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A topological characterization of generalized stable sets

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Abstract

The purpose of this note is to provide a topological characterization for the existence of the generalized stable set introduced by Van Deemen (Soc Choice Welf 8:255–260, 1991) as a generalization of the Von Neumann–Morgenstern stable set.

1 Introduction

The notion of a stable sets solution[1] introduced by Von Neumann and Morgenstern in their classical work Theory of Games and Economic Behavior (Von Neumann and Morgenstern 1947), is an important tool in the field of Decision Theory. In a social choice process, even if the best social choices are not available, there may be social choices that are acceptable as a possible outcome. Von Neumann and Morgenstern refer to sets of such acceptable social states as "standards of behavior". The notion of stable set formalizes the idea of a standard of behavior of a social economy. While the authors describe this economy as a game of *n* participants with imputation-based payoffs, they also refer to a more general framework for their stable sets. This more general framework constitutes a theory of social phenomena based on the effective preferences between the various states of society (see Von Neumann and Morgenstern 1947, Sects. 4.4.3 and 4.6). The core is contained in each Von Neumann–Morgenstern stable set. The theory of stable sets has a significant flaw in that it can be empty in the case of odd cycles. For example, if *X* is a collection of three social states *x*, *y*, *z* and *R* is a preference structure on *X* so that $xRyRzRx$, then for this preference structure there are no stable sets. More generally, if $X = \{x_1, ..., x_n\}$ is a collection of social states with *n* odd and *R* is a preference structure on *X*, then

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[1] Originally stable sets were called "*general solutions*," but today we use the term "*general solution*" or "*general solution concept*" for any one of the many solution ideas which have been proposed in economic and game theories.

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X has no stable sets if there is a unique cycle $x_1 R x_2 R x_3 R, ..., x_n R x_1$ (see Van Deemen 2013, Theorem 4.11). To avoid this particular problem, Van Deemen introduced the notion of the generalized stable set which for finite sets of alternatives is able to produce a solution for every possible cyclic dominance relation.

In this note, we prove that a feasible set X endowed with a dominance relation R has a characterization of the generalized stable set with respect to R if a compact topology τ on X exists such that R is upper tc-semicontinuous. This is done in a general framework for which dominance relation refers to an arbitrary binary relation defined on a set of alternatives that is not necessarily finite. Since any topology on a finite set X is compact and any dominance relation R is upper tc-semicontinuous (any subset of X is open), the results of this paper generalize corresponding results given by Van Deemen (1991).

2 Notations and definitions

An abstract decision problem consists of two parts. One is an arbitrary set X of alternatives (called the *ground set*) from which an individual or group must select. In most cases, there are at least two alternatives to choose from. Otherwise, there is no need to make a decision. The other is a dominance relation over this set, which reflects preferences or evaluations for different alternatives. Preferences or evaluations over X are modelled by a binary relation R. When representing abstract decision problems, the pair (X, R) is used. We sometimes abbreviate $(x, y) \in R$ as xRy. For any subset $Y \subseteq X$, let $R|_Y$ denote the restriction of R to Y. That is, for any $x, y \in X$, $xR|_Y y$ if xRy and $x, y \in Y$. The *transitive closure* of R is the relation $\overline{R}$ defined as follows: For all $x, y \in X$, $(x, y) \in \overline{R}$ if and only if there exist $K \in \mathbb{N}$ and $x_0, ..., x_K \in X$ such that $x = x_0, (x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_K = y$. A subset $D \subseteq X$ is *R-undominated* if and only if for no $x \in D$ there is a $y \in X \setminus D$ such that yRx. A subset $Y \subseteq X$ is an *R-cycle* if for all $x, y \in Y$, we have $(x, y) \in \overline{R}$ and $(y, x) \in \overline{R}$. We say that R is *acyclic* if there does not exist an R-cycle. An alternative $x \in X$ is *R-maximal* with respect to an acyclic binary relation R, if $(y, x) \in R$ for no $y \in X$. A subset F of (X, R) is called a *Von Neumann-Morgenstern stable set* if (i) no alternative in F is dominated with respect to R by another alternative in F, and (ii) any alternative outside F is dominated with respect to R by an alternative inside F. The first property is called *internal stability of domination* and the second property *external stability of domination*. A subset F of X is called a *generalized stable set* if it is a Von Neumann-Morgenstern stable set of X with respect to the transitive closure of R. The following definitions are taken from Van Deemen (1991). An abstract decision problem (X, R) is called *strongly connected* if xRy for all $x, y \in X$. A *strong component* of an abstract decision problem (X, R) is an abstract decision problem $(Y, R|_Y), Y \subseteq X$, satisfying the following properties: (i) $(Y, R|_Y)$ is strongly connected; (ii) no abstract decision problem $(Y', R|_{Y'})$ with $Y' \supset Y$ is strongly connected. Note that when an element x is not on any cycle, it forms a singleton strongly connected component $\{x\}$

