Social Choice and Welfare (2021) 57:181–195
https://doi.org/10.1007/s00355-021-01316-z

ORIGINAL PAPER

Dominance in spatial voting with imprecise ideals

Mathieu Martin[1] · **Zéphirin Nganmeni**[2] · **Craig A. Tovey**[3]

Received: 11 September 2019 / Accepted: 9 February 2021 / Published online: 5 March 2021

Abstract
We introduce a dominance relationship in spatial voting with Euclidean preferences, by treating voter ideal points as balls of radius δ. Values $\delta > 0$ model imprecision or ambiguity as to voter preferences from the perspective of a social planner. The winning coalitions may be any consistent monotonic collection of voter subsets. We characterize the minimum value of δ for which the δ-core, the set of undominated points, is nonempty. In the case of simple majority voting, the core is the yolk center and δ is the yolk radius.

1 Introduction

A classical problem in social choice theory is the existence of a core in voting games. The core is the set of undominated alternatives, those for which there is no winning coalition, that is to say a group of individuals which can enforce a decision, that wants to replace them by other alternatives. A non-empty core assures the stability and predictability of the collective choice. In this paper, we consider spatial voting, where the alternatives are the points in real m-dimensional space, and each voter has an ideal point in that space. In the standard spatial model individual preferences are Euclidean and winning coalitions are defined by majority rule. Voters with ideal points $\{q^i\}$ have Euclidean preferences if i strictly prefers alternative x to alternative y when q^i is closer to x than to y. According to majority rule for n voters, the winning coalitions are all subsets of more than $\frac{n}{2}$ voters.

This research has been developed within the center of excellence MME-DII (ANR-11-LBX-0023-01). Research supported by National Science Foundation Grant # 1335301.

✉ Mathieu Martin
mathieu.martin@cyu.fr

1 THEMA (UMR 8184), Cergy Paris Université, Cergy, France

2 Laboratoire d’Economie Dionysien (EA 3391), Paris 8 University, Saint-Denis, France

3 School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, USA

For all dimensions $m \geq 2$, the standard spatial model rarely admits of a nonempty core. For n odd, (Plott 1967) and (Davis et al. 1972) prove that under majority rule, the core is nonempty if and only if all the median hyperplanes have a common point of intersection. It follows that the set of configurations of ideal points for which the core is not empty has measure zero. For n even, results by (Rubinstein 1979), (Banks 1995), (Banks et al. 2006), and (Saari 2014) imply the same measure zero phenomenon for all dimensions $m \geq 3$. For n even and dimension $m = 2$, the set of configurations with nonempty core has strictly positive measure[1]. Nonetheless, the probability of a nonempty core is less than $\frac{2\sqrt{\pi n}}{e^{\frac{1}{2}(n-\frac{1}{3n})}}$ (Tovey 2010b) which converges rapidly to zero as n increases.

There are many results about the existence or non-existence of the core of more general spatial voting games (for an interesting review of literature, see (Miller 2015)). One line of research considers preferences more general than Euclidean such as convex quadratic or smooth convex (Banks 1995; Schofield 1983; Grandmont 1978). Another line of research considers more general forms of the set of winning coalitions. Supermajority voting is a generalization of majority voting, in which an incumbent alternative can only be defeated by a vote of at least a fraction $\alpha > \frac{1}{2}$ of the voters. (Schofield and Tovey 1992) show that a core exists with probability converging to 1 for $\alpha = \frac{1}{2} + \sqrt{\frac{m \log d}{n}}$ and $\alpha = 1 - \frac{1}{e}$, respectively, when n ideal points are sampled from a centered distribution and a concave distribution, respectively. (Greenberg 1979) (among many others) analyzes the more general quota voting, where each voter v has positive weight w_v and a set S of voters is winning if and only if the sum of their weights $\sum_{v \in S} w_v$ is at least the quota requirement q. (Saari 2014) proves that in quota voting the core is the intersection of convex hulls of the ideal points of every minimal winning coalition. Even more generally, for arbitrary monotonic sets of winning coalitions, (Nakamura 1979) shows that the core of a voting game is non-empty if and only if the size of the space m is less or equal to the Nakamura number, which is the minimum number of winning coalitions with an empty intersection.

Many solution concepts for the spatial model in $m \geq 2$ dimensions have been proposed in order to provide predictive and/or prescriptive guidance when the core is empty. We have the minmax or Simpson-Kramer point, the point against which the maximum possible coalition is minimal (Kramer 1977); the strong point or Copeland winner, the point dominated by the least m-dimensional volume of points (Owen 1990), the uncovered set, those points not defeated in either one or two steps by another point (Miller 1980); the yolk[2], a smallest ball that intersects all median hyperplanes (McKelvey 1986; Ferejohn et al. 1984); the finagle point, a point from which the least distance must be traveled to dominate

[1] Place $n-1$ ideal points at the vertices of a regular $(n-1)$-polygon, and place the nth ideal point at the polygon center. Then for any small perturbation of the points, the nth point remains undominated.

[2] Stricly speaking, the yolk is not a solution concept since it was built as a tool to enclose the uncovered set. However, following Owen (Owen 1990), we interpret the yolk as a near-core concept.

any other point (Wuffle et al. 1989). Other solution concepts include the epsilon-core (Shubik and Wooders 1983; Eban and Stephen 1990; Tovey 2010a; Tovey 1991; Bräuninger 2007), the heart (Schofield 1995), and the soul (Austen-Smith 1996).

All of these solution concepts coincide with the core when the core is nonempty. As a rule, solution concepts are clearly motivated by a plausible rationale. The minmax and strong points are those points that are most like a core point, in terms of voter opposition and dominating alternatives, respectively. The finagle point and epsilon core are those most like a core point, in terms of ambiguity of the alternative's location, and voter resistance to change, respectively. The uncovered set is the set of possible outcomes of one-step lookahead strategic sequential majority voting. The yolk is an interesting exception to this rule. McKelvey invented the yolk to get a handle on the uncovered set, and proved that the former contains the latter, if the radius is inflated by a factor of four (McKelvey 1986).

