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Individual preferences and democratic processes: two theorems with implications for electoral politics

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Abstract

The paper provides a complete characterization of Nash equilibria for games in which n candidates choose a strategy in the form of a platform, each from a circle of feasible platforms, with the aim of maximizing the stretch of the circle from where the candidate's platform will receive support from the voters. Using this characterization, it is shown that if the sum of all players' payoffs is 1, the Nash equilibrium payoff of each player in an arbitrary Nash equilibrium must be restricted to the interval $[1/2(n-1), 2/(n+1)]$. This implies that in an election with four candidates, a candidate who is attracting less than one-sixth of the voters can do better by changing his or her strategy.

1 Introduction

The subject of human welfare and political institutions has been an ancient field of inquiry. However, we owe more to Kenneth Arrow than anybody else the bringing of formal reasoning and axiomatic rigor to the discipline (Arrow 1950, 1951). There had been other related works before that. There was the famous 1929 paper by Harold Hotelling, Arrow's PhD adviser, who provided the tools for a purely positive exercise concerning the outcomes of democratic elections. There were the works of Bergson (1938) and Samuelson (1947) that created the concept of a social welfare function, which would lend rigor to early ideas pertaining to utilitarianism. This led to a substantial literature on the normative and positive aspects of using democratic processes to aggregate individual preferences (Sen 1970; Pattanaik 1971; d'Aspremont and Gev-

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ers 1977; Maskin 1978; Suzumura 1983). Our aim in this paper is to make a small but pointed contribution to this broad area of divergent preferences and democratic aggregation.

Starting with Hotelling's (1929) classic paper, the economics of location, for instance, where and how firms position themselves on a line or on a surface, or the ideological space which political parties try to capture, has become an important tool for analyzing product quality differentiation and antitrust law, retail marketing and the theory of vendor positioning and, most prominently, political economy and electoral politics. In the area of political economy, it has led to the celebrated median voter theorem of Black (1948) for majority decisions, and of Downs (1957) for a representative democracy, which enabled us to take on some of the big questions of electoral democracy, and development (see, for instance, Stokes 1963; Osborne 1995; Congleton 2002; Acemoglu and Robinson 2005).

The present paper is an exercise in the pure theory of location choice over a circle or rotary. Given the wide use of location analysis in economics and political science it is hoped the paper will provide foundations for different kinds of practical studies on electoral democracy or retail trading. Each of these entail further complications. In the case of retail stores choosing their location, the analysis is complicated by the fact that they can, typically, choose the price as well. In electoral politics, personal attributes (for example, the appearance) of the candidate may matter as well and not just the ideological platform the candidate offers. However, the theorems reported in this paper should be of value for all such models. For the most part, we refer to the problem of electoral democracy and use its associated language, although our study is best viewed at a more abstract level.

The importance of the pivotal voter was recognized as early as 1785 by Condorcet. However, following the influential work of Hotelling, the bulk of the analysis got focused on voters spread over a line. At one level, our paper is a reminder that there can be other formations among voters and these can lead to related but distinct results. In our set up, for instance, it is not the case that candidates invariably converge on the same electoral platform but, instead, we get the result that no matter how many candidates contest an election, there will never be more than two candidates announcing the same platform.[1]

As Stokes (1963, p. 368) had noted over a half a century ago, "The use of spatial ideas to interpret party competition is a universal phenomenon of modern politics. Such ideas are the common coin of political journalists…" Congleton (2002) points out in his survey of the median voter theorem, "Most analytical work in public choice is based upon relatively simple models of majority decision making. These models are widely used even though the researchers know that real political settings are more complex than the models seem to imply. The … simple models provide us with engines of analysis that allow a variety of hypotheses about more complex phenomena to be developed, many of which would be impossible (or uninteresting) without the frame of reference provided by simple models." It is in this spirit that we venture into investigating electoral democracy in this paper.