by itself. Clearly, the set of strongly connected components form a partition of the space (X, R). The *contraction* of (X, R) is an abstract decision problem $(\Xi, \widetilde{R})$ where

1. $\Xi = \{X_i | i \in I\}$ is the collection of ground sets of the strong components of (X, R);
2. for any $X_i, X_j \in \Xi$, $X_i \widetilde{R} X_j$ if there are $x \in X_i, y \in X_j$ with xRy. Clearly, $\widetilde{R}$ is acyclic by definition.

In what follows, $\mu(\Xi, \widetilde{R}) = \{X_i^* | i \in I\}$ denotes the family of ground sets which are $\widetilde{R}$-maximal in Ξ.

Let R be a binary relation defined on a topological space (X, τ). The relation R is *upper* (resp. *lower*) *semicontinuous* if for all $x \in X$ the set $xR = \{y \in X | xRy\}$ (resp. $Rx = \{y \in X | yRx\}$) is open. According to Alcantud ([1999](#), p 181) a binary relation R on a topological space (X, τ) is *upper* (resp. *lower*) *tc-semicontinuous* if its transitive closure is upper semicontinuous; i.e., if for all $x \in X$ the set $x\overline{R} = \{y \in X | x\overline{R}y\}$ (resp. $\overline{R}x = \{y \in X | y\overline{R}x\}$) is open. The acronym *tc* refers to *transitive closure*. Upper semicontinuity implies upper tc-semicontinuity, and both concepts are equivalent for transitive binary relations. We say that a topological space (X, τ) is *compact* if for each collection of open sets which covers X there exists a finite subcollection that also covers X.

3 The main result

The following theorem gives a characterization of the generalized stable set of an abstract decision problem.

Theorem 3.1 *Let (X, R) be an abstract decision problem, and let τ be a compact topology on X. Suppose that R is upper tc-semicontinuous. Then, F is a generalized stable set of (X, R) if and only if it contains exactly one alternative of the ground set of each $\widetilde{R}$-maximal strong component.*

Proof Let (X, R) denote an abstract decision problem satisfying the assumptions of the theorem. Let $\{X_i^* | i \in I\}$ be the family of all ground sets which are $\widetilde{R}$-maximal in Ξ and let $x_i \in X_i^*$ for each $i \in I$. We prove that $F = \{x_i | i \in I\}$ is a generalized stable set. If (X, R) is strongly connected, then $X_i^* = X$ for all $i \in I$. In this case, for each $x \in X$, $\{x\}$ is a generalized stable set of (X, R). Suppose that $X_{i'}^* \neq X_{i''}^*$ for at least one pair $(x_{i'}, x_{i''}) \in I \times I$. We first prove that F satisfies internal stability of domination. Indeed, let $x_{i'}, x_{i''} \in F$. We suppose, by way of contradiction, that $(x_{i'}, x_{i''}) \in \overline{R}$. Then, there exists $N \in \mathbb{N}$ and $x_1, x_2, ..., x_N \in X$ such that $x_{i'} R x_1 R x_2 R ... R x_N R x_{i''}$. Therefore, there are $X_1, X_2, ..., X_N \in \Xi$ with $x_n \in X_n$ for all $n \in \{1, 2, ..., N\}$ satisfying $X_{i'}^* \widetilde{R} X_1 \widetilde{R} X_2 \widetilde{R} ... \widetilde{R} x_N \widetilde{R} X_{i''}^*$. Therefore, $X_{i''}^*$ cannot be maximal. This contradiction shows that $(x_{i'}, x_{i''}) \notin \overline{R}$. Similarly we can prove that $(x_{i''}, x_{i'}) \notin \overline{R}$. Hence, F satisfies internal stability of domination.

