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ORIGINAL PAPER

Liberal political equality does *not* imply proportional representation

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Abstract
In their article 'Liberal political equality implies proportional representation', which was published in *Social Choice and Welfare* 33(4):617–627 in 2009, Eliora van der Hout and Anthony J. McGann claim that any seat-allocation rule that satisfies certain 'Liberal axioms' produces results essentially equivalent to proportional representation. We show that their claim and its proof are wanting. Firstly, the Liberal axioms are only defined for seat-allocation rules that satisfy a further axiom, which we call *Independence of Vote Realization* (*IVR*). Secondly, the proportional rule is the *only* anonymous seat-allocation rule that satisfies *IVR*. Thirdly, the claim's proof raises the suspicion that reformulating the Liberal axioms in order to save the claim won't work. Fourthly, we vindicate this suspicion by providing a seat-allocation rule which satisfies reformulated Liberal axioms but which fails to produce results essentially equivalent to proportional representation. Thus, the attention that their claim received in the literature on normative democratic theory notwithstanding, van der Hout and McGann have *not* established that liberal political equality implies proportional representation.

1 Introduction

How can democracy be justified? Typically, such justifications are rooted in spelling out the consequences of liberal principles, notably equality (Kolodny 2014). Arrhenius (2015: 15) summarizes:

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> In what we roughly could characterise as the received view of democracy, the democratic ideal is conceived in terms of some sort of equality among citizens, often expressed by the slogan 'one person, one vote', in combination with the idea of majority rule.

For this view, May's (1952) theorem showing that majority rule can be justified in terms of fundamental democratic notions is key. May showed that the majority rule is the unique binary decision rule which satisfies the following axioms: *anonymity*, *neutrality* and *positive responsiveness*. Indeed, van der Hout and McGann (2009a, b) interpret May's axioms as expressing liberal political equality. They claim to provide an analogous justification for proportional representation (PR), which we call the 'LP claim'. Informally, they state it as follows (van der Hout and McGann 2009a: 617):

> This article provides an axiomatic justification of proportional representation (PR), similar to May's (1952) theorem for majority rule. It shows that any single-vote seat-allocation rule that treats all voters equally must produce results essentially equivalent to pure PR.

The LP claim has received considerable attention in the literature on normative democratic theory. For one, the LP claim plays a central role in McGann's monograph *The Logic of Democracy* (2006), which is described by Iain McLean as ranking "with Riker and Mackie as one of the most important works in democratic theory of the last 30 years" (McGann 2006: fourth cover). For another, the LP claim is discussed at length in Lagerspetz's *Social Choice and Democratic Values* (2016: 129), where it is described as "a new breakthrough".

Van der Hout and McGann obtain their result in a framework that conceives of the democratic electoral process in two stages. In the first stage, votes are cast and seats are allocated on the basis of a seat-allocation rule. In the second stage, a government has to be formed. The government must consist of a coalition of parties that, conjointly, receives a majority of the seats. Van der Hout and McGann claim to show that any seat-allocation rule that satisfies May-style 'Liberal axioms' induces the same *winning coalitions*, i.e. coalitions that receive a majority of the seats, as the proportional seat-allocation rule does.

We show that the Liberal axioms used by van der Hout and McGann are ill-defined because they switch between the two stages too quickly. Informally, the axioms overlook that *different* ballot profiles in the first stage, can give rise to the *same* coalitional profile in the second stage. To illustrate the problem, consider the following two ballot profiles:

(i) Luc, Cas and Sef vote for parties a, b and c respectively.
(ii) Luc, Cas and Sef vote for parties a, a and c respectively.

Although (i) and (ii) are different ballot profiles, they give rise to the same *coalition profile*, as in both (i) and (ii), Luc and Cas (implicitly) vote for coalition $\{a, b\}$ whereas Sef (implicitly) votes for coalition $\{c\}$. Hence, coalitional profiles are 'multiply realizable'.

The Liberal axioms require to *derive* a coalitional seat-allocation rule F from a given seat-allocation rule f, where F and f take coalitional profiles and ballot profiles as their respective inputs. In order to 'derive an F from an f' one should allot a coalition "the sum of the seats allocated, by f, to each party in the coalition" (2009a: 621). However, owing to the multiple realisability of a coalitional profile, this sum need not be (uniquely) defined. For instance, in the above example, allotting coalition $\{a, b\}$ the sum of seats that f allots to a and b on the basis of (i) will typically be different from allotting on the basis of (ii). Hence, typically one cannot derive an F from an f, which renders the Liberal axioms ill-defined.

The Liberal axioms are ill-defined because *typically*, we cannot derive an F from an f. But when a further assumption is satisfied, one that we call 'Independence of Vote Realization' (*IVR*), one *can* do so. And yet, invoking *IVR* to modify and rescue the LP claim won't do. For one, it is hard to see how *IVR* is to be interpreted in liberal terms. For another, we show that the proportional rule P is the *only* anonymous seat-allocation rule that satisfies *IVR*. So, when the ill-defined Liberal axioms are augmented with *IVR* in order to render them well-defined, the Liberal axioms effectively become redundant for justifying proportionality.

Thus, *IVR* specifies conditions under which the Liberal axioms, *as formulated by van der Hout and McGann*, are well-defined. As a strategy for modifying and rescuing the LP claim, appealing to *IVR* does not work. We also explore another strategy, one that *reformulates* the Liberal axioms. In particular, reformulating the requirement that one derives an F from an f looks promising. Now, relative to a ballot profile such as (i) or (ii) we *can* allot seats to a coalition by summing the seats that f allots to the members of that coalition. And so, although f does not derive an F, it does derive, as we will say, a *coalitional aggregate*, which allocates seats to coalitions on the basis of ballot profiles. Our reformulation strategy basically seeks to reformulate the Liberal axioms and LP claim by replacing appeals to ill-defined coalitional seat-allocation rules with appeals to well-defined coalitional aggregates.

Although our reformulation strategy is *prima facie* promising, an inspection of the proof of the LP claim raises the suspicion that this reformulation strategy won't rescue it. For, even if the Liberal axioms *would* be well-defined, further steps are needed to derive the LP claim on their basis. The steps taken by van der Hout and McGann to do so allow for a natural, *well-defined*, reconstruction involving a premise (LP_4) which, intuitively, says that by allotting more seats to coalitions that receive more votes, one ensures that if a coalition receives a majority of votes, it receives a majority of seats.

Premise LP_4 plays a crucial role in our reconstruction of steps taken by van der Hout and McGann's to derive the LP claim. Although LP_4 is well-defined and does not require to derive an F from an f, it is also false, as we will demonstrate. Owing to the falsity of LP_4, which is independent of the ill-definedness of the Liberal axioms, the prospects for the reformulation strategy in term of coalitional aggregates looks far less promising than at first glance. And indeed they are: we will show that seat-allocation rules may satisfy properly reformulated 'Liberal axioms for coalitional aggregates' and yet fail to induce the same winning coalitions as the proportional rule.

So, the attention that their claim received in the literature on normative democratic theory notwithstanding, van der Hout and McGann have *not* established that liberal political equality implies proportional representation. Moreover, plausible attempts to repair or reformulate the LP claim do not yield a justification of proportional representation in liberal terms. Hence, while our paper contains novel results pertaining to proportional seat-allocation in single-vote elections, the main conclusion of our paper is a negative one: Liberal Political Equality does *not* imply Proportional Representation.

The structure of this paper is as follows. In Sect. 2.1 we set out the formal model used throughout this paper and in Sect. 2.2 we present the definitions of anonymity, neutrality and positive responsiveness for seat-allocation rules.

In Sect. 3.1 we give the definition of the LP claim, in Sect. 3.2 we explain that the Liberal axioms are ill-defined and in Sect. 3.3 we explain LP_4's crucial role in our reconstruction of the argument for the LP claim and show that LP_4 is false.

In Sect. 4.1 we define *IVR*, in Sect. 4.2 we characterize P in terms of *IVR* and anonymity and in Sect. 4.3 we comment on the normative upshot of our characterization.

In Sect. 5.1 we present the rationale of reformulating the LP claim in terms of coalitional aggregates, in Sect. 5.2 we actually reformulate the LP claim and in Sect. 5.3 we show that the reformulated LP claim is false. In Sect. 5.4 we show that *IVR* is equivalent to 'Vote Shuffle Invariance' (*VSI*), which is an axiom pertaining to coalitional aggregates. We note that the proportional rule can also be characterized in terms of *VSI* and anonymity for coalitional aggregates.

In Sect. 6 we conclude by briefly reflecting on the upshot of this paper.

2 Allocating seats in single-vote elections

2.1 The model

Suppose that elections are run under a single-vote electoral system, as specified by the following definition.

Definition 1 (*Elections and (ballot) profiles*) An *election* is a triple $\mathcal{E} = (E, N, \mathcal{A})$ where $E \in \mathbb{R}_+$ is the *estate*, representing a number of seats, to be divided amongst the alternatives in $\mathcal{A} = \{1, \dots, k\}$ on the basis of votes cast by the voters in $N = \{1, \dots, n\}$

A *(ballot) profile* $\mathbf{P}$ for an election $\mathcal{E}$ is an $|N| \times |\mathcal{A}|$ matrix which respects the following conditions:

$$\mathbf{P}_{ij} \in \{0, 1\} \qquad \sum_{j \in \mathcal{A}} \mathbf{P}_{ij} \leq 1 \qquad \sum_{j \in \mathcal{A}} \sum_{i \in N} \mathbf{P}_{ij} \geq 1 \tag{1}$$

In the above, $\mathbf{P}_{ij} = 1$ indicates that voter i votes for alternative j. So, (1) states that each voter can cast *at most* one vote, but that she may also abstain from voting. Further, (1) states that there is at least one voter who does not abstain.[1]

The following election will serve as a running example in this paper.

Example 1 (*ELECT and two of its profiles*) A new parliament, consisting of 9 members, has to be elected. The electorate consists of 18 voters who can cast a single vote for party a, b or c. Throughout this paper, we will use $ELECT = (9, \{1, 2, \dots, 18\}, \{a, b, c\})$ to refer to this election. There are many possible ways in which the electorate *could* vote, i.e. there are many possible *profiles* for *ELECT*. Here are just two examples:

Profile A. Voters $1, 2, \dots, 8$ vote for party a, voters $9, 10, \dots, 14$ vote for party b and voters 15, 16, 17 and 18 vote for party c.

Profile B. Voters $1, 2, \dots, 8$ vote for party a, voters $9, 10, \dots, 15$ vote for party b and voters 16, 17 and 18 vote for party c.