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[1] Similar issues of cluster and spread arise in the literature on spatial location and agglomeration (Fujita and Thisse 1996; Pal 1998; Matsushima 2001).
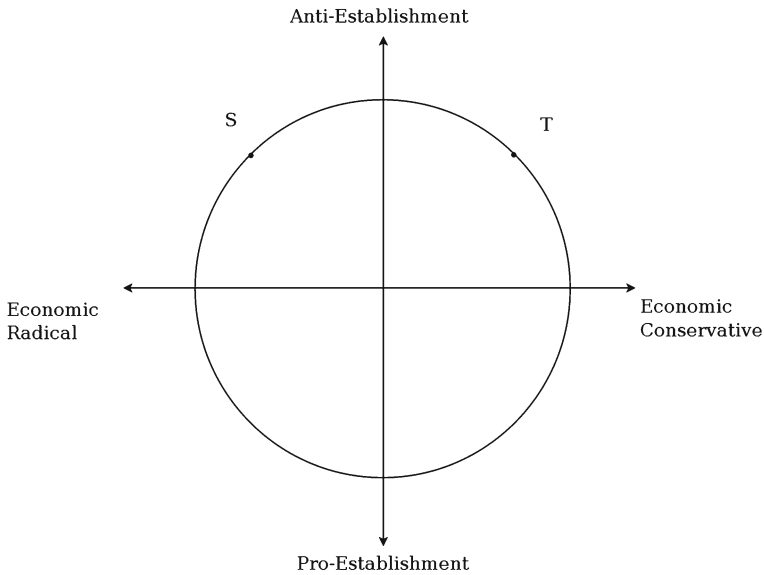
Fig. 1 Two dimensional attributes

To understand the idea of using location analysis to investigate electoral democracy, suppose voters are interested in two aspects of candidates running for office: how anti-establishment they are, and their economic ideology, for instance, their position on economic conservatism. Clearly, each voter's ideal can be represented as a point in a two-dimensional space. Suppose all voters ideals lie on a circle, of the kind shown in Fig. 1, and suppose each candidate has to announce his or her agenda which is basically a point on the circle. Thus we can think of candidates S and T announcing their respective agenda as points S and T on the circle as shown. From the vertical perspective, S and T look identical but they are far apart in terms of economic ideology, measured on the horizontal axis. (Any resemblance of S and T to Sanders and Trump is entirely spurious.) If we assume that voters opt for the candidate closest to his or her ideal, the closeness being measured in terms of distance along the circle, then we have a classic location-choice game.[2]

Before delving into the analysis, it is useful to make a few pertinent remarks about our choice of the circle as the space of voter ideals, and the related choice of the arc distance along the circle as the measure of the closeness of a candidate's platform to a voter's ideal.

The most general assumption would be to think of a finite number of individuals scattered arbitrarily in a specified subset of an *M* dimensional Euclidean space (where

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[2] Alternatively, we could think of the circle as a rotary along which people in a town live. The only way to travel in this town is along the rotary (in any direction). Two retail vendors, selling goods which have exogenously fixed prices, have to set up shops at points on the rotary. If we make the reasonable assumption that customers will go to the nearest store, and each vendor is interested in maximizing the number of customers she has, then again we have a classic game theory problem. Similar arguments apply to brand proliferation (see d'Aspremont et al. 1979; Schmalensee 1978; Salop 1979; Basu 1993; Gabszewicz et al. 1986; Brander and Spencer 2015).

M is the number of distinct issues that a voter cares about); and then, to have a finite number of candidates choose their platforms as points in that subset. [For $M = 2$, the specified subset could be a circle, a disc, or the entire two-dimensional Euclidean space]. In general, that would be very difficult to analyze and might be riddled with non-existence problems. Hotelling moved away from that framework by considering an infinite number of voters distributed on a finite length straight line (thereby focusing on a single over-riding issue that a voter cares about). That became such an innate part of our thinking that we seem to be somewhat rudderless in analyzing the rise of politics in which some candidate is conservative in terms of economics but politically a radical and vice versa for another candidate.