To prove that F satisfies external stability of domination, let $x^* \in X$ with $x^* \notin F$. We have two cases to consider: when (1) $x^* \in \bigcup_{i \in I} X_i^* \setminus F$ and when (2) $x^* \in X \setminus \bigcup_{i \in I} X_i^*$. In case (1), $x^* \in X_{i'}^*$ for some $i' \in I$. Then, either $X_{i'}^*$ consists of a single element or not. Since $x^* \notin F$, $X_{i'}^*$ cannot consist of a single element. Hence, $X_{i'}^*$ must be an R-cycle. Therefore, $(x_{i'}, x^*) \in \overline{R}$.

In case (2), let $x^* \in X \setminus \bigcup_{i \in I} X_i^*$. We prove that there exists $i \in I$ such that $(x_i, x^*) \in \overline{R}$, which proves the desired property of external stability of domination.

Suppose to the contrary that:

$$\text{For each } i \in I \text{ it holds that } (x_i, x^*) \notin \overline{R}. \tag{1}$$

Then, $(y, x^*) \notin \overline{R}$ holds for every $y \in \bigcup_{i \in I} X_i^*$. Indeed, assume, by contradiction, that $(y, x^*) \in \overline{R}$ for some $y \in X_j^*$ with $j \in I$. Then, since $(x_j, y) \in \overline{R}$, we have that $(x_j, x^*) \in \overline{R}$, a contradiction to (1). Therefore,

$$\text{for each } y \in \bigcup_{i \in I} X_i^* \text{ we have } (y, x^*) \notin \overline{R}. \tag{2}$$

Let

$$A_{x^*} = \left\{ y \in X \mid \emptyset \subset \overline{R}y \subseteq \overline{R}x^* \right\}. \tag{3}$$

We show that $A_{x^*} \neq \emptyset$, that is $y_1 \overline{R} y_0 \overline{R} x^*$ for some $y_0, y_1 \in X \setminus \bigcup_{i \in I} X_i^*$.

We can always find a $y_0 \in X \setminus \bigcup_{i \in I} X_i^*$ such that $y_0 \overline{R} x^*$. Indeed, suppose that $(y, x^*) \notin \overline{R}$ for all $y \in X \setminus \bigcup_{i \in I} X_i^*$. Then, since as a consequence of (2), $(y, x^*) \notin \overline{R}$ for all $y \in \bigcup_{i \in I} X_i^*$, we have that y_0 would be an R-undominated element. Hence, $\{y_0\} = X_i^*$ and thus $y_0 = x_i$, for some $i \in I$, an absurdity under the fact that $y_0 \in X \setminus \bigcup_{i \in I} X_i^*$. Therefore,

$$\text{there exists } y_0 \in X \setminus \bigcup_{i \in I} X_i^* \text{ such that } y_0 \overline{R} x^*. \tag{4}$$

To prove that there exists an $y_1 \in X \setminus \bigcup_{i \in I} X_i^*$ such that $y_1 \overline{R} y_0$, we simply need to demonstrate that y_0 possesses the properties of x^*. All that is required is for the property (2) that satisfies x^* to also satisfy y_0. That is, for all $y \in \bigcup_{i \in I} X_i^*$ it holds that $(y, y_0) \notin \overline{R}$. Indeed, suppose that $(y', y_0) \in \overline{R}$ for some $y' \in X_{i_0}^*$, $i_0 \in I$, then from $(x_{i_0}, y') \in \overline{R}$, $(y', y_0) \in \overline{R}$ and $(y_0, x^*) \in \overline{R}$ we have that $(x_{i_0}, x^*) \in \overline{R}$, a contradiction to (1). Therefore, there exists a $y_1 \in X \setminus \bigcup_{i \in I} X_i^*$ such that $y_1 \overline{R} y_0$. It follows that

$$y_1 \overline{R} y_0 \overline{R} x^* \text{ with } y_0, y_1 \in X \setminus \bigcup_{i \in I} X_i^*. \tag{5}$$

By (5) we conclude that A_{x^*} is non-empty. We now show that A_{x^*} is closed with respect to τ. Suppose that t belongs to the closure of A_{x^*}. Then, there exists a net $(t_k)_{k \in K}$ in A_{x^*} with $t_k \to t$. We have to show that $t \in A_{x^*}$, i.e., $\overline{R}t \subseteq \overline{R}x^*$. Take any