In this paper we propose a new solution concept for spatial voting with Euclidean preferences and arbitrary monotonic winning coalitions, motivated by imprecision or ambiguity of voter ideal point locations. We treat ideal points as balls of radius $\delta \geq 0$ and derive a δ-dominance relationship from the viewpoint of a prudent mechanism designer or social planner. We characterize the minimal value of δ for which there is an undominated point, the δ-core, using a mathematical approach like in (Saari 2014) and (Martin and Nganmeni 2016). We then focus on the basic case of majority rule. We show that in this case the δ-core is the yolk center and the minimal δ is the yolk radius.

The paper is organized as follows. Section 2 formally defines the spatial voting game and associated basic terminology. Section 3 motivates and defines the δ-dominance relationship and δ-core. It also compares them with the ϵ-core and its implicit dominance relationship. Section 4 characterizes the threshold value of δ for non-emptiness of the δ-core, and shows its correspondence to the yolk in the case of majority rule. All proofs are deferred to the Appendix.

2 Definitions and notation

A spatial voting model V in m dimensions may be written $V = (N, W, \{q^i\}_{i \in N})$ where $N = \{1, ..., n\}$ is a finite set of individuals (voters), $W \subset 2^N$ is the set of winning coalitions (that is, the set of groups of individuals that can enforce a decision), and $q^i \in \mathbb{R}^m$ is the ideal point of individual i. The set of winning coalitions must satisfy these conditions:

1) $\emptyset \notin W; N \in W$ (nonvacuous and nonempty)
2) $S \in W \Rightarrow N \setminus S \notin W$ (consistent)
3) $\forall S, T \in 2^N : S \subset T, S \in W \Rightarrow T \in W$ (monotonic).

As stated previously, V employs *quota voting* if there exist positive weights $w_i : i \in N$ and a quota q such that $S \in W \Leftrightarrow \sum_{i \in S} w_i \geq q$, such a game is noted $V = [q; w_1, ..., w_n]$. Supermajority voting with $\frac{1}{2} < \alpha \leq 1$ is the special case $w_i = 1$ $\forall i$, $q = \lceil \alpha n \rceil$. Majority rule is the special case $w_i = 1 \, \forall i$, $q = \lceil \frac{n+1}{2} \rceil$.

The set of alternatives is $\mathbb{R}^m$. Denote by $d(x, y)$, the Euclidean distance between points x and y in $\mathbb{R}^m$. For any set of points $\mathcal{F} \subset \mathbb{R}^m$, let $Conv(\mathcal{F})$ denote the convex hull of $\mathcal{F}$. To simplify notation, for a set of voters $S \subseteq N$, we also let $Conv(S)$ denote the convex hull of their ideal points, which otherwise would entail the cumbersome notation $Conv(\cup_{i \in S} q^i)$.

According to Euclidean preferences, voter $i \in N$ is indifferent between alternatives x and y when $d(q^i, x) = d(q^i, y)$, and strictly prefers x to y when $d(q^i, x) < d(q^i, y)$. Alternative y is *dominated by alternative x via coalition S*, denoted $y \prec_S x$, if S is a winning coalition and all individuals in S strictly prefer x to y:

1) $S \in W$
2) $\forall i \in S, d(q^i, x) < d(q^i, y)$

The point $y \in \mathbb{R}^m$ is *dominated by* x, denoted $y \prec x$, if $y \prec_S x$ for some (winning) coalition S. The point y is *undominated* if it is not dominated by any x. The core of a spatial voting game, denoted $C(V)$, is the set of alternatives that are undominated.

A hyperplane $H = \{x \in \mathbb{R}^m : c \cdot x = c_0\}$ is *median* if both of the closed halfspaces it defines contain the ideal points of at least $\frac{n}{2}$ voters. That is, H is a median hyperplane if $|\{i \in N : c \cdot q^i \leq c_0\}| \geq \frac{n}{2}$ and $|\{i \in N : c \cdot q^i \geq c_0\}| \geq \frac{n}{2}$. The *yolk* is a smallest ball with center c and radius r, that intersects all median hyperplanes. Equivalently, following (McKelvey and Tovey 2010), define the *yolk radius* $r(x)$ of alternative x as the minimum value such that $B(x, r(x)) \cap H \neq \emptyset$ for all median hyperplanes H. Then the yolk is a ball with center $c = \arg\min r(x)$ and radius $r = r(c)$.

To define the ϵ-core, one posits a change in the voters' behavior: the Euclidean preferences are replaced by a new preference relation which we call the ϵ-preference. Formally, voter i strictly ϵ-prefers an alternative x to an alternative y if $d(x, q^i) < d(y, q^i) - \epsilon$ and i is ϵ-indifferent between the alternatives if $|d(x, q^i) - d(y, q^i)| \leq \epsilon$. Alternative x is ϵ-dominated if there are an alternative y and a winning coalition S such that every member of S, strictly ϵ-prefers y to x. The ϵ-core of a spatial voting game is the set of alternatives which are not ϵ-dominated.

3 The δ-core as a new core concept

The idea of our proposed δ-domination solution concept is to treat ideal points as balls of radius δ. The motivation for doing so comes from two assumptions:

Assumption 1 The social choice is made by a social planner who cannot determine individual ideal points with perfect accuracy and precision. Inaccuracy may be ascribed to imperfect preference elicitation, which is notoriously inexact with respect to multiple objectives. For example, (Borcherding et al. 1991) report that two-thirds of their test subjects were inconsistent. Reasons such as subject fatigue, framing bias, and time variation have been proposed to explain inconsistencies (Weber and Borcherding 1993; Eisenführ et al. 2009). Imprecision may be ascribed to ambiguity or uncertainty as to a voter's placement. As Serra (Serra 2009) argues, individuals may often be uncertain themselves about their own placement. After all, the concept of an ideal point is itself an idealization, a mathematical model much simpler and more tractable than the underlying reality.

