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Fair cake-cutting among families

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Abstract

We study the fair division of a continuous resource, such as a land-estate or a time-interval, among pre-specified groups of agents, such as families. Each family is given a piece of the resource and this piece is used simultaneously by all family members, while different members may have different value functions. Three ways to assess the fairness of such a division are examined. (a) Average Fairness means that each family's share is fair according to the "family value function", defined as the arithmetic mean of the value functions of the family members. (b) Unanimous Fairness means that all members in all families feel that their family's share is fair according to their personal value function. (c) Democratic Fairness means that in each family, at least a fixed fraction (e.g. a half) of the members feel that their family's share is fair. We compare these criteria based on the number of connected components in the resulting division and on their compatibility with Pareto-efficiency.

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1 Introduction

Fair division of heterogeneous resources among agents with different preferences has been an important issue since Biblical times. Today it is an active area of research in the interface of computer science (Robertson and Webb 1998; Procaccia 2015; Brânzei 2015; Lindner and Rothe 2016) and economics (Moulin 2004; Thomson 2011). Its applications range from politics (Brams and Taylor 1996; Brams 2007) to multi-agent systems (Chevaleyre et al. 2006).

In most fair division problems, the resource is divided among n individual agents, and the fairness of a division is assessed based on their individual preferences. A common fairness criterion is **proportionality**. It requires that each agent receives a share that is at least as good as $1/n$ of the total endowment, according to the agent's individual preferences.[1]

In practice, however, goods are often owned and used by groups. As an example, consider a land-estate inherited by k families, a river that has to be divided among k states, or the usage-time of a conference room that has to be divided among k meeting groups. The resource (whether land or time) should be divided into k pieces, one piece per group. Each group's share is then used by *all* its members simultaneously. The land-plot allotted to a family is inhabited by the entire family. The share of the river allotted to a state becomes a national park open to all its citizens. In the time-slot allotted to a group, the conference room is used by all group members.[2]

The happiness of each group member depends on his/her valuation of the entire share of the group. But, in each group there are different members with different valuations. The group's share can be valued by some of its members as at least $1/k$ of the total and by others as less than $1/k$ of the total. How, then, should the fairness of a division be assessed?

The present paper studies this question in the classic setting of *cake-cutting*, introduced by Steinhaus (1948). In this setting, there is a measurable space (e.g. an interval or a polygon) called the *cake*, and the preferences of each agent are represented by a value-measure on the cake.[3]

We study three ways to assess the fairness of a division.

First, it is possible to aggregate the valuations in each family to a single *family valuation*. Following the utilitarian tradition (Bentham 1789), the family-valuation can be defined as the sum or (equivalently) the arithmetic average of the valuations of

[1] The condition of receiving at least $1/n$ of the total endowment was introduced by Steinhaus (1948). Economists often call it *fair-share guarantee* (Bogomolnaia et al. 2017). Computer scientists often call it *proportionality* (Robertson and Webb 1998). This term is motivated by a generalization of the fair division problem in which different agents may have different entitlements (see page 4 below). In this setting, proportionality guarantees to each agent a value lower-bound that is in direct *proportion* to his/her entitlement.

[2] In economic terms, the allotted piece becomes a "club good" (Buchanan 1965).

[3] The assumption that agents' preferences can be represented by measures is a strong one. It implies that the model is applicable only for the special case of linear preferences—the sum of values of two disjoint pieces equals the value of their union. Equivalently, the agents have constant marginal utilities (Chambers 2005)—the marginal utility of a land-plot for an agent does not depend on the other land-plots owned by that agent. Although most papers on the cake-cutting problem assume linearity, there are some notable exceptions; they are surveyed in Sect. 8.5.

all family members. We call a division **fair-on-average** if it is fair according to these family valuations. In particular, a division is proportional-on-average if every family receives a share with an average value (averaged over all family members) of at least $1/k$ of its average value of the entire endowment.

By this definition, the family-division problem is easy to solve. Since the average of measures is itself a measure, each family can be represented by a single agent, and the problem reduces to fair division among the k representatives. Classic results imply that proportional-on-average allocations exist (Sect. 3).

Average fairness makes sense only when the numerical values of the agents' valuations are meaningful and they are all measured in the same units, e.g. in dollars (see chapter 3 of Moulin (2004) for some real-life examples of such situations). However, if the valuations represent individual happiness measures that cannot be put on a common scale, then their sum is meaningless, and other fairness criteria should be used.

A second option is to require that all members of every family agree that the division is fair. We call a division **unanimous fair** if it is fair according to every individual valuation. In particular, a division is unanimously-proportional if every agent values his/her family's share as at least $1/k$ of the total value . The advantage of this definition is that it does not need to assume that all valuations share a common scale. Even though it is a very strong requirement, we prove that unanimously-proportional allocations exist (Sect. 4).

A disadvantage of unanimous fairness, compared to average fairness, is that unanimously-fair divisions might be highly fractioned. When an interval is divided, there always exists an proportional-on-average division that is also *connected*—the share of each family is a single interval (Sect. 3). However, there might not exist connected unanimously-proportional divisions. Moreover, in some cases, the number of intervals in any unanimously-proportional division is at least n—the number of individual agents (Sect. 4). When the number of agents is large, as in the case of dividing land among states, such divisions might be impractical.

In democratic societies, decisions are almost never unanimous. In fact, when the number of citizens is large, it may be impossible to attain unanimity on even the most trivial issue, and decisions are often made by voting. Therefore we suggest a third fairness criterion. Given a fraction $h \in [0, 1]$, we call a division h**-democratic fair** if at least a fraction h of the members in each family consider it fair. Unanimous-fairness is equivalent to 1-democratic fairness. The case $h = \frac{1}{2}$ is particularly interesting. $\frac{1}{2}$-democratic fairness can be justified by the following process. After a division is proposed, each family conducts a voting process in which each member approves the division if he/she values the family's share as at least $1/k$ of the total. The division is implemented only if, in every family, a (weak) majority of the population approves it.

Democratic-fairness combines some advantages of unanimous-fairness and average-fairness. It is similar to unanimous-fairness in that it does not need to assume that all valuations share a common scale. It is similar to average-fairness in that, when $h \leq \frac{1}{k}$, h-democratic fairness among k families can be attained with connected pieces. In particular, with $k = 2$ families, there always exists an allocation in which each family receives a single connected piece, and at least a weak majority in each family considers the allocation fair. An additional advantage of democratic fairness in this case is that it can be computed efficiently (Sect. 5).[4]

Although democratic-fairness might leave some citizens unhappy, this may be unavoidable in real-life situations. This is understandable in light of Winston Churchil's dictum: "democracy is the worst form of government, except all the others that have been tried".[5]

While the geometric requirement of having a connected division is practically important, an even more important requirement from an economic perspective is Pareto-efficiency. All three variants of proportionality are compatible with Pareto-efficiency. However, connectivity and Pareto-efficiency are incompatible even without fairness considerations (Sect. 6).

The proportionality criterion can be generalized to a situation in which each family has a different entitlement. Suppose that each family j is entitled to a fraction t_j of the resource (where $\sum_{j=1}^{k} t_j = 1$). An allocation is unanimously-proportional if each member in family j believes that the share of family j is worth at least a fraction t_j of the total. In a similar way it is possible to generalize average-proportionality and democratic-proportionality. In the simplest setting, the families have equal entitlements, i.e, for each $j \in \{1, \ldots, k\}$: $t_j = 1/k$. Equal entitlements make sense, for example, when k siblings inherit their parents' estate. While an heir will probably like to take his family's preferences into account when selecting a share, each heir is entitled to $1/k$ of the estate regardless of the family size.

In general, each family may have a different entitlement. The entitlement of a family may depend on its size but may also depend on other factors. For example, consider several families who jointly buy a vacation apartment. The apartment can host one family at a time, so the families have to divide the year (a time-interval) among them. The entitlement of each family naturally depends on the amount of money it contributed to the purchase, rather than on the family's size.[6] The results presented in Sects. 3–6 consider proportionality both with equal and with different entitlements.

An alternative fairness criterion that is very common in economics is **envy-freeness**. In the context of individual agents, it means that each agent receives a share that is at least as good as the share of any other agent, according to the first

4 In contrast, average-fairness and unanimous-fairness cannot be computed by *any* finite protocol. See Remark 1 in page 9.

5 A fourth fairness criterion that could be considered is **individual fairness**. In particular, an allocation is *individually-proportional* if the allocation $X = (X_1, \ldots, X_k)$ admits a refinement $Y = (Y_1, \ldots, Y_n)$, where for each family F_j, $\cup_{i \in F_j} Y_i = X_j$, such that for each agent i, $V_i(Y_i) \geq 1/n$. Individually-fair allocations always exist and can be found by using any classic fair division procedure on the individual agents, disregarding their families. Individual-fairness makes sense if, after the division of the land among the families, each family intends to further divide its share among its members. However, often this is not the case. When an inherited land-estate is divided between two families, the members of each family intend to live and use their entire share together, rather than dividing it among them. Therefore, the happiness of each family member depends on the entire value of his family's share, rather than on the value of a potential private share he would get in a hypothetic sub-division.