$z \in \overline{R}t$. Since $\{w \in X | z\overline{R}w\}$ is an open neighborhood of t, there exists $k' \in K$ such that for each $k \geq k'$, $z\overline{R}t_k$ holds. On the other hand, for each $k \geq k'$, $t_k \in A_{x^*}$. Hence, $z \in \overline{R}t_k \subseteq \overline{R}x^*$. It follows that $\overline{R}t \subseteq \overline{R}x^*$ which implies that $t \in A_{x^*}$. Therefore, A_{x^*} is a closed subset of X.

We continue with the ultimate goal of proving that (1) does not hold true. This will ensure us the existence of an $i^* \in I$ such that $(x_{i^*}, x^*) \in \overline{R}$ with $x^* \in X \setminus \bigcup_{i \in I} X_i^*$. In this direction, we first show that,

$$\text{for each } t \in A_{x^*} \text{ there exists } y \in X \setminus \bigcup_{i \in I} X_i^* \text{ such that } (y, t) \in \overline{R}. \qquad (6)$$

Indeed, if $t \in A_{x^*}$, then $t \in \{y' \in X | \emptyset \subset \overline{R}y' \subseteq \overline{R}x^*\}$. It follows that $\overline{R}t \neq \emptyset$ and $\overline{R}t \subseteq \overline{R}x^*$. Therefore, there exists $y \in X$ such that $y\overline{R}t$ and $y\overline{R}x^*$. Because of (2), we conclude that $y \in X \setminus \bigcup_{i \in I} X_i^*$. Therefore, Eq. (6) holds.

Based on the validity of (6), we have two cases to consider: when (a) $y = t$ and when (b) $y \neq t$.

In case (a), there exists an R-cycle $\mathcal{C} \subseteq X \setminus \bigcup_{i \in I} X_i^*$ which contains t. Assume $(\mathcal{C}_\gamma)_{\gamma \in \Gamma}$, $\mathcal{C}_\gamma \subseteq X \setminus \bigcup_{i \in I} X_i^*$ is the family of all R-cycles that contains t. Let $\mathfrak{C} = (\mathcal{C}_\beta)_{\beta \in B}$, $B \subseteq \Gamma$, be any chain in $(\mathcal{C}_\gamma)_{\gamma \in \Gamma}$ so that any two sets belonging to the chain are related by set inclusion. Clearly, $\bigcup_{\beta \in B} \mathcal{C}_\beta$ is an upper bound for $\mathfrak{C}$. However, because $\bigcap_{\beta \in B} \mathcal{C}_\beta$ contains t, $\bigcup_{\beta \in B} \mathcal{C}_\beta$ is also an R-cycle. Therefore, by the Lemma of Zorn,[2] the family of all R-cycles $(\mathcal{C}_\gamma)_{\gamma \in \Gamma}$, $\mathcal{C}_\gamma \subseteq X \setminus \bigcup_{i \in I} X_i^*$, which contains t has a maximal element, let $\mathcal{C}_{\gamma_0}$. Then, $\mathcal{C}_{\gamma_0} \in \Xi$. Since $\mathcal{C}_{\gamma_0} \in X \setminus \bigcup_{i \in I} X_i^*$ we have that $\mathcal{C}_{\gamma_0}$ is not a maximal element with respect to $\widetilde{R}$ in Ξ. Thus, there exists $\hat{i} \in I$ such that $X_{\hat{i}}^* \widetilde{R} \mathcal{C}_{\gamma_0}$. It follows that $(x, y) \in R$ for some $x \in X_{\hat{i}}^*$ and some $y \in \mathcal{C}_{\gamma_0}$. Since $X_{\hat{i}}^*$ is an R-cycle, $x, x_{\hat{i}} \in X_{\hat{i}}^*$, $(x, y) \in R$ and $y \in A_{x^*}$ we conclude that $(x_{\hat{i}}, x^*) \in \overline{R}$, a contradiction to (1).

