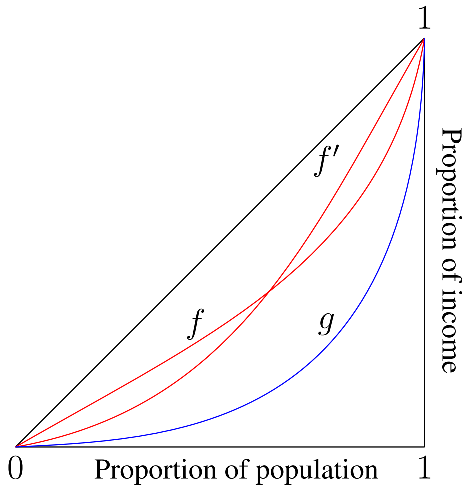
We actually can rewrite index I in a weighted average of $I(\overline{F})$ and $I(\underline{F})$.

$$I(F) = \lambda_F I(\overline{F}) + (1-\lambda_F) I(\underline{F}).$$

Using this, we can express $\xi(F)$as

$$\xi(F) = \lambda_F\left[\mu(\overline{F})(1-I(\overline{F}))\right] + (1-\lambda_F)\left[\mu(\underline{F})(1-I(\underline{F}))\right].$$

Fig. 1 $\{f, f'\}$ Lorenz dominates g



As previously mentioned, the function ξ itself or any increasing transformation function of it can be regarded as a regular SWF, and thus implies and is implied by other inequality indices. Thus, ξ implies and is implied inequality indices. However, $\xi(F)$ is not directly implied by $I(F)$, but rather through $I(\overline{F})$, $I(\underline{F})$ and $I(F)$. The welfare function used in the index is represented as an increasing function of a weighted sum of two products: the mean of the upper limit distribution and the shortfall of its inequality index from unity, and the mean of the lower limit distribution and the shortfall of its inequality index from unity. This type of welfare function is referred to as a boundary reduced-form welfare function because its arguments summarize the entire distribution in terms of the mean and inequality of the upper and lower limit distributions.

2.3 Lorenz dominance and coarse inequality

Lorenz (1905) uses a Lorenz curve to present deterministic allocation profile in an illuminating fashion. The Lorenz domination criterion is widely acknowledged as a fundamental principle to rank alternative profiles in terms of comparative inequality. In this subsection, we explore the extension of Lorenz domination from deterministic profiles to general profiles and develop its relation with SWF.

Recall that a deterministic profile f is said to *Lorenz dominate* g,[7] if

[7] The classic definition of Lorenz domination, such as Atkinson (1970) and Dasgupta et al. (1973), assumed that the compared profiles have the same mean, which does not fit in our setting. Therefore, our definition is an extension of their concept, which has referred to as generalized Lorenz dominance by Shorrocks (1980).

$$\frac{1}{n\mu(f)}\sum_{i=1}^{k}\tilde{f}_i \geq \frac{1}{n\mu(g)}\sum_{i=1}^{k}\tilde{g}_i,$$

for all $k = 1, 2, \ldots, n$. That is, f Lorenz dominates g if the Lorenz curve of f is nowhere below the Lorenz curve of g. Now we extend this definition on deterministic profiles to the general profiles.

Definition 2 A profile F *Lorenz dominates* another profile G, write as $F \succsim_L G$, if for every $f \in F$ and $g \in G$, f Lorenz dominates g (Fig. 1).

A profile F Lorenz dominates G if every feasible deterministic profile in F Lorenz dominates every deterministic allocation in G. Thus, as we can see in Figure (1), if $F = \{f, f'\}$ and $G = \{g\}$, then F Lorenz dominates G. However, the ranking of profiles generated by the Lorenz domination comparison is incomplete since, assuming $F' = \{f, g\}$ and $G' = \{f', g\}$, we cannot rank F' and G' by the Lorenz domination criterion. Though, $\succsim_L$ is incomplete, but it satisfies transitivity. Below we state the relation between the Lorenz domination criterion and social welfare functions.

Proposition 2 *Suppose that social welfare function W is regular. Let F and G be two profiles such that $\min_{f\in F}\mu(f) \geq \max_{g\in G}\mu(g)$. Then $F \succsim_L G$ if and only if $W(F) \geq W(G)$, and $W(f) \geq W(g)$ for each $f \in F$ and $g \in G$.*

This result states that a regular SWF will rank a profile and any deterministic profiles within it higher than another profile and any deterministic profiles within it, respectively, if and only if the Lorenz curves of the first profile are nowhere lower than those of the latter profile. This implies that a regular SWF is compatible with the Lorenz domination criterion. Therefore, it is reasonable to focus on regular SWFs when developing a robust inequality index.

2.4 Two robust indices

In this section, we extend two of the most popular indices, namely, the Atkinson index and the Gini index, to the robust indices.[8] To discuss about the two specific indices, we need to restrict our robust inequality index I further. An inequality index I is a *relative* or scale invariant index if for all $F \in \mathcal{F}$ and $c > 0$, $I(cF) = I(F)$. To make I a relative index,[9] further assumption on SWF W is required. We say W is *homothetic* if for all F, $W(F) = \Phi(\hat{W}(F))$, where $\hat{W}$ is linear homogeneous, i.e. $\hat{W}(cF) = c\hat{W}(F)$ for $c > 0$, and Φ is an increasing transformation.

[8] We refer to chapter 2 of Moulin (1991) for a discussion of two classic indices developed on AKS approach.

[9] We refer to Blackorby and Donaldson (1980) for detailed discussion about relative index.

Proposition 3 *A robust index of inequality I defined as in eq (1) is a relative index if and only if W is homothetic.*

Since the following indices we consider are relative, we restrict our attention to SWF that is both regular and homothetic.

Robust Atkinson index

We first consider a regular and homothetic SWF, so-called *robust Atkinson SWF*, which would characterize a robust Atkinson index, namely,

$$W_A(F) = \alpha \sum_{I=1}^{n} u(\overline{F}_i) + (1-\alpha) \sum_{i=1}^{n} u(\underline{F}_i). \tag{2}$$

where $0 \leq \alpha \leq 1$ and $u : \mathcal{X} \to \mathbb{R}$ is defined by

$$u(x) = \begin{cases} a + b \cdot \dfrac{x^r}{r} & \text{for } 0 < r < 1, \\ a + b \cdot \log x & \text{for } r = 0; \end{cases} \tag{3}$$

with constant number a and positive number b. Using the SWF above, we get the explicit form of the *robust Atkinson index* according to eq (1):

$$I_A(F) = \begin{cases} 1 - \alpha\Big[\dfrac{1}{n} \cdot \displaystyle\sum_{i=1}^{n} \big(\dfrac{\overline{F}_i}{\mu(\overline{F})}\big)^r\Big]^{1/r} - (1-\alpha)\Big[\dfrac{1}{n} \cdot \displaystyle\sum_{i=1}^{n} \big(\dfrac{\underline{F}_i}{\mu(\underline{F})}\big)^r\Big]^{1/r} & \text{for } 0 < r < 1, \\ 1 - \alpha\Big[\displaystyle\prod_{i=1}^{n} \big(\dfrac{\overline{F}_i}{\mu(\overline{F})}\big)^{1/n}\Big] - (1-\alpha)\Big[\displaystyle\prod_{i=1}^{n} \big(\dfrac{\underline{F}_i}{\mu(\underline{F})}\big)^{1/n}\Big] & \text{for } r = 0. \end{cases} \tag{4}$$

The Gini index is perhaps the most commonly used measure of inequality, and our robust Gini index offers a means of measuring the Gini index when the allocation profile is not deterministic. Once again, the parameter α can be interpreted as the weight of confidence that society places on the upper limit distribution in a given profile.

Robust Gini indexWe now consider a SWF that characterizes a robust Gini index.

$$\begin{aligned} W_G(F) &= \alpha\Big\{\mu(\overline{F}) - \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} |\overline{F}_i - \overline{F}_j|}{2n^2}\Big\} + (1-\alpha)\Big\{\mu(\underline{F}) - \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} |\underline{F}_i - \underline{F}_j|}{2n^2}\Big\} \\ &= \alpha \cdot \frac{\sum_{i=1}^{n}\big[2(n-i)+1\big] \cdot \widetilde{\overline{F}}_i}{n^2} + (1-\alpha) \cdot \frac{\sum_{i=1}^{n}\big[2(n-i)+1\big] \cdot \widetilde{\underline{F}}_i}{n^2}, \end{aligned} \tag{5}$$

where $0 \leq \alpha \leq 1$. Hence, the robust Gini index defined below corresponds to the above SWF.

$$
\begin{aligned}
I_G(F) &= \alpha \cdot \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}|\overline{F}_i - \overline{F}_j|}{2n^2\mu(\overline{F})} + (1-\alpha)\cdot \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}|\underline{F}_i - \underline{F}_j|}{2n^2\mu(\underline{F})} \\
&= 1 - \alpha \cdot \frac{\sum_{i=1}^{n}\left[2(n-i)+1\right]\cdot \tilde{\overline{F}}_i}{n^2\mu(\overline{F})} - (1-\alpha)\cdot \frac{\sum_{i=1}^{n}\left[2(n-i)+1\right]\cdot \tilde{\underline{F}}_i}{n^2\mu(\underline{F})}
\end{aligned} \tag{6}
$$

The Gini index might be the most widely used index of inequality and our robust Gini index provides a way to measure Gini index whenever allocation profile is not deterministic. The parameter α, once again, can be regarded as the confident weight that society assigns to upper limit distribution in a profile.