6 See Cseh and Fleiner (2018) for a recent account of fair division among individual agents with different entitlements.

agent's individual valuation.[7] In the context of families, three variants of envy-freeness can be defined analogously to the three variants of proportionality (for families with equal entitlements): average-envy-freeness, unanimous-envy-freeness and democratic-envy-freeness (Sect. 7).

From a geometric perspective, these three variants behave similarly to their proportionality counterparts, that is:

– Connected envy-free-on-average allocations always exist;
– Connected unanimously-envy-free allocations are not guaranteed to exist even for two families;
– Connected $\frac{1}{k}$-democratically-envy-free allocations are guaranteed to exist for k families. In particular, connected $\frac{1}{2}$-democratically-envy-free allocations are guaranteed to exist for two families (but not for three or more families).

However, from an efficiency perspective, envy-freeness behaves differently:

– Pareto-efficient envy-free-on-average allocations always exist;
– Pareto-efficient unanimously-envy-free allocations are guaranteed to exist for two but not for three or more families;
– Pareto-efficient $\frac{1}{2}$-democratically-envy-free allocations are guaranteed to exist for two but not for five or more families (we do not know whether they always exist for three or four families).

The paper is organized as follows. Most of the paper focuses on the *proportionality* criterion. Section 2 formally presents the model. Sections 3, 4 and 5 study average, unanimous and democratic proportionality respectively. We study this criterion both for families with equal entitlements and for families with different entitlements.

Section 6 studies the three variants of proportionality in combination with Pareto-efficiency. Section 7 studies family fairness based on the envy-freeness criterion, explaining the differences between the results for proportionality and for envy-freeness. Finally, Sect. 8 compares our work to previous and ongoing related work.

2 Model and notation

2.1 Resource and agents

In the usual cake-cutting setting, there is a resource C ("cake") that has to be divided. For simplicity it is assumed that C is an interval in $\mathbb{R}$. A realistic example of such a resource is time: consider a conference room that can host a single meeting at a time. It is available between 8:00 and 20:00, and this time-interval must be divided among all those who want to use the room. Another realistic example is the shoreline of a sea or a river: while usually not a straight line, it can be easily mapped to an interval.

There is a set of agents $N = \{1, \ldots, n\}$. Each agent $i \in N$ has a value measure V_i, defined on the Borel subsets of C. The V_i are assumed to be nonatomic, so that all

[7] The condition of receiving at least as much as any other agent was introduced by Gamow and Stern (1958) and Foley (1967). Economists often call it *no envy* (Bogomolnaia et al. 2017). Computer scientists often call it *envy-freeness* (Robertson and Webb 1998).

singular points have a value of 0 to all agents. As the term measure implies, the V_i are *additive*—the value of a union of two disjoint pieces is the sum of the values of the pieces. The value measures are normalized such that $\forall i: V_i(\emptyset) = 0,\ V_i(C) = 1$.

2.2 Families and entitlements

In our setting, there is a set of families $F = \{F_1, \dots, F_k\}$. We use the term "family" to emphasize that the partition of agents to groups is fixed in advance and cannot be modified during the division process.

The number of agents in F_j is denoted n_j. Each agent $i \in N$ is a member of exactly one family $F_j \in F$, so $n = \sum_{j=1}^{k} n_j$.

For each family F_j, there is a positive number t_j representing the entitlement of this family. The sum of all entitlements is one: $1 = \sum_{j=1}^{k} t_j$. In the special case in which all families have equal entitlements, we have for each $j \in \{1, \dots, k\}$: $t_j = 1/k$.

2.3 Allocations and components

An *allocation* is a vector of k pieces, $X = (X_1, \dots, X_k)$, one piece per family, such that the X_j are pairwise-disjoint and $\cup_j X_j = C$.

Each piece is a finite union of intervals. We denote by $\textsc{Comp}(X_j)$ the number of connected components (intervals) in the piece X_j, and by $\textsc{Comp}(X)$ the total number of components in the allocation X, i.e:

$$\textsc{Comp}(X) = \sum_{j=1}^{k} \textsc{Comp}(X_j)$$

Ideally, we would like that each piece be connected, i.e, $\forall i : \textsc{Comp}(X_i) = 1$ and $\textsc{Comp}(X) = k$. This requirement is especially meaningful when the divided resource is a time-interval or a land-resource (e.g. a river-bank), since a contiguous piece of time or land is much easier to use than a collection of disconnected patches.

However, we will show that a fair division with connected pieces is not always possible.[8] In case a division with connected pieces is not possible, it is still desirable that the number of connectivity components—$\textsc{Comp}(X)$—be as small as possible. When dividing an interval, the components are sub-intervals and their number is one plus the number of *cuts*. Hence, the number of components is minimized by minimizing the number of cuts (Robertson and Webb 1995; Webb 1997; Shishido and Zeng 1999; Barbanel and Brams 2004, 2014). In a realistic, 3-dimensional world, the additional

[8] This impossibility appears not only in our one-dimensional theoretic model but also in practical, two-dimensional land division situations. A striking example was the India-Bangladesh border. According to Wikipedia page *India– Bangladesh enclaves*, up to 2015, "Within the main body of Bangladesh were 102 enclaves of Indian territory, which in turn contained 21 Bangladeshi counter–enclaves, one of which contained an Indian counter–counter–enclave... within the Indian mainland were 71 Bangladeshi enclaves, containing 3 Indian counter–enclaves". Another example is *Baarle-Hertog*—a Belgian municipality made of 24 separate parcels of land, most of which are exclaves in the Netherlands. For more details and examples see the Wikipedia page *List of enclaves and exclaves*. We are grateful to Ian Turton for the references.

dimensions can be used to connect the components, e.g, by bridges or tunnels. Still, it is desirable to minimize the number of components in the original division in order to reduce the number of required bridges/tunnels.[9]

2.4 Fairness criteria

We first define the *family-valuation* functions:

$$W_j^{\text{avg}}(X_j) = \frac{\sum_{i \in F_j} V_i(X_j)}{n_j} \quad \text{for,} \quad j \in \{1, \ldots, k\}.$$

Now, an allocation X is called:

$$\begin{array}{ll} \text{proportional-on-average} & \text{if } \forall j \in \{1, \ldots, k\} : W_j^{\text{avg}}(X_j) \geq t_j; \\ \text{unanimously-proportional} & \text{if } \forall j \in \{1, \ldots, k\} : \forall i \in F_j : V_i(X_j) \geq t_j; \\ h-\text{democratically-proportional} & \text{if } \forall j \in \{1, \ldots, k\}, \\ \text{for at least a fraction } h \text{ of the members } i \in F_j : V_i(X_j) \geq t_j & . \end{array}$$

A property of an allocation is called *feasible* if for every k families and n agents there exists an allocation satisfying this property. Otherwise, the property is called *infeasible*. In the following sections we will study the feasibility of the above fairness criteria.

Note that unanimous-proportionality obviously implies both average-proportionality and h-democratic-proportionality for any $h \in [0, 1]$. The other two do not imply each other, as shown in the following example.

Example 1 Consider an interval consisting of four sub-intervals. It has to be divided between two families: (1) {Alice,Bob,Chana} and (2) {David,Esther,Frank}. The families have equal entitlements, i.e, $t_1 = t_2 = 1/2$. Each member's valuation of each sub-interval is shown in the table below:

Alice	60	30	3	3
Bob	50	40	3	3
Chana	10	80	3	3
David	3	3	60	30
Esther	3	3	60	30
Frank	3	3	0	90

Note that the value of the entire interval is 96 for all agents. Therefore, proportionality implies that each family should get a value of at least 48.

[9] The goal of minimizing the number of components is pursued not only in cake-cutting papers but also in real-life politics. Going back to India and Bangladesh, after many years of negotiations they finally started to exchange most of their enclaves during the years 2015–2016. This reduced the number of components from 200 to a more reasonable number.

If the two leftmost subintervals are given to family 1 and the two rightmost subintervals are given to family 2, then the division satisfies *unanimous-proportionality*, since each member of each family feels that his family's share is worth 90. Of course, it also satisfies average-proportionality and democratic-proportionality.

If only the single leftmost subinterval is given to family 1 and the other three are given to family 2, then the division still satisfies $\frac{1}{2}$*-democratic-proportionality*, since Alice and Bob feel that their family received more than 48. However, Chana feels that her family received only 10, so the division does not satisfy unanimous-proportionality. Moreover, the division does not satisfy average-proportionality since the average valuation of family 1 is only $(60 + 50 + 10)/3 = 40$.