In case (b), we have that for each $t \in A_{x^*}$ there exists $y \in X \setminus \bigcup_{i \in I} X_i^*$ with $y \neq t$ and $(y, t) \in \overline{R}$. It follows that $y \in A_{x^*}$ $(\overline{R}y \subseteq \overline{R}x^*)$. Therefore, for each $t \in A_{x^*}$, the sets $\{y \in X | y\overline{R}t, y \neq t\} \cap A_{x^*}$ are open in the relative topology of A_{x^*}, due to upper tc-semicontinuity of R.

Thus, the collection $(\{t \in X | y\overline{R}t, y \neq t\} \cap A_{x^*})_{y \in A_{x^*}}$ is an open cover of A_{x^*}, that is,

$$A_{x^*} = \bigcup_{y \in A_{x^*}} \left(\{t \in X | y\overline{R}t, y \neq t\} \cap A_{x^*} \right).$$

Since A_{x^*} is compact there exist $y_1, y_2, ..., y_n \in X$ such that

$$A_{x^*} = \bigcup_{i=1,2,...,n} \left(\{t \in X | y_i \overline{R}t, y_i \neq t\} \cap A_{x^*} \right).$$

We show that among the elements $y_1, y_2, ..., y_n$ there must be an R-cycle. First note that if $i^* \in \{1, 2, ..., n\}$, then y_{i^*} is an element of one of the covering sets

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$\{t \in X| \, y_i \overline{R}t, y_i \neq t\} \cap A_{x^*}$, $i = 1, 2, ...n$. If $y_{i^*} \in \{t \in X| \, y_i \overline{R}t, y_i \neq t\} \cap A_{x^*}$, then we would have an R-cycle. Otherwise, for each $i, j \in \{1, 2, ..., n\}$, $i \neq j$, $y_i \in \{t \in X| \, y_j \overline{R}t, y_j \neq t\} \cap A_{x^*}$. Without loss of generality, we assume that $y_1 \in \{t \in X| \, y_2 \overline{R}t, y_2 \neq t\} \cap A_{x^*}$. Now, for an arbitrary i, we have just the case as we did for $i = 1$, that $y_i \in \{t \in X| \, y_{i+1} \overline{R}t, y_{i+1} \neq t\} \cap A_{x^*}$. Then, $y_n \in \{t \in X| \, y_k \overline{R}t, y_k \neq t\} \cap A_{x^*}$ with $k \in \{1, 2, ..., n\}$. Thus, we would have an R-cycle.

Therefore, there exists an R-cycle $\widetilde{C} \subseteq X \setminus \bigcup_{i \in I} X_i^*$ which contain the elements of a set $M \subseteq \{y_1, y_2, ..., y_n\}$. By the Lemma of Zorn, the family of all R-cycles $(\widetilde{C}_\gamma)_{\gamma \in \Gamma}$, $\widetilde{C}_\gamma \subseteq X \setminus \bigcup_{i \in I} X_i^*$, which contain M has a maximal element, which we call $\widetilde{C}_{\gamma_0}$. Therefore, as in the family C_γ of the case (a) above, there exists $\widetilde{i} \in I$ such that $X_{\widetilde{i}}^* \widetilde{R} \widetilde{C}_{\gamma_0}$ which implies that $(x_i, x^*) \in \overline{R}$, a contradiction to (1). Because the assumption referred to in (2) always leads to a contradiction, we conclude that F satisfies external stability of domination.

Conversely, let F be a generalized stable set and let $x_0 \in F$. Let also $\{X_i^* | i \in I\}$ be the set of maximal elements in $(\Xi, \widetilde{R})$. We prove that $x_0 \in X_{\mathfrak{i}}^*$ for some $\mathfrak{i} \in I$. Put

$$B_{x_0} = \left\{ t \in X | \overline{R}t \subseteq \overline{R}x_0 \right\}.$$