3 Axiomatization

In this section, we will discuss the axioms that must be satisfied by a society in order to have a robust Atkinson or a robust Gini SWF. We will use the characterization and transformation method introduced in the previous section to derive the robust Atkinson index and the robust Gini index.

Formally, let $\succsim \subset \mathcal{F} \times \mathcal{F}$ be a social preference over a set of allocation profiles. We say that a SWF $W : \mathcal{F} \to \mathbb{R}$ represents the social preference $\succsim$ if for all $F, G \in \mathcal{F}$, $W(F) \geq W(G)$ if and only if $F \succsim G$.

3.1 Regular axioms

We first state five regular axioms. These axioms with respect to deterministic profiles are widely assumed in the inequality literature. Also the five axioms are necessary for both robust Atkinson and robust Gini SWF.

A1 (*Weak order*) $\succsim$ is complete and transitive.
A2 (*Continuity*) For all $F \in \mathcal{F}$, the sets $\{G : G \succsim F\}$ and $\{G : F \succsim G\}$ are closed in $\mathcal{F}$ with respect to Hausdorff distance.

A1 is commonly required conditions and do not need further elaboration. A2 generalizes traditional continuity for deterministic profiles and can be interpreted in a similar manner.

For a permutation $\pi : \mathcal{N} \to \mathcal{N}$ and $F \in \mathcal{F}$, define $\pi \circ F \in \mathcal{F}$ by $(\pi \circ F)_i = F_{\pi(i)}$ for every $i \in \mathcal{N}$.

A3 (*Symmetry*) For all $F, G \in \mathcal{F}$, if there is a permutation π such that $F = \pi \circ G$, then $F \sim G$.

A3 states that any permutation of individual labels should be considered allocation-equivalent. This axiom ensures that the social ranking depends solely on the allocated variable and not on any other characteristic that might be distinguishable among members of society. Therefore, under symmetry, the identities of individuals are entirely irrelevant to the social decision-making process. Although not self-evident, this axiom is widely accepted in the literature.

A4 (*Unanimity*) For all $F, G \in \mathcal{F}$, if $F_i \geq G_i$ for all i, then $F \succsim G$. Furthermore, if $F_j > G_j$ for some j, then $F \succ G$.

A4 states that if, for every individual, the worst allocation in F is better than the best allocation in G, then society prefers F to G. Moreover, if there exists an individual for whom the worst allocation in F is strictly better than the best allocation in G, then society strictly prefers F to G.

We say that profile F *dominates* profile G if (i) for every $f \in F$, there exists a $g \in G$ such that $f \succsim g$, and (ii) for every $g \in G$, there exists $f \in F$ such that $f \succsim g$. In other words, if profile F dominates G, then for any deterministic allocation in F, there must exist a worse deterministic allocation in G, and for any deterministic allocation in G, there must exist a better deterministic profile in F. The next axiom simply states that a dominant profile is always preferred to a dominated profile.

A5 (*Dominance*.) If profile F dominates profile G, then $F \succsim G$.

The above five axioms are intuitive assumptions in the inequality literature. Below we discuss further the very axioms that would characterize either robust Atkinson SWF or robust Gini SWF.

3.2 Robust Atkinson SWF

We now want to state the required axioms that characterize robust Atkinson SWF. To state next axiom, we need some notation first. If $F \in \mathcal{F}$ and $T \subset \mathcal{N}$, we write $F_T = (F_i)_{i \in T}$ and $F_{T^c} = (F_i)_{i \in \mathcal{N} \setminus T}$.

A6 (*Separability*) For all $F, G \in \mathcal{F}$ and nonempty $T \subset \mathcal{N}$, if $(F_T, F_{T^c}) \succsim (G_T, F_{T^c})$, then $(F_T, G_{T^c}) \succsim (G_T, G_{T^c})$

Separability basically means that when considering social welfare ordering, if two profiles only differ in a subset T of individuals, then the allocation of the rest of the individuals would not affect social ordering. In other words, social rankings are independent of non-concerned individuals. Along with the first four axioms, separability implies that the social welfare function has an additively separable form, which is defined as follows.

Definition 3 We say a SWF $W : \mathcal{F} \to \mathbb{R}$ is *additively separable* if there exist an increasing function $u : \mathcal{X} \to \mathbb{R}$ (*i.e.* $Y, Y' \in \mathcal{X}$ and $Y \geq Y'$ imply $u(Y) \geq u(Y')$), such that, for all $F \in \mathcal{F}$,

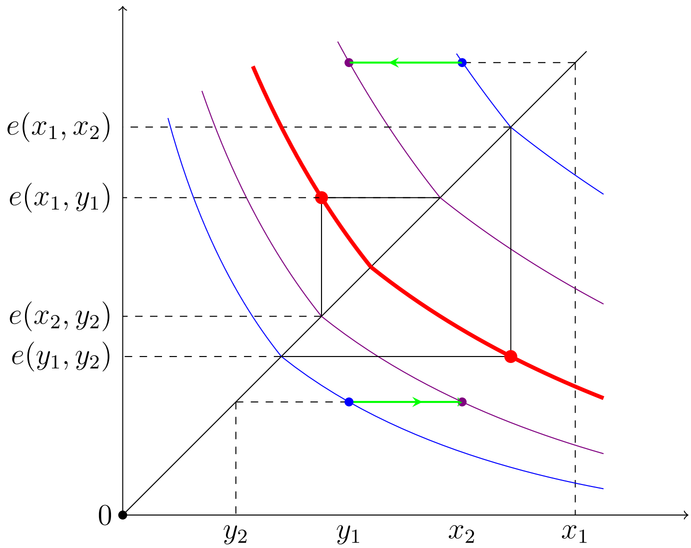


Fig. 2 Commutativity

$$W(F) = \sum_{i=1}^{n} u(F_i).$$

Proposition 4 *A social preference* $\succsim$ *satisfies* A1-4 *and* A6 *if and only if social preference* $\succsim$ *is represented by an additively separable SWF.*

This result says that a social preference that satisfies A1-4 and A6 is equivalent to the existence of a utility function defined on a set of possible allocation $\mathcal{X}$ such that any allocation profile is evaluated by the utility sum over every individual allocation. Furthermore, this utility function is increasing in $\mathcal{X}$. In contrast, the classic additively separable SWF is defined over deterministic allocation profile. Our result can be regarded as a direct extension of classic one.

Moreover, together with A5 (Dominance), it turns out that function $u(X)$ only depends on the maximum and minimum values of X. In fact, if A4 is replaced by an axiom saying that $\overline{F} \geq \overline{G}$ implies $F \succsim G$, then function $u(X)$ only depends on the minimum values of X. Similarly axiom can makes $u(X)$ only depends on the maximum values of X.

Actually, robust Atkinson SWF is additively separable in which function u has the following form: there exists $\alpha \in [0, 1]$ such that for $Y \in \mathcal{X}$,

$$u(Y) = \alpha \max_{x \in Y} u(x) + (1 - \alpha) \min_{x \in Y} u(x).$$

Along with A5, the next axiom will characterize function u with the above expression. The last two axioms will guarantee function u on X has the expression as in eq (3).

For $Y \in \mathcal{X}$, we say an allocation $e(Y) \in X$ is *equivalent* to Y, if profile $(Y, \ldots, Y) \sim (e(Y), \ldots, e(Y))$. In words, if a profile has the same allocation Y for every individual, then a deterministic profile with allocation $e(Y)$ for every individual is socially equivalent.

A7 (*Commutativity.*) For $x_1, x_2, y_1, y_2 \in X$, if $x_1 \geq \{x_2, y_1\} \geq y_2$, then $F \sim G$ whenever $F_i = \{e(x_1, x_2), e(y_1, y_2)\}$ and $G_i = \{e(x_1, y_1), e(x_2, y_2)\}$ for all i.

To better understand the commutativity, see Fig. 2 for the indifference curves over profiles $(Y, \ldots, Y)$ in which Y contains at most two values. Any point (x, y) in the quadrant represents profile $(Y, \ldots, Y)$ where $Y = \{x, y\}$. Therefore, the diagonal represents the deterministic profiles $(c, \ldots, c)$.

Consider a coarse dataset consisting of four possible allocations: x_1, x_2, y_1, and y_2, where x_1 represents the best allocation and y_2 represents the worst allocation. How should society evaluate such a coarse dataset? One possible approach is to first divide the coarse dataset into two groups: a “good” group containing the best allocation and a “bad” group containing the worst allocation. Then, the two groups are evaluated separately to find their respective equivalent allocations. Finally, society finds deterministic allocations that are equivalent to these two group-based equivalent allocations.

A7 requires that in this situation, the way in which the two groups are formed should not affect how society evaluates this coarse dataset, as long as the good group contains the best allocation and the bad group contains the worst allocation. In other words, A7 states that society’s evaluation of the coarse dataset should be independent of the division of groups.

A8 (*Scale Invariance*) For all deterministic profiles $f, g \in X^n$ and all $\lambda > 0$, if $f \succsim g$, then $\lambda f \succsim \lambda g$.