Thus, we suppose that the social planner can only locate each ideal point with a margin of error that is a positive real number δ. This assumption induces an imperfect perception of the voter's ideal point as a ball of radius δ. We call $B(q^i, \delta)$ the *imprecise ideals locale* of voter i. This is similar but not equivalent to the use of intervals rather than single numbers for preference weights (Steuer 1976).

Assumption 2 The social planner makes changes to the status quo cautiously; the planner replaces the status quo y by x only when certain y is dominated by x whatever the locations of the voters in their imprecise ideal locales.

For any spatial voting game $V = (N, W, \{q^i\}_{i\in N})$ and any real positive number δ, the pair (V, δ) is the game associated with V, in which the voters have imprecise ideal points: the ideal point of the voter i can be any point in $B(q^i, \delta)$. The convex hull of the imprecise ideals locales of the voters belonging to a coalition S is denoted $Conv(S, \delta)$. We can introduce a new domination relation called the δ-domination, which is a direct generalization of the classical one.

We say that $y \in \mathbb{R}^m$ is δ-dominated by $x \in \mathbb{R}^m$ via a coalition S, denoted $y \prec_{S,\delta} x$ if:

1) $S \in W$
2) $\forall i \in S, \forall z \in B(q^i, \delta), d(x, z) < d(y, z)$

As said in *Assumption* 1, a prudent social mechanism designer, knowing that voters flip-flop on some issues, allows policy change from x to y only if the voters in the winning coalition would consistently prefer y to x. We say that $y \in \mathbb{R}^m$ is $\delta-$dominated by $x \in \mathbb{R}^m$, denoted $y \prec_{\delta} x$, if there exists $S \in W$ such that $y \prec_{S,\delta} x$. The δ-core of a spatial voting game, denoted $C(V, \delta)$, is the subset of $\mathbb{R}^m$ of δ-undominated alternatives.

3.1 Individual preferences

Figure 1a illustrates the usual case for $m = 2$ dimensions in which the individual preferences of a voter i are entirely determined by her/his single ideal point q^i. Voter i is indifferent between a given alternative x and another alternative y such that $d(y, q^i) = d(x, q^i)$. These points y form the circle of center q^i and radius $d(x, q^i)$. Any alternative l such that $d(l, q^i) < d(x, q^i)$, is strictly preferred to x, while x is strictly preferred to any alternative z such that $d(z, q^i) > d(x, q^i)$.

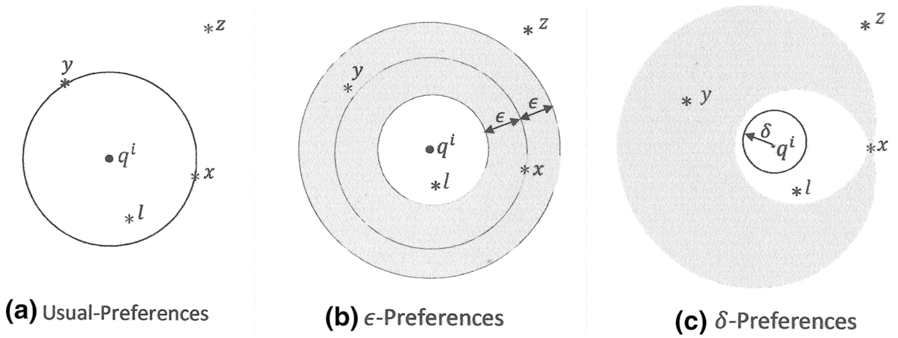

Fig. 1 Comparison of individual preferences

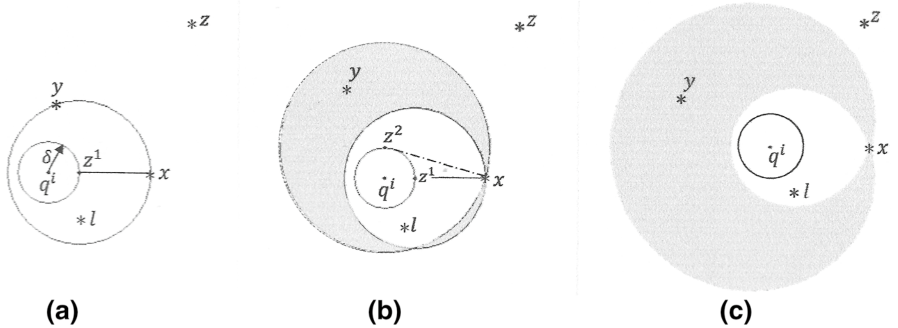

Fig. 2 Determination of individual δ-Preferences

Figure 1b shows the case that leads to the ϵ-core. The individual is indifferent between two alternatives x and y belonging to the dark area since $\left|d(y,q^i) - d(x,q^i)\right| \leq \epsilon$. S/he strictly prefers l to x since $d(x,q^i) - d(l,q^i) > \epsilon$. Finally, s/he prefers x to any alternative z outside the dark area since $\epsilon < d(z,q^i) - d(x,q^i)$.

Figure 1c corresponds to the case that leads to the δ-core. The social planner, unable to observe the ideal points perfectly, is unsure as to which points voter i prefers to x. We illustrate the uncertainty in Fig. 2. In Fig. 2a, the disc with center q^i and radius δ represents the imprecise ideals locale of the voter i. If z^1 were the ideal point of voter i they would prefer any point l in the open disc of center z^1 and radius $d(x,z^1)$ to x. But if z^2 were the ideal point of voter i, Fig. 2b shows that they would prefer any point l in the open disc (with dashed border) of center z^2 and radius $d(x,z^2)$ to x. Now, if we know that the ideal point of voter i is an element of the set $\{z^1,z^2\}$ then, with respect to our approach, any point l belonging to the intersection of these two discs is better than x for individual i. Within the dark region of Fig. 2b, voter i is indifferent between x and y since for the point z^1, we have $d(x,z^1) < d(y,z^1)$ but for the point z^2 we have $d(y,z^2) < d(x,z^2)$.