If the three leftmost subintervals are given to family 1 and only the rightmost one is given to family 2, then the division satisfies *average-proportionality*, since family 2's average valuation of its share is $(30 + 30 + 90)/3 = 50$. However, it does not satisfy unanimous-proportionality nor even $\frac{1}{2}$-democratic-proportionality, since David and Esther feel that their share is worth only 30. □

3 Average fairness

With average fairness, the family cake-cutting problem can be reduced to the classic problem of cake-cutting among individuals. This gives the following results.

Theorem 1 *(a) When families have equal entitlements, average-proportionality with connected pieces (and k components) is feasible.*

(b) When families have different entitlements, average-proportionality with connected pieces is infeasible. Moreover, in some cases, any proportional-on-average allocation has at least $2k - 1$ *components.*

(c) When families have different entitlements, average-proportionality with at most $O(k \log k)$ *components is feasible.*

Proof The positive results—parts (a) and (c)—are based on the following reduction. For each family F_j, define a *representative agent* A_j whose valuation is the function W_j^{avg} defined in Sect. 2.4 above. Note that, since the V_i are all nonatomic measures, the k family-valuations W_j^{avg} are nonatomic measures too. By classic results (Steinhaus 1948; Even and Paz 1984), when there are k agents with equal entitlements, there always exists a connected proportional division. As shown in Segal-Halevi (2019), when there are k agents with different entitlements, there always exists a proportional allocation with at most $2k \log_2 \widehat{k} - 2\widehat{k} + 2$ cuts, where $\widehat{k} := 2^{\lceil \log_2 k \rceil} = k$ rounded up to the nearest power of two. These cuts create $2k \log_2 \widehat{k} - 2\widehat{k} + 3$ components. By definition, such a division satisfies average-proportionality.

The negative result (b) follows immediately from an identical negative result for individual agents (Segal-Halevi 2019), by considering k one-member families. □

Remark 1 Fairness for individuals and average-fairness for families are equivalent only from an existential perspective; from a computational perspective they are quite different. A proportional division among k individual agents with equal entitlements can be found by asking the agents $O(k \log k)$ queries (Even and Paz 1984). However, a proportional-on-average division cannot be found using a finite number of queries even when there are $k = 2$ families with two agents in each family, and even without any restrictions on the number of components.

Proof The proof is by a reduction from the problem of *equitable cake-cutting* among individual agents. In this problem, the goal is to find an allocation X such that for every two agents i and j, $V_i(X_i) = V_j(X_j)$. Procaccia and Wang (2017), extending a previous result by Cechlárová and Pillárová (2012), showed that an equitable allocation cannot be computed by a finite number of queries, even when there are only two individual agents and no connectivity constraints. We show that the same is true for average-proportionality.

Given an instance of equitable cake-cutting with two agents with value-measures V_1 and V_2, construct an instance of average-proportionality cake-cutting with two families, where in each family there are two members with value-measures V_1 and V_2. We claim that an allocation is equitable in the original problem if-and-only-if it is proportional-on-average in the new problem:

$$\begin{aligned}
&\text{The allocation } (X_1, X_2) \text{ is equitable among the individuals}\\
\iff &V_1(X_1) = V_2(X_2)\\
\iff &V_1(X_1) = 1 - V_2(X_1) \quad (\text{since } V_2(X_1) + V_2(X_2) = V_2(C) = 1)\\
\iff &[V_1(X_1) + V_2(X_1)]/2 = 1/2\\
\iff &[V_1(X_1) + V_2(X_1)]/2 \geq 1/2 \quad \text{and} \quad [V_1(X_1) + V_2(X_1)]/2 \leq 1/2\\
\iff &[V_1(X_1) + V_2(X_1)]/2 \geq 1/2 \quad \text{and} \quad [V_1(X_2) + V_2(X_2)]/2 \geq 1/2\\
&\qquad (\text{since } V_i(X_1) + V_i(X_2) = V_i(C) = 1)\\
\iff &\text{The allocation } (X_1, X_2) \text{ is proportional-on-average among the families.}
\end{aligned}$$

Hence, if we could find any proportional-on-average division by a finite number of queries, we could also find an equitable division by a finite number of queries—a contradiction to Procaccia and Wang (2017). □

4 Unanimous fairness

Before presenting our results, we note that unanimous-proportionality, like average-proportionality, can also be defined using family-valuation functions. Define:

$$W_j^{\min}(X_j) := \min_{i \in F_j} V_i(X_j) \quad \text{for} \quad j \in \{1, \ldots, k\}.$$

Then, a division is unanimously-proportional if and only if:

$$\forall j : W_j^{\min}(X_j) \geq t_j$$

However, in contrast to the functions W^{avg} defined in Sect. 3, the functions $W^{\min}$ are in general not additive. For example, suppose C is an interval with three subintervals and a family has the following valuations:

	C_1	C_2	C_3	$C_1 \cup C_2 \cup C_3$
Alice	1	1	1	$3 = 1 + 1 + 1$
Bob	0	2	1	$3 = 0 + 2 + 1$
Chana	0	1	2	$3 = 0 + 1 + 2$
$W^{\min}$	0	1	1	$3 > 0 + 1 + 1$

While the individual valuations are additive, $W^{\min}$ is not additive (it is not even subadditive). Therefore, the classic results we used in Theorem 1 are inapplicable here, and different techniques are needed.

4.1 Exact division

Initially, we assume that the entitlements are equal, i.e: $t_j = 1/k$ for all j. We relate unanimous-proportionality to the problem of finding an *exact division*: [10].

Definition 1 Exact (N, K) is the following problem. Given N agents and an integer K, divide C into K pieces, such that each of the N agents assigns exactly the same value to all pieces:

$$\forall j = 1, \ldots, K : \quad \forall i = 1, \ldots, N : \ V_i(X_j) = 1/K.$$

From an economic perspective, there is little intrinsic value in the concept of exact division. However, in this section we will prove that it is closely linked to the concept of unanimously-fair division. In fact, we will prove that the existence of a solution to each of these problems implies a solution to the other problem.

Below, we denote by UnanimousPR(n, k) the problem of finding a unanimously-proportional division when there are n agents grouped in k families with equal entitlements.

4.2 UnanimousPR $\implies$ Exact

Lemma 1 *For every pair of integers $N \geq 1, K \geq 1$, a solution to UnanimousPR $(N(K-1)+1, K)$ implies a solution to Exact (N, K) with the same number of components.*

Proof Given an instance of Exact(N, K) (N agents and a number K of required pieces), create K families. Each of the first $K - 1$ families contains N agents with the

[10] The definition uses capital N and K to distinguish the parameters of exact division from the parameters of unanimous-fair division.

same valuations as the given agents. The K-th family contains a single agent whose valuation is V^*, defined as the average of $V_1, \ldots, V_N$:

$$V^* = \frac{1}{N}\sum_{i=1}^{N} V_i.$$

The total number of agents in all K families is $N(K-1)+1$. Use UnanimousPR $(N(K-1)+1,\ K)$ to find a unanimously-proportional division, X. By definition of unanimous fairness, for each agent i in family j: $V_i(X_j) \geq 1/K$.

By construction, each of the first $K-1$ families has an agent with valuation V_i. Hence, all N agents value each of the first $K-1$ pieces as at least $1/K$ and:

$$\forall i = 1, \ldots, N: \quad \sum_{j=1}^{K-1} V_i(X_j) \geq \frac{K-1}{K}.$$

Hence, by additivity, every agent values the K-th piece as at most $1/K$:

$$\forall i = 1, \ldots, N: \quad V_i(X_K) \leq 1/K.$$

The piece X_K is given to the agent with value measure V^*, so by proportionality: $V^*(X_K) \geq 1/K$. By construction, $V^*(X_K)$ is the average of the $V_i(X_K)$. Hence:

$$\forall i = 1, \ldots, N: \quad V_i(X_K) = 1/K.$$

Again by additivity:

$$\forall i = 1, \ldots, N: \quad \sum_{j=1}^{K-1} V_i(X_j) = \frac{K-1}{K}.$$

Hence, necessarily:

$$\forall i = 1, \ldots, N, \quad \forall j = 1, \ldots, K-1: \quad V_i(X_j) = 1/K.$$

So we have found an exact division and solved Exact(N, K) as required. □

Alon (1987) proved that for every N and K, an Exact(N, K) division might require at least $N(K-1)+1$ components. Combining this result with the above lemma implies the following negative result:

Theorem 2 *For every N, K, let $n = N(K-1)+1$. A unanimously-proportional division for n agents grouped into K families might require at least n components.*

In particular, unanimous-proportionality with connected pieces is infeasible.