Then, since the space is compact, as in the case of A_{x^*} in (3) above, there exists a finite subset $\{t_1, t_2, ..., t_n\}$ of B_{x_0} where among its element there must be an R-cycle, let $\widehat{C}$. By the Lemma of Zorn, the family of all R-cycles $(\widehat{C}_\gamma)_{\gamma \in \Gamma}$, $\widehat{C}_\gamma \subseteq B_{x_0}$, which contain $\{t_1, t_2, ..., t_n\}$ has a maximal element, let $\widehat{C}_{\gamma_0}$. To prove that $x_0 \in X_{\mathfrak{i}}^*$ for some $\mathfrak{i} \in I$, we have two cases to consider: when (1) $B_{x_0} = \emptyset$ and when (2) $B_{x_0} \neq \emptyset$. In case (1), we have that x_0 cannot be a member of a cycle. But then, x_0 is an R-undominated element. Indeed, suppose to the contrary that xRx_0 for some $x \in X$. Since F satisfies internal stability of domination, $x\overline{R}x_0$ implies that $x \notin F$. Then, because of external stability of domination, there exists $y \in F$ such that $y\overline{R}x$. It follows that $y\overline{R}x_0$, a contradiction to internal stability of domination. Therefore, x_0 is an R-undominated element. We now show that $\{x_0\} = X_{\widehat{i}}^*$ for some $\widehat{i}$. Indeed, since $(x, x_0) \notin \overline{R}$ for all $x \in X$, by definition of $\widetilde{R}$, we cannot have $X_i^* \widetilde{R} \{x_0\}$ for some $i \in I$. Thus, $X_{\widehat{i}}^* = \{x_0\}$ is an $\widetilde{R}$-maximal element of Ξ with $x_0 \in X_{\widehat{i}}^*$.

In case (2), we only have to prove that $\widehat{C}_{\gamma_0}$ is an $\widetilde{R}$-maximal element in Ξ and $x_0 \in \widehat{C}_{\gamma_0}$. In order to prove that $\widehat{C}_{\gamma_0}$ is maximal with respect to $\widetilde{R}$ in Ξ, suppose to the contrary that there exists a X_i^* for some $i \in I$ such that $X_i^* \widetilde{R} \widehat{C}_{\gamma_0}$. Then, there exist a $x \in X_i^*$ and a $y \in \widehat{C}_{\gamma_0}$ such that xRy by definition of $\widetilde{R}$. Since $y \in \widehat{C}_{\gamma_0} \subseteq A_{x_0}$ and $\overline{R}y \subseteq \overline{R}x_0$ we conclude that $x\overline{R}x_0$. Therefore, because of the internal stability of domination, x cannot belong to F. Thus, there exists $t \in F$ such that $t\overline{R}x$. It follows that $t\overline{R}x_0$, a contradiction to internal stability of domination. Hence, $\widehat{C}_{\gamma_0}$ is maximal with

respect to $\widetilde{R}$ in Ξ, that is, $\widehat{\mathcal{C}}_{r_0} = X_i^*$ for some $i \in I$. It remains to prove that $x_0 \in X_i^*$. Suppose to the contrary that $x_0 \notin X_i^*$. By $B_{x_0} \neq \emptyset$, there exists $y_0 \in B_{x_0}$ such that $y_0 \overline{R} x_0$. Then, because of the internal stability of domination, by $y_0 \overline{R} x_0$, we conclude that $y_0 \notin F$. Therefore, external stability of domination implies that $t \overline{R} y_0$ for some $t \in F$. It follows that $t \overline{R} x_0$, a contradiction to internal stability of domination. Hence, $x_0 \in X_i^*$ and the proof is over. $\qquad\square$

The following corollary, which is the main result in Van Deemen (1991), is an immediate result of Theorem 3.1.

Corollary 3.2 (Van Deemen 2013, Theorem 2). *Let X be a nonempty finite set and let R be an asymmetric binary relation on X. Let $\mu(\Xi, \widetilde{R}) = \{X_1^*, X_2^*, ..., X_n^*\}$ and $x_1, x_2, ..., x_n \in X$. Then, $\{x_1, x_2, ..., x_n\}$ is a generalized stable set if and only if $x_1 \in X_1^*, x_2 \in X_2^*, ..., x_n \in X_n^*$.*

Proof Since any topology on a finite set X is compact and any dominance relation R is upper tc-semicontinuous (any subset of X is open), the corollary is an immediate consequence of Theorem 3.1. $\qquad\square$

The following example serves two purposes: it demonstrates that Theorem 3.1 generalizes Van Deemen's corresponding result for infinite cases, and results in a strict extension of the stable set to the generalized stable set.