Profiles **A** and **B** are, officially, 1-0 matrices that are specified as follows:

$$\begin{cases} \mathbf{A}_{ia} = 1 & \text{iff } i = 1, \dots, 8 \\ \mathbf{A}_{ib} = 1 & \text{iff } i = 9, \dots, 14 \\ \mathbf{A}_{ic} = 1 & \text{iff } i = 15, \dots, 18 \end{cases} \qquad \begin{cases} \mathbf{B}_{ia} = 1 & \text{iff } i = 1, \dots, 8 \\ \mathbf{B}_{ib} = 1 & \text{iff } i = 9, \dots, 15 \\ \mathbf{B}_{ic} = 1 & \text{iff } i = 16, \dots, 18 \end{cases} \qquad \square$$

A *seat-allocation rule*[2] specifies how to allocate the available seats for each profile, i.e. a seat-allocation rule solves *single-vote problems*:

Definition 2 (*Single-vote problems, seat-allocations, and rules*) A *single-vote problem* of an election $\mathcal{E} = (E, N, \mathcal{A})$ is a pair $(\mathcal{E}, \mathbf{P})$ consisting of the election $\mathcal{E}$ together with a profile $\mathbf{P}$ for $\mathcal{E}$.

A *seat-allocation* x for $(\mathcal{E}, \mathbf{P})$ is an element of $\mathbb{R}_+^{\mathcal{A}}$, with x_j interpreted as the amount of seats that are allocated to alternative $j \in \mathcal{A}$ and which respects the following constraints:

(i) *No votes no seats*: if $\sum_{i \in N} \mathbf{P}_{ij} = 0$ then $x_j = 0$, for all $j \in \mathcal{A}$
(ii) *Efficiency*: $\sum_{j \in \mathcal{A}} x_j = E$

[1] We demand that there is always at least one voter who does not abstain from voting in order to avoid the cumbersome, merely technical, question as to how to allocate the seats when no one votes.

[2] Note that van der Hout and McGann work with seat-share functions (on which they do not impose *any* condition other than that their entries are in [0, 1]), not seat-allocation rules. Now every seat-allocation can be translated into seat-shares but not vice versa: in order to translate shares into seats we need to know the total number E of seats under consideration. All the results of this paper can be trivially reformulated in terms of seat-share functions. However, to work with seat-allocation rules is more convenient and general.

So in a seat-allocation x, (i) an alternative does not receive seats when it receives no votes, and (ii) all available seats are allocated.

A *seat-allocation rule* for $\mathcal{E}$ is a function f that assigns an allocation $f(\mathcal{E}, \mathbf{P})$ to each single-vote problem $(\mathcal{E}, \mathbf{P})$. We will write $f(\mathbf{P})$ instead of $f(\mathcal{E}, \mathbf{P})$ whenever doing so cannot lead to confusions.

Proportional representation is the idea that seats in parliament should be divided proportional to votes, which is realized by the *proportional seat-allocation rule P*:

$$P(\mathbf{P})_j = \frac{\sum_{i \in N} \mathbf{P}_{ij}}{\sum_{y \in \mathcal{A}} \sum_{i \in N} \mathbf{P}_{iy}} \cdot E \qquad \text{for all } j \in \mathcal{A}$$

Thus, P recommends to allocate the seats proportional to the number of votes received by the parties. For profiles **A** and **B** of *ELECT*, this yields the following allocations:

$$P(\mathbf{A}) = (4,\ 3,\ 2) \qquad P(\mathbf{B}) = (4,\ 3.5,\ 1.5)$$

Indeed, seat-allocation rules, and so in particular the proportional rule, may recommend to allocate *fractions* of seats, as e.g. $P(\mathbf{B})$ illustrates: for profiles such as **B**, proportional representation is, strictly speaking, unrealisable. How to best realize proportional representation when the ideal of proportionality is not attainable, i.e. when the ideal recommends to allocate fractions of seats? This important question we address elsewhere[3]. Here we follow van der Hout and McGann in simply assuming divisible seats, so as to fully focus on justifying the *ideal* of proportional representation.

The proportional rule P is a prominent seat-allocation rule, but many different seat-allocation rules exist. To give just one example, the *Squared proportional rule S* divides seats proportional to squared vote totals:

$$S(\mathbf{P})_x = \frac{(\sum_{i \in N} \mathbf{P}_{ix})^2}{\sum_{y \in \mathcal{A}} (\sum_{i \in N} \mathbf{P}_{iy})^2} \cdot E \qquad \text{for all } x \in \mathcal{A}$$

When applied to profiles **A** and **B** of *ELECT*, the squared proportional rule yields the following allocations.

$$S(\mathbf{A}) = (4.97,\ 2.79,\ 1.24) \qquad S(\mathbf{B}) = (4.72,\ 3.61,\ 0.66)$$

[3] An *apportionment method* specifies how to divide parliamentary seats, or other indivisible goods, when the ideal of proportional division is, strictly speaking, not attainable. *Apportionment theory* (cf. Balinski and Young 2001) studies the wide variety of apportionment methods that exist systematically and *axiomatically*, that is in terms of the (elementary) properties, or axioms, that these methods fulfil. Per definition, apportionment methods all share the property that when the ideal of proportionality *is* attainable, as it is in **A**, they realize it: each apportionment method recommends allocation (4, 3, 2) for **A**. In Wintein and Heilmann (2018) we discuss the relation between Broome's (1990) theory of fairness and apportionment theory.

As stated in the introduction, van der Hout and McGann obtain their justification in a framework that conceives of the democratic process as consisting of *two* stages. In the first stage, votes are cast and seats are allocated on the basis of a seat-allocation rule. In the second stage, a government has to be formed which, in order to govern, must consist of a *coalition of parties* that, conjointly, receives a majority of the seats. In order to study the coalitions in the second stage, we associate, with each election $\mathcal{E} = (E, N, \mathcal{A})$ and each *partition* $\mathcal{C}$ of the alternatives in $\mathcal{A}$, the *coalitional election* $\mathcal{E}(\mathcal{C})$, which is defined as follows.

Definition 3 (*Coalitional elections*) Let $\mathcal{E} = (E, N, \mathcal{A})$ be an election and $\mathcal{C} = \{C_1, \dots, C_m\}$ a *partition* of $\mathcal{A}$:

$$\bigcup_{i=1,\dots,m} C_i = \mathcal{A} \quad \text{and} \quad \text{if } i \neq j \text{ then } C_i \cap C_j = \emptyset$$

Then $\mathcal{E}(\mathcal{C}) = (E, N, \mathcal{C})$ defines an election, called the (coalitional) *election* $\mathcal{E}$ *on* $\mathcal{C}$, which has the same voters N and seats E as election $\mathcal{E}$, but whose alternatives are given by $\mathcal{C}$, interpreted as a set of disjoint coalitions of parties in $\mathcal{A}$.

Now a coalitional election $\mathcal{E}(\mathcal{C})$ is, formally, just an election, i.e. it respects Definition 1. Thus, the definition of a *profile* and of a *seat-allocation rule* for $\mathcal{E}(\mathcal{C})$ are given by Definitions 1 and 2 respectively. For instance, the proportional rule P and squared proportional rule S straightforwardly define seat-allocation rules for coalitional elections. Nevertheless, it will be convenient to refer to seat-allocation rules for $\mathcal{E}(\mathcal{C})$ as *coalitional* seat-allocation rules. Also, we use 'f' and 'F' for an arbitrary seat-allocation rule for $\mathcal{E}$ and an arbitrary coalitional seat-allocation rule for $\mathcal{E}(\mathcal{C})$ respectively.

The following example illustrates the notion of a coalitional election.

Example 2 (*ELECT* on $\{\{a\}, \{b, c\}\}$ *and a profile*) Partition $\{\{a\}, \{b, c\}\}$ of the parties of *ELECT* gives rise to coalitional election *ELECT* on $\{\{a\}, \{b, c\}\}$ in which nine seats have to be allocated amongst the coalitions $\{a\}$ and $\{b, c\}$ and in which each of the 18 voters can cast a single vote for one of these two coalitions. Here is an example of a profile for this coalitional election:

Profile C. Voters $1, 2, \dots, 8$ vote for coalition $\{a\}$, voters $9, 10, \dots, 18$ vote for coalition $\{b, c\}$.

Applying the proportional rule and squared proportional rule to **C** yields allocations $P(\mathbf{C}) = (4, 5)$ and $S(\mathbf{C}) = (3.51, 5.49)$ respectively. □

2.2 May's axioms for seat-allocation rules

Van der Hout and McGann seek to provide "an axiomatic justification of proportional representation (PR), similar to May's (1952) theorem for majority rule." Now, May (1952) gave an axiomatic characterization of the majority rule, the well-known *social* *decision rule* that is often used for deciding between two alternatives a and b: decide in favour of a just in case a majority prefers a to b. May showed that the majority rule is the unique binary decision rule which satisfies the following axioms: *anonymity*, *neutrality* and *positive responsiveness*.[4]

May's theorem has received considerable attention in, most notably, the literature on normative democratic theory. The reason for this is that although majority rule may be intuitively appealing, May's theorem shows that majority rule can be justified in terms of fundamental democratic notions. For, some authors (e.g. Dahl (1956) but also van der Hout and McGann (2009a, b)) interpret May's axioms as expressing liberal political equality. Indeed, as Goodin and List (Goodin and List 2006: 942) put it:

> Because its proof is relatively straightforward, May's theorem may count only as a minor classic in formal social choice theory, but it has been received as a major finding in democratic theory more generally.

Hence, May's axiomatic characterization via anonymity, neutrality and positive responsiveness may be interpreted as providing a liberal justification for majority rule. Now, the idea of Van der Hout and McGann is to provide a similar, liberal justification for proportional representation. In order to do so, they translate May's axioms, which are defined for social decision functions, into corresponding axioms for seat-allocation rules. In this section we present the seat-allocation counterparts of May's axioms.

According to *anonymity* the names, or identities, of the voters should have no bearing on the election result: it is the *votes* that should count, not *who* voted.

Definition 4 (*Anonymity*) Let f be a seat-allocation rule for an election $\mathcal{E} = (E, N, \mathcal{A})$ and let σ be any permutation of the voters in N, represented by a permutation of the rows of a profile. We say that f is *anonymous* iff for each profile $\mathbf{P}$, $f(\mathbf{P}) = f(\sigma\mathbf{P})$.

To illustrate anonymity, let σ be the following permutation of voters of *ELECT*.

$$\sigma(i) = \begin{cases} i+8 & \text{for } i = 1, \dots, 8 \\ i-8 & \text{for } i = 9, \dots, 16 \\ i & \text{for } i = 17, 18 \end{cases}$$

When we apply σ to profile $\mathbf{A}$ (cf. Example 1) we obtain profile $\sigma\mathbf{A}$:

Profile $\sigma\mathbf{A}$. Voters $9, 10, \dots, 16$ vote for party a, voters $1, 2, \dots, 6$ vote for party b and voters 7, 8, 17 and 18 vote for party c.