4.3 Exact $\Longrightarrow$ UnanimousPR

Lemma 2 *For every pair of integers $n \geq 2$, $k \geq 1$, a solution to Exact $(n-1, k)$ implies a solution to UnanimousPR (n, k) for any grouping of the n agents to k families.*

Proof Suppose we are given an instance of UnanimousPR(n, k), i.e, we are given some n agents grouped into k families. Select $n - 1$ agents arbitrarily. Use Exact$(n - 1, k)$ to find a partition of C into k pieces, such that each of the $n - 1$ agents values each of these pieces at exactly $1/k$. Ask the n-th agent to choose a piece with a maximal value for him/her. The average value of a piece is $1/k$, so the piece of maximal value is worth for the n-th agent at least $1/k$. Give that piece to the family of the n-th agent. Give the other $k - 1$ pieces arbitrarily to the remaining $k - 1$ families. The resulting division is unanimously-proportional. □

Alon (1987) proved that for every N and K, Exact(N, K) has a solution with at most $N(K - 1) + 1$ components (at most $N(K - 1)$ cuts). Combining this result with the above lemma implies the following positive result:

Theorem 3 *Given n agents in k families with equal entitlements, a unanimously-proportional division with $(n - 1) \cdot (k - 1) + 1$ components is feasible.*

For $k = 2$ families, the number of components in Theorem 3 is n, which matches the lower bound of Theorem 2. For $k > 2$ families, the number of components can be made smaller, as explained below.

4.4 Less components: equal entitlements

The purpose of this subsection is to find a unanimously-proportional allocation with fewer components than the guarantee of Theorem 3, when all families have equal entitlements.

We start with an example. Assume there are $k = 4$ families. As in Theorem 3, using $3(n - 1)$ cuts, C can be divided into 4 subsets which are considered equal by $n - 1$ members. But for a unanimously-proportional division, it is not required that all members think that all pieces are equal, it is only required that all members believe that their family's share is worth at least $1/4$. This can be achieved as follows:

- Divide C into two subsets which all n agents value at exactly $1/2$. This is equivalent to solving Exact$(n, 2)$, which by Alon (1987), can be done with at most n cuts. Call the two resulting subsets West and East.
- Assign arbitrary two families to West and the other two families to East. Mark by n_W the total number of members in the families assigned to West and by n_E the total number of members assigned to East.
- Divide the West into two pieces which all n_W agents value at exactly $1/4$; this can be done with n_W cuts. Give a piece to each family. Divide the East similarly using n_E cuts.

The first step requires n cuts and the second step requires $n_W + n_E = n$ cuts too. Hence the total number of cuts required is only $2n$, rather than $3n - 1$.

In fact, two cuts can be saved in each step by excluding two members (from two different families) from the exact division. These members will not think that the division is equal, but they will be allowed to choose their favorite piece for their family. Thus only $2(n-2)$ cuts are required. A simple inductive argument shows that whenever k is a power of 2, $(\log_2 k) \cdot (n - k/2)$ cuts are required.

When k is not a power of 2, a result by Stromquist and Woodall (1985) can be used. They prove that, for every fraction $r \in [0, 1]$, it is possible to cut a piece of C such that all n agents agree that its value is exactly r using at most $2n-2$ cuts.[11] This can be used as follows:

- Select integers $l_1, l_2 \in \{1, \ldots, k-1\}$ such that $l_1 + l_2 = k$.
- Apply Stromquist and Woodall (1985) with $r = l_1/k$: using $2n-4$ cuts, cut a piece X_1 that $n-1$ agents value at exactly l_1/k. This means that these $n-1$ agents value the other piece, X_2, at exactly l_2/k.
- Let the n-th agent choose a piece for his family; assign the other families arbitrarily such that l_1 families are assigned to piece X_1 and the other l_2 families to piece X_2.
- Recursively divide piece X_1 to its l_1 families and piece X_2 to its l_2 families.

After a finite number of recursion steps, the number of families assigned to each piece becomes 1 and the procedure ends. The number of cuts in each level of the recursion is at most $2n-4$. The depth of recursion can be bounded by $\lceil \log_2 k \rceil$ by dividing k to halves (if it is even) or to almost-halves (if it is odd; i.e. take $l_1 = (k-1)/2$ and $l_2 = (k+1)/2$). Hence:

Theorem 4 *Given n agents in k families with equal entitlements, a unanimously-proportional division with* $\lceil \log_2 k \rceil \cdot (2n-4) + 1$ *components is feasible.*

Note that Theorems 3 and 4 both give upper bounds on the number of components required for unanimous-proportionality. The bound of Theorem 3 is stronger when k is small and the bound of Theorem 4 is stronger when k is large.

4.5 Less components: different entitlements

The purpose of this subsection is to find a unanimously-proportional allocation with fewer components than the guarantee of Theorem 3, when families may have different entitlements.

When families have different entitlements, the procedure of the previous subsection cannot be used. We cannot let the n-th agent select a piece for his family, since the pieces are different. For example, suppose there are two families with entitlements $t_1 = 1/3$, $t_2 = 2/3$. We can divide C into two pieces X_1, X_2 such that $n-1$ agents value X_1 as 1/3 and X_2 as 2/3. So all of them agree that X_1 should be given to family 1 and X_2 should be given to family 2. But, the n-th agent might select the wrong piece for his family. Therefore, the procedure should be modified as follows.

- Select an integer $l \in \{1, \ldots, k-1\}$.

[11] They prove that, if C is a circle, the number of connected components is $n-1$. Hence, the number of cuts is $2n-2$. This is also true when C is an interval, although the number of connected components in this case is n.

- Divide the families into two subsets: $F_1, \ldots, F_l$ and $F_{l+1}, \ldots, F_k$.
- Apply Stromquist and Woodall (1985) with $r = \sum_{j=1}^{l} t_j$: using $2n-2$ cuts, cut a piece X_1 which all n agents value at exactly $\sum_{j=1}^{l} t_j$. This means that all n agents value the other piece, X_2, at exactly $\sum_{j=l+1}^{k} t_j$.
- Recursively divide piece X_1 to $F_1, \ldots, F_l$ and piece X_2 to $F_{l+1}, \ldots, F_k$.

Here, the number of cuts in each level of the recursion is at most $(2n-2)$. The depth of recursion can be bounded by $\lceil \log_2 k \rceil$ by choosing $l = k/2$ (if k is even) or $l = (k-1)/2$ (if k is odd). Hence:

Theorem 5 *Given n agents in k families with different entitlements, a unanimously-proportional division with $\lceil \log_2 k \rceil \cdot (2n-2) + 1$ components is feasible.*

To conclude the analysis of unanimous-proportionality, recall that, even for $k = 2$ families, unanimous-proportionality is as difficult as exact division and might require the same number of components—n. In the worst case, we might need to give a disjoint component to each member, which negates the concept of division to families. Therefore we now turn to the analysis of an alternative fairness criterion that yields more useful results.

5 Democratic fairness

Like unanimous-proportionality (Sect. 4), h-democratic-proportionality too can be defined using family-valuation functions. For example, for $h = \frac{1}{2}$:

$$W_j^{\text{med}}(X_j) := \frac{\text{median}_{i \in F_j} V_i(X_j)}{n_j} \quad \text{for} \quad j \in \{1, \ldots, k\}.$$

A division is $\frac{1}{2}$-democratically-proportional if and only if:

$$\forall j : W_j^{\text{med}}(X_j) \geq t_j$$

However, the W^{med} functions are not additive,[12] so again the classic results referred to in Theorem 1 are inapplicable.

5.1 A division procedure

We start with a positive result for families with equal entitlements, which shows that democratic-proportionality is substantially easier than unanimous-proportionality.

Theorem 6 *For every integer $k \geq 2$, when there are k families with equal entitlements, $\frac{1}{k}$-democratic-proportionality with connected pieces is feasible and can be found by an efficient protocol.*

[12] See the example in the beginning of Sect. 4. In that example W^{med} is identical to W^{min}.

Proof The Dubins-Spanier moving-knife protocol (Dubins and Spanier 1961) can be adapted to families as follows. A knife moves continuously over the cake from left to right. Whenever in a certain family at least $1/k$ of its members believe that the cake to the left of the knife is worth at least $1/k$, they shout "stop", the cake is cut at the knife location, and the shouting family receives the cake to its left. The division so far is proportional for $1/k$ of the members in this family.

In the remaining $k-1$ families, at least $(k-1)/k$ of the members believe the remaining cake is worth at least $(k-1)/k$ of its original value. Dividing the remaining cake recursively using the same procedure yields a division that $1/(k-1)$ of $(k-1)/k$ of the members in each remaining family value as at least $1/(k-1)$ of $(k-1)/k$ of the original value; in other words, the division is proportional for at least $1/k$ of the members in each family. □

Theorem 6 is particularly useful for $k = 2$ families. It implies the existence of a connected division that is considered fair by at least a weak majority in each family.