Extending this reasoning to all the points of the imprecise ideals locale, we obtain Fig. 2c, which corresponds to Fig. 1c. Individual i prefers x to any

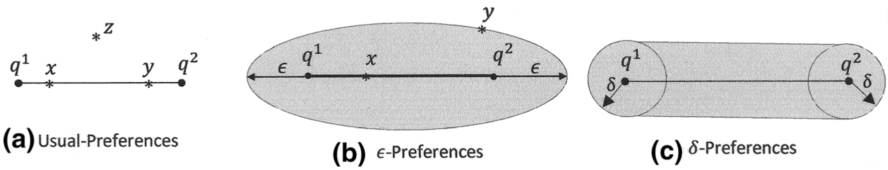

Fig. 3 Comparison of collective preferences, case of two voters

alternative z located on the outer white region and prefers any alternative l located on the inner white region to x. Individual i is indifferent (simply denoted $\sim$) between the point x and any point located on the dark region. The dark region comprises all the points to which i is indifferent compared to x.

The indifference relation $\sim$ is reflexive, that is, $x \sim x$ for all x. However, the relation is not transitive. (In two dimensions, let the imprecise ideal locale be $\{x : d(x,0) \leq 1\}$. Then $(1,0) \sim (1,2)$ because $d((1,0),(0,1)) = d((1,2),(0,1))$. Also, $(1,2) \sim (\sqrt{5},0)$ because the two points are equidistant from (0, 0). However, individual i strictly prefers (1, 0) to $(\sqrt{5},0)$, contradicting transitivity of the $\sim$ relation.)

3.2 Collective preferences

Given a coalition $S \subseteq N$, we want to characterize the set of alternatives that are dominated via S and those that are not dominated via S. To do this, we will give a result that works with the δ-preferences. Clearly, $\delta = 0$ corresponds to the usual case, that is why this result remains valid for classical preferences.

Proposition 1 *Let (V,δ) be a spatial voting game with δ the radius of the imprecise ideals locale. An alternative $z \in \mathbb{R}^m$ is not δ-dominated via $S \subseteq N$ if and only if $z \in Conv(S,\delta)$.*

Proof All the proofs are in the appendix. □

The set of δ-Pareto-optimal alternatives for a coalition S comprises those alternatives that are not δ-dominated via S. For $\delta = 0$, we have $Conv(S,\delta) = Conv(S)$ and this proposition means that the set of Pareto-optimal alternatives for the members of a coalition S is the convex hull of the ideal points of the voters belonging to S. Fig. 3a, illustrates the situation with two ideal points. The convex hull corresponds to the straight line segment defined by these two points. From a collective point of view, the two voters are indifferent between two alternatives x and y belonging to the segment whereas any alternative z outside the segment is dominated.

Consider Fig. 3b, we can geometrically show that y is on the border of the set of undominated alternatives if and only if there is x belonging to the segment$[q^1,q^2]$ such that $d(x,q^i) = d(y,q^i) - \epsilon,\ i = 1,2$. We have $d(x,q^1) + d(x,q^2) = d(q^1,q^2)$, which is a constant, say D. Thus $d(y,q^1) - \epsilon + d(y,q^2) - \epsilon = D$ and $d(y,q^1) + d(y,q^2) = D + 2\epsilon$ which means that the undominated area is an ellipse with foci q^1 and q^2. We obtain Fig. 3b where any point in the grey area is undominated via the coalition $\{1, 2\}$.

When $\delta > 0$, the set of δ-Pareto-optimal alternatives for a coalition S is the convex hull of the imprecise ideal locales on S. $Conv(S, \delta)$ is geometrically deduced from $Conv(S)$ by the enlargement of the boundaries over a distance δ. It corresponds to Fig. 3c.

Observe that when a coalition S contains more than two voters, it is not simple to characterize the set of alternatives that are not ϵ-dominated via S.

The next basic result is a direct consequence of Proposition 1.

Corollary 1 *For every spatial voting game* $V \equiv \left(N, W, \{q^i\}_{i\in N}, \delta\right)$, *we have*: $C(V, \delta) = \bigcap_{S\in W} Conv(S, \delta)$.

3.3 Non-emptiness of the δ-core

In this section we are looking for the minimum value of δ that ensures a non-empty δ-core. The essential idea turns out to be one employed previously by the authors (Martin and Nganmeni 2016) to demonstrate the non-uniqueness of the yolk in $m \geq 3$ dimensions. The yolk may be characterized as a smallest ball that intersects the convex hull of every subset of at least half of the ideal points. The relevant subsets are precisely the minimal winning coalitions under majority rule. (The non-minimal subsets are obviously irrelevant.) We seek to extend from majority rule to voting rules in general. Accordingly, we define the generalized yolk (g-yolk) as follows.

Definition 1 A generalized yolk, or g-yolk, is a smallest ball that meets the convex hull of every minimal winning coalition.

The non-emptiness of the δ-core is then characterized as follows:

Theorem 1 *Let* (V, δ) *be a spatial voting game with imprecise ideals locales of radius* δ.

Then the δ*-core is nonempty*, $C(V, \delta) \neq \emptyset$, *if and only if* $\delta \geq r$, *where* r *is the radius of a g-yolk.*

The following example illustrates the applicability of Definition 1 and Theorem 1 to simple majority voting and to other voting rules.