According to anonymity, seats should be allocated on the basis of the vote-totals received by the parties, and information as to which specific voters realized these vote-totals should be neglected. For instance, in both $\mathbf{A}$ and $\sigma\mathbf{A}$, parties a, b and c receive 8, 6 and 4 votes respectively. As anonymous seat-allocation rules can only use these

[4] May's (May 1952) positive responsiveness condition is a relatively strong monotonicity condition and contested (see e.g. Coleman and Ferejohn 1986).

vote-totals as the basis for their recommendation, anonymous rules prescribe the same allocation for **A** and σ**A**.

Anonymity ensures that elections are not biased towards the names, or identities, of *voters*. *Neutrality* ensures that elections are not biased towards the names, or identities, of *alternatives*.

Definition 5 (*Neutrality*) Let f be a seat-allocation rule for an election $\mathcal{E} = (E, N, \mathcal{A})$ and let π be any permutation of the alternatives in $\mathcal{A}$, represented by a permutation of the columns of a profile. We say that f is *neutral* just in case, for each profile **P**, we have $\pi f(\mathbf{P}) = f(\pi\mathbf{P})$.

To illustrate neutrality, let σ be the following permutation of the alternatives of *ELECT*.

$$\pi(a) = b, \qquad \pi(b) = a, \qquad \pi(c) = c$$

When we apply σ to profile **A** we become profile π**A**:

Profile π**A**. Voters $1, 2, \ldots, 8$ vote for party b, voters $9, 10, \ldots, 14$ vote for party c and voters 15, 16, 17 and 18 vote for party c.

Under π, parties a and b are permuted so that all voters who vote for a (b) in **A** vote for b (a) in π**A** and vice versa. Neutrality dictates that the permutation of parties should be reflected in the seat-allocation, meaning that if f recommends e.g. allocation (4, 3, 2) for **A** it should, if it is neutral, recommend (3, 4, 2) for π**A**.

Finally, positive responsiveness spells out the idea that, 'all else being equal, if an alternative receives more votes, it should receive more seats'. Van der Hout and McGann (2009b) and McGann (2006) spell out this notion as follows.[5]

Definition 6 (*Positive responsiveness*) A seat-allocation rule f for an election $\mathcal{E}$ is *positive responsive* iff for all profiles **P** and **Q** for which:

(i) for all $i \in N$: if $\mathbf{P}_{ix} = 1$ then $\mathbf{Q}_{ix} = 1$, and
(ii) for some $i \in N$: $\mathbf{Q}_{ix} = 1$ and $\mathbf{P}_{ix} = 0$, and
(iii) for all $i \in N$: if $\mathbf{Q}_{ix} = 0$ then, for all $y \neq x$: $\mathbf{Q}_{iy} = 1$ iff $\mathbf{P}_{iy} = 1$,

we have, for all $y \neq x$: if $f(\mathbf{P})_x = f(\mathbf{P})_y$ then $f(\mathbf{Q})_x > f(\mathbf{Q})_y$.

So, (i) states that all who vote for x in **P** do so in **Q** as well, whereas (ii) states that some vote for x in **Q** but not so in **P**. We say that (i) and (ii) conjointly state that **Q** is obtained from **P** via a *change favouring* x. Moreover, (iii) states that 'all else is equal'. That is, voters who do not—in passing from **P** to **Q**—change their mind

5 Note that van der Hout and McGann (2009a) adapt a different definition of positive responsiveness. In appendix A, we explain that and why the definition of positive responsiveness of Definition 6 is preferable to that of van der Hout and McGann (2009a). Van der Hout and McGann also investigate the weaker condition of non-negative responsiveness (also in two different versions), which we skip here for the sake of brevity and ease of exposition. With a few simple and natural adjustments, our results can be rephrased in terms of non-negative responsiveness.

in favour of x, vote in **Q** exactly as they do in **P**. When **P** and **Q** satisfy (i), (ii) and (iii), we say that **Q** is obtained from **P** via an *order-preserving* change favouring x. For instance, profile **B** is obtained from profile **A** (cf. Example 1) via an order-preserving change favouring b.

The definition of positive responsiveness can then be paraphrased as follows. In **Q**, alternative x should receive *more* seats than y, if: in **P**, alternatives x and y receive the same amount of seats and if **Q** is obtained from **P** via an order-preserving change favouring x.

The above definitions of anonymity, neutrality and positive responsiveness are presented as pertaining to seat-allocation rules f for elections $\mathcal{E}$. However, as coalitional elections $\mathcal{E}(\mathcal{C})$ are species of elections, with the alternatives being the coalitions in partition $\mathcal{C}$ of $\mathcal{A}$, the definitions straightforwardly apply to coalitional seat-allocation rules F (cf. §2.1) as well. As we will discuss in the next section, van der Hout and McGann use the 'coalitional versions' of anonymity, neutrality and positive responsiveness in order to define the LP claim.

3 The Liberal Proportionality claim

3.1 The LP claim

Van der Hout and McGann (2009a: 617) seek to provide "an axiomatic justification of proportional representation (PR), similar to May's (May 1952) theorem for majority rule, which shows that any single-vote seat-allocation rule that treats all voters equally must produce results essentially equivalent to pure PR". More precisely, they seek to prove the Liberal Proportionality claim, which has the following form:

The Liberal Proportionality claim (LP claim): If a seat-allocation rule f satisfies the Liberal axioms, then f induces the same winning coalitions as the proportional rule P.

Van der Hout and McGann define *the Liberal axioms*, as well as the notion of a seat-allocation rule that *induces the same winning coalitions as P*, as follows:

The Liberal axioms. A seat-allocation rule f for $\mathcal{E} = (E, N, \mathcal{A})$ *satisfies the Liberal axioms* $=_{def}$ for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional seat-allocation rule F for $\mathcal{E}(\mathcal{C})$ that is derived from f is anonymous, neutral and positive responsive.

Inducing the same winning coalitions as P. A seat-allocation rule f for $\mathcal{E} = (E, N, \mathcal{A})$ *induces the same winning coalitions as* $P =_{def}$ for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional seat-allocation rule F for $\mathcal{E}(\mathcal{C})$ that is derived from f allots—on each profile for $\mathcal{E}(\mathcal{C})$—a majority of seats to a coalition $C \in \mathcal{C}$ iff C receives a majority of votes.

Indeed, "under pure PR, a coalition wins a majority of seats iff it has more than 50% of the vote" (2009a:622). The LP claim states that seat-allocation rules which satisfy the Liberal axioms also have this property, so that such rules “produce results essentially equivalent to pure PR”.

3.2 The LP claim is ill-defined

The LP claim refers to the coalitional seat-allocation rule F for $\mathcal{E}(\mathcal{C})$ that is *derived* from f. As we will explain below, the LP claim is ill-defined as, typically, one cannot derive F from an f.

Van der Hout and McGann (2009a: 621) stipulate that the coalitional seat-allocation rule F that is derived from f allots, to a coalition $C \in \mathcal{C}$, “the sum of the seats allocated, by f, to each party in the coalition C”. It is this stipulation that makes it that the LP claim is ill-defined. For clearly, f only allocates seats to parties on the basis of a *profile* for an election $\mathcal{E}$. And equally clearly, a profile for a coalitional election $\mathcal{E}(\mathcal{C})$ can be *realized* via various profiles for $\mathcal{E}$.

Definition 7 (*Realizing a coalitional profile*) Let $\mathcal{E} = (E, N, \mathcal{A})$ be an election and let $\mathcal{C}$ be partition of $\mathcal{A}$. A profile $\mathbf{P}$ for $\mathcal{E}$ *realizes* profile $\mathbf{X}$ for coalitional election $\mathcal{E}(\mathcal{C})$ when:

$$\text{For every } i \in N : \mathbf{X}_{iC} = 1 \iff \mathbf{P}_{ij} = 1 \text{ and } j \in C$$

So, when $\mathbf{P}$ realizes $\mathbf{X}$, i votes for C in $\mathbf{X}$ iff i votes for some $j \in C$ in $\mathbf{P}$.

Coalitional profiles are multiply-realizable. To see this, consider Examples 1 and 2 and note that profile $\mathbf{C}$ for *ELECT* on $\{\{a\}, \{b, c\}\}$ is realized by both profiles $\mathbf{A}$ and $\mathbf{B}$ for *ELECT*. Indeed, in both $\mathbf{A}$ and $\mathbf{B}$, voters $1, 2, \dots, 8$ vote for a party in $\{a\}$ whereas voters $9, 10, \dots, 18$ vote for a party in $\{b, c\}$. That is, $1, 2, \dots, 8$ can be said to vote for coalition $\{a\}$ and $9, 10, \dots, 18$ for coalition $\{b, c\}$, exactly as they do in $\mathbf{C}$.

So then, given a seat-allocation rule f, how to obtain the output of the derived F on $\mathbf{C}$? Should we derive it via $\mathbf{A}$, or via $\mathbf{B}$? That is, should the output be $F^{\mathbf{A}}(\mathbf{C})$ or $F^{\mathbf{B}}(\mathbf{C})$, where:

$$F^{\mathbf{A}}(\mathbf{C}) = (f(\mathbf{A})_a, f(\mathbf{A})_b + f(\mathbf{A})_c)$$
$$F^{\mathbf{B}}(\mathbf{C}) = (f(\mathbf{B})_a, f(\mathbf{B})_b + f(\mathbf{B})_c)$$

Clearly, typically $F^{\mathbf{A}}(\mathbf{C}) \neq F^{\mathbf{B}}(\mathbf{C})$ so that *the* coalitional seat-allocation rule F that is derived from seat-allocation rule f does not exist. As a concrete illustration, note that

$$\begin{aligned}(S(\mathbf{A})_a,\ S(\mathbf{A})_b + S(\mathbf{A})_c) &= (4.97,\ 4.03)\\ \neq (S(\mathbf{B})_a,\ S(\mathbf{B})_b + S(\mathbf{B})_c) &= (4.72,\ 4.28)\end{aligned} \tag{2}$$

So then, typically one cannot derive F from f. Hence, the LP claim is ill-defined. In addition, there are further problems with the argument for the LP claim which are independent of it being ill-defined.

3.3 An argument for the LP claim

Here is a reconstruction of the argument by which Van der Hout and McGann seek to deduce the LP claim. Our reconstruction consists of four premises, the first two of which are the definitions of the Liberal axioms and of inducing the same winning coalitions as *P*, as discussed in Sect. 3.1 above.