Unfortunately, this positive result cannot be improved—it is impossible to guarantee the support of a weak majority when there are three or more families, and it is impossible to guarantee the support of larger majority when there are two families. This is proved in the following subsection.

5.2 Three or more families: an impossibility result

This subsection presents a lower bound on the number of components required for a democratic-fair division. The lower bound holds not only for proportionality but even for a much weaker fairness notion called *positivity*.

Given a specific division of C among families, define a *zero* agent as an agent who values his family's share as 0 and a *positive* agent as an agent who believes his family received a share with a positive value. Note that proportionality implies positivity but not vice-versa. The following lower bound holds even for positivity, hence it also holds for proportionality.

Lemma 3 *Assume there are $n = mk$ agents, divided into k families with m members in each family. To guarantee that at least q members in each family are positive, the total number of components may need to be at least:*

$$k \cdot \frac{kq - m}{k - 1}$$

Proof Number the families by $j = 0, \ldots, k-1$ and the members in each family by $i = 0, \ldots, m-1$. Assume that C is the interval $[0, mk]$. In each family j, each member i wants only the following interval: $(ik+j,\ ik+j+1)$. Thus there is no overlap between desired intervals of different members. The table below illustrates the construction for $k = 2,\ m = 3$. The families are {Alice,Bob,Chana} and {David,Esther,Frank}:

Suppose the piece X_j (the piece given to family j) is made of $l \geq 1$ components. We can make l members of F_j positive using l intervals of positive length inside their desired areas. However, if $q > l$, we also have to make the remaining $q - l$ members

Alice	1	0	0	0	0	0
Bob	0	0	1	0	0	0
Chana	0	0	0	0	1	0
David	0	1	0	0	0	0
Esther	0	0	0	1	0	0
Frank	0	0	0	0	0	1

positive. For this, we have to extend $q - l$ intervals to length k. Each such extension totally covers the desired area of one member in each of the other families. Overall, each family forces $q - l$ zero members in each of the other families. The number of zero members in each family is thus $(k-1)(q-l)$. Adding the q members who must be positive in each family gives the necessary condition: $(k-1)(q-l)+q \le m$. This is equivalent to:

$$m + l(k-1) \ge kq$$
$$\implies l \ge \frac{kq - m}{k-1}$$

The total number of components is $k \cdot l$, which is the claimed expression. □

By setting $q = hm$ in Lemma 3, we get the following lower bound on the number of cuts in an h-democratically-proportional division:

Theorem 7 *For any $h \in [0, 1]$, in an h-democratically-proportional division with n agents grouped into k families, the number of components may need to be at least*

$$n \cdot \frac{hk - 1}{k - 1}$$

In a unanimously-proportional division $h = 1$, so the number of components is at least n, which coincides with the lower bound of Theorem 2. On the other hand, when $h = 1/k$ the lower bound is 0, and indeed we already saw that in this case a connected allocation is feasible (Theorem 6). However, when $h > 1/k$, for sufficiently large n, the expression in Theorem 7 is larger than k, which implies that a connected division might not exist. In particular, we cannot guarantee a connected $\frac{1}{2}$-democratically-proportional division for three or more families, and we cannot guarantee a connected h-democratically-proportional division even for two families if $h > 1/2$.

5.3 Three or more families: positive results

Suppose we do want a $\frac{1}{2}$-democratically-proportional division for three or more families. How many components are sufficient?

As a first positive result, we can use Theorem 5, substituting $n/2$ instead of n: select half of the members in each family arbitrarily, then find a division which is unanimously-proportional for them while ignoring all other members. This leads to:

Theorem 8 *Given n agents in k families with different entitlements,* $\frac{1}{2}$*-democratic-proportionality with* $1 + \lceil \log_2 k \rceil \cdot (n-2)$ *components is feasible.*

However, for families with equal entitlements we can do much better.

Theorem 9 *Given n agents in k families with equal entitlements,* $\frac{1}{2}$*-democratic-proportionality with at most*

$$\min\left(2 + (\lceil k/2 \rceil - 1) \cdot (n/2 - 2), \quad 2 + \lceil \log_2 \lceil k/2 \rceil \rceil \cdot (n-8)\right).$$

components is feasible.

Proof The proof is summarized in Algorithm 1. □

Algorithm 1 $\frac{1}{2}$-democratically-proportional division for $k \geq 2$ families.

INPUT:
- C:= the unit interval [0, 1].
- n additive agents, all of whom value C as 1.
- A grouping of the agents to k families, $F_1, ..., F_k$.

OUTPUT:
A $\frac{1}{2}$-democratically-proportional division of C into k pieces.

ALGORITHM:
Step 1: Halving
- Each agent $i = 1, ..., n$ selects an $x_i \in [0, 1]$ such that $V_i([0, x_i]) = \frac{\lceil k/2 \rceil}{k}$ (this means $\frac{1}{2}$ if k is even and $\frac{k+1}{2k}$ if k is odd). Note: $V_i([x_i, 1]) = \frac{\lfloor k/2 \rfloor}{k}$.
- For each family $j = 1, ..., k$, find the median of its members' selections: $M_j = \text{median}_{i \in F_j} x_i$.
- Order the families in increasing order of their medians. Find the median of the family-medians: $M^* = M_{\lceil k/2 \rceil}$. Cut C at $x = M^*$.
Step 2: Sub-division
- Define the *western families* as the F_j with $j = 1, ..., \lceil k/2 \rceil$. Let n_W be the total number of members in these families. Divide the interval $[0, M^*]$ among the western families using UnanimousPR$(n_W/2, \lceil k/2 \rceil)$.
- Similarly, define the *eastern families* as the F_j with $j = \lceil k/2 \rceil + 1, ..., k$. There are $\lfloor k/2 \rfloor$ such families. Let n_E be their total number of members. Divide the interval $(M^*, 1]$ among the eastern families using UnanimousPR$(n_E/2, \lfloor k/2 \rfloor)$.

The algorithm works in two steps.

Step 1: Halving. For each family, a location M_j is calculated such that, if C is cut at M_j, half the family members value the interval $[0, M_j]$ as at least $\frac{\lceil k/2 \rceil}{k}$ and the other half value the interval $[M_j, 1]$ as at least $\frac{\lfloor k/2 \rfloor}{k}$. Then, C is cut in M^*—the median of the family medians. The $\lceil k/2 \rceil$ "western families"—for which $M_j \leq M^*$—are assigned to the western interval of C—$[0, M^*]$. By construction, at least half the members in each of the western families value $[0, M^*]$ as at least $\frac{\lceil k/2 \rceil}{k}$. We say that these members are "happy". Similarly, the $\lfloor k/2 \rfloor$ eastern families—for which $M_j \geq M^*$—are assigned to the eastern interval $(M^*, 1]$; at least half the members in each of these families are "happy", i.e, value the interval $(M^*, 1]$ as at least $\frac{\lfloor k/2 \rfloor}{k}$.

If there are only two families ($k = 2$), then we are done: there is exactly one western family and one eastern family ($\lceil k/2 \rceil = \lfloor k/2 \rfloor = 1$). For each family $j \in \{1, 2\}$, at least half the members of each family value their family's share as at least $1/2$. Hence, the allocation of X_j to family j is $\frac{1}{2}$-democratically-proportional.

If there are more than two families ($k > 2$), an additional step is required.

Step 2: Sub-division. Each of the two sub-intervals should be further divided among the families assigned to it. In each family F_j, at least $n_j/2$ members are happy. So for each F_j, select exactly $n_j/2$ members who are happy. Our goal now is to make sure that these agents remain happy. This can be done using a unanimously-proportional allocation, where only $n_j/2$ happy members in each family (hence $n/2$ members overall) are counted.

The unanimously-proportional allocation guarantees that every western-happy-member believes that his family's share is worth at least $\frac{\lceil k/2 \rceil}{k} \cdot \frac{1}{\lceil k/2 \rceil} = \frac{1}{k}$. Similarly, every eastern-happy-member believes that his family's share is worth at least $\frac{\lfloor k/2 \rfloor}{k} \cdot \frac{1}{\lfloor k/2 \rfloor} = \frac{1}{k}$. Hence, the resulting division is $\frac{1}{2}$-democratically-proportional.

We now calculate the number of components in the resulting division. One cut is required for the halving step. For the unanimously-proportional division of the western interval, the number of required cuts is at most $(\lceil k/2 \rceil - 1) \cdot (n_W/2 - 1)$ by Theorem 3, and at most $\lceil \log_2 \lceil k/2 \rceil \rceil \cdot (n_W - 4)$ by Theorem 4. Similarly, for the eastern interval the number of required cuts is at most the minimum of $(\lfloor k/2 \rfloor - 1) \cdot (n_E/2 - 1)$ and $\lceil \log_2 \lfloor k/2 \rfloor \rceil \cdot (n_E - 4)$. The total number of cuts is thus at most $1 + (\lceil k/2 \rceil - 1) \cdot (n/2 - 2)$ and at most $1 + \lceil \log_2 \lceil k/2 \rceil \rceil \cdot (n - 8)$. The total number of components is larger by one.