Under scale invariance axiom, it does not matter whether we measure allocation in euros or dollars as long as the unit is the same for each individual allocation.

A9 (*Pigou-Dalton principle*) For all deterministic profiles $f, g \in X^n$, if there are $i, j \in \mathcal{N}$ such that $f_k = g_k$ for $k \notin \{i, j\}$ and $f_i + f_j = g_i + g_j$ and $|f_i - f_j| < |g_i - g_j|$, then $f \succ g$.

A9 simply states that a transfer between two individual allocation, in such a way that their allocation difference is reduced, will result in a strictly social preferred allocation profile. This principle demonstrates that redistributions from the rich to the poor would improve the social welfare.

Theorem 1 *A social preference $\succsim$ on $\mathcal{F}$ satisfies* A1-9 *if and only if social preference $\succsim$ is represented by a robust Atkinson SWF as in eq.* (2).

This result presents a characterization of the robust Atkinson Social Welfare Function (SWF) in cases where individual allocation may not be deterministic. As a result, a social preference that respects the A1-9 criteria considers the welfare loss due to both inequality and imprecision in each allocation profile. Additionally, by applying a mathematical transformation to the SWF as shown in eq. (1), a robust Atkinson index can be derived, as depicted in eq. (4).

Now, let's briefly examine how this result is established, which allows us to observe the independence of each axiom. Firstly, axioms 1-3 and axiom 6 collectively imply that the SWF is additively separable, a result corresponds to the theorem by Debreu (1960). Axiom 4 necessitates that the SWF must exhibit monotonicity. Axiom 5 focuses on the utility function u defined on the set $Y \in \mathcal{X}$. Dominance suggests that u depends solely on the maximal and minimal equal distributions within set Y. Subsequently, Axiom 7 stipulates that u is the weighted sum of the values associated with maximal and minimal equal distributions. Finally, when combined with Scale Invariance and the Pigou-Dalton Principle, this implies that u conforms to the desired function defined in eq. (3).

3.3 Robust Gini SWF

We now want to characterize robust Gini SWF. As we see from eq. (5), robust Gini SWF is not additively separable. It is additive with respect to order-preserving. Formally, two deterministic allocation profiles $f, g \in X^n$ are *order-preserving* if $f_i \geq f_j \Leftrightarrow g_i \geq g_j$ for all $i, j \in \mathcal{N}$. For $F, G \in \mathcal{F}$, we say F and G are *order-preserving* (in boundary) if both $\overline{F}, \overline{G}$ and $\underline{F}, \underline{G}$ are order-preserving. For every F, G, we define $F + G$ by for each $i \in \mathcal{N}$,

$$(F+G)_i = \{f_i + g_i \, : f_i \in F_i \text{ and } g_i \in G_i\}.$$

Note that if F, G, H are pairwisely order-preserving profiles, then $F + H$ and $G + H$ are also pairwisely order-preserving.

A6' (*Order-preserving Independence.*) For all $F, G, H \in \mathcal{F}$, if F, G, H are pairwisely order-preserving, then $F \succsim G \Leftrightarrow F + H \succsim G + H$.

This axiom states that the social ranking of two profiles F and G, which agree on the ordering of upper and lower limits, respectively, should be invariant to the addition of another order-preserving profile H. The inspiration for it may best be seen through the cases it precludes: if, for instance, two profiles $F + H$ and $G + H$ are the addition of a common profile H; and F and G are not order-preserving, then the overall judgement between $F + H$ and $G + H$ is not completely determined by a comparison of F and G. Suppose individual i is the richest in F, but the poorest in G. On the contrary, individual j is the poorest in F, but the richest in G. If H is a profile with high allocation for i, but low allocation for j, then addition of F and H may make the difference between i and j even larger. As a result, profile $F + H$ is more unequal than F. At the same time, the addition of G and H would reduce the difference between i and j and is more equal than H. Therefore, it is not promising to insist the invariance to addition of the common profile. This asymmetric impact on inequality may give rise to preference reversal. A6' only requires that if the profiles are order-preserving, then preference reversal should not occur. Also, this axiom can be regarded as a generalization of traditional order-preserving independence over deterministic profiles (See Weymark (1981)).

We state the last three axioms to derive classic Gini SWF defined on deterministic profiles. The next two axioms are first proposed by Elchanan and Itzhak (1994). For $f \in X^n$ and $i, j \in \mathcal{N}$, we say i precedes j in f if $f_i \leq f_j$ and there is no $k \in \mathcal{N}$ such that $f_i < f_k < f_j$.

A7' (*Transfer Invariance.*) For all $f, g, f' g' \in X^n$ and $i, j \in \mathcal{N}$, if the following are satisfied:

(i) i precedes j in $f, g, f' g'$;
(ii) $f_i = f'_i + c$, $f_j = f'_j - c$ and $g_i = g'_i + c$, $g_j = g'_j - c$ for some $c > 0$;
(iii) $f_k = f'_k$ and $g_k = g'_k$ for $k \notin \{i, j\}$,

then $f \succsim g$ if and only if $f' \succsim g'$.

A7' requires that there is no preference reversal if there is same amount of transfer between two preceded individuals i, j. However, it is indeed a strong claim since it is possible that i, j are poor in f, but rich in g.

A8' (*Inequality Aversion.*) For all $f, g \in X^n$ and $i \in \mathcal{N}$, if $\tilde{f}_i = \tilde{g}_i + c$ and $\tilde{f}_{i+1} = \tilde{g}_{i+1} - c$ for some $c > 0$ and $\tilde{f}_j = \tilde{g}_j$ for $j \notin \{i, i+1\}$, then $f \succ g$.

A8' simply says that it is socially preferred that if we transfer an amount of money from an individual to the next richest one without changing the ordering. This axiom is a weaker version of Dalton-Pigou principle, in which any transfer from rich to poor is preferred.

A9' (*Tradeoff.*) For all $c > 0$ and $k \in \mathcal{N}$,

$$(kc, 0, \ldots, 0) \sim (\underbrace{\frac{c}{k}, \ldots, \frac{c}{k}}_{k \text{ individuals}}, 0, \ldots, 0)$$

Let's consider a scenario where a society initially possesses a wealth of c. There exist two methods for distributing this wealth: (i) Equally dividing it among k individuals, thereby leaving nothing for the remaining $n - k$ individuals. (ii) Entrusting all the wealth to a single individual, who would then generate wealth that is k times greater, but the remaining $n - 1$ individuals would receive nothing. In A9', it is demonstrated that the society regards both of these distribution methods as equally valid. However, it becomes apparent that the equality of the first method increases as k grows larger, while the second method requires the creation of additional wealth for the society to remain indifferent between the two alternatives.

Theorem 2 *A social preference* $\succsim$ *on* $\mathcal{F}$ *satisfies* $A1$-5 *and* $A6'$-$9'$ *if and only if social preference* $\succsim$ *is represented by a robust Gini SWF as in eq.* (5).

This result provides a complete characterization of the robust Gini Social Welfare Function (SWF). This SWF is not additively separable but order-preserving additive. It is worth noting that when restricted to deterministic profiles, it becomes the classic Gini SWF. However, our characterization improves upon the results of Elchanan and Itzhak (1994) since their findings are restricted to deterministic profiles with fixed total income.

Aaberge (2001) suggests an axiomatic characterization of the classic Gini SWF based on Lorenz curve orderings, which was initiated by Yaari (1988). However, his result is built on the assumption that the Lorenz curve is convex, which may not hold in our framework. Hence, we provide the first complete characterization of the classic Gini index as a by-product.

4 Concluding remark

In recent years, there has been a growing recognition that inequality affects nearly every aspect of economics. Numerous studies have been conducted over the past few decades to measure inequality, but they have typically assumed that each individual allocation can be precisely estimated. However, many widely used datasets only provide imprecise estimations, which poses conceptual and practical challenges in measuring inequality.

This paper presents a novel approach to measuring inequality in the face of indeterministic allocation profiles. Our methodology extends the classic Atkinson and Gini indices to their robust counterparts, and we provide an axiomatic justification for the associated SWFs. While this innovation corrects some of the shortcomings of traditional methods, it also has some limitations. Continuing to improve upon these measures is important and needs more work on it.

Appendix: Proofs

A Proof of Section 2

A.1 Proof of Proposition 1

To prove Proposition 1, suppose that function $I : \mathcal{F} \to \mathbb{R}$ is defined as in eq. (1) and the associated SWF W is regular.

Proof of (i) By definition, for $F \in \mathcal{F}$,

$$I(F) = \alpha_F I(\overline{F}) + (1 - \alpha_F) I(\underline{F}).$$

Since $\alpha_F \in [0, 1]$, it is immediate to see that $\min\{I(\overline{F}), I(\underline{F})\} \leq I(F) \leq \max\{I(\overline{F}), I(\underline{F})\}$.