To analyze such cases, we need (for a start) to deal with at least two *independent* issues that voters care about. That is, it should not be possible to *predict* the voter's view on the second issue by knowing her view on the first issue (otherwise, Hotelling's formulation would suffice, as there would be only one over-riding issue). Our paper can be viewed as a first cut at formulating such a framework of analysis. If the space of voter ideals is a circle, it captures an essential aspect of this problem; as illustrated in Fig. 1, platforms S and T both reflect anti-establishment positions, but it is not possible to predict the economic ideology of the platforms from this, as that might vary widely.

We are, of course, aware that by treating the circle as the space of voter ideals we are using a strong assumption, which has some unintended features. To remedy the situation, one might explore in future research formulations in which (for instance) one moves from the unit circle as the space of voter ideals to the closed unit disc (which includes all points on and inside the unit circle). A pragmatic way to do this would be (as a generalization of our current framework), to consider the space of voter ideals to be a finite number of concentric circles, a framework which approaches the unit closed disc model as the number of concentric circles becomes large. The crucial issue one would have to face in such a formulation is how to define a reasonable notion of closeness between platforms which lie on distinct concentric circles.[3]

It may also be pointed out that the set up we use has some important history in industrial organization theory, law and economics, as exemplified by the papers of Schmalensee (1978) and Salop (1979). Since the abstract results in our paper have as much bearing on political economy as on industrial organization, the paper may be viewed as a contribution to the latter as well.

The specific problem that we want to address in this paper is the following. If there are n candidates in an election how will they position themselves on the circle in equilibrium? And, once politicians have taken up positions, is there an easy way to check if this is an equilibrium? The entire exercise in this paper is done by defining equilibrium in the sense of Nash. What the paper shows is that there is a surprisingly easy answer to the question just posed (see Theorem 1).

Hotelling's classic location game (for two vendors with exogenously fixed prices) has a unique Nash equilibrium. However, for many non-cooperative games, there is a multiplicity of Nash equilibria, and this is seen as a principal drawback of the

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[3] A voter whose ideal is on an inner concentric circle might feel perfectly aligned with the platform of the corresponding radial point on the unit circle (closest to it). The view might be that the two platforms differ only in "intensity" of preference on the two issues, but not in "direction". In this case, we recover precisely the model with only the unit circle as the space of voter ideals.

equilibrium concept, as its predictive power can be limited as a result. In our setting too, there can be many (in fact, a continuum of) Nash equilibria. However, instead of focusing on equilibrium outcomes, if we shift our focus to equilibrium payoffs, a rather general result on the *distribution of payoffs* in Nash equilibria emerges in this model. We find that the entire set of Nash equilibria is subject to rather stringent bounds on the equilibrium payoffs. Specifically, if the sum of all players' payoffs is equal to 1, the Nash equilibrium payoff of each player must lie in the interval:

$$\Pi = \left[\frac{1}{2(n-1)}, \frac{2}{n+1} \right]$$

for an arbitrary number *n* of players, and for an *arbitrary* Nash equilibrium (see Theorem 2). As an implication, in a four-way race, no candidate can get less than one-sixth of the vote in a Nash equilibrium. This makes one-sixth a useful benchmark to decide whether to change one's strategy in a four-way race.

2 The model

2.1 Preliminaries

We consider a democratic election scenario in the following way. There are a number (say *n*) of political *candidates*. Each candidate has a political *platform*. When each candidate has a distinct platform, candidates can be identified by their political *platforms*, so that these words can be used interchangeably. Clearly, this is an abstraction, since there can be more to a candidate than her political platform, even from the point of view of an election. Further, the official platform can often be different from the actual platform, or the platform as perceived by the voters.