5.4 Comparison and open questions

Table 1 compares the three variants of proportionality, focusing on families with equal entitlements. Recall that n is the total number of agents in all families.

The case of $k = 2$ families is well-understood. The results for all fairness criteria are tight: by all fairness definitions, we know that a fair division exists with the smallest possible number of connectivity components.

The case of $k > 2$ families opens some questions:

- Is unanimous-proportionality with n components feasible for all k? (particularly, with $k = 3$ families, is the number of required components n as in the lower bound, or $2n - 1$ as in the upper bound?).
- Is $\frac{1}{2}$-democratic-proportionality with $n \cdot \frac{k/2-1}{k-1}$ components feasible for all k? (particularly, with $k = 3$ families, is the number of required components $n/4$ as in the lower bound, or $n/2$ as in the upper bound?).

The case of different entitlements is much less understood even for individual agents (Segal-Halevi 2019), let alone for families.

What fairness notion is the most practical? The table shows that it depends on the total number of agents (n). When n is small (as is common when dividing an estate among heirs), it is reasonable to try to attain a unanimously-fair division. However, when n is large (as is common when dividing disputed lands among states), unanimous fairness quickly becomes impractical, as the number of components might grow

Table 1 Properties of a proportional division with different family-fairness notions and different number of families (k). The rightmost column considers the compatibility of the fairness notion with Pareto-efficiency

Fairness criterion	k	#Connectivity Components		Compatible with PE
		Lower	Upper	
average-proportionality (Sec. 3)	k	k	k (connected)	Yes
unanimous-proportionality	2	n	n	Yes
	3	n	$2n-1$	
	4	n	$2n-3$	
(Sec. 4)	k	n	$\min(1+\lceil\log_2 k\rceil\cdot(2n-4),\ (k-1)\cdot(n-1)+1)$	
$\frac{1}{2}$-democratic-proportionality	2	2	2 (connected)	Yes
	3	$n/4$	$n/2$	
	4	$n/3$	$n/2$	
(Sec. 5)	k	$n\cdot\frac{k/2-1}{k-1}$	$\min(2+\lceil\log_2\lceil k/2\rceil\rceil\cdot(n-8),\ 2+(\lceil k/2\rceil-1)\cdot(n/2-2))$	

linearly with n. In this case, we must settle for a weaker fairness criterion. When $k = 2$, we can find a democratically-fair allocation that is also connected. When $k > 2$, democratic fairness too might be impractical, and we may have to settle for average-fairness.

6 Pareto-efficiency

So far, we studied the compatibility of fairness criteria with a *geometric* requirement—reducing the number of connectivity components. In this section we replace the geometric requirement with an *economic* requirement—Pareto efficiency. An allocation is called *Pareto-efficient (PE)* if no other allocation is weakly better for all individual agents and strictly better for some individual agents. Fortunately, PE is compatible even with the strongest variant of the proportionality criterion:

Theorem 10 *There always exists an allocation that satisfies both PE and unanimous-proportionality (hence also average-proportionality and democratically-proportional), even when families have different entitlements.*

Proof We use a famous theorem of Dubins and Spanier (1961), which is a special case of a measure-theoretic theorem by Dvoretzky et al. (1951).

For every partition X of C into k pieces, let $M(X)$ be its *value-matrix*—an n-by-k matrix M where $\forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, k\} : M_{i,j} = V_i(X_j)$. Let $\mathbb{M}_C$ be the set of all matrices that correspond to such partitions:

$$\mathbb{M}_C := \{M(X) | X \text{ is a partition of } C \text{ into } k \text{ pieces}\}$$

Theorem 1 of Dubins and Spanier (1961) implies that the set $\mathbb{M}_C$ is compact.

Define a second set of matrices representing the unanimously-proportional condition:

$$\mathbb{M}_{PR} := \{M \text{ is an } n \times k \text{ matrix} | \forall j \in \{1, \dots, k\} : \forall i \in F_j : M_{i,j} \geq t_j\}$$

Finally, define $\mathbb{M}_{CPR} := \mathbb{M}_C \cap \mathbb{M}_{PR}$. This set represents all value-matrices of allocations of C that are unanimously-proportional. By Theorem 3, $\mathbb{M}_{CPR}$ is non-empty. Since $\mathbb{M}_C$ is compact and $\mathbb{M}_{PR}$ is closed, their intersection $\mathbb{M}_{CPR}$ is compact.

Define the following function $U : \mathbb{M}_{CPR} \to \mathbb{R}$:

$$U(M) := \prod_{j=1}^{k} \prod_{i \in F_j} M_{i,j}$$

This is a continuous function, so it has a maximum point in $\mathbb{M}_{CPR}$; let's call it M^*. This matrix corresponds to an allocation X^* that maximizes, among all unanimously-proportional allocations, the product of valuations of all agents: $\prod_{j=1}^{k} \prod_{i \in F_j} V_i(X_j)$. This product is strictly increasing with the value of each agent $i \in N$, so the allocation

X^* is Pareto-efficient in the set $\mathbb{M}_{CPR}$. Since every Pareto-improvement of an allocation in $\mathbb{M}_{CPR}$ is also in $\mathbb{M}_{CPR}$, the allocation X^* is also Pareto-efficient in general. □

Remark 2 So far we considered two pairs of requirements: fairness+connectivity and fairness+efficiency. This raises the natural question of whether connectivity+efficiency are compatible. We provide two answers.

(a) There might not exist a connected allocation that is Pareto-efficient *in the set of all allocations*, even without any fairness considerations, and even with only two individual agents (two singleton families). Moreover, the number of components in such allocation might be unbounded.

Proof For any integer M, suppose C is the interval $[0, 2M]$. Suppose agent 1 assigns a value of 1 to the intervals [0, 1], [2, 3], [4, 5], etc., and a value of 0 to the rest of C, and agent 2 assigns a value of 1 to the intervals [1, 2], [3, 4], [5, 6], etc., and a value of 0 to the rest of C. Then, any Pareto-efficient allocation has $2M$ connected components. □

On the other hand:

(b) There always exists a connected allocation that is Pareto-efficient *in the set of all allocations with at most d components*, where $d \geq 1$ is any fixed integer. Existence is guaranteed for any number of families with any number of agents, and even with a unanimous-proportionality requirement.

Proof Suppose w.l.o.g. that C is the interval [0, 1]. Each division with at most d components can be represented by a vector $x_1, \ldots, x_d$ where $\forall i \in [d] : x_i \in [0, 1]$ and $\sum_{i=1}^{d} x_i = 1$ (where x_i represents the length of the i-th component). Hence the set of all such divisions can be represented by the $(d-1)$-dimensional standard simplex in $\mathbb{R}^d$. This set is compact and convex. Hence, similarly to Theorem 10, this set contains an allocation that is both PE and unanimously-proportional. □

7 Envy freeness

So far, we used proportionality as our individual fairness criterion. Another criterion that is very common in economics is *envy-freeness*. We study this criterion for families with equal entitlements.

Analogously to the definitions in Sect. 2.4, we call an allocation X—

envy-free-on-average if $\forall j, j' \in \{1, \ldots, k\} : W_j^{\text{avg}}(X_j) \geq W_j^{\text{avg}}(X_{j'})$;
unanimously-envy-free if $\forall j, j' \in \{1, \ldots, k\} : \forall i \in F_j : V_i(X_j) \geq V_i(X_{j'})$;
h − democratically-envy-free if $\forall j, j' \in \{1, \ldots, k\}$,
for at least a fraction h of the members $i \in F_j : V_i(X_j) \geq V_i(X_{j'})$.