Example 3.3 Let $X = A \cup B$ where A, B are disjoint sets of social states endowed with a compact topology. Given a cover $(U_i)_{i \in I}$ of X, it is in particular, a cover of A and B. By compactness, we can cover each of A and B individually by finitely many $A_i = \{U_i \cap A | i = 1, 2, 3, ..., n\}$ and $B_i = \{U_i \cap B | i = 1, 2, 3, ..., n\}$. Together, they must cover the union, which is $A \cup B = X$. We define a binary relation R on X such that xRy if and only if for each $i \in \{1, 2, 3, ..., n\}$ one of the following conditions holds: (i) $x, y \in A_i$ or $x, y \in B_i$; (ii) $x \in A_i, y \in B_{i+1}$; (iii) $x \in B_{i+1}$ and $y \in B_i$. In all other cases x and y are non-comparable. For each $x \in X$, the set $K_x = \{y \in X | x\overline{R}y\}$ is open. Indeed, if x belongs to some A_i with $i \in \{1, 2, 3, ..., n-1\}$, then $K_x = A_i \cup B_1 \cup B_2 \cup ... \cup B_{i+1}$ and $K_x = A_i$ if $i = n$. If x belongs to some B_i with $i \in \{2, ..., n\}$, then $K_x = B_1 \cup ... \cup B_i$ and $K_x = B_i$ if $i = 1$. Therefore, R is upper tc-semicontinuous. It follows that (X, R) satisfies the assumptions of Theorem 3.1 and thus it has at least a generalized stable set. The set $M = \{x_1, x_2, ..., x_n\}$ where $x_i \in A_i$ for all $i \in \{1, 2, ..., n\}$ can serve as a confirmation of this fact. Since the members of $A_i, A_j, i \neq j$, are not comparable with respect to R, M is internally stable. For each $y \in X \setminus M$ there exists $x_i \in M$ such that $x_i \overline{R} y$. Hence, we have that M is a generalized stable set with respect to R in X. Because the elements of B_1 are non-dominated with respect to R from the elements of A and the elements of A are non-dominated with respect to R from the elements of B, (A, R) has no stable set.

4 Conclusions

The theory of generalized stable sets is strongly related to the Von Neumann–Morgenstern theory of stable sets, which are flawed in that they can fail to yield a solution in the case of odd cycles. This is addressed by the theory of generalized stable sets, which is able to yield a solution for any possible cyclic social preference. Another advantageous feature is that if the set of best social choices is not empty, the recommended solution set will include it. Defining general solutions concepts over an infinite space requires us to think more deeply about the choice process, with choice sets being subsets of the alternative space rather than simply orderings of alternatives. In social choice, every generalization is not straightforward because certain characterizations of a finite set of alternatives are lost in the case of infinite sets of alternatives. Since each finite set can be considered as a discrete topological space, it makes sense for a social science model to be generalized through topology. On the other hand, the concept of compactness is an extension of the benefits of finiteness to infinite sets. Most properties of compact sets in social science are analogous to the properties of finite sets which are quite trivial. A crucial aspect in proving the validity of Theorem 3.1, for instance, is the demonstration of the existence of maximal elements for the ground sets of the strong components of (X, R). If the feasible set is finite, a sufficient condition for the existence of such a maximal element is that preference must be acyclic. However, the same statement remains true if finiteness is replaced by compactness and the dominance relation is upper semicontinuous. With these two notions as the main assumptions in Theorem 3.1, the transition from finite to infinite is generally secured because the validity of applications from the finite case is preserved in the infinite one. Indeed, when finiteness is replaced by compactness and upper tc-semicontinuity, given the validity of Theorem 3.1, it is straightforward to verify the validity of all theorems and conclusions of Van Deemen (1991) for the case of infinite sets of alternatives (Van Deemen uses asymmetric binary relations). If, as required in Theorem 3.1, the binary relation is arbitrary (i.e. symmetry is allowed), all of the results in Van Deemen (1991) are still true, with the exception that in Theorem 4, sec. 3, the set of maximal elements is replaced by the set of R-undominated elements. Finally, in abstract decision problems (X, R) where X is finite and R is asymmetric, there are important differences and commonalities between the generalized stable set and the other solution theories in the social choice literature. It will be interesting to be explored if there are analogous generalizations of other general solution concepts to the case of an infinite set of alternatives and to determine if their differences and similarities with the generalized stable set, as presented in Theorem 3.1, follow the same logic as those of the finite case.

Declarations

Conflict of interest The author declare that there is no conflict of interest.

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