Platforms are typically multi-dimensional (as we have illustrated in Sect. 1), although in prevalent political discussions, one often discusses platforms in one (major) dimension (or in one dimension at a time), since this simplifies the argument one is trying to make. Actually, the distortion produced in such analysis is substantial, and this has long been recognized in the political science literature that has followed since the uni-dimensional median voter theorem.[4]

We might describe a platform by a list of *issues*; when speaking of candidates, one might use the term *attributes* instead. Thus, for example, one might specify a platform by a list of seven major issues, by indicating a candidate's stand on those seven issues. Unlike in the economics of location, where there would be fairly clear cut numerical measures to identify locations of vendors, and prices charged at those locations, the problem of quantitative representation of political issues can be a substantial one. On some issues (such as, what should be the magnitude of the capital gains tax), there might be clear-cut numerical measures to summarize a candidate's stand on the issue. But, on others (such as, the extent of gun control), there might not be such summary measures. However, even in such cases, it is common to use not only ordinal

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[4] For an extensive discussion of, and empirical evidence on, this issue, see Stokes (1963, pp. 370–371).

measures (preference for more gun-control to less), but also measures which would be interpreted as cardinal (strongly opposed to gun control). While being aware of the abstraction involved, we choose to represent a platform by a point in M dimensional real space, if there are M clearly defined issues.[5]

Not all points in M dimensional space may be considered to be platforms. It is more reasonable to suppose that there is a *feasible set* of platforms, $\mathcal{F}$, (a subset of $\mathbb{R}^M$) which is well understood by the candidates as well as the electorate.[6] Because we need to understand simpler structures first, in order to usefully investigate more complex and realistic structures later, we simplify by choosing $M = 2$; whereas M can in reality take many values, there is a special significance to moving from 1 to any larger value, including 2. It immediately allows for political situations where candidates differ on some issues and agree on others. Further, again in the interest of simplification, we consider the feasible set of platforms $\mathcal{F}$ to be a circle (with, say, origin at (0, 0) and circumference equal to 1), just as represented in Figs. 2, 3, 4, 5, 6, 7 and 8 in Sects. 3 and 4 below. The circle is capable of including fine distinctions between similar platforms, as well as diametrically opposing platforms.[7]

We turn now to the electorate. We suppose a voter has preferences over the set of feasible platforms, in the form of an "ideal platform", which is best from the voter's point of view, other platforms being inferior. A voter votes for the candidate whose platform is "closest" to the voter's ideal platform. We define "closest" in terms of the length of the arc between the voter's ideal and the candidate's platform. [We elaborate on this measure in Sect. 3.1.3].

Finally, we need to consider the distribution of voters, or more precisely, the distribution of the ideal platforms of voters. As an extreme case, consider (in the context of Fig. 1) that even though there is a lot of variety of feasible platforms on a circle, voters don't really care about whether a candidate is pro-establishment or anti-establishment. They would then be bunched up (in terms of their ideal platforms) at two points on the circle on the horizontal axis. Because we want to emphasize the importance of multi-dimensional platforms, we do not wish to investigate such extreme cases. A more reasonable assumption is that the voters ideal platforms are distributed with *full support* on the circle. Assuming a uniform distribution on the circle (which we do in our model below) is a useful place to start to keep the model tractable.

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[5] The list of important issues can of course change over time. So, the same "fixed structure" [to use the terminology of Stokes (1963)] might not be applicable at different points in time (political eras).

[6] This is not an innocent supposition. Stokes (1963, p. 370) notes: "What is more, when our respondents are asked directly to describe the parties in terms of the liberal-conservative distinction, nearly half confess that the terms are unfamiliar. And the bizarre meanings given the terms by many of those who do attempt to use them suggest that we are eliciting artificial answers that have little to do with the public's everyday perceptions of the parties."

[7] The circle has been an object of intellectual interest from pre-Socratic and pre-Pythagorean times. It was analyzed in some depth by Thales of Miletus. Born in Miletus in Asia Minor (currently Turkey), in 624 BC, he was 54 years senior to Pythagoras, and is often regarded as the first "Greek philosopher" that proved some theorems concerning right-angled triangles and the circle, using pure deductive reasoning. In that tradition, quite apart from its implications for electoral politics and location economics, this paper may be viewed as an exercise in pure abstract reasoning.