With individual agents, it is well known that envy-freeness implies proportionality (with equal entitlements). With two individual agents, envy-freeness and proportionality are equivalent. The same implications are true with families. Each variant of envy-freeness implies the corresponding variant of proportionality.[13] When there are $k = 2$ families, each variant of envy-freeness is equivalent to the corresponding variant of proportionality.[14]

Most of our results for proportionality with equal entitlements are also valid for envy-freeness. For average-envy-freeness, we can use classic results proving the existence of envy-free allocations with connected pieces among individual agents (Stromquist 1980; Su 1999). Applying the same reduction as in Theorem 1 we get:

Theorem 1′ *For any k families, average-envy-freeness with connected pieces is feasible.*

Since envy-freeness implies proportionality, the negative results are still valid:

Theorem 2′ *For every* N, K*, let* $n = N(K-1)+1$*. A unanimously-envy-free division for n agents grouped into K families might require at least n components.*

Some positive results remain valid too. Lemma 2 is based on an exact division. Therefore it holds, with the same proof, even if we replace unanimously-proportional with unanimously-envy-free. Therefore:

Theorem 3′ *Given n agents in k families, a unanimously-envy-free division with* $(n-1)\cdot(k-1)+1$ *components is feasible.*

However, the recursive-halving procedure of Theorem 4 cannot be used here. Suppose we divide C into two subsets, West and East, which all n agents value at exactly $1/2$. Then, we assign arbitrary $k/2$ families to West and the other families to East. We find an exact division of the West among the western families and an exact division of the East among the eastern families. While this division is proportional, it is not necessarily envy-free, since the agents in the west might envy families in the east and vice versa. Therefore, while the number of components required for unanimously-proportional division is in $O(n \log k)$, the best we can currently say about the number of components required for unanimously-envy-free is that it is in $O(nk)$.

The positive result of Theorem 6 holds for envy-freeness too:

Theorem 6′ *For every integer* $k \geq 2$*, there exists a connected division among k families, that is envy-free for at least* $1/k$ *of the members in each family.*

Proof Su (1999) presents a procedure (attributed to Simmons) for finding a connected envy-free division among k individual agents. It is based on presenting various connected partitions to the agents, and asking each agent which of the k pieces is the best.

[13] Suppose an agent $i \in F_j$ thinks the division is envy-free. Then $V_i(X_j)$ is equal to $\max_{j'=1}^{k} V_i(X_{j'})$. The maximum is at least as large as the average, so $V_i(X_j)$ is at least as large as an average value of a piece, which is $V_i(C)/k$.

[14] Suppose an agent $i \in F_j$ thinks the division is proportional. Then $V_i(X_j) \geq 1/2$. By additivity, for the other family $j' \neq j$, $V_i(X_{j'}) \leq 1/2$. Hence $V_i(X_j) \geq V_i(X_{j'})$, so agent i thinks the division is envy-free.

He proves that there exists a partition in which each agent gives a different answer; that partition corresponds to an envy-free allocation. He also shows a procedure for finding a sequence of partitions that converges (after possibly infinitely many steps) to that envy-free allocation.

The Simmons-Su procedure can be adapted to families in the following way. Whenever a family is asked "which of the k pieces is better?", it answers by doing a *plurality voting* among its members. Then, in the final division, each family receives a piece that is considered the best by a plurality of its members, which is at least a fraction $1/k$ of its members. Therefore, at least $1/k$ of each family's members feel that the final allocation is envy-free. □

Similarly, the negative result for h-democratic-proportionality in Theorem 7 is valid for h-democratic-envy-freeness too.

Theorem 9 about $\frac{1}{2}$-democratic-proportionality does not hold as-is for $\frac{1}{2}$-democratic-envy-freeness, but it can be adapted by adapting Algorithm 1. Step 1—the halving step—remains the same. Step 2—the subdivision step—should be modified to use an exact division, as follows:

– Using Exact($n/2$, $\lceil k/2 \rceil$) , find an exact division of the interval $[0, M^*]$ into $\lceil k/2 \rceil$ pieces, such that all $n/2$ happy agents find the pieces equal. Assign these pieces to the *western families*—the F_j with $j = 1, \ldots, \lceil k/2 \rceil$.
– Using Exact($n/2$, $\lfloor k/2 \rfloor$) , find an exact division of the interval $(M^*, 1]$ into $\lfloor k/2 \rfloor$ pieces, such that all $n/2$ happy agents find the pieces equal. Assign the pieces to the *eastern families*—F_j with $j = \lceil k/2 \rceil + 1, \ldots, k$.

The halving step requires a single cut. The two exact divisions require $(n/2) \cdot (k-2)$ cuts. Therefore the total number of components is $n(k-2)/2 + 2$:

Theorem 9′ *Given n agents in k families with equal entitlements, $\frac{1}{2}$-democratic-envy-freeness with at most $n(k-2)/2 + 2$ components is feasible.*

Table 2 summarizes our results for envy-free division and shows some remaining gaps.

We now consider the combination of envy-freeness with Pareto-efficiency. Some of our positive results from Sect. 6 are still valid:

Theorem 10′ *(a) With $k = 2$ families, there always exists an allocation that satisfies both PE and unanimous-envy-freeness (hence also h-democratic-envy-freeness for any $h \in [0, 1]$).*
(b) With any number of families, there always exists an allocation that satisfies both PE and average-envy-freeness.

Proof (a) With $k = 2$ families, envy-freeness and proportionality are equivalent, so this follows directly from Theorem 10.
(b) We use the same reduction as in Theorem 1 and the same compactness argument as in Theorem 10. For each family F_j, define a representative agent A_j whose valuation is W_j^{avg}. There exists an allocation X^* that maximizes the product of valuations of the representatives: $\prod_{j=1}^{k} W_j^{\text{avg}}(X_j)$.

Table 2 Properties of envy-free division with different family-fairness notions and different number of families (k). The rightmost column considers the compatibility of the fairness notion with Pareto-efficiency

Fairness criterion	k	#Connectivity Components		Compatible with PE
		Lower bound	Upper bound	
average-envy-freeness	Any	k	k (connected)	Yes
unanimous-envy-freeness	2	n	n	Yes
(Sec. 4)	3	n	$2n-1$	No
	4	n	$3n-2$	No
	Any	n	$(k-1)\cdot(n-1)+1$	No
$\frac{1}{2}$-democratic-envy-freeness	2	2	2 (connected)	Yes
(Sec. 5)	3	$n/4$	$n/2+2$	?
	4	$n/3$	$n+1$	?
	Any	$n\cdot\frac{k/2-1}{k-1}$	$n\cdot(k-2)/2+2$	No ($k\geq 5$)

(Segal-Halevi and Sziklai 2018a, Section 5) prove, in the setting of cake-cutting among individuals, that every allocation maximizing the product of values is envy-free. Therefore, in the allocation X^*, there is no envy among the representatives. Therefore X^* satisfies average-envy-freeness.

The product $\prod_{j=1}^{k} W_j^{\text{avg}}(X_j)$ is strictly increasing with the value of each individual agent $i \in N$. Therefore, the allocation X^* maximizing this product is Pareto-efficient. □

Next, consider Remark 2 regarding connectivity and efficiency. Part (a) holds regardless of fairness considerations. Part (b) implies that there always exists a unanimously-proportional allocation that is PE in the set of connected allocations; this positive result is not true when we replace proportionality with envy-freeness, even with singleton families. This is proved by Example 5.1 in Segal-Halevi and Sziklai (2018b).

Even without connectivity constraints, Pareto-efficiency is incompatible with unanimous-envy-freeness and democratic-envy-freeness.

The incompatibility between PE and unanimous-envy-freeness appears even when we take a minimal step forward from the case of two families: there are three families, only one of which is a couple and the other two are singles.

Theorem 11 *With three or more families, there might be no allocation that is both PE and unanimously-envy-free.*

Proof The proof is based on an example used by Bade and Segal-Halevi (2018) in the context of fair division of homogeneous goods. C is an interval composed of two sub-intervals Y and Z of length 1. C has to be divided among three families—a couple and two singles—with the following valuations:

	Y	Z
Alice	1	1
George	7	1
Bob	2	1
Esther	5	1

Suppose that we have a unanimously-envy-free allocation of C among the three families. Denote by Y_{AG}, Z_{AG} the lengths of Y, Z given to Alice+George, and similarly Y_B, Z_B, Y_E, Z_E are the lengths given to Bob and Esther. Then:

$$7Y_{AG} + Z_{AG} \geq 7Y_B + Z_B \quad \text{(George does not envy Bob)}$$
$$2Y_B + Z_B \geq 2Y_{AG} + Z_{AG} \quad \text{(Bob does not envy George)}$$
$$(7-2)Y_{AG} \geq (7-2)Y_B \quad \text{(from the above inequalities)}$$
$$(*) \quad Y_{AG} \geq Y_B$$
$$1Y_{AG} + Z_{AG} \geq 1Y_B + Z_B \quad \text{(Alice does not envy Bob)}$$

$$2Y_B + Z_B \geq 2Y_{AG} + Z_{AG} \qquad \text{(Bob does not envy Alice)}$$
$$(1-2)Y_{AG} \geq (1-2)Y_B \qquad \text{(from the above inequalities)}$$
$$(**) \qquad Y_{AG} \leq Y_B$$
$$(***) \qquad Y_{AG} = Y_B \qquad \text{(from * and **)}$$
$$\Longrightarrow Z_{AG} = Z_B \qquad \text{(Bob and Alice+George do not envy)}$$

We proved that, in any unanimously-envy-free allocation, the share given to Alice+George is identical to the share given to Bob (i.e, the same lengths of both subintervals). The proof does not depend on the exact valuation functions—it only depends on the fact that $1 < 2 < 7$, i.e, Bob's valuation of Y is strictly between Alice's and George's valuations. Hence, exactly the same proof works for Esther, i.e: $Y_{AG} = Y_E$ and $Z_{AG} = Z_E$. Therefore, the shares given to all three families are identical.