Example 1 Consider 5 ideal points $q^1 = (1, 3)$; $q^2 = (2, 0)$; $q^3 = (0, 2)$; $q^4 = (2, 2)$ and $q^5 = O = (0, 0)$, Fig. 4 depicts the g-yolks for five points in $\mathbb{R}^2$ with respect to three different sets of minimal winning coalitions.

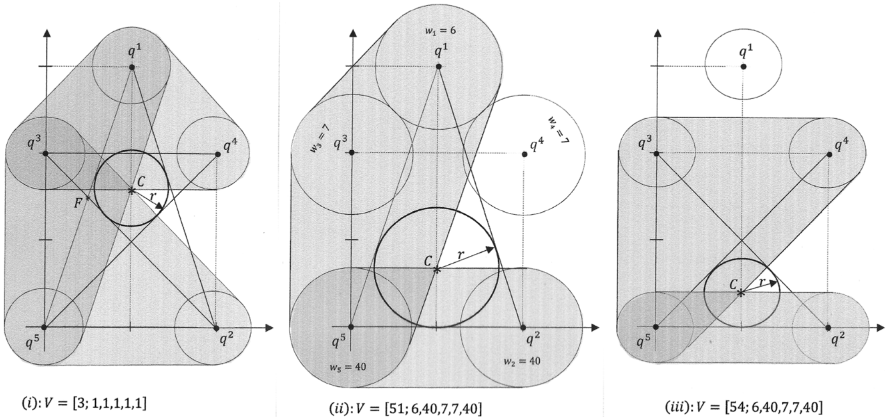

Fig. 4 The center C and radius r of the g-yolk depend on the set of winning coalitions

Let $c = (c_1, c_2)$ and r denote the coordinates and radius, respectively, of C. Fig. 4-(i) depicts the case of simple majority voting. The coordinates and radius are $c = (1, \frac{1+\sqrt{5}}{2})$ and $r = \frac{\sqrt{10}-\sqrt{2}}{4}$. Appendix B contains all computations.

Figure 4-(ii) depicts a case of weighted voting, wherein each voter has a weight, as given in Fig. 4. A coalition is winning if the sum of the weights of its members equals or exceeds a threshold value of 51. The coordinates and radius are $c = (1, \frac{\sqrt{10}-1}{3})$ and $r = \frac{\sqrt{10}-1}{3}$. All computations are in Appendix B.

Figure 4-(iii) depicts another case of weighted voting, with the same weights as in 4-(ii), but with higher threshold 54. The coordinates and radius are $c = (1, r)$ and $r = \sqrt{2} - 1$. This example illustrates how, depending on the winning coalitions, the g-yolk can differ and thus the delta-core may differ as well.

Note that the core is non-empty thanks to the degree of imprecision in the choice of the ideal points. In other words, the lack of information implies more stability in the collective choice. From a geometrical point of view, it is intuitive: without information on the preferences, the radius of the ideal points tends to infinity and then the intersection of the convex hulls cited before is of course non-empty. The particularity of our result is to propose a minimal degree of imprecision which guarantees a non-empty core.

Corollary 2 *Let (V, δ) be a spatial voting game for which the g-yolk is unique and has radius r. Then for $\delta = r$ the δ-core is the singleton point $\{C\}$, the center of the g-yolk. If there are multiple g-yolks (with common radius r) then for $\delta = r$ the δ-core is the set of centers of the g-yolks.*

Focusing now on majority rule, we restate Corollary 2 as our promised rationale for the yolk.

Corollary 3 *In the standard spatial model with Euclidean preferences and majority rule, the set of yolk centers is the minimal nonempty δ-core and the yolk radius equals the threshold value of δ.*

4 Conclusion

Several final observations can be made. Firstly, it may seem counter-intuitive that having less information (i.e., $\delta > 0$) yields a better social choice outcome. In optimization, better information can only increase productivity, theoretically speaking. In this paper, however, the goal is stability, not optimization. Secondly, our δ-dominance relation elevates the yolk center as a meaningful solution concept in its own right, rather than as the center of a bound on the uncovered set. It also generalizes the yolk to Euclidean preferences and arbitrary monotonic winning coalitions. Thirdly, to further enhance the theoretical properties of the yolk, it would be nice to have a dynamical justification of the yolk, a proof that the natural forces of majority voting tend to drive the group decision toward it. A dynamic justification would complement the essentially normative nature of our δ-dominance justification. Many other well-known solution concepts possess a dynamical justification. For example, (Kramer 1977) showed that repeated proposals by competing vote-maximizing parties will produce sequences converging to the minmax (Simpson-Kramer) set. (Miller 1980) showed that for many agenda rules, the outcome of strategic voting will be in the uncovered set. (Tovey 1991) showed that if $\epsilon > 2r$ any sequence of ϵ-voting from arbitrary starting point x will reach the ϵ-core in at most $\frac{|||x-c||-r|}{\epsilon-2r}$ votes. (Ferejohn et al. 1984) gave a partial result for the yolk: if proposals are made at random with majority voting, then the process never terminates, but the incumbent proposal will be in the yolk with frequency more than $\frac{1}{2}$ or even $\frac{2}{3}$ for some parameter values. It seems likely to us that the addition of some "stickiness"from $\delta > 0$, incorporated into Ferejohn *et al.*'s analysis, would yield an upper bound on the expected number of votes until reaching and remaining in the yolk.

Last, if *Assumption 1* (imprecision in the location of ideal points) is accepted, it affects all solution concepts, whether Either the voters themselves have imprecise knowledge, or the social choice mechanism cannot elicit their preferences exactly, or both. The effects depend heavily on what is chosen for *Assumption 2*. We see two opposite alternatives. The first is a conservative social planner or mechanism that is cautious to change the status quo, as in this paper. The second is an optimistic or charismatic candidate at x, who believes s/he can defeat y if there is a winning coalition all of whose members have at least one point in their imprecise ideal locale that favors x over y. The first leads to greater stability while the second leads to less stability, because points become harder and easier to dislodge, respectively. The winsets shrink in the first case, and expand in the second case. Thus the δ-core may be empty, and the strong point, Simpson-Kramer point, and uncovered set may change. Finagling becomes more difficult in the first case and easier in the second case. We leave these issues to further study.