LP_1
A seat-allocation rule *f* for $\mathcal{E} = (E, N, \mathcal{A})$ satisfies the Liberal axioms iff for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional seat-allocation rule *F* for $\mathcal{E}(\mathcal{C})$ that is derived from *f* is anonymous, neutral and positive responsive.
LP_2
A seat-allocation rule *f* for $\mathcal{E} = (E, N, \mathcal{A})$ induces the same winning coalitions as *P* iff for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional seat-allocation rule *F* for $\mathcal{E}(\mathcal{C})$ that is derived from *f* allots a majority of seats to a coalition $C \in \mathcal{C}$ iff *C* receives a majority of votes.
LP_3
A coalitional seat-allocation rule *F* for $\mathcal{E}(\mathcal{C})$ which is anonymous, neutral and positive responsive satisfies *plurality ranking*.[6]
LP_4
A coalitional seat-allocation rule *F* for $\mathcal{E}(\mathcal{C})$ that satisfies plurality ranking allots a majority of seats to a coalition $C \in \mathcal{C}$ iff *C* receives a majority of votes.
$\therefore$
If a seat-allocation rule *f* satisfies the Liberal axioms, then *f* induces the same winning coalitions as *P*: the LP claim is true.

Note that the above argument is *valid*, meaning that *if* its premisses are true *then* its conclusion, the LP claim, is true as well. However, the argument is not *sound*, as not all of its premisses are true. We have already discussed LP_1 and LP_2: these premisses are both ill-defined, and hence not true, owing to their reference to the seat-allocation rule *F* for $\mathcal{E}(\mathcal{C})$ that is derived from *f*. The other two premisses of the argument, LP_3 and LP_4, also refer to a coalitional seat-allocation rule *F*, but do not derive this *F* from an underlying seat-allocation rule *f*. In fact, LP_3 and LP_4 are well-defined, meaningful claims.

The valid argument just presented is our *reconstruction* of the argument by Van der Hout and McGann. Whereas LP_1 and LP_2 basically spell out the meaning of the Liberal axioms and of the LP claim, LP_3 and LP_4 provide further steps needed to derive the LP claim from the Liberal axioms. LP_3 is a literal translation from a lemma that is used by Van der Hout and McGann (2009b:753) in their derivation. LP_4, however, is a natural, well-defined version of a corresponding, ill-defined, claim by Van der Hout and McGann (2009a: 626, b: 754) which appeals to the coalitional seat-allocation rule *F* that is obtained from an underlying *f*: premise LP_4 is obtained by removing the illegitimate reference to the "*F* from an *f*" in Van der Hout and McGann's claim.

[6] That is, *F* allots more seats to coalitions that receive more votes. The formal definition of plurality ranking is given below.

It is one thing for a claim to be meaningful, to be true is another. It turns out, as we will explain next, that LP_3 is true but that LP_4 is false. That LP_3 is true follows from Proposition 1 below, which exploits the following definition[7] of plurality ranking:

Definition 8 (*Plurality ranking*) A seat-allocation rule f for an election $\mathcal{E} = (E, N, \mathcal{A})$ satisfies *plurality ranking* iff, for any profile $\mathbf{P}$

$$\text{If } \sum_{i \in N} \mathbf{P}_{ix} > \sum_{i \in N} \mathbf{P}_{iy} \text{ then } f(\mathbf{P})_x > f(\mathbf{P})_y \quad \text{with } x, y \in \mathcal{A} \tag{3}$$

Proposition 1 (*May-Plurality ranking*) *Any seat-allocation rule f for an election $\mathcal{E}$ that is anonymous, neutral and positive responsive satisfies plurality ranking*.

Proof See van der Hout and McGann (2009a, b). □

In contrast to LP_3, premise LP_4 is false, as established by the following counterexample.

Example 3 (*Illustrating that LP_4 is false*) Consider coalitional election *ELECT* on the singleton partition $\{\{a\}, \{b\}, \{c\}\}$. It is readily verified that the squared proportional rule S is anonymous, neutral and positive responsive,[8] so that it follows from Proposition 1 that S satisfies plurality ranking on any profile for $ELECT(\{\{a\}, \{b\}, \{c\}\})$. In particular, it does so for profile $\mathbf{A}^\star$, which is obtained from profile $\mathbf{A}$ (cf. Example 1) by stipulating that i votes for $\{x\}$ in $\mathbf{A}^\star$ iff i votes for x in $\mathbf{A}$. Applying S to $\mathbf{A}^\star$ yields:

$$S(\mathbf{A}^\star) = (4.97,\ 2.79,\ 1.24) \tag{4}$$

Note that, in $\mathbf{A}^\star$, coalitions $\{a\}$, $\{b\}$ and $\{c\}$ receive 8, 6 and 4 votes respectively. Hence, (4) shows that, indeed, S respects plurality ranking on $\mathbf{A}^\star$. If LP_4 were true, S should allot a majority of seats to a coalition in $\{\{a\}, \{b\}, \{c\}\}$ just in case that coalition receives a majority of seats. But (4) shows that S fails to do this. For, $\{a\}$ receives 4.97 seats, which is a majority of the total number of 9 seats. However, $\{a\}$ receives 8 votes, which is a minority of the total number of 18 votes. Hence, LP_4 is false. □

So in Example 3, $\{a\}$ is a winning coalition according to S, while it receives a minority of the votes. This conflicts with the behaviour of the proportional rule, according to which winning coalitions always receive a majority of votes. In particular this is illustrated by profile $\mathbf{A}^\star$. None of the coalitions in $\{\{a\}, \{b\}, \{c\}\}$ receives a majority of votes in $\mathbf{A}^\star$ and, as $P(\mathbf{A}^\star) = (4,\ 3,\ 2)$, none of them is winning according to P.

[7] We formulate Definition 8 and Proposition 1 in terms of seat-allocation rules f for election $\mathcal{E}$ for sake of consistency with the definitions given in Sect. 4.2, which received similar formulations. Naturally, Definition 8 and Proposition 1 apply to coalitional seat-allocation rules F for elections $\mathcal{E}(\mathcal{C})$ as well.

[8] Anonymity and neutrality are immediate. Positive responsiveness can be established along the lines of Proposition 5 if needed.

Although LP_4 is false, a restricted version of the claim, one that applies to elections with two alternatives (be it parties or coalitions) is easily shown to be true.

Proposition 2 (The Binary Liberal Proportionality claim) *If a seat-allocation rule f for a binary election* $\mathcal{E} = (E, N, \{a, b\})$ *satisfies anonymity, neutrality and positive responsiveness then f allots a majority of seats to a* (*b*) *iff a* (*b*) *receives a majority of the votes.*

Proof We have to show that for any profile **P** for $\mathcal{E}$:

$$f(\mathbf{P})_a > f(\mathbf{P})_b \iff \sum_{i \in N} \mathbf{P}_{ia} > \sum_{i \in N} \mathbf{P}_{ib} \tag{5}$$

The right-to-left direction of (5) follows immediately from Proposition 1. We prove the left-to-right direction by contraposition. So suppose that it is not the case that $\sum_{i \in N} \mathbf{P}_{ia} > \sum_{i \in N} \mathbf{P}_{ib}$. We distinguish two cases. Firstly suppose that $\sum_{i \in N} \mathbf{P}_{ia} = \sum_{i \in N} \mathbf{P}_{ib}$. Then, it readily follows from anonymity and neutrality of f that[9] $f(\mathbf{P})_a = f(\mathbf{P})_b$. Secondly, suppose that $\sum_{i \in N} \mathbf{P}_{ia} < \sum_{i \in N} \mathbf{P}_{ib}$. It then follows from Proposition 1 that $f(\mathbf{P})_a < f(\mathbf{P})_b$. Conjointly, the two cases establish the left-to-right direction of (5). □

May (1952) presented an axiomatic characterization of the majority rule: he showed that the majority rule is the unique *binary* decision rule which satisfies anonymity, neutrality and positive responsiveness. For decision rules that apply to three alternatives or more, May's axioms do not characterize a decision rule. Instead, when there are three alternatives or more, a multitude of decision rules exist that satisfy May's axioms. *Mutatis mutandis*, the results of this section, in particular Proposition 1 and Example 3, strongly suggest that the status of May's axioms for seat-allocation rules is rather similar.

4 *P* via Independence of Vote Realization

4.1 Independence of Vote Realization

In Sect. 3.2 we explained that the LP claim is ill-defined as *typically*, one cannot derive F from f. However, there are some circumstances under which one *can* unambiguously derive F from f. For, if the sum of the seats allocated, by f, to each party in the coalition C is the same for each profile on the basis of which we calculate this sum, then an unambiguous derivation exists. That is, we can derive F from f if, and only if, f satisfies the *Independence of Vote Realization* axiom.

[9] For a proof, see Lemma 1 of Van der Hout and McGann (2009a:624), where they show that anonymity and neutrality of f ensure that f has the "cancellation property", i.e. f allots an equal amounts of seats to parties who receive an equal amounts of votes.

Definition 9 (*Independence of Vote Realization (IVR)*) Let $\mathcal{E} = (E, N, \mathcal{A})$ be an election and let f be a seat-allocation rule for $\mathcal{E}$. With $\mathcal{C}$ a partition of $\mathcal{A}$, we say that *f is IVR on* $\mathcal{C}$ iff, for any profile $\mathbf{X}$ for $\mathcal{E}(\mathcal{C})$, and for any two profiles $\mathbf{P}$, $\mathbf{Q}$ for $\mathcal{E}$ that realize $\mathbf{X}$, we have:

$$\sum_{j \in C} f(\mathbf{P})_j = \sum_{j \in C} f(\mathbf{Q})_j \quad \text{for all } C \in \mathcal{C} \tag{6}$$

We say that *f is IVR* just in case *f* is *IVR* on $\mathcal{C}$ for *any* partition $\mathcal{C}$ of $\mathcal{A}$.

As profile $\mathbf{A}$ and $\mathbf{B}$ for *ELECT* realize the same coalitional profile for *ELECT* on $\{\{a\}, \{b, c\}\}$, seat-allocation rules that are *IVR* are such that

$$f(\mathbf{A})_a = f(\mathbf{B})_a \qquad f(\mathbf{A})_b + f(\mathbf{A})_c = f(\mathbf{B})_b + f(\mathbf{B})_c \tag{7}$$

So, as testified by (2), the squared proportional rule S fails to satisfy *IVR*. The proportional rule P satisfies *IVR* as is readily verified. Below, we will show that any anonymous seat-allocation rule that satisfies *IVR is P*. That is, we will characterize P in terms of *IVR* and anonymity.

4.2 *P* from *IVR* and anonymity

As discussed in Sect. 2.2, anonymity ensures that seats are allocated on the basis of the received vote-totals, such that information as to which specific voters realize these vote-totals is neglected. As such, anonymous seat-allocation rules can be understood as acting on *tallied-vote problems* rather than on single-vote problems.