Proof of (ii) For $F \in \mathcal{F}$, $W(F) = W(\xi(F) \cdot \mathbb{1})$. By continuity of W, ξ is also a continuous function. By monotonicity of W, for any $F, G \in \mathcal{F}$,

$$W(F) \geq W(G) \Longleftrightarrow \xi(F) \geq \xi(G).$$

Since W is Schur concave with respect to deterministic profiles, for any bistochastic matrix M of order n,

$$W(fM) \geq W(f).$$

For any $c > 0$, $W(c \cdot \mathbb{1}) = c$. So, we have

$$W(fM) = \xi(fM) \geq W(f) = \xi(f).$$

Note that $\mu(fM) = \mu(f)$. Therefore,

$$\begin{aligned} I(fM) &= 1 - \frac{\xi(fM)}{\mu(fM)} \\ &= 1 - \frac{\xi(fM)}{\mu(f)} \\ &\leq 1 - \frac{\xi(f)}{\mu(f)} \\ &= I(f) \end{aligned}$$

Proof of (iii) Consider bistochastic matrix $\hat{M} = (m_{ij})$ with $m_{ij} = 1/n$ for all $i, j \in \mathcal{N}$. Then for any $f \in \mathcal{F}$, Schur concavity implies that

$$W(f\hat{M}) = \mu(f) \geq W(f).$$

Therefore, for $F \in \mathcal{F}$, we have

$$0 \leq \frac{\xi(\overline{F})}{\mu(\overline{F})}, \frac{\xi(\underline{F})}{\mu(\underline{F})} \leq 1.$$

Therefore, according to the definition of I,

$$0 \leq I(\overline{F}), I(\underline{F}) \leq 1.$$

So, for any $\alpha_F \in [0, 1]$,

$$I(F) = \alpha_F I(\overline{F}) + (1 - \alpha_F) I(\underline{F}) \in [0, 1].$$

Furthermore, if $I(F) = 0$, then $I(\overline{F}) = I(\underline{F}) = 0$. This means that $\xi(F) = \mu(F)$, which leads to $F = c \cdot \mathbb{1}$ for some $c > 0$. Conversely, if $F = c \cdot \mathbb{1}$, we have $\xi(F) = \mu(F)$, which leads to $I(F) = 0$.

A.2 Proof of Proposition 2

Since the proof of necessity part is straightforward, we only prove sufficiency part. Suppose $F \succsim_L G$. By definition, for all $f \in F$ and $g \in G$, $f \succsim_L g$. Lorenz dominance requires that for all $k = 1, 2, \dots, n$

$$\frac{1}{n\mu(f)} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{1}{n\mu(g)} \sum_{i=1}^{k} \tilde{g}_i.$$

Since $\mu(\underline{F}) = \min_{f \in F} \mu(f) \geq \max_{g \in G} \mu(g) = \mu(\overline{G})$, we have

$$\frac{\mu(\underline{F})}{n\mu(f)} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{\mu(\overline{G})}{n\mu(g)} \sum_{i=1}^{k} \tilde{g}_i,$$

which implies

$$\frac{1}{n} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{1}{n} \sum_{i=1}^{k} \tilde{g}_i.$$

Now, according to Marshall et al. (1979) (64 pp.), there must exist a bistochastic matrix M such that $\tilde{f} \geq \tilde{g}M$. Then, monotonicity of W implies $W(\tilde{f}) \geq W(\tilde{g}M)$. Furthermore, Schur-concavity implies that $W(\tilde{g}M) \geq W(\tilde{g})$. Notice that Schur-concavity implies symmetry, hence, $W(\tilde{f}) = W(f)$ and $W(\tilde{g}) = W(g)$. As a result, we have $W(f) \geq W(g)$. Since this inequality holds for any $f \in F$ and $g \in G$,

$$\min_{f \in F} W(f) \geq \max_{g \in G} W(g).$$

Monotonicity requires that $W(F) \geq \min_{f \in F} W(f)$ and $\max_{g \in G} W(g) \geq W(G)$, which implies $W(F) \geq W(G)$.

A.3 Proof of Proposition 3

We first show the necessity part: suppose that I is a relative index. By definition, we have

$$\begin{aligned}\xi(F) &= \lambda_F \xi(\overline{F}) + (1 - \lambda_F)\xi(\underline{F}) \\ &= \lambda_F \mu(\overline{F})(1 - I(\overline{F})) + (1 - \lambda_F)\mu(\underline{F})(1 - I(\underline{F}))\end{aligned}$$

Since index I is homogeneous of degree zero, linear homogeneity of mean μ implies linear homogeneity of ξ.

$$W(F) = W(\xi(F) \cdot \mathbb{1})$$
$$= \Phi(\xi(F)),$$

where Φ is increasing in its argument. Hence, W is homothetic.

Now we show the sufficiency part: suppose that W is homothetic. Then, there exist an increasing function Φ and a linearly homogeneous function $\hat{W}$ such that for $F \in \mathcal{F}$,

$$W(F) = \Phi(\hat{W}(F)).$$

Since $\hat{W}$ is linearly homogeneous, we have

$$\xi(F) = \frac{\hat{W}(F)}{\hat{W}(\mathbb{1})}.$$

Therefore, ξ is also linearly homogeneous. Since μ is also linearly homogeneous, robust index I defined as above becomes homogeneous of degree zero. Thus, I is a relative index.

B Proof of Section 3

B.1 Proof of Proposition 4

The necessity part is straightforward. We only prove the sufficiency part. Suppose $\succsim$ satisfies A1-4 and A6.

First, restricted $\succsim$ to set of deterministic profiles X^n. Since X is connected and separable, and $\succsim$ satisfies conditions of Debreu (1960) separable Theorem, there exists a continuous function $u_i : X \to \mathbb{R}$ such that the sum of u_i represents $\succsim$. Symmetry further requires that each u_i has to be identical. Therefore, there is a continuous function $u : X \to \mathbb{R}$ such that $f \succsim g \Leftrightarrow \sum_{i=1}^n u(f_i) \geq \sum_{i=1}^n u(g_i)$. Furthermore, A4 unanimity implies that u is also increasing in X.

Now, we extend u from domain X to $\mathcal{X}$ in the following way. For $Y \in \mathcal{X}$ and $c \in X$, we define $u(Y) = u(c)$ if $F \sim f$ whenever $F_i = Y$ and $f_i = c$ for all i. Since Y is compact, there exist $a, b \in X$ such that $a \geq Y \geq b$. Unanimity implies that equally distributed profiles must satisfy the preferences: $(a, \dots, a) \succsim (Y, \dots, Y) \succsim (b, \dots, b)$. Therefore, by continuity, there exists a unique c such that $(c, \dots, c) \sim (Y, \dots, Y)$. Hence, u on $\mathcal{X}$ is well-defined.

Pick any $F = (Y_1, \dots, Y_n) \in \mathcal{F}$. Let $c_1, \dots, c_n$ in X be such that $u(Y_i) = u(c_i)$ for all i. To prove the additive separability, it suffices to show that $F \sim (c_1, \dots, c_n)$. We prove it by induction.

Claim 1 *For any $i \in \mathcal{N}$, $(c_1, \dots, c_{i-1}, Y_i, c_{i+1}, \dots, c_n) \sim (c_1, c_2, \dots, c_n)$.*

Proof of Claim By A3 symmetry, it suffices to prove that $(Y_1, c_2, \dots, c_n) \sim (c_1, \dots, c_n)$. Furthermore, by separability, we only need to show the case where

$(Y_1, c_1, \dots, c_1) \sim (c_1, c_1, \dots, c_1)$. Suppose such indifference relation does not hold. Assume first that

$$(Y_1, c_1, \dots, c_1) \succ (c_1, \dots, c_1).$$

Then, separability implies that $(Y_1, \dots, Y_1) \succ (c_1, Y_1, \dots, Y_1)$. According to definition, $(c_1, \dots, c_1) \sim (Y_1, \dots, Y_1)$, which implies that

$$(Y_1, c_1, \dots, c_1) \succ (c_1, Y_1, \dots, Y_1).$$

By symmetry, it is equivalent to $(c_1, Y_1, c_1, \dots, c_1) \succ (c_1, Y_1, \dots, Y_1)$. Applying separability again, we have

$$(Y_1, Y_1, c_1, \dots, c_1) \succ (Y_1, \dots, Y_1, Y_1) \succ (c_1, Y_1, \dots, Y_1).$$

Similarly, we can use separability and symmetry again to get

$$(Y_1, Y_1, Y_1, c_1 \dots, c_1) \succ (Y_1, \dots, Y_1, Y_1) \succ (c_1, Y_1, \dots, Y_1).$$

Repeat this process, we finally have $(Y_1, \dots, Y_1, c_1) \succ (Y_1, \dots, Y_1, Y_1)$, which contradicts to our assumption.

Now, if we assume the other possibility that $(c_1, \dots, c_1) \succ (Y_1, c_1, \dots, c_1)$, it is similar to show the contradiction. □

Claim 2 *If* $(Y_1, \dots, Y_t, c_{t+1}, \dots, c_n) \sim (c_1, \dots, c_n)$, *then* $(Y_1, \dots, Y_{t+1}, c_{t+2}, \dots, c_n) \sim (c_1, \dots, c_n)$.

Proof of Claim By separability, it suffices to prove that if $(Y_1, \dots, Y_t, c, \dots, c) \sim (c_1, \dots, c_t, c, \dots, c)$ for some t, then it holds for $t + 1$. Since $(Y_1, \dots, Y_t, c, \dots, c) \sim (c_1, \dots, c_t, c, \dots, c)$, separability implies that

$$(Y_1, \dots, Y_{t+1}, c, \dots, c) \sim (c_1, \dots, c_t, Y_{t+1}, c, \dots, c).$$

By Claim 1, $(c_1, \dots, c_t, Y_{t+1}, c, \dots, c) \sim (c_1, \dots, c_{t+1}, c, \dots, c)$. Hence, this claim holds. □

By Claim 1 and 2, for any $F \in \mathcal{F}$, we define $W : \mathcal{F} \to \mathbb{R}$ by $W(F) = \sum_{i=1}^{n} u(F_i)$, which clearly represents $\succsim$.