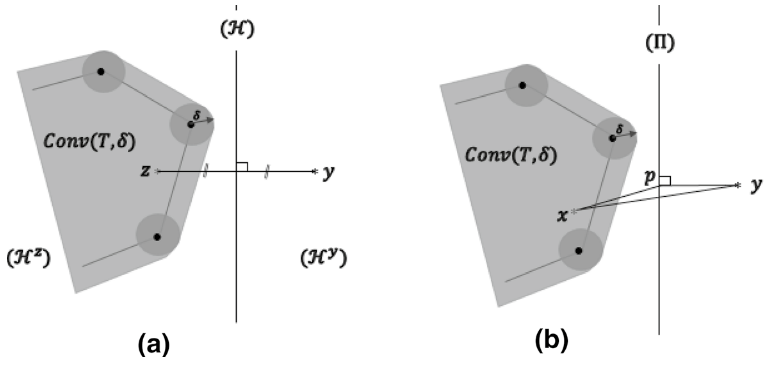

Fig. 5 δ-dominated positions

Appendix

A- Proofs

Throughout this section, $\left(N, W, \{q^i\}_{i\in N}, \delta\right) \equiv (V, \delta)$ is a m–dimensional spatial voting game where $B(q^i, \delta)$ is the imprecise ideals locale of the voter $i \in N$.

Proof of Proposition 1

Suppose y is δ-dominated by some point z with respect to a coalition $T \in W$. Let $\mathcal{H} = \{x : ||x - z|| = ||x - y||\}$ be the hyperplane orthogonal to and bisecting segment $\left[y, z\right]$, Fig. 5a. Let $\mathcal{H}^z$ (respectively $\mathcal{H}^y$) be the open halfspace defined by $\mathcal{H}$ that contains z (respectively y). By definition of δ-dominance, $\forall i \in T$, $B(q^i, \delta) \subset \mathcal{H}^z$. Hence the set $\mathcal{S} \equiv \bigcup_{i\in T} B(q^i, \delta) \subset \mathcal{H}^z$. Since $\mathcal{H}^z$ is convex, the convex hull $Conv(T, \delta) \subset \mathcal{H}^z$ as well. Then $y \in \mathcal{H}^y \Rightarrow y \notin \mathcal{H}^z \Rightarrow y \notin Conv(T, \delta)$.

Conversely, suppose $y \notin Conv(T, \delta)$. The set $\mathcal{S}$ is a finite union of compact sets and therefore is compact. It is known from Caratheodory's Theorem (see e.g. Theorem 11.1.8.6 in (Berger 1978)) that in $\mathbb{R}^m$ the convex hull of any compact set is compact. Hence the convex hull $Conv(T, \delta)$ is compact. By (Rockafellar 1970, Corollary 11.4.2 p. 99), there exists a hyperplane $\Pi = \{x : \pi \cdot x = \pi_0\}$ strictly separating $\{y\}$ from $Conv(T, \delta)$, such that $\pi \cdot y < \pi_0$ and $\pi \cdot x > \pi_0$, $\forall x \in Conv(T, \delta)$, Fig. 5-(b).

Let p be the projection of y onto Π, so $\pi \cdot p = \pi_0$ and $p = y + \alpha\pi$ for some scalar $\alpha > 0$.

Geometrically one can see that p is closer than y to every point in $Conv(T, \delta)$ because $p \in \Pi$ and y is on the other side of Π from $Conv(T, \delta)$. Algebraically, for all $x \in Conv(T, \delta)$,

$$\begin{aligned}||p-x||^2 &= (p-x)\cdot(y+\alpha\pi-x)\\ &= \alpha\pi\cdot p-\alpha\pi\cdot x+p\cdot y+||x||^2-x\cdot p-x\cdot y\\ &< \alpha\pi_0-\alpha\pi_0+p\cdot y+||x||^2-x\cdot p-x\cdot y\\ &< \alpha\pi\cdot x-\alpha\pi\cdot y+p\cdot y+||x||^2-x\cdot p-x\cdot y\\ &= (x-y)\cdot(x+\alpha\pi-p)=||y-x||^2.\end{aligned}$$

Therefore, p δ-dominates y with respect to coalition T, which completes the proof. □

Proof of Corollary 1

The proof of the Corollary follows immediately from Proposition 1 and the definition of δ-core. □

Proof of Corollary 2

Consider the function ρ defined from $\mathbb{R}^m$ to $\mathbb{R}$ by: $\forall z\in\mathbb{R}^m$, $\rho(z)=\max_{S\in W} d(z, Conv(S))$. Finding a minimum for the function ρ may be restricted to $Conv(N)$, the convex hull of $\left\{q^i\right\}_{i\in N}$. Indeed, if $z\notin Conv(N)$, then there is a point l such that for all $S\in W$, $d(l, Conv(S)) < d(z, Conv(S))$. However for all $S\in W$, $d(l, Conv(S)) < d(z, Conv(S))$ implies that $\max_{S\in W} d(l, Conv(S)) < \max_{S\in W} d(z, Conv(S))$, i.e. $\rho(l) < \rho(z)$.