Definition 10 (*Tallied-vote problems, seat-allocations, and rules*) A *tallied-vote problem* for an election $\mathcal{E} = (E, N, \mathcal{A})$ is a pair $(\mathcal{E}, v)$ consisting of the election $\mathcal{E}$ together with a *vote vector* for $\mathcal{E}$, i.e. a vector $v \in \mathbb{N}^{\mathcal{A}}$ such that $v_i \geq 0$ and $1 \leq \sum_{j \in \mathcal{A}} v_j \leq |N|$.

A *seat-allocation* x for $(\mathcal{E}, v)$ is an element of $\mathbb{R}^{\mathcal{A}}_{+}$, with x_j interpreted as the amount of seats that are allocated to alternative $j \in \mathcal{A}$ which satisfies *No votes no seats* and *Efficiency*.

A *tallied seat-allocation rule* for $\mathcal{E}$ is a function r that assigns an allocation $r(\mathcal{E}, v)$ to each tallied-vote problem $(\mathcal{E}, v)$. We will write $r(v)$ instead of $r(\mathcal{E}, v)$ whenever doing so cannot lead to confusions.

Any profile $\mathbf{P}$ induces a unique vote vector p, where $p_j = \sum_{i \in N} \mathbf{P}_{ij}$. Conversely, a vote vector v can be induced by many profiles. For instance, both profile $\mathbf{A}$ and $\sigma\mathbf{A}$ for *ELECT* induce vote vector (8, 6, 4). For any vote vector v, let us write $[v]$ for the set of all profiles that induce v:

$$[v] = \left\{ \mathbf{P} \mid \sum_{i \in N} \mathbf{P}_{ij} = v_j \text{ for all } j \in \mathcal{A} \right\}.$$

Per definition, an *anonymous* seat-allocation rule f must output the same seat-allocation on each $\mathbf{P} \in [v]$. Hence an anonymous seat-allocation rule f can be identified with a tallied seat-allocation rule, r:

$$r(v) = f(\mathbf{P}) \text{ for some } \mathbf{P} \in [v]$$

Working with tallied-vote problems is convenient. Not only because they are simpler objects than single-vote problems, but also because tallied-vote problems are intimately related to *claims problems*, which have been studied in great detail.[10]

A *claims problem* is a triple (E, N, c) where $E > 0$ is an estate that has to be divided amongst $N = \{1, \ldots n\}$ claimants, where the *claims vector* $c \in \mathbb{R}^N$ specifies the claim $c_i \geq 0$ of each claimant $i \in N$ and where $\sum_{i \in N} c_i > E$, i e. the sum of claims exceeds the estate.

A *division rule* d maps each claims problem (E, N, c) to an allocation $x \in \mathbb{R}^N$ which is such that $0 \leq x_i \leq c_i$ and $\sum_{i \in N} x_i = E$.

So indeed, tallied-vote problems are closely related to claims problems. There are three differences though:

1. A claims vector c has non-negative real numbers as its entries, whereas the entries of a vote vector v are non-negative integers.
2. We have $1 \leq \sum_{j \in \mathcal{A}} v_j \leq |N|$ as at least someone votes and as at most everyone votes, but there are no corresponding restrictions on the sum of the elements of a claims vector.
3. We have $\sum_{i \in N} c_i > E$ as there is not enough 'to go around' in a claims problem, but the number of seats may exceed the sum-total of votes cast in a tallied-vote problem.

We will show that any *anonymous* seat-allocation rule that satisfies *IVR* is the proportional rule. To do so, we rely on the literature on claims problems. Firstly, we prove a lemma that shows that an anonymous seat-allocation rule f satisfies *IVR* just in case the corresponding tallied seat-allocation rule r satisfies *no advantageous transfer*. Secondly, we invoke a result that is familiar from the literature on claims problems: the proportional rule is the only division rule that satisfies *no advantageous transfer*.

Definition 11 (*No advantageous transfer*) Let $\mathcal{E} = (E, N, \mathcal{A})$ be an election. A tallied seat-allocation rule r for $\mathcal{E}$ satisfies *no advantageous transfer* iff for each $C \subseteq \mathcal{A}$ and for all vote vectors p, q for $\mathcal{E}$:

$$\text{If } \sum_{j \in C} q_j = \sum_{j \in C} p_j \text{ and } q_j = p_j \text{ for } j \notin C, \text{ then } \sum_{j \in C} r(p)_j = \sum_{j \in C} r(q)_j \tag{8}$$

So r satisfies no advantageous transfer when no coalition C can gain seats by reallocating votes amongst its members.

Here is the promised lemma.

[10] Thomson (2019) is a state-of-the-art review of the literature on claims problems. In Wintein and Heilmann (2021) we provide a review of claims problems and their relation to fairness and fair division.

Lemma 1 (*IVR*-no advantageous transfer equivalence) *Let* $\mathcal{E} = (E, N, \mathcal{A})$ *be an election. Let f be an anonymous seat-allocation rule for* $\mathcal{E}$ *and let r be the associated tallied-vote seat-allocation rule*: *f satisfies IVR iff r satisfies no advantageous transfer.*

Proof Suppose that *f* satisfies *IVR*. Let p, q be two vote vectors for $\mathcal{E}$ and let $C \subseteq \mathcal{A}$. We need to show that *r* satisfies (8). To do so, suppose that $\sum_{j\in C} p_j = \sum_{j\in C} q_j$ and that $p_j = q_j$ for $j \notin C$. Let $\mathcal{C} = \{C, \{j\} \mid j \notin C\}$ be a partition of $\mathcal{A}$. Let $\mathbf{X}$ be any profile for $\mathcal{E}(\mathcal{C})$ for which $\sum_{i\in N} \mathbf{X}_{iC} = \sum_{j\in C} p_j$ and $\sum_{i\in N} \mathbf{X}_{i\{j\}} = p_j$ for $j \notin C$. Let $\mathbf{P}$ and $\mathbf{Q}$ be any two profiles for $\mathcal{E}$ that realize $\mathbf{X}$. Then, as *f* is *IVR* on $\mathcal{C}$, we get:

$$\sum_{j\in C} f(\mathbf{P})_j = \sum_{j\in C} f(\mathbf{Q})_j \tag{9}$$

Per construction of $\mathbf{X}$, the vote vectors that are induced by $\mathbf{P}$ and $\mathbf{Q}$ are p and q respectively, so that it follows from (9) that $\sum_{j\in C} r(p)_j = \sum_{j\in C} f(q)_j$.

Suppose that *r* satisfies *no advantageous transfer*. Let $\mathcal{C}$ be an arbitrary partition of $\mathcal{A}$, let $\mathbf{X}$ be profile for $\mathcal{E}(\mathcal{C})$, and let $\mathbf{P}$ and $\mathbf{Q}$ be any two profiles for $\mathcal{E}$ that realize $\mathbf{X}$. We have to show that, for any $C \in \mathcal{C}$, Eq. (9) is satisfied. To do so, let $C \in \mathcal{C}$ and let $\overline{C} = \mathcal{A} - C$ be the complement of C in $\mathcal{A}$. Define profile $\mathbf{S}$ as follows:

$$\text{For all } i \in N, j \in C : \mathbf{S}_{ij} = \mathbf{Q}_{ij} \qquad \text{For all } i \in N, j \in \overline{C} : \mathbf{S}_{ij} = \mathbf{P}_{ij}$$

Let p, q and s be the vote vectors of, respectively, $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{S}$. Note that p, s and C satisfy the antecedent of (8) so that, by *no advantageous transfer*,

$$\sum_{j\in C} r(p)_j = \sum_{j\in C} r(s)_j \tag{10}$$

Also, q, s and $\overline{C}$ satisfy the antecedent of (8) so that:

$$\sum_{j\in \overline{C}} r(q)_j = \sum_{j\in \overline{C}} r(s)_j \tag{11}$$

It follows from *Efficiency* of *r* that, in (11), we can replace $\overline{C}$ with C, after which it follows from (10) and (11) that

$$\sum_{j\in C} r(p)_j = \sum_{j\in C} r(q)_j \tag{12}$$

And so, per definition of *r*, (12) yields the desired (9). □

Variants of the following proposition can be found at various places in the literature on claims problems.[11] For our purposes, the following presentation of the proposition due to Thomson (2019) is most suitable.

[11] For instance, Moulin (1985), Chun (1988) or Ju et al. (2007).

Proposition 3 (P from no advantageous transfer) *Let* $\mathcal{E} = (E, N, \mathcal{A})$ *be an election with* $|\mathcal{A}| \geq 3$ *and let r be a tallied seat-allocation rule for* $\mathcal{E}$. *Then r satisfies no advantageous transfer if and only if r is the proportional rule P.*

Proof See Thomson (2019: 184), for a proof that a division rule d satisfies no advantageous transfer if and only if d is the proportional rule. The proof carries over 1:1 to tallied seat-allocation rules as Thomson's proof only exploits the *Efficiency* and *No votes no seats* axiom for division rules. It does not use the fact that the sum of claims in a claims problem exceeds the estate. For sake of completeness, we write down Thomson's proof in Appendix B. □

Proposition 4 (*P from IVR and anonymity*) *Let* $\mathcal{E} = (E, N, \mathcal{A})$ *be an election with* $|\mathcal{A}| \geq 3$ *and let f be a seat-allocation rule for* $\mathcal{E}$: *f is the proportional rule P if and only if f satisfies anonymity and IVR.*

Proof Immediate from Lemma 1 and Proposition 3. □

We have thus characterized P in terms of *IVR and* anonymity. That the anonymity axiom cannot be dispensed with in this characterization is testified by a *D*ictatorship. The seat-allocation rule D^i allots all the seats to the party voted for by the dictator i, if the dictator *does* vote. If i abstains from voting, D^i allocates the seats proportionally:

$$D^i(\mathbf{P})_x = \begin{cases} E & \text{if } \mathbf{P}_{ix} = 1 \\ 0 & \text{if } \mathbf{P}_{ix} = 0 \text{ but } \mathbf{P}_{iy} = 1 \text{ for some } y \in \mathcal{A} \\ P(\mathbf{P})_x & \text{if } \mathbf{P}_{iy} = 0 \text{ for all } y \in \mathcal{A} \end{cases}$$

Now, D^i is *IVR* on any partition $\mathcal{C}$ of the alternatives of an election $\mathcal{E}$. To see this, let $\mathbf{X}$ be a profile for $\mathcal{E}(\mathcal{C})$ and let $\mathbf{P}$ and $\mathbf{Q}$ be any two profiles that realize $\mathbf{X}$. Firstly, suppose that i votes for $C' \in \mathbf{X}$. Then, for both $\mathbf{P}$ and $\mathbf{Q}$, i must vote for some alternative in C', so that:

$$\begin{cases} \sum_{j \in C'} D^i(\mathbf{P})_j = \sum_{j \in C'} D^i(\mathbf{Q})_j = E \\ \sum_{j \in C} D^i(\mathbf{P})_j = \sum_{j \in C} D^i(\mathbf{Q})_j = 0 \text{ for all } C \in \mathcal{C}, C \neq C' \end{cases}$$

Hence, (6) is satisfied. Secondly, suppose that in $\mathbf{X}$, the dictator i abstains from voting, i.e. that i does not vote for any coalition in $\mathcal{C}$. It readily follows that i must abstain from voting in both $\mathbf{P}$ and $\mathbf{Q}$ so that D^i recommends, for both $\mathbf{P}$ and $\mathbf{Q}$, to allocate the seats proportionally. Hence, as P is *IVR*, it follows that (6) is satisfied also when the dictator abstains from voting.