We now prove that this allocation cannot be PE. We consider three cases.

- Case 1: $Y_B = 0$. Then also $Y_E = Y_{AG} = 0$ so Y remains unallocated and the allocation is not PE.
- Case 2: $Z_E = 0$. Then also $Z_B = Z_{AG} = 0$ so Z remains unallocated and the allocation is not PE.
- Case 3: Y_B and Z_E are positive. Let $\epsilon = \min(Y_B, Z_E/3)$. Suppose that Bob gives ϵ of his Y to Esther, and gets in exchange 3ϵ of her Z. Then, Bob's value increases by $3\epsilon - 2\epsilon$; Esther's value increases by $5\epsilon - 3\epsilon$; and the values of Alice and George are unchanged. This means that the original allocation was not Pareto-efficient. □

□

Weakening unanimously-envy-free to democratically-envy-free does not help when there are 5 or more families.

Theorem 12 *With five or more families, there might be no allocation that is both PE and democratically-envy-free.*

Proof The proof is based on an extension of the example of Theorem 11, where there are five families—one triplet and four singles—with the valuations:

	Y	Z
Alice	1/4	1
Dina	1	1
George	4	1
Bob	1/2	1
Chana	1/3	1
Esther	2	1
Frank	3	1

Suppose that we have a democratically-envy-free allocation of C among the families. By definition of democratically-envy-free, all singles must not feel any envy. Moreover, in the first family, at least two members must not feel any envy. There are three options for the identity of these non-envious members.

Option A: Alice and Dina feel no envy. We consider Bob and Chana. The value of Y for each of them is strictly between the value of Y for Alice and the value of Y for Dina. Therefore, similar arguments as in the proof of Theorem 11 imply that the three allocations of Chana, Bob, and Alice+Dina+George are identical. Now there are three cases:

- Case A1: $Y_C = 0$. Then also $Y_B = Y_{ADG} = 0$. Each of Chana, Bob, and Alice+Dina+George receive at most $1/3$ of Z, so Dina's value is at most $1/3$. Since Dina does not envy Esther and Frank, each of them must receive at most $1/3$ of Y. This means that at least $1/3$ of Y remains unallocated, so the allocation is not PE.
- Case A2: $Z_B = 0$. Then also $Z_C = Z_{ADG} = 0$. Again Dina's value is at most $1/3$, so Esther and Frank must receive at most $1/3$ of Z, so at least $1/3$ of Z remains unallocated, so the allocation is not PE.
- Case A3: Y_C and Z_B are positive. Then, Bob can give $\epsilon/2.5$ of his Z to Chana in exchange for ϵ of her Y (for some small $\epsilon > 0$) and attain a Pareto improvement, so the original allocation is not PE.

Option B: George and Dina feel no envy. We consider Esther and Frank. The value of Y for each of them is strictly between the value of Y for George and the value of Y for Dina. Therefore the three allocations of Esther, Frank, and Alice+Dina+George are identical. The rest of the proof is analogous to Option A.

Option C: Alice and George feel no envy. The value of Y for all the singles is strictly between the value of Y for Alice and the value of Y for George. Therefore, the allocations of all five families must be identical. The rest of the proof is analogous to Theorem 11. □

An interesting question that is left open by Theorems 10′ and 12 is what happens when there are 3 or 4 families—does there always exist an allocation that is both PE and democratically-envy-free?

8 Related work

There are numerous papers about fair division in general and fair cake-cutting in particular. We mentioned some of them in the introduction. Here we survey some work that is more closely related to family-based fairness.

8.1 Dividing other kinds of resources among families

In a contemporaneous and independent line of work, several authors have studied the problem of fairly dividing *discrete* items among families (Manurangsi and Suksompong 2017; Suksompong 2018a, b; Kyropoulou et al. 2019).

They focused on unanimous fairness. They proved that, in many cases, unanimously-fair allocations do not exist. These results complement our impossibility results for unanimous fairness in dividing a *continuous* resource. After the publication of their work and our working paper, we joined forces to study democratic-fair allocation of discrete goods among families (Segal-Halevi and Suksompong 2018).

Recently, Ghodsi et al. (2018) studied fair division of *rooms and rent* among families, using three notions of fairness which they term strong, aggregate and weak. Their strong fairness is our unanimous fairness; their aggregate fairness is our average fairness; their weak fairness means that at least one agent does not envy.

In another recent work, Bade and Segal-Halevi (2018) studied fair and efficient allocation of *homogeneous* divisible resources among families.

8.2 Group-envy-freeness and on-the-fly coalitions

Berliant et al. (1992), Hüsseinov (2011), Todo et al. (2011) study the concept of *group-envy-freeness*. They assume the standard model of fair division among *individuals* (and not among families). They define a group-envy-free division as a division in which no coalition of individuals can take the pieces allocated to another coalition with the same number of individuals and re-divide the pieces among its members such that all members are weakly better-off. Coalitions in cake-cutting are also studied by Dall'Aglio et al. (2009), Dall'Aglio and Di Luca (2014).

In our setting, the families are pre-determined and the agents do not form coalitions on-the-fly. In an alternative model, in which agents *are* allowed to form coalitions based on their preferences, the family-fair-division problem becomes easier. For instance, it is easy to achieve a unanimously-proportional division with connected pieces between two coalitions: ask each agent to mark its median line, find the median of all medians, then divide the agents to two coalitions according to whether their median line is to the left or to the right of the median-of-medians.

8.3 Fair division with public goods

In our setting, the piece given to each family is considered a "public good" in this specific family. The existence of fair allocations of homogeneous goods when some of the goods are public has been studied e.g. by Diamantaras (1992), Diamantaras and Wilkie (1994), Diamantaras and Wilkie (1996), Guth and Kliemt (2002). In these studies, each good is either private (consumed by a single agent) or public (consumed by all agents). In the present paper, each piece is consumed by all agents in a single family—a situation not captured by existing public-good models.

8.4 Matching markets

Besides fair division, family preferences are important in matching markets, too. For example, when matching doctors to hospitals, usually a husband and a wife who are both doctors want to be matched to the same hospital. This issue poses a substantial challenge to stable-matching mechanisms (Klaus and Klijn 2005, 2007; Kojima et al. 2013; Ashlagi et al. 2014).

The idea of satisfying a certain fairness notion for only a certain fraction of the population, rather than unanimously, has also been studied in the context of matching markets. For example, Ortega (2018) proves that, when two matching markets are merged and a stable matching mechanism is run, it is impossible to attain monotonic improvement for everyone, but it is possible to attain monotonic improvement for at least half the population.

8.5 Non-additive utilities

As explained in Sects. 4 and 5, the difficulty with unanimous-proportionality and democratic-proportionality is that the associated family-valuation functions are not additive. It is therefore interesting to compare our work to other works on cake-cutting with non-additive valuations.

Berliant et al. (1992); Maccheroni and Marinacci (2003); Dall'Aglio and Maccheroni (2005) focus on sub-additive, or concave, valuations, in which the sum of the values of the parts is *more* than the value of the whole. These works are not applicable to our setting, because the family-valuations are not necessarily sub-additive—the sum of values of the parts might be smaller than the value of the whole (see the example in the beginning of Sect. 4).

Sagara and Vlach (2009), Dall'Aglio and Maccheroni (2009), Hüsseinov and Sagara (2013) consider general non-additive value functions. They provide pure existence proofs and do not say much about the nature of the resulting divisions (e.g, the number of connectivity components), which we believe is important in practical division applications.

Su (1999) presents a protocol for envy-free division with connected pieces which does not assume additivity of valuations. However, when the valuations are non-additive, there are no guarantees about the value per agent. In particular, with non-additive valuations, the resulting division is not necessarily proportional.

Mirchandani (2013) suggests a division protocol for non-additive valuations using non-linear programming. However, the protocol is practical only when the resource to divide is a collection of a small number of *homogeneous* components, where the only thing that matters is what fraction of each component is allocated to each agent. In contrast, in our model the resource is a single *heterogeneous* good.

Finally, Berliant and Dunz (2004), Caragiannis et al. (2011), Segal-Halevi et al. (2017) study specific non-additive value functions which are motivated by geometric considerations (location, size and shape). The present paper contributes to this line of work by studying specific non-additive value functions which are motivated by a different consideration: handling the different preferences of family members. A possible future research topic is to find fair division rules that handle these considerations simultaneously, as both of them are important for fair division of land.

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