The function ρ is continuous as it is the maximum of finitely many continuous functions. By Caratheodory's Theorem $Conv(N)$ is compact. It follows that ρ attains its minimum at a point z^*. Let $\delta^*=\rho(z^*)=\min_Z\left(\max_S d(z, Conv(S))\right)$. By definition, δ^* coincides with r the radius of a g-yolk.

$$\begin{aligned} Moreover, \delta^*=\rho(z^*)=\max_{S\in W} d(z^*, Conv(S)) &\iff \forall S\in W, d(z^*, Conv(S))\le\delta^*\\ &\iff \forall S\in W, z^*\in Conv(S,\delta^*)\\ &\iff z^*\in\bigcap_{S\in W} Conv(S,\delta^*)=C(V,\delta^*)\end{aligned}\tag{1}$$

Therefore, $C(V,\delta^*)\neq\emptyset$. Furthermore, if $\delta\ge\delta^*$ then $C(V,\delta^*)\subseteq C(V,\delta)$ i.e. $C(V,\delta)\neq\emptyset$.

Now, for a $\delta\ge 0$, assume that $C(V,\delta)\neq\emptyset$. Then there exists $x\in C(V,\delta)=\bigcap_{S\in W} Conv(S,\delta)$, such that for all $S\in W$, $d(x, Conv(S))\le\delta$. It follows that $\rho(x)=\max_{S\in W} d(x, Conv(S))\le\delta$. We already know that $\delta^*\le\rho(x)$. Then $\delta^*\le\delta$ i.e. $C(V,\delta)\neq\emptyset\Rightarrow\delta\ge\delta^*$. ■

Proof of Corollary 3

According to the proof of Corollary 2, $z\in C(V,\delta^*)=C(V,r)$, where r is the radius of a g-yolk if and only if for all $S\in W$, $B(z,r)\cap Conv(S)\neq\emptyset$. The radius r of all g-yolks is unique since it is minimal. If there is a unique center c such that $B(c, r)$ is a g-yolk, then by definition the $\delta-core$ is the singleton c. ■

B- an example

In this example, we find the center C and radius r of the g-yolk depend on the set of winning coalitions. For $q^1 = (1,3)$; $q^2 = (2,0)$; $q^3 = (0,2)$; $q^4 = (2,2)$ and $q^5 = O = (0,0)$, we show that : (i) for simple majority voting $c = (1, \frac{1+\sqrt{5}}{2})$ and $r = \frac{\sqrt{10}-\sqrt{2}}{4}$; ($ii$) for weighted voting game $V = [51;6,40,77,40]$, $c = (1, \frac{\sqrt{10}-1}{3})$ and $r = \frac{\sqrt{10}-1}{3}$ and (iii) for weighted voting game $V = [54;6,40,77,40]$, $c = (1, r)$ and $r = \sqrt{2} - 1$. Figures 4(i), (ii), iii) provided earlier in Sect. 3.3 relate to this example.

(i) Let $F = (0.5, 1.5) = \left(q^1q^5\right) \cap \left(q^2q^3\right)$. The radius of the inscribed circle of triangle q^1q^2F is the ratio of its area to semiperimeter. A unitary orientation vector of the line $\left(q^1q^2\right)$ is $\frac{\overrightarrow{q^1q^2}}{\left\|\overrightarrow{q^1q^2}\right\|} = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$. A normal vector of $\left(q^1q^2\right)$ is $\vec{u} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$. Line $\left(q^1q^2\right)$ is the set of points $p = (x_1, x_2)$ that check the normalized equation $\vec{u} \cdot \overrightarrow{q^2p} = 0$ i.e. $\vec{u} \cdot \overrightarrow{Op} = \vec{u} \cdot \overrightarrow{Oq^2}$. It follows that $\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) \cdot (x_1, x_2) = \frac{6}{\sqrt{10}}$. The distance from F to $\left(q^1q^2\right)$ is therefore $\frac{6}{\sqrt{10}} - \vec{u} \cdot \overrightarrow{OF} = (\frac{1}{\sqrt{10}})(6 - (1.5 + 1.5)) = \frac{3}{\sqrt{10}}$. Therefore the area of the triangle q^1q^2F is $\frac{1}{2}(\frac{3}{\sqrt{10}})\overline{q^1q^2} = \frac{3}{2}$. Its semiperimeter is $\frac{1}{2}(\sqrt{10} + \frac{3}{2}\sqrt{2} + \frac{1}{2}\sqrt{10})$ $= \frac{3}{4}(\sqrt{10} + \sqrt{2})$. The radius is $r = \frac{2}{\sqrt{10}+\sqrt{2}} = \frac{1}{4}(\sqrt{10} - \sqrt{2})$.

The normalized equation for $\left(q^2q^3\right)$ is $\frac{1}{\sqrt{2}}(x_1 + x_2) = \sqrt{2}$. The c_1 coordinate of the yolk center is 1, by symmetry. Let the yolk center have coordinates $(1, c_2)$. It must satisfy $\frac{1}{\sqrt{2}}(1 + c_2) = \sqrt{2} + r$. Hence $\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2} = \sqrt{2} + \frac{\sqrt{10}}{4} - \frac{\sqrt{2}}{4}$ $\Rightarrow c_2 = \sqrt{2}\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{10}}{4}\right) = \frac{1+\sqrt{5}}{2}$.

To summarize, the yolk's center is $c = (1, \frac{1+\sqrt{5}}{2})$ and its radius is $r = \frac{\sqrt{10}-\sqrt{2}}{4}$. It also must be verified that this circle intersects the median line defined by the segment $\left[q^3, q^4\right]$. We require $\frac{1+\sqrt{5}}{2} + r \geq 2$. The calculation is $\frac{1}{4}(2 + 2\sqrt{5} + \sqrt{10} - \sqrt{2}) \approx \frac{8.21}{4} > 2$.