So D^i is *IVR* and as (the non-anonymous) D^i is clearly distinct from P, anonymity cannot be dispensed with in our characterization of P.

Proposition 4 tells us that, *for single-vote elections with at least three alternatives*, P is the only seat-allocation rule that satisfies anonymity and *IVR*. For single-vote elections with two alternatives, i.e. for *binary elections*, characterizing P in terms of *IVR* and anonymity would not do. Indeed, for binary elections, *IVR* is trivially satisfied for *any* seat-allocation rule whatsoever. For binary elections, Proposition 2 shows that seat-allocation rules which satisfy May's axioms induce the same winning coalitions as P.

4.3 The normative appeal of *IVR*

The Liberal axioms seek to provide a justification of P in terms of *liberal political equality*. Now, although anonymity expresses an important aspect of liberal political equality, it seems clear to us that *IVR* does not. As such, our characterization of P in terms of *IVR* and anonymity does not yield a justification of P in terms of liberal political equality. But the space of justifications is not exhausted by the political liberal ones. Hence, we may ask what kind of justification for P, if one at all, our characterization via *IVR* and anonymity and *IVR* yields.

IVR is more than just a condition for the well-definedness of the LP claim. Indeed, Lemma 1 explains that, for anonymous seat-allocation rules, *IVR* is equivalent to *no-advantageous transfer*. The latter condition has normative appeal in the present electoral context, as we will now demonstrate via an example. Consider profile $\mathbf{A}$ for *ELECT* and remember that a, b and c receive 8, 6 and 4 votes respectively and that the squared proportional rule recommends allocation $S(\mathbf{A}) = (4.97,\ 2.79,\ 1.24)$ for this profile. So according to S, party a receives a majority of the seats and the result of the election is that a will govern. Now suppose that after the election, b and c team up, form a coalition and complain that the result of the election is unfair. Their argument is as follows:

> The rules of the election prescribe that seats are divided proportional to squared vote totals. Now we, i.e. $\{b, c\}$, have received a total of ten votes whereas a has received eight votes. Dividing the nine seats proportional to squared vote totals results in $\frac{64}{164} \cdot 9 = 3.51$ seats for a and in $\frac{100}{164} \cdot 9 = 5.49$ seats for us. Hence, we are entitled to a majority of the seats and we should govern instead of a.

The argument advanced by $\{b, c\}$ illustrates that S violates no advantageous transfer.[12] We take it that there's something to the argument of $\{b, c\}$ against S. To our mind, the argument points to a certain inconsistency in the way that S treats coalitions in allocating seats. Also, to the desirability of anonymous seat-allocation rules that *do* satisfy *IVR*. That is, we do think that *IVR* can be invoked to justify P. But again, we do not think that this justification is to be understood in terms of liberal political equality. In what terms, exactly, the justification has to be understood then, is beyond the scope of this article.

[12] In terms of Definition 11, we can understand the situation as a reallocation by $\{b, c\}$ of b-votes to c-votes.

5 Reformulating the Liberal Proportionality claim

5.1 The LP claim for coalitional aggregates

In Sect. 3.2, we explained that the LP claim is ill-defined: owing to the multiple realisability of coalitional profiles, f typically does not induce a coalitional seat-allocation rule F. However, given a partition $\mathcal{C}$ of the parties in $\mathcal{A}$, a seat-allocation rule f *does* induce, via (13), the *coalitional aggregate* $f^{\mathcal{C}}$:

$$f^{\mathcal{C}}(\mathbf{P})_C = \sum_{i \in C} f(\mathbf{P})_i \quad \text{for all } C \in \mathcal{C} \tag{13}$$

A coalitional seat-allocation rule F and a coalitional aggregate $f^{\mathcal{C}}$ take different types of profiles as their input: F takes profiles for a coalitional election $\mathcal{E}(\mathcal{C})$ as its input and $f^{\mathcal{C}}$ profiles for a (regular) election $\mathcal{E}$. However, both coalitional seat-allocation rules and coalitional aggregates yield the same output: allocations of seats to coalitions. As such, coalitional seat-allocation rules and coalitional aggregates are intimately related. This intimate relation, then, suggests to reformulate the LP claim in terms of coalitional aggregates:

The LP claim for coalitional aggregates: If a seat-allocation rule f satisfies the Liberal axioms for coalitional aggregates, then f induces the same winning coalitional aggregates as the proportional rule P.

The Liberal axioms for coalitional aggregates, as well as the notion of *inducing the same winning coalitional aggregates as P*, are defined as follows:

The Liberal axioms for coalitional aggregates. A seat-allocation rule f for $\mathcal{E} = (E, N, \mathcal{A})$ *satisfies the Liberal axioms for coalitional aggregates* $=_{def}$ for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional aggregate $f^{\mathcal{C}}$ is anonymous, neutral and positive responsive.

Inducing the same winning coalitional aggregates as P. A seat-allocation rule f for $\mathcal{E} = (E, N, \mathcal{A})$ *induces the same winning coalitional aggrgegates as* $P =_{def}$ for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional aggregate $f^{\mathcal{C}}$ allots—on each profile for $\mathcal{E}$—a majority of seats to a coalition $C \in \mathcal{C}$ iff C receives a majority of votes.

The content of the reformulated Liberal axioms and, by extension, of the reformulated *LP* claim, thus depends on a specification of what it means for a coalitional aggregate $f^{\mathcal{C}}$ to be anonymous, neutral and positive responsive. In Sect. 5.2 we will present such a specification.

The reformulated LP claim is, in contrast to the original LP claim, meaningful and well-defined. However, as we know from Sect. 3.3, it is one thing to be meaningful and well-defined, and another to be true. Although our reformulation repairs the ill-definedness of the LP claim, ill-definedness is only one of the two flaws that we discussed in Sect. 3. The other flaw (cf. § 3.3) is that anonymity, neutrality and positive responsiveness may ensure plurality ranking, but that this is no guarantee for yielding the same winning coalitions as P. As this flaw is not addressed by our reformulation, we expect the reformulated LP claim to be false. This expectation is vindicated in Sect. 5.3.

In Sect. 5.4 we will show that our characterization of P in terms of anonymity and *IVR* can be reformulated in terms of coalitional aggregates.

5.2 May's axioms for coalitional aggregates

According to anonymity, the outcome of an election should not depend on the names, or identities, of the voters. Anonymity requires that a permutation of the names of the voters does not change the outcome. So then, in case the outcome is a coalitional aggregate, anonymity is formulated as follows:

Definition 12 (*Anonymity for coalitional aggregates*) Let f be a seat-allocation rule for an election $\mathcal{E} = (E, N, \mathcal{A})$ and let $\mathcal{C}$ be a partition of $\mathcal{A}$. We say that $f^{\mathcal{C}}$ *is anonymous* iff for any profile $\mathbf{P}$ and for any permutation σ of the voters in N, we have: $f^{\mathcal{C}}(\mathbf{P}) = f^{\mathcal{C}}(\sigma\mathbf{P})$.

According to neutrality, an electoral system should not discriminate on the basis of the names, or identities of the alternatives. Neutrality requires that a permutation π of the names of the alternatives is reflected by a corresponding permutation of the outcome. To reflect π in the outcome recorded by a coalitional aggregate $f^{\mathcal{C}}$, we will apply π to sets of alternatives $C \subseteq \mathcal{A}$, where $\pi C = \{\pi(x) \mid x \in C\}$ and extend this application to partitions $\mathcal{C}$ of $\mathcal{A}$ by letting $\pi\mathcal{C} = \{\pi C \mid C \in \mathcal{C}\}$. Using this notation, we propose to define the notion of neutrality for coalitional aggregates as follows.

Definition 13 (*Neutrality for coalitional aggregates*) Let f be a seat-allocation rule for an election $\mathcal{E} = (E, N, \mathcal{A})$ and let $\mathcal{C}$ be a partition of $\mathcal{A}$. We say that $f^{\mathcal{C}}$ *is neutral* iff for any profile $\mathbf{P}$ and for any permutation[13] of alternatives π we have: $f^{\mathcal{C}}(\pi\mathbf{P}) = f^{\pi\mathcal{C}}(\mathbf{P})$.

Positive responsiveness allows for a straightforward reformulation in terms of coalitional aggregates. It requires that a coalition C should receive more seats than a coalition C' in profile $\mathbf{Q}$, if: $\mathbf{Q}$ is obtained from $\mathbf{P}$ via an order-preserving change favouring C, and in $\mathbf{P}$, coalitions C and C' receive the same amount of seats. More precisely, positive responsiveness for coalitional aggregates is defined as follows.

Definition 14 (*Positive responsiveness for coalitional aggregates*) Let f be a seat-allocation rule for an election $\mathcal{E} = (E, N, \mathcal{A})$, let $\mathcal{C}$ be a partition of $\mathcal{A}$ and let $f^{\mathcal{C}}$ be the coalitional aggregate induced by f and $\mathcal{C}$. We say that $f^{\mathcal{C}}$ is *positive responsive* iff for all profiles $\mathbf{P}$ and $\mathbf{Q}$ for which:

(i) for all $i \in N$ and all $x \in C$: if $\mathbf{P}_{ix} = 1$ then $\mathbf{Q}_{ix} = 1$, and
(ii) for some $i \in N$ and some $x \in C$: $\mathbf{Q}_{ix} = 1$ and $\mathbf{P}_{ix} = 0$, and

[13] We say that a permutation of alternatives π *respects a partition* $\mathcal{C}$ just in case, for all $C \in \mathcal{C}$, $x \in C$ iff $\pi(x) \in C$, i.e. iff $C = \pi C$. So for permutations π that respect $\mathcal{C}$, neutrality requires that $f^{\mathcal{C}}(\pi\mathbf{P}) = f^{\mathcal{C}}(\mathbf{P})$.

(iii) for all $i \in N$: if $\mathbf{Q}_{ix} = 0$ for all $x \in C$ then, for all $y \notin C$: $\mathbf{Q}_{iy} = 1$ iff $\mathbf{P}_{iy} = 1$,

we have, for all $C' \neq C$: if $f^{\mathcal{C}}(\mathbf{P})_C = f^{\mathcal{C}}(\mathbf{P})_{C'}$ then $f^{\mathcal{C}}(\mathbf{Q})_C > f^{\mathcal{C}}(\mathbf{Q})_{C'}$.