B.2 Proof of Theorem 1

Sufficiency Part:

Suppose that $\succsim$ on $\mathcal{F}$ satisfies A1-9. Our strategy to prove that robust Atkinson SWF represents $\succsim$ is following: First, we consider only the profiles that every individual have identical and binary values. We show that there exists unique $\alpha \in (0, 1)$ such that for any $x > y$ in X, $u(\{x, y\}) = \alpha u(x) + (1 - \alpha)u(y)$. Second, we consider the profiles that every individual have identical, but arbitrarily many

outcomes. We show that for any $Y \in \mathcal{X}$, $u(Y) = \alpha u(\max_{x \in Y} x) + (1-\alpha)u(\min_{y \in Y} y)$.
Third, we show that A8 scale invariance and A9 Pigou-Dalton principle imply that u on X has either power function or log function form. Finally, combined with Proposition 4, A5 dominance implies that for any $F \in \mathcal{F}$,

$$W(F) = \alpha \sum_i u(\overline{F}_i) + (1-\alpha) \sum_i u(\underline{F}_i)$$

represents $\succsim$.

To start, notice that proposition 4 implies the existence of u on $\mathcal{X}$. Define $\succsim^*$ on X^2 by

$$(a,b) \succsim^* (c,d) \Leftrightarrow u(\{a,b\}) \geq u(\{c,d\}).$$

Lemma B1 *For all $a,b,c \in X$, if $a \geq b$, then $(a,c) \succsim^* (b,c)$.*

Proof Take $a,b,c \in X$ with $a \geq b$. Let $Y = \{a,c\}$ and $Z = \{b,c\}$. So profile $(Y,\dots,Y)$ dominates profile $(Z,\dots,Z)$. By A5, $(Y,\dots,Y) \succsim (Z,\dots,Z)$. Proposition 4 implies $u(Y) \geq u(Z)$. Hence, by definition, $(a,b) \succsim^* (b,c)$. □

Let $0 < \ell \leq \ell' < \infty$. Consider $\succsim^*$ restricted to $[0,\ell] \times [\ell', +\infty)$. We show that this restricted preference has an additive conjoint structure, hence has a additively separable utility representation.

Lemma B2 *$\succsim^*$ restricted to $[0,\ell] \times [\ell', +\infty)$ satisfies the following conditions*:

$A1^*$ (weak order): $\succsim^*$ is complete and transitive.

$A2^*$ (Independence): $(x,b') \succsim^* (y,b')$ implies $(x,x') \succsim^* (y,x')$; also, $(b,x') \succsim^* (b,y')$ implies $(x,x') \succsim^* (x,y')$.

$A3^*$ (Thomsen): $(x,z') \sim^* (z,y')$ and $(z,x') \sim^* (y,z')$ imply $(x,x') \sim^* (y,y')$.

$A4^*$ (Essential): There exist $b,c \in [0,\ell]$ and $a \in [\ell', +\infty)$ such that $(b,a) \nsim^* (c,a)$, and $b' \in [0,\ell]$ and $a',c' \in [\ell', +\infty)$ such that $(b',a') \nsim^* (b',c')$.

$A5^*$ (Solvability): If $(x,x') \succsim^* (y,y') \succsim^* (z,x')$, then there exist $a \in [0,\ell]$ such that $(a,x') \sim (y,y')$; if $(x,x') \succsim^* (y,y') \succsim^* (x,z')$, then there exists $a' \in [\ell', +\infty)$ such that $(x,a') \sim^* (y,y')$.

$A6^*$ (Archimedean): For all $x,x' \in [0,\ell]$ and $y,z \in [\ell', +\infty)$, if $(x,y) \succsim^* (x',z)$, then there exists a, b in $[0,\ell]$ satisfying $(x,y) \succsim^* (a,y) \sim^* (b,z) \succ^* (b,y) \succsim^* (x',z)$. A similar statement holds with the roles of $[0,\ell]$ and $[\ell', +\infty)$ reversed.

Proof By definition, $\succsim^*$ is a weak order. It is easy to show all the axioms except Thomsen condition. Below, we show Thomsen condition.

Suppose that $(x, z') \sim^* (z, y')$ and $(z, x') \sim^* (y, z')$. By definition, this is equivalent to $u(x, z') = u(z, y')$ and $u(z, x') = u(y, z')$. To show that $u(x, x') = u(y, y')$, there are three cases to consider: $z' \geq \{x', y'\}$, $y' \geq \{x', z'\}$ and $x' \geq \{y', z'\}$.

Suppose first that $z' \geq \{x', y'\}$. Since $\{x', y'\} \geq \{x, y, z\}$, Lemma B1 implies that $(x', z') \succsim^* (x, z')$ and $(y', z') \succsim^* (y, z')$. Thus, A7 commutativity implies

$$u(e(x, x'), e(z', z')) = u(e(x, z'), e(x', z')).$$

Note that $(x, z') \sim^* (z, y')$ implies $e(x, z') = e(z, y')$. Therefore,

$$u(e(x, z'), e(x', z')) = u(e(z, y'), e(x', z')).$$

Applying commutativity again, we have

$$u(e(z, y'), e(x', z')) = u(e(z, x'), e(y', z')).$$

Note again that $(z, x') \sim^* (y, z')$ implies $e(z, x') = e(y, z')$. Therefore,

$$u(e(z, x'), e(y', z')) = u(e(y, z'), e(y', z')).$$

Commutativity implies that

$$u(e(y, z'), e(y', z')) = u(e(y, y'), e(z', z')).$$

Therefore, we have $u(e(x, x'), e(z', z')) = u(e(y, y'), e(z', z'))$, which implies, by Lemma B1, $e(x, x') = e(y, y')$. That is, $u(x, x') = u(y, y')$.

For the other two cases, similar arguments will lead to the same results. □

Lemma B3 *There exist two real-valued functions ϕ and φ on X such that for all $x, x', y, y' \in X$ with $x \leq y$ and $x' \leq y'$,*

$$(x, y) \succsim^* (x', y') \iff \phi(x) + \varphi(y) \geq \phi(x') + \varphi(y').$$

Furthermore, if there are ϕ', φ'represents $\succsim^*$ instead of ϕ, φ, respectively, then there exist $\gamma > 0$ and β_1, β_2 such that $\phi' = \gamma\phi + \beta_1$ and $\varphi' = \gamma\varphi + \beta_2$.

Proof Let $a > 0$. Lemma B2 implies that $\succsim^*$ restricted to $[0, a] \times [a, +\infty)$ is an additive conjoint structure. Thus, by Theorem 2 of Chapter 6 in Krantz et al. (2006), there exist two function ϕ_a on $[0, a]$ and φ_a on $[a, +\infty)$ represent $\succsim^* \subset [0, a] \times [a, +\infty)$, i.e. for all $x, x' \in [0, a]$ and $y, y' \in [a, +\infty)$

$$(x, y) \succsim^* (x', y') \iff \phi_a(x) + \varphi_a(y) \geq \phi_a(x') + \varphi_a(y').$$

By uniqueness of representation, we can normalize ϕ_a and φ_a such that

$$u(a) = \phi_a(a) + \varphi_a(a).$$

If $b > a$, since $\succsim^* \subset [0, b] \times [b, +\infty)$ is also an additive conjoint structure, then there exist functions ϕ_b on $[0, b]$ and φ_b on $[b, +\infty)$ that represent such preferences. Due to the uniqueness of representation, we can normalized ϕ_b in the way such that $\phi_b(a) = \phi_a(a)$. By similar method, if $c \in (0, a)$, since $\succsim^* \subset [0, c] \times [c, +\infty)$ is also an additive preference structure, then there exist functions ϕ_c on [0, c] and φ_c on $[c, +\infty)$ that represent such preferences. Again, φ_c is normalized in the way that $\varphi_c(a) = \varphi_a(a)$.