(ii) For the weighted voting case with threshold 51 the minimal coalitions are $\{2,5\}$, $\{1,3,5\}$, $\{1,2,4\}$, and also $\{3,4,5\}$ and $\{2,3,4\}$. The first three are met by the inscribed circle of triangle $q^1q^2q^5$. The triangle's area is $\frac{1}{2}(2)(3) = 3$. Its semiperimeter is $\frac{1}{2}\left(\sqrt{10} + \sqrt{10} + 2\right) = \sqrt{10} + 1$. The radius is therefore $r = \frac{3}{\sqrt{10}+1} = \frac{\sqrt{10}-1}{3}$. The center has coordinates $(1, \frac{\sqrt{10}-1}{3})$ because the circle is tangent to the convex combinations of the winning coalition $\{2,5\}$.

(iii) For the weighted voting case with threshold 54, voter 1 is not a member of any minimal winning coalition. Those coalitions are $\{2,5\}$, $\{3,4,5\}$, and $\{2,3,4\}$. The yolk is the inscribed circle of the triangle with vertices q^2, q^5, and $\left(q^2q^3\right) \cap \left(q^4q^5\right) = (1,1)$. The triangle area is 1 since it is $\frac{1}{4}$ of 2^2 by symmetry of the square $q^3q^4q^2q^5$. The semiperimeter is $\frac{1}{2}(2 + 2\sqrt{2}) = \sqrt{2} + 1$. The radius is therefore $r = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$. As in the previous case, the circle must be tangent to $\left(q^2 q^5\right)$, so its center is $(1, \sqrt{2} - 1)$. ■

References

Austen-Smith D (1996) Refinements of the heart. In: Collective decision-making: social choice and political economy, pages 221–236. Springer
Banks JS (1995) Singularity theory and core existence in the spatial model. J Math Econ 24:523–536
Banks JS, Duggan J, Le Breton M (2006) Social choice and electoral competition in the general spatial model. J Econ Theory 126:194–234
Berger M (1978) *Géométrie, vol 3 : Convexes et polytopes, polyèdres réguliers, aires et volumes*. Fernand-Nathan, Paris
Borcherding K, Eppel T, Von Winterfeldt D (1991) Comparison of weighting judgments in multiattribute utility measurement. Manag Sci 37(12):1603–1619
Bräuninger T (2007) Stability in spatial voting games with restricted preference maximizing. J Theoretical Politics 19:173–191
Davis OA, DeGroot MH, Hinich MJ (1972) Social preference orderings and majority rule. Econometrica 40:147–157
Eban G, Stephen WS (1990) Predicting committee behavior in majority rule voting experiments. RAND J Econ 21:293–313
Eisenführ F, Weber M, Langer T (2009) Rational decision making. Springer, Berlin
Ferejohn JA, McKelvey R, Packel E (1984) Limiting distributions for continuous state Markov models. Soc Choice Welf 1:45–67
Grandmont JM (1978) Intermediate preferences and the majority rule. Econometrica 46:317–330
Greenberg J (1979) Consistent majority rules over compact sets of alternatives. Econometrica 47:627–636
Kramer GH (1977) A dynamical model of political equilibrium. J Econ Theory 16:310–334
Martin M, Nganmeni Z (2016) Tovey craig a on the uniqueness of the yolk. Soc Choice Welf 47:511–518
McKelvey RD (1986) Covering, dominance, and institution free properties of social choice. Am J Political Sci 30:283–314
McKelvey RD, Tovey CA (2010) Approximating the yolk by the LP yolk. Math Soc Sci 1:102–109
Miller NR (1980) A new solution set for tournaments and majority voting. Am J Political Sci 24:68–96
Miller NR (2015) The spatial model of social choice and voting. In: Heckelman JC, Miller NR (eds) Handbook of social choice and voting, chapter 10. Edward Elagar, Cheltenham, pp 163–181
Nakamura K (1979) The vetoers in a simple game with ordinal preferences. Int J Game Theory 8:55–61
Owen G (1990) Stable outcomes in spatial voting games. Math Soc Sci 19:269–279
Plott C (1967) A notion of equilibrium and its possibility under majority rule. Am Econ Rev 57:787–806
Rockafellar RT (1970) Convex analysis. Princeton University Press, Princeton
Rubinstein A (1979) A note about the 'nowhere denseness' of societies having an equilibrium under majority rule. Econometrica 47:511–514
Saari DG (2014) Unifying voting theory from Nakamura's to Greenberg's theorems. Math Soc Sci 69:1–11
Schofield N (1983) Generic instability of majority rule. Rev Econ Stud 50:695–705
Schofield N (1995) Coalition politics: a formal model and empirical analysis. J Theoretical Politics 7:245–281
Schofield N, Tovey C (1992) A Probability and convergence for supra-majority rule with euclidean preferences. Math Comput Modell 16(8–9):41–58
Serra J (2009) Uncertain Ideal Points and the Impact of Electoral Dimensions on the Vote Choice. *Paper presented at the annual meeting of the Midwest Political Science Association 67th Annual National Conference, The Palmer House Hilton, Chicago, IL*,
Shubik M, Wooders MH (1983) Approximate cores of replica games and economies Part II: set-up costs and firm formation in coalition production economies. Math Soc Sci 6:285–306
Steuer R (1976) E Multiple objective linear programming with interval criterion weights. Manag Sci 23(3):305–316

Tovey C. A (1991) Some foundations for empirical study in spatial euclidean social choice models. In W. Barnett, editor, Political Economy: Institutions, Competition, and Representation, chapter 7. Cambridge University Press, . presented at the Seventh International Symposium in Economic Theory and Econometrics

Tovey CA (2010) The instability of instability of centered distributions. Math Soc Sci 59:53–73

Tovey CA (2010) The probability of majority rule instability in the 2d euclidean model with an even snumber of voters. Soc Choice Welf 35:705–708

Weber M, Borcherding K (1993) Behavioral influences on weight judgments in multiattribute decision making. Euro J Oper Res 67(1):1–12

Wuffle A, Feld SL, Owen G, Grofman B (1989) Finagle's law and the Finagle point: A new solution concept for two-candidate competition in spatial voting games without a core. Am J Political Sci 33:348–375