This ends our reformulation of May's axioms for coalitional aggregates. Hence, we now have fully specified the content of *the LP claim for coalitional aggregates*, described in Sect. 5.1.

5.3 The LP claim for coalitional aggregates is false

We now go on to show that the reformulated LP claim is false. In order to do so, the following proposition establishes that the squared proportional rule S satisfies the Liberal axioms for coalition aggregates.

Proposition 5 (S satisfies the Liberal axioms for coalition aggregates) *Let* $\mathcal{E} = (E, N, \mathcal{A})$ *be an election and* $\mathcal{C}$ *a partition of* $\mathcal{A}$. *Then, the coalitional aggregate* $S^{\mathcal{C}}$ *of* S *is anonymous, neutral and positive responsive.*

Proof It follows immediately from the definitions that $S^{\mathcal{C}}$ is anonymous and neutral. To show that $S^{\mathcal{C}}$ is positive responsive, let $\mathbf{P}$ and $\mathbf{Q}$ be two profiles for $\mathcal{E}$ which satisfy condition (i), (ii) and (iii) of Definition 14.

Let $C \neq C'$ be such that $S^{\mathcal{C}}(\mathbf{P})_C = S^{\mathcal{C}}(\mathbf{P})_{C'}$, from which it readily follows that:

$$\sum_{x\in C}\left(\sum_{i\in N}\mathbf{P}_{ix}\right)^2 = \sum_{x\in C'}\left(\sum_{i\in N}\mathbf{P}_{ix}\right)^2 \tag{14}$$

From condition (i), (ii), we get:

$$\sum_{x\in C}\left(\sum_{i\in N}\mathbf{Q}_{ix}\right)^2 > \sum_{x\in C}\left(\sum_{i\in N}\mathbf{P}_{ix}\right)^2 \tag{15}$$

Now suppose that $\mathbf{Q}_{ic} = 1$ for some $c \in C'$. Then, as the intersection of C' and C is empty and as each voter can cast at most one vote, we have $\mathbf{Q}_{ix} = 0$ for all $x \in C$. It then follows from condition (iii) that for all $y \in C'$ we have $\mathbf{Q}_{iy} = 1$ iff $\mathbf{P}_{iy} = 1$ so that in particular $\mathbf{P}_{ic} = 1$. Hence, it follows that

$$\sum_{x\in C'}\left(\sum_{i\in N}\mathbf{Q}_{ix}\right)^2 \leq \sum_{x\in C'}\left(\sum_{i\in N}\mathbf{P}_{ix}\right)^2 \tag{16}$$

By combining (14), (15) and (16), we get that:

$$\sum_{x\in C}\left(\sum_{i\in N}\mathbf{Q}_{ix}\right)^2 \leq \sum_{x\in C'}\left(\sum_{i\in N}\mathbf{Q}_{ix}\right)^2, \tag{17}$$

so that $S^{\mathcal{C}}(\mathbf{Q})_C > S^{\mathcal{C}}(\mathbf{Q})_{C'}$, which is what we had to show. □

As S satisfies the Liberal axioms for coalitional aggregates it should, according to the reformulated LP claim, induce the same winning coalitional aggregates as P. However, it readily follows from an inspection of Example 3 that it does not. Hence, the Liberal claim for coalitional aggregates is false.

So, reformulating the Liberal axioms in terms of coalitional aggregates neither yields a liberal nor other justification of proportional representation. Next, we will present a characterization of P in terms of coalitional aggregates which, as we will show, is equivalent to our characterization of P in terms of *IVR* and anonymity.

5.4 *P* from vote-shuffles and anonymity

We say that profiles $\mathbf{P}$ and $\mathbf{Q}$ for an election $\mathcal{E} = (E, N, \mathcal{A})$ are related by a *vote-shuffle that respects* $\mathcal{C}$ just in case, for each voter $i \in N$:

$$\text{For all } C \in \mathcal{C}: \ \mathbf{P}_{ix} = 1 \text{ for some } x \in C \iff \mathbf{Q}_{ix} = 1 \text{ for some } x \in C$$

So $\mathbf{P}$ and $\mathbf{Q}$ are related by a vote-shuffle that respects $\mathcal{C}$ iff each voter votes for some party of a coalition $C \in \mathcal{C}$ in $\mathbf{P}$ iff she does so in $\mathbf{Q}$. So then, when $\mathbf{P}$ and $\mathbf{Q}$ are related by a vote-shuffle that respects $\mathcal{C}$, each coalition in $\mathcal{C}$ receives the same number of votes in $\mathbf{P}$ as it does in $\mathbf{Q}$, although the distribution of the votes may differ widely. To require that, for any two profiles that are related by a vote-shuffle that respects $\mathcal{C}$, the sum of seats that a coalition of $\mathcal{C}$ receives is the same, is equivalent to the following: to require that a seat-allocation rule satisfies *vote-shuffle invariance* (*VSI*) *for coalitional aggregates*.

Definition 15 (*VSI for coalitional aggregates*) Let f be a seat-allocation rule for an election $\mathcal{E} = (E, N, \mathcal{A})$ and let $\mathcal{C}$ be a partition of $\mathcal{A}$. We say that $f^{\mathcal{C}}$ is vote-shuffle invariant (*VSI*) iff $f^{\mathcal{C}}(\mathbf{P}) = f^{\mathcal{C}}(\mathbf{Q})$ for any two profiles $\mathbf{P}$ and $\mathbf{Q}$ that are related by a vote-shuffle that respects $\mathcal{C}$.

Profile $\mathbf{A}$ and $\mathbf{B}$ for *ELECT* are related by a vote-shuffle that respects $\{\{a\}, \{b, c\}\}$: the profiles only differ in the distribution of the ten votes that are cast for $\{b, c\}$ amongst b and c. As such, $\mathbf{A}$ and $\mathbf{B}$ induce the same coalitional profile, $\mathbf{C}$, for $ELECT(\{\{a\}, \{b, c\}\})$. More generally, profiles are related by a vote-shuffle just in case they realize the same coalitional profile, from which it readily follows that *VSI* is equivalent to *IVR*, as recorded by the following lemma.

Lemma 2 (*VSI-IVR*lemma) *A seat-allocation rule f for an election $\mathcal{E} = (E, N, \mathcal{A})$ is IVR iff, for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional aggregate $f^{\mathcal{C}}$ is VSI.*

Proof The lemma readily follows from the following two observations that pertain to an arbitrary partition $\mathcal{C}$ of $\mathcal{A}$:

(i) Two profiles for $\mathcal{E}$ realize the same profile for $\mathcal{E}(\mathcal{C})$ iff they are related by a vote-shuffle that respects $\mathcal{C}$.
(ii) f, $\mathbf{P}$ and $\mathbf{Q}$ satisfy Eq. (6) iff $f^{\mathcal{C}}(\mathbf{P}) = f^{\mathcal{C}}(\mathbf{Q})$. □

Lemma 2 allows us to reformulate our characterization of P in terms of *VSI* and anonymity for coalitional aggregates, as recorded by the following proposition.

Proposition 6 (P from *VSI* and anonymity for coalitional aggregates) *Let $\mathcal{E} = (E, N, \mathcal{A})$ be an election with $\mid \mathcal{A} \mid \geq 3$ and let f be a seat-allocation rule for $\mathcal{E}$: f is the proportional rule P if and only if for each partition $\mathcal{C}$ of $\mathcal{A}$, the coalitional aggregate $f^{\mathcal{C}}$ is anonymous and VSI.*

Proof It follows from Lemma 2 that f is *IVR* iff $f^{\mathcal{C}}$ is *VSI* for each partition $\mathcal{C}$. We will show that f is anonymous iff $f^{\mathcal{C}}$ is anonymous for each partition $\mathcal{C}$. Having done so, the result immediately follows from Proposition 4.

Suppose that f is anonymous so that $f(\mathbf{P}) = f(\sigma\mathbf{P})$ for any permutation σ of N. From $f(\mathbf{P}) = f(\sigma\mathbf{P})$ it follows that $f^{\mathcal{C}}(\mathbf{P}) = f^{\mathcal{C}}(\sigma\mathbf{P})$ for any partition $\mathcal{C}$ of $\mathcal{A}$. Conversely, suppose that $f^{\mathcal{C}}$ is anonymous for each partition $\mathcal{C}$. So in particular, $f^{\mathcal{C}}$ is anonymous for the singleton partition $\mathcal{S} = \{\{x\} \mid x \in \mathcal{A}\}$, meaning that $f^{\mathcal{S}}(\mathbf{P}) = f^{\mathcal{S}}(\sigma\mathbf{P})$ for any permutation σ of individuals in N from which it follows that $f(\mathbf{P}) = f(\sigma\mathbf{P})$ for any permutation σ so that f is anonymous. □

Although *VSI* is thus equivalent with *IVR*, we feel that their normative content is more clearly expressed by *VSI*. In particular, the relation of *VSI* with no advantageous transfer is more easily seen from an inspection of its definition. As such, we will not present a separate account of the normative appeal of *VSI*, but note that such an account can be developed similar to the one sketched in Sect. 4.3.

6 Concluding remarks

To summarize, we analyzed van der Hout and McGann's (2009a, b) 'LP claim' that any seat-allocation rule that satisfies certain 'Liberal axioms' produces results essentially equivalent to proportional representation. We showed that the LP claim and its proof are wanting. In Sect. 3 we explained that (i) the LP claim is ill-defined and that (ii) the argument for the claim contains a further problematic step, independent of its ill-definedness. In Sect. 4, we pointed out that the LP claim is only defined when the *IVR* condition is satisfied and showed that P is the unique anonymous seat-allocation rule that satisfies *IVR*. We pointed out that anonymity and *IVR* are conjointly equivalent to 'no advantageous transfer' and that this latter property is normatively appealing in an electoral context. The normative appeal, though, is not one of liberal political equality. In Sect. 5 we investigated plausible ways to reformulate, and rescue, the LP claim in terms of 'coalitional aggregates'. The reformulated, well-defined, LP claim turns out to be false, which was to be expected in the light of (ii).

In conclusion, the attention that the LP claim has received in the literature on normative democratic theory notwithstanding, the LP claim does not yield a justification for proportional representation from liberal axioms. Our conclusions are thus mainly negative. And yet, some of our results are novel, and we think that the general outlook is anything but clouded. Perhaps the small scope and sparse structure of May's

(1952) initial result meant that it never was the right springboard for justifying proportional representation to begin with. We are confident that the rich literature on fairness, proportionality, and claims problems in both philosophy and economics does harbour resources which can be of use for such a justification. A 'proportional response' to the well-established 'majority rule' is possible. Formulating this response, though, we have to leave for another day.