Now, define $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \phi_x(x) & \text{if } x > 0; \\ \phi_a(0) & \text{if } x = 0. \end{cases}$$

Similarly, define $\varphi : X \to \mathbb{R}$ by

$$\varphi(y) = \begin{cases} \varphi_y(y) & \text{if } y > 0; \\ u(0) - \phi_a(0) & \text{if } y = 0. \end{cases}$$

Therefore, ϕ and φ on X are uniquely specified. According to continuity and unanimity, $\varphi(0) < \varphi(y)$ for all $y > 0$. Take arbitrary $0 < y \leq x$. There always exists a, b such that $x < a$ and $0 < b < y$. Therefore,

$$\begin{aligned} x \geq y &\Leftrightarrow (x, a) \succsim^* (y, a) \\ &\Leftrightarrow \phi_a(x) + \varphi_a(a) \geq \phi_a(y) + \phi_a(a) \\ &\Leftrightarrow \phi_x(x) \geq \phi_y(y) \\ &\Leftrightarrow \phi(x) \geq \phi(y) \end{aligned}$$

Similarly, we have

$$\begin{aligned} x \geq y &\Leftrightarrow (b, x) \succsim^* (b, y) \\ &\Leftrightarrow \phi_b(b) + \varphi_b(x) \geq \phi_b(b) + \phi_b(y) \\ &\Leftrightarrow \varphi_b(x) \geq \varphi_b(y) \\ &\Leftrightarrow \varphi(x) \geq \varphi(y) \end{aligned}$$

Therefore $x \geq y \Leftrightarrow \phi(x) + \varphi(x) \geq \phi(y) + \varphi(y)$. We show that ϕ and φ have the properties above. Let $x \leq y$ and $x' \leq y'$. Suppose that $(x, y) \succsim^* (x', y')$. There are two cases: either $x \geq y'$ or $x < y'$. First, assume that $x \geq y'$. Then, continuity and unanimity imply that there exists a and b such that $(x, y) \sim^* (a, a)$ and $(b, b) \sim^* (x', y')$.

$$\begin{aligned} (x, y) \sim^* (a, a) &\Leftrightarrow \phi(x) + \varphi(y) = \phi(a) + \varphi(a), \\ (x', y') \sim^* (b, b) &\Leftrightarrow \phi(x') + \varphi(y') = \phi(b) + \varphi(b). \end{aligned}$$

Note that $a \geq b$, which is $\phi(a) + \varphi(a) \geq \phi(b) + \varphi(b)$. Therefore,

$$(x, y) \succsim^* (x', y') \Leftrightarrow \phi(x) + \varphi(y) \geq \phi(x') + \varphi(y').$$

The uniqueness of representation follows immediately from the definition of ϕ and φ. □

Lemma B4 *There exists* $0 \leq \alpha \leq 1$ *such that for all* $x \geq y$,

$$u(\{x, y\}) = \alpha u(x) + (1 - \alpha)u(y).$$

Proof It suffices to show that there are constants $\beta > 0$ such that $\varphi(x) = \beta\phi(x)$. If $a > 0$, define

$$\phi_{1a}(x) = \phi(e(a, x)) \quad \text{and} \quad \varphi_{1a}(x) = \varphi(e(a, x)), \text{ for } x \geq a;$$
$$\phi_{2a}(x) = \phi(e(x, a)) \quad \text{and} \quad \varphi_{2a}(x) = \varphi(e(x, a)), \text{ for } x \leq a.$$

For $\{x, y, z, w\} \geq a$, if $x \leq y$ and $z \leq w$, then $(a, y) \succsim^* (a, x)$ and $(a, w) \succsim^* (a, z)$. Therefore,

$$\begin{aligned}(z, w) \succsim^* (x, y) &\Leftrightarrow \phi(z) + \varphi(w) \geq \phi(x) + \varphi(y)\\ &\Leftrightarrow e(z, w) \geq e(x, y)\\ &\Leftrightarrow (a, e(z, w)) \succsim^* (a, e(x, y))\end{aligned}$$

The last equivalence is implied by Lemma B1. Commutativity implies that $(a, e(z, w)) \sim^* (e(a, z), e(a, w))$ and $(a, e(x, y)) \sim^* (e(a, x), e(a, y))$. Therefore,

$$\begin{aligned}(z, w) \succsim^* (x, y) &\Leftrightarrow \phi(z) + \varphi(w) \geq \phi(x) + \varphi(y)\\ &\Leftrightarrow (e(a, z), e(a, w)) \succsim^* (e(a, x), e(a, y))\\ &\Leftrightarrow \phi(e(a, z)) + \varphi(e(a, w)) \geq \phi(e(a, x)) + \varphi(e(a, y))\\ &\Leftrightarrow \phi_{1a}(z) + \varphi_{1a}(w) \geq \phi_{1a}(x) + \varphi_{1a}(y).\end{aligned}$$

If $x \leq y \leq a$ and $z \leq w \leq a$, then similarly we have

$$(z, w) \succsim^* (x, y) \Leftrightarrow \phi_{2a}(z) + \varphi_{2a}(w) \geq \phi_{2a}(x) + \varphi_{2a}(y).$$

Thus ϕ_{1a} and φ_{2a} represent $\succsim^*$ on $[0, a] \times [a, +\infty)$. By uniqueness of representation, there are $k_1, k_2 > 0$ and k_{11} and k_{12} such that for $a, b > 0$,

$$\phi_{1a}(x) = k_1(a)\phi(x) + k_{11}(a) \quad \text{and} \quad \varphi_{2b}(y) = k_2(b)\varphi(y) + k_{12}(b).$$

Notice that if φ is constant, then it is trivially that $u(\{x, y\}) = \phi(x) = u(x)$, which is $\alpha = 1$. Similarly, if ϕ is constant, then $u(\{x, y\}) = u(y)$, which is $\alpha = 0$. Now, suppose both ϕ and φ are non-constant. Take $w \geq \{y, z\} \geq x$. Lemma B1 implies that $(z, w) \succsim^* (x, y)$ and $(y, w) \succsim^* (z, x)$. According to commutativity,

$$\begin{aligned}&(e(x, y), e(z, w)) \sim^* (e(x, z), e(y, w))\\ &\Leftrightarrow \phi(e(x, y)) + \varphi(e(z, w)) = \phi(e(x, z)) + \varphi(e(y, w))\\ &\Leftrightarrow k_1(x)\phi(y) + k_{11}(x) + k_2(w)\varphi(z) + k_{12}(w) = k_1(x)\phi(z) + k_{11}(x) + k_2(w)\varphi(y) + k_{12}(w)\\ &\Leftrightarrow k_1(x)(\phi(y) - \phi(z)) = k_2(w)(\varphi(y) - \varphi(z)).\end{aligned}$$

Since the above equations are satisfied for all x, y, z, w with $w \geq \{y, z\} \geq x$, there exist positive constants λ, δ such that

$$k_1(x) = \lambda \quad \text{and} \quad k_2(y) = \delta.$$

Thus, for all $y, z > 0$,

$$\lambda(\phi(y) - \phi(z)) = \delta(\varphi(y) - \varphi(z)).$$

Hence, there are $\beta > 0$ such that $\phi(x) = \beta\varphi(x)$ for all x. Let $\alpha = \frac{1}{1+\beta}$. Clearly $0 < \alpha < 1$. According to unique representation, we can normalize $u(x) = \frac{\phi(x)}{\alpha}$. Therefore, for $x \leq y$,

$$\phi(x) + \varphi(y) = \alpha u(x) + (1 - \alpha)u(y).$$

□

Lemma B5 *There exist $a \in \mathbb{R}$ and $b > 0$ such that for every $x \in X$,*

$$u(x) = \begin{cases} a + b \cdot \dfrac{x^r}{r} & \text{for } 0 < r < 1 \\ a + b \cdot \log x & \text{for } r = 0. \end{cases}$$

Proof Restricted $\succsim$ to deterministic profiles. Since $\succsim$ is continuous and separable on X^n, Roberts (1980) demonstrates that scale invariance implies that function u has the following forms: there are constant a and positive b such that

$$u(x) = \begin{cases} a + b \cdot \dfrac{x^r}{r} & \text{for } r > 0 \\ a - b \cdot \dfrac{x^r}{r} & \text{for } r < 0 \\ a + b \cdot \log x & \text{for } r = 0. \end{cases}$$

Note that Pigou-Dalton principle means that for $x, y, z, w \in X$, if $x + y = z + w$ and $|x - y| < |z - w|$, then $u(x) + u(y) \geq u(z) + u(w)$. This is equivalent to for all $x < y$ and all $c > 0$

$$u(x + c) - u(x) \geq u(y + c) - u(y),$$

which implies that u is concave on X. Thus, concavity of u requires that $r \leq 1$. Furthermore, unanimity requires that $r \geq 0$. Therefore, u must have the expression stated at this lemma. □

For $Y \in \mathcal{X}$, denote $y^* = \max_{y \in Y} y$ and $y_* = \min_{y \in Y} y$.

Lemma B6 *For $Y \in \mathcal{X}$, $u(Y) = u(y^*, y_*)$.*

Proof Take $Y \in \mathcal{X}$. Since $\{y^*, y_*\} \subseteq Y$, we know $(Y, \ldots, Y)$ dominates $(\{y^*, y_*\}, \ldots, \{y^*, y_*\})$. By definition of y^* and y_*, it is immediate that $(\{y^*, y_*\}, \ldots, \{y^*, y_*\})$ also dominates $(Y, \ldots, Y)$. Therefore, according to dominance axiom, $(\{y^*, y_*\}, \ldots, \{y^*, y_*\}) \sim (Y, \ldots, Y)$. This is equivalent to $u(y^*, y_*) = u(Y)$.

□

Necessity Part:

Suppose that $\succsim$ is represented by a robust Atkinson SWF W. We want to prove that this preference satisfies A1-9. We only demonstrate commutativity axiom since the rest axioms are straightforward.