Appendices

Appendix A: on positive responsiveness$_I$

Van der Hout and McGann (2009a, b) present *different* definitions of positive responsiveness. The notion of positive responsiveness that we discussed in the body of this paper is the one used by van der Hout and McGann (2009b) and McGann (2006). In this appendix, we will present positive responsiveness$_I$, the notion used by van der Hout and McGann (2009a) and explain why, to our minds, this notion is not normatively compelling. Here is the definition of positive responsiveness$_I$.

Definition 16 (*Positive responsiveness$_I$*) A seat-allocation rule f for an election $\mathcal{E} = (E, N, \mathcal{A})$ is *positive responsive$_I$* iff for all profiles $\mathbf{P}$ and $\mathbf{Q}$ for which:

(i) for all $i \in N$: if $\mathbf{P}_{ix} = 1$ then $\mathbf{Q}_{ix} = 1$, and
(ii) for some $i \in N$: $\mathbf{Q}_{ix} = 1$ and $\mathbf{P}_{ix} = 0$,

we have: $f(\mathbf{Q})_x > f(\mathbf{P})_x$.

Now, when (i) everyone who votes for x in $\mathbf{P}$ also votes for x in $\mathbf{Q}$ whereas (ii) some vote for x in $\mathbf{Q}$ but not in $\mathbf{P}$, we say that $\mathbf{Q}$ *is obtained from* $\mathbf{P}$ *by a change favouring* x. So, when $\mathbf{Q}$ is obtained from $\mathbf{P}$ by a change favouring x, a positive responsive$_I$ f allots more seats to x in $\mathbf{Q}$ than in $\mathbf{P}$.

Although the definition of positive responsiveness$_I$ is clear enough, its conceptual underpinning is not. For, remember that in the single-vote elections under consideration, individuals can also *abstain* from voting. As such, when $\mathbf{Q}$ is obtained from $\mathbf{P}$ by a change favouring x, the additional support for x in $\mathbf{Q}$, relative to $\mathbf{P}$, may co-exist with stronger additional support in $\mathbf{Q}$ for other parties. Under these circumstances, it is not reasonable to require, as positive responsiveness$_I$ does, that x receives more seats in $\mathbf{Q}$ than in $\mathbf{P}$. For a concrete illustration of our qualms with the notion of positive responsiveness$_I$, consider the following two profiles for *ELECT*.

Profile D. Voters $1, 2, \dots, 8$ vote for party a, voter 10 votes for party b, all other voters abstain from voting.

Profile E. Voters $1, 2, \dots, 9$ vote for party a, voter $10, 11, \dots 18$ vote for party b.

Party a receives more support in **E** than in **D**. However, in **D**, where the voter turnout is only 50%, nearly all of those who do vote, vote for a. In contrast, in **E**, voter turnout is 100% with only half of the voters voting for a. It seems reasonable that party a receives more seats in **D** than in **E**, which conflicts with the requirements of positive responsiveness$_I$. Hence, positive responsiveness$_I$ is not normatively compelling.

Moreover, note that the proportional rule allots, to party a, 8 seats in **D**, and 4.5 seats in **E**. Hence, the profiles **D** and **E** testify that *the proportional rule violates positive responsiveness$_I$*. Now this is somewhat odd, as van der Hout and McGann (2009a) define the notion of positive responsiveness$_I$ with the purpose of giving an axiomatic justification of P: they seek to justify P in terms of an axiom that P does not satisfy. Although this is odd, it is not absurd. For, van der Hout and McGann only seek to define axioms whose satisfaction is *sufficient* for a seat-allocation rule to induce the same winning coalitions as P. To be sure, P trivially induces the same winning coalitions as P. However, this does not entail that P needs to satisfy axioms which constitute a sufficient—in contrast to a necessary—condition for inducing the same winning coalitions as P. Nevertheless, we do think that positive responsiveness$_I$ is a less compelling notion than the notion of positive responsiveness used by van der Hout and McGann (2009b) and McGann (2006), which is why we chose to work with the latter notion in the body of the paper.

Appendix B: proof of Proposition 3

For sake of completeness, we give the proof of Proposition 3. Our proof closely follows the proof of Thomson (2019: 184), but we elaborate on a couple of proof-steps for the convenience of the reader. There is one notable distinction between our proof and that of Thomson, though. In his proof, Thomson uses a theorem on Cauchy's functional equation, which applies to functions from $\mathbb{R}$ to $\mathbb{R}$. As Proposition 3 is concerned with tallied-vote problems rather than with claims problems, we need a similar result to the one invoked by Thomson, but which pertains to functions from $\mathbb{N}$ to $\mathbb{R}$. The result that we need is the following lemma, whose proof we present for the sake of completeness.

The Cauchy lemma.

Let $V > 0$ be an integer and let $\varphi : \{0, 1, \ldots, V\} \to \mathbb{R}$ be a function which satisfies the following (Cauchy) equation for all $x, y \in \{0, 1, \ldots, V\}$:

$$\varphi(x) + \varphi(y) = \varphi(x + y) \tag{18}$$

Then there exists a $\lambda \in \mathbb{R}$ such that:

$$\text{For all } x \in \{0, 1, \ldots, V\} : \varphi(x) = \lambda \cdot x \tag{19}$$

Proof We claim that $\lambda = \varphi(1)$ satisfies (19) and we will establish this claim by induction on $x \in \{0, 1, \ldots, V\}$.

Induction base. If $x = 0$, it follows from (18) that $2 \cdot \varphi(0) = \varphi(0)$ so that $\varphi(0) = 0$. So for $x = 0$, $\varphi(x) = \lambda \cdot x$ is satisfied for any λ whatsoever and so in particular for $\lambda = \varphi(1)$.

Induction step. Suppose that $\lambda = \varphi(1)$ satisfies (19) for some $x \in \{0, 1, \dots, V-1\}$. We may then show that $\lambda = \varphi(1)$ also satisfies (19) for $x + 1$, and hence establish our lemma, as follows:

$$\varphi(x+1) = \varphi(x) + \varphi(1) = \varphi(1) \cdot x + \varphi(1) = \varphi(1) \cdot (x+1) \tag{20}$$

The first equality in (20) follows from (18), the second from the induction hypothesis and the last from elementary algebra. □

Proposition 3 (*P from no advantageous transfer*) Let $\mathcal{E} = (E, N, \mathcal{A})$ be an election with $|\ \mathcal{A}\ | \geq 3$ and let r be a tallied seat-allocation rule for $\mathcal{E}$. Then r satisfies *no advantageous transfer* if and only if r is the proportional rule P.

Proof It is readily verified that P satisfies no advantageous transfer. We will show that any tallied seat-allocation rule r which satisfies no advantageous transfer is the proportional rule. Let V be an integer > 0 and let $(\mathcal{E}, v)$ be any tallied-vote problem such that $\sum_{j \in \mathcal{A}} v_j = V$. We first establish that the following four claims are true:

$$r(v)_1 + r(v)_2 = r(v_1 + v_2, 0, v_3, \dots, v_k)_1 \tag{21}$$

When party 1 and party 2 reallocate the votes that they receive in v, this does not affect the sum of seats received by the parties in virtue of *no advantageous transfer*. When 1 and 2 reallocate by giving all votes to 1 and none to 2, party 2 receives no seat in virtue of *no votes no seats* so that, after this reallocation, 1 get all the seats that 1 and 2 jointly received before the reallocation, which is what (21) expresses.

$$r(v)_1 = r(v_1, 0, V - v_1, 0, \dots, 0)_1, \quad r(v)_2 = r(0, v_2, V - v_2, 0, \dots, 0)_2 \tag{22}$$

When all parties in $\mathcal{A} - \{i\}$ reallocate the votes amongst them, the sum of seats they receive should remain the same in virtue of *no advantageous transfer*. In virtue of *efficiency* then, party i receives E minus the sum of seats allotted to the parties in $\mathcal{A} - \{i\}$ before and after reallocation. This is what is expressed by (22) for $i = 1, 2$ and for the reallocation in which the parties in $\mathcal{A} - \{i\}$ transfer all their votes to party 3.

$$r(v)_2 = r(0, v_2, V - v_2, 0, \dots, 0)_2 = r(v_2, 0, V - v_2, 0, \dots, 0)_1 \tag{23}$$

The left-most equality of (23) follows from (22). As for the right-most equality of (23): the two vote vectors are related to one another by a reallocation of votes by party 1 and 2 so the sum of seats they receive on the basis of these vote vectors is the same in virtue of *no advantageous transfer*. For the vote vector displayed on the left-hand side of the equality, party 1 gets no votes so that, by *no votes no seats*, party 2 gets the full sum of seats. Similarly, for the vote vector displayed on the right-hand side of the equality, it is party 1 who gets the full sum. So party 1 and party 2 indeed get the same amount of seats for the vote vectors of the right-most equality of (23).

$$r(v_1 + v_2, 0, v_3, \dots, v_k)_1 = r(v_1 + v_2, 0, V - v_1 - v_2, 0, \dots, 0)_1 \tag{24}$$

The truth of (24) is established similar to that of (22).

We define the function $\varphi : \{0, \dots V\} \mapsto \mathbb{R}$, by letting:

$$\varphi(t) := r(t, 0, V - t, 0, \dots, 0)_1$$

From the definition of φ, (22) and (23) we get:

$$r(v)_1 = \varphi(v_1), \qquad r(v)_2 = \varphi(v_2) \tag{25}$$

From the definition of φ and (24) we get:

$$r(v_1 + v_2, 0, v_3, \dots, v_k)_1 = \varphi(v_1 + v_2) \tag{26}$$

From (25), (26) and (21) we get:

$$\varphi(v_1) + \varphi(v_2) = \varphi(v_1 + v_2) \tag{27}$$

Remember that V is fixed and that we are considering an *arbitrary* tallied-vote problem $(\mathcal{E}, v)$ which respects the constraint that $\sum_{j \in \mathcal{A}} v_j = V$. So then, it follows from (27) that we have effectively established that for all $x, y \in \{0, 1, \dots, V\}$ we have:

$$\varphi(x) + \varphi(y) = \varphi(x + y) \tag{28}$$

It follows from (28) and the Cauchy lemma that there is a $\lambda \in \mathbb{R}$ such that for each $x \in \{0, \dots, V\}$ we have:

$$\varphi(x) = \lambda \cdot x \tag{29}$$

Following proof-steps similar used to those for establishing (25), we get:

$$\text{For all } i \in \mathcal{A} : r(v)_i = \varphi(v_i) \tag{30}$$

It follows from (29) and (30) that

$$\text{For all } i \in \mathcal{A} : r(v)_i = \lambda \cdot v_i \tag{31}$$

It follows from (31) and the *efficiency* of r that:

$$\lambda = \frac{E}{\sum_{i \in \mathcal{A}} v_i} \tag{32}$$

So that it follows from (31) and (32) that r is P. □

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