Consider $x_1, x_2, y_1, y_2 \in X$ where $x_1 \geq \{x_2, y_1\} \geq y_2$. Let $F \in \mathcal{F}$ be such that $F_i = \{e(x_1, x_2), e(y_1, y_2)\}$ for all i. Also, let $G \in \mathcal{F}$ be such that $G_i = \{e(x_1, y_1), e(x_2, y_2)\}$ for all i. According to the representation function, we have

$$\begin{aligned} u(e(x_1, x_2), e(y_1, y_2)) &= \alpha u(x_1, x_2) + (1-\alpha)u(y_1, y_2) \\ &= \alpha[\alpha u(x_1) + (1-\alpha)u(x_2)] + (1-\alpha)[\alpha u(y_1) + (1-\alpha)u(y_2)] \\ &= \alpha[\alpha u(x_1) + (1-\alpha)u(y_1)] + (1-\alpha)[\alpha u(x_2) + (1-\alpha)u(y_2)] \\ &= \alpha u(x_1, y_1) + (1-\alpha)u(x_2, y_2) \\ &= u(e(x_1, y_1), e(x_2, y_2)). \end{aligned}$$

B.3 Proof of Theorem 2

Since the necessity part is straightforward, we only show the sufficiency part. Suppose that $\succsim$ satisfies A1-5 and A6'-9'. Our strategy is first to show that $\succsim$ restricted to deterministic profile have Gini SWF. Then we show that if $\succsim$ restricted to the profiles in which individual 1 has binary values and all the rest individuals have singleton value, then $\succsim$ has a robust Gini SWF representation. Finally, we extend this result to the whole set of profiles.

Lemma C1 *Let $\succsim$ restrict to X^n. Then there exists ϕ on X^n such that*

$$\phi(f) = \mu(f) - \frac{\sum_i \sum_j |f_i - f_j|}{2n^2},$$

represents $\succsim$ on X^n.

Proof It is clear to see that $\succsim$ on X^n_c also satisfies A1-4 and A6'-8'. Therefore, according to Theorem D of Elchanan and Itzhak (1994), there exist $0 < \delta < \frac{1}{n(n-1)}$ and ϕ on X^n such that for $f \in X^n$

$$\phi(f) = \mu(f) - \delta \cdot \sum_i \sum_j |f_i - f_j|,$$

represents $\succsim$ on X^n. Pick $c > 0$ and $k \in \mathcal{N}$. By A9' tradeoff, we know $(kc, 0, \dots, 0) \sim (c/k, \dots, c/k, 0, \dots, 0)$. The above ϕ function implies that

$$\frac{kc}{n} - \delta(n-1)kc = \frac{c}{n} - \delta \cdot 2k(n-k)\frac{c}{k}.$$

Therefore, the only solution is

$$\delta = \frac{1}{2n^2}.$$

□

We denote

$$\mathcal{F}^1 = \{F \in \mathcal{F} : F_1 = \{a, b\} \text{ and } F_i = \{c\}, \forall i \neq 1 \text{ and } a, b, c \in X \text{ with } a, b \geq c\}.$$

the set of all profiles in which individual 1 is the richest with two possible allocations in the society, and the rest in the society have deterministic and equalized allocation. Note that for $F \in \mathcal{F}^1$, if $a = b$, then F is a deterministic profile; and if $a = b = c$, then F is a deterministic equally distributed profile.

Lemma C2 *If $F, G \in \mathcal{F}^1$, then F and G are order-preserving.*

Proof This follows immediately from the definition of order-preserving. □

Lemma C3 *For F,$f \in \mathcal{F}^1$, if $F \sim f$, then $\alpha F + (1-\alpha) f \sim f$ for all $\alpha \in (0, 1)$.*

Proof Pick $F \in \mathcal{F}^1$ be such that $F_1 = \{a, b\}$, $F_i = \{c\}$ for $i \neq 1$ and $a \geq b \geq c$. If there is a deterministic profile $f \in \mathcal{F}^1$ be such that $F \sim f$, we should have $\frac{F}{2} \sim \frac{f}{2}$. To see this, suppose not. Assume that $\frac{F}{2} \succ \frac{f}{2}$. Since $\frac{F}{2}, \frac{f}{2} \in \mathcal{F}^1$, A6' order-preserving independence implies that

$$\frac{F}{2} + \frac{F}{2} \succ \frac{F}{2} + \frac{f}{2} \succ \frac{f}{2} + \frac{f}{2} = f.$$

Notice that

$$\left(\frac{F}{2} + \frac{F}{2}\right)_1 = \left\{a, \frac{a+b}{2}, b\right\} \quad \text{and} \quad \left(\frac{F}{2} + \frac{F}{2}\right)_i = c.$$

Recall that the representation of $\succsim$ restricted on deterministic profile can also be written as

$$\phi(f) = \frac{1}{n^2} \sum_{i=1}^{n} [(2(n-k) + 1]\tilde{f}_i$$

Therefore, $\overline{F} = (a, c, \dots, c)$ is the most preferred deterministic profile in both F and $\frac{F}{2} + \frac{F}{2}$, i.e.

$$\overline{F} = (a, c, \dots, c) \in \arg\max_{f \in F} \phi(f) \qquad \text{and} \qquad \overline{F} = (a, c, \dots, c) \in \arg\max_{f \in \frac{F}{2} + \frac{F}{2}} \phi(f);$$

and $\underline{F} = (b, c, \dots, c)$ is the least preferred deterministic profile in both F and $\frac{F}{2} + \frac{F}{2}$. Hence, F and $\frac{F}{2} + \frac{F}{2}$ dominates each other. According to A6', $F \sim \frac{F}{2} + \frac{F}{2}$, which contradicts the assumption that $F \sim f$. Now assume that $\frac{f}{2} \succ \frac{F}{2}$. We repeat the similar process as above and lead to a contradiction. Hence, $F \sim f$ implies $\frac{F}{2} \sim \frac{f}{2}$.

Proceeding with induction, we have for every integer $k = 1, 2, \dots$

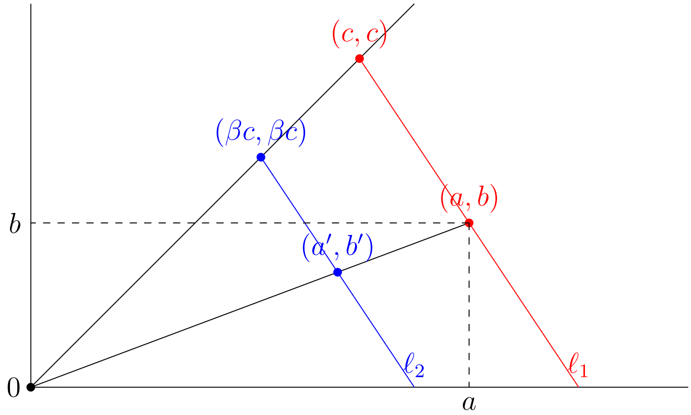


Fig. 3 Indifference curve on $\mathcal{F}^1(0)$

$$\frac{F}{k} \sim \frac{f}{k}.$$

Also, by A6’,

$$F \sim f \Rightarrow F + F \sim F + f \sim f + f = 2f.$$

Observe that $(2F)_1 = \{2a, 2b\}$, $(F + F)_1 = \{2a, a + b, 2b\}$ and $(2F)_i = (F + F)_i = \{2c\}$ for $i \neq 1$. Since $a \geq b$, it is immediate that $2F$ and $F + F$ dominates each other, therefore, $2F \sim F + F$. Hence $F \sim f$ implies $2F \sim 2f$. By induction, we have for every integer k,

$$kF = kf.$$

Combine the results abover, for every positive rational number α, we have

$$\alpha F \sim \alpha f.$$

Continuity implies that the above result holds for every positive real number α. Now, take any $\alpha \in (0, 1)$ and apply A6’,

$$\alpha F \sim \alpha f \Leftrightarrow \alpha F + (1 - \alpha)f \sim f.$$

□

Recall that for $F \in \mathcal{F}$, $\overline{F}$ and $\underline{F}$ represents the upper limit and lower limit distribution in F, respectively.

Lemma C4 *There exists* $\alpha \in [0, 1]$ *such that for all* $F \in \mathcal{F}^1$,

$$F \sim \alpha\overline{F} + (1 - \alpha)\underline{F}.$$

Proof If $x \in X$, we define $\mathcal{F}^1(x) = \{F \in \mathcal{F}^1 : F_i = \{x\} \text{ for all } i \neq 1\}$ denote the collection of profiles in $\mathcal{F}^1$ in which, except individual 1, every individual have equal allocation. Therefore, $\mathcal{F}^1 = \cup_{x\in\mathbb{R}}\mathcal{F}(x)$. We first show that the result holds on the restricted domain $\mathcal{F}^1(0)$.

Referring to Fig. 3. For $f \in \mathcal{F}^1(0)$ with $F_1 = \{a, b\}$, F can be identified by the point (a, b) if $a > b$. Similarly, for $F \in \mathcal{F}^1(0)$ with $F_1 = \{c\}$, F can be identified by the point (c, c). Therefore, there is one-to-one correspondence between set $\mathcal{F}^1(0)$ and the points between horizontal axis and diagonal. For every $F, G \in \mathcal{F}^1(0)$, where $F_1 = \{a, b\}$ and $G_1 = \{c, d\}$, we define

$$(a, b) \succsim (c, d) \Leftrightarrow F \succsim G.$$

Take $a > b$. We have

$$(a, a) \succ (b, b).$$

By definition, we know that profile (a, a) dominates (a, b) and (a, b) dominates (b, b). Therefore, A6' implies

$$(a, a) \succsim (a, b) \succsim (b, b).$$

Continuity implies that there exists $\alpha \in [0, 1]$ such that

$$(\alpha a + (1-\alpha)b, \alpha a + (1-\alpha)b) \sim (a, b).$$

Let $\alpha a + (1-\alpha)b = c$. Lemma C3 implies that any points on the straight line between (c, c) and (a, b) are indifferent. Therefore, every indifferent curve must be a straight line.

Now, we need to show that every indifferent lines parallel to each other. Take any point (a', b'). Connect points (a', b') and $(0, 0)$ by a straight line. Without loss of generality, suppose this line intersects the indifference curve, line between (c, c) and (a, b), at point (a, b). Therefore, there exists unique $\beta > 0$ such that

$$(a', b') = (\beta a, \beta b).$$

Since $(a, b) \sim (c, c)$, Lemma C3 implies that

$$(\beta a, \beta b) \sim (\beta c, \beta c).$$

Therefore, $(a', b') \sim (\beta c, \beta c)$, which means that two indifferent curves ℓ_1, ℓ_2 parallel to each other.

To finish our proof, we now extend the result from domain $\mathcal{F}^1(0)$ to $\mathcal{F}^1$. Pick any a, b, c such that $a \geq b \geq c > 0$. Consider a profile $F \in \mathcal{F}^1$ being such that $F_1 = \{a-c, b-c\}$ and $F_i = \{0\}$ for $i \neq 1$. Clearly, such F belongs to $\mathcal{F}^1(0)$ and, therefore,

$$(a-c, b-c) \sim (\alpha(a-c) + (1-\alpha)(b-c), \alpha(a-c) + (1-\alpha)(b-c)).$$

Now, adding constant deterministic profile $(c, \dots, c)$ on both proifles, A6' implies that

$$F \sim (\alpha a + (1-\alpha)b, c, \dots, c).$$

Since $\overline{F} = (a, c, \dots, c)$ and $\underline{F} = (b, c \dots, c)$, we have $F \sim \alpha\overline{F} + (1-\alpha)\underline{F}$. □

We now define a real valued function W on $\mathcal{F}^1$ by, for $F \in \mathcal{F}$,

$$W(F) = \phi(\alpha\overline{F} + (1-\alpha)\underline{F}).$$

It is immediate to see that W represents $\succsim$ restricted on $\mathcal{F}^1$. Notice that for each $F \in \mathcal{F}$, $\overline{F}$ and $\underline{F}$ are order-preserving. By the order-preserving additivity and homogeneity of ϕ, we have

$$W(F) = \alpha\phi(\overline{F}) + (1-\alpha)\phi(\underline{F})$$

represents $\succsim$ on $\mathcal{F}^1$.

Lemma C5 *For $F \in \mathcal{F}$ and $G \in \mathcal{F}^1$, if $\overline{F} \sim \overline{G}$ and $\underline{F} \sim \underline{G}$, then $F \sim G$.*

Proof Since both F and G dominate each other, it is immediate that $F \sim G$ according to A5. □

Now, we can extend real-valued function W to the whole set $\mathcal{F}$ by for $F \in \mathcal{F}$ if there is $G \in \mathcal{F}^1$ such that $\overline{F} \sim \overline{G}$ and $\underline{F} \sim \underline{G}$, then

$$W(F) = \alpha\phi(\overline{F}) + (1-\alpha)\phi(\underline{F}).$$

We claim that W represents the $\succsim$ on $\mathcal{F}$. To see this, note that by continuity, for every $F \in \mathcal{F}$, there must exist $F^1 \in \mathcal{F}^1$ such that $\overline{F} \sim \overline{F^1}$ and $\underline{F} \sim \underline{F^1}$. Take any $F, G \in \mathcal{F}$. According to Lemma C5, we have

$$\begin{aligned} F \succsim G &\iff F^1 \succsim G^1 \\ &\iff W(F^1) \geq W(G^1) \\ &\iff \alpha\phi(\overline{F^1}) + (1-\alpha)\phi(\underline{F^1}) \geq \alpha\phi(\overline{G^1}) + (1-\alpha)\phi(\underline{G^1}) \\ &\iff \alpha\phi(\overline{F}) + (1-\alpha)\phi(\underline{F}) \succsim \alpha\phi(\overline{G}) + (1-\alpha)\phi(\underline{G}) \\ &\iff W(F) \geq W(G). \end{aligned}$$

References

Aaberge Rolf (2001) Axiomatic characterization of the Gini coefficient and Lorenz curve orderings. J Econ Theory 101(1):115–132. https://doi.org/10.1006/jeth.2000.2749

Atkinson Anthony (1970) On the measurement of inequality. J Econ Theory 2(3):244–263. https://doi.org/10.1016/0022-0531(70)90039-6

Atkinson Anthony B, Thomas Piketty, Emmanuel Saez (2011) Top incomes in the long run of history. J Econ Literat 49(1):3–71

Ben-Porath Elchanan, Gilboa Itzhak, Schmeidler David (1997) On the measurement of inequality under uncertainty. J Econ Theory 75(1):194–204. https://doi.org/10.1006/jeth.1997.2280

Charles Blackorby, David Donaldson (1980) A theoretical treatment of indices of absolute inequality. Int Econ Rev. https://doi.org/10.2307/2526243
Chew Soo Hong, Sagi Jacob (2012) An inequality measure for stochastic allocations. J Econ Theory 147(4):1517–1544. https://doi.org/10.1016/j.jet.2011.05.002
Corrado Gini (1921) Measurement of inequality of incomes. Econ J 31(121):124–126
Cowell Frank (2011) Measuring inequality. Oxford University Press, Oxford
Dasgupta Partha, Sen Amartya, Starrett David (1973) Notes on the measurement of inequality. J Econ Theory 6(2):180–187. https://doi.org/10.1016/0022-0531(73)90033-1
David Schmeidler (1989) Subjective probability and expected utility without additivity. Econometrica 57(3):571–587
Debreu Gerard (1960) "Topological methods in cardinal utility theory." *Mathematical Economics: Twenty Papers of Gerard Debreu Econometric Society Monographs*, 120-132. Cambridge University Press. https://doi.org/10.1017/CCOL052123736X.010
Elchanan Ben Porath, Itzhak Gilboa (1994) Linear measures, the Gini index, and the income-equality trade-off. J Econ Theory 64(2):443–467. https://doi.org/10.1006/jeth.1994.1076
Gajdos Thibault, Maurin Eric (2004) Unequal uncertainties and uncertain inequalities: an axiomatic approach. J Econ Theory 116(1):93–118. https://doi.org/10.1016/j.jet.2003.07.008
Gajdos Thibault, Tallon Jean-Marc (2002) Fairness under uncertainty. Econ Bull 4(18):1–7
Ghirardato Paolo, Maccheroni Fabio, Marinacci Massimo (2004) Differentiating ambiguity and ambiguity attitude. J Econ Theory 118(2):133–173. https://doi.org/10.1016/j.jet.2003.12.004
Gul Faruk, Pesendorfer Wolfgang (2015) Hurwicz expected utility and subjective sources. J Econ Theory 159:465–488. https://doi.org/10.1016/j.jet.2015.05.007
Heitjan Daniel F (1989) Inference from grouped continuous data: a review. Statis Sci 4:164–179
Henson MF (1967) *Trends in the Income of Families and Persons in the United States, 1947-1964.* U. S. Department of Commerce, Bureau of the Census. https://books.google.fr/books?id=yJiXxgEACAAJ
Kolm Serge-Christophe (1969) "The optimal production of social justice." *Public economics*, 145–200
Krantz David H, Luce Robert Duncan, Suppes Patrick, Tversky Amos (2006) *Foundations of Measurement Volume I: Additive and Polynomial Representations.* Vol. 1, Dover Publications
Lorenz MO (1905) Methods of measuring the concentration of wealth. Am Statis Assoc 9(70):209–219
Marshall Albert W, Ingram Olkin, Arnold Barry C (1979) Inequalities: theory of majorization and its applications, vol 143. Springer, Berlin
Moulin Hervi (1991) Axioms of cooperative decision making. Cambridge University Press
Roberts Kevin WS (1980) Interpersonal comparability and social choice theory. Rev Econ Stud 47(2):421–439
Rodríguez Juan Gabriel, Salas Rafael (2014) The Gini coefficient: Majority voting and social welfare. J Econ Theory 152:214–223. https://doi.org/10.1016/j.jet.2014.04.012
Salas Rafael, Rodríguez Juan Gabriel (2013) Popular support for social evaluation functions. Social Choice Welfare 40(4):985–1014. https://doi.org/10.1007/s00355-012-0669-z
Sen Amartyas (1973) On economic inequality. Oxford University Press
Shorrocks AF (1980) The class of additively decomposable inequality measures. Econometrica 48(3):613–625
Theil Henri (1967) Economics and information theory. North-Holland, Amsterdam, the Netherlands
Weymark John (1981) Generalized Gini inequality indices. Math Social Sci 1(4):409–430. https://doi.org/10.1016/0165-4896(81)90018-4
Wojciech Olszewski (2007) Preferences over sets of lotteries. Rev Econ Stud 74(2):567–595
Yaari Menahem E (1988) A controversial proposal concerning inequality measurement. J Econ Theory 44(2):381–397. https://doi.org/10.1016/0022-0531(88)90010-5