2.2 Description of the game

There are n players (political candidates). Each player has to set up a platform at some point on the circle (with origin at $(0, 0)$ and circumference equal to 1). Each voter has an ideal platform on the circle and votes for the candidate whose platform is closest to his own. If a voter is equi-distant to several players' locations, he chooses randomly from them by assigning equal probability to each of those players. There is a continuum of voters, and their ideal platforms are distributed uniformly on the circle.

Each player's (political candidate's) payoff is equal to the expected number of voters who vote for her and her aim is to maximize her expected payoff.

Players choose their platforms simultaneously. A choice of platform by each player will be called an n-tuple of strategies or simply a *placement*. The aim is to analyze the Nash equilibria of this game, that is a placement such that no player can do better (increase her payoff) through a unilateral deviation to another platform.

It is useful to coin a few more terms before we begin the analysis. A placement in which there is no platform that is chosen by more than one player will be referred to as a *scattered placement*. All other kinds of placements are called *clustered placements*. Given a placement, a stretch of the circle where no platform has been chosen by any of the players will be called an *empty stretch*. A Nash equilibrium in which the placement is scattered will be called a *scattered Nash equilibrium*. All other Nash equilibria are called *clustered Nash equilibria*.

3 Scattered Nash equilibria

This section provides a complete characterization of scattered Nash equilibria. If $n = 2$ it is easy to see that all placements are Nash equilibria. Henceforth, and without further mention, we will focus on cases where $n \geq 3$.

3.1 Notation and definitions of concepts

Recall that we want to present a criterion, such that given a placement, the criterion can be used to easily check whether the placement is (or is not) a Nash equilibrium. For this purpose, some notation is needed to formalize the concepts introduced above.

3.1.1 Circle and arcs

We consider a circle with circumference equal to C.[8] An *arc* of a circle is any connected set of points of the circle. Associated with any arc are its boundary points. The boundary points need not belong to the arc itself; for example, the arc can be open relative to the circle. If the closure of the arc is not the entire circle, it will have two boundary points.

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[8] This circumference is specified precisely, as soon as we pick a radius of the circle, a positive real number.

Given any two distinct points A and B on the circle, two arcs can be associated with them. One arc is the set of points traversed as one moves from A to B clockwise; we denote this by $arc[A, B]$. [This set will be understood to include the points A and B].The other arc is the set of points traversed as one moves from A to B counter clockwise. This is, of course, the same as the set of points traversed as one moves from B to A clockwise. We denote this by $arc[B, A]$. The set $arc[A, B]\backslash\{A, B\}$ is denoted by $arc(A, B)$. Similar obvious notation applies to arcs containing B, but not A, and to arcs containing A, but not B. Note that $arc(A, B)$ is *disjoint* from $arc[B, A]$, and therefore from $arc(B, A)$, $arc(B, A]$ and $arc[B, A]$.

The *arc length* of $arc[A, B]$ is defined to be equal to $(C\theta/360)$, if θ is the angle (in degrees) which $arc[A, B]$ subtends at the centre of the circle. This is also defined to be the arc length of $arc(A, B)$. The arc length of $arc[A, B]$ is denoted by $a[A, B]$.[9]When the two arc lengths $a[A, B]$ and $a[B, A]$ are not equal, the arc with the larger arc length is called the *major arc*, and the arc with the smaller arc length is called the *minor arc*. Thus, the arc length of the minor arc is $\min\{a[A, B], a[B, A]\}$, and we denote it by $ma[A, B]$; it is also equal to $ma[B, A]$. In what follows, without loss of generality, we take the circumference C of the circle to be equal to 1, and therefore the radius of the circle to be $r = (1/2\pi)$.

3.1.2 Candidates, platforms and neighborhoods

In a scattered placement, each candidate $i \in I \equiv \{1, \ldots, n\}$ can be identified with a distinct point (platform) on the circle. Thus, the placement itself can be described by an n-tuple of platforms $\{s_1, \ldots, s_n\}$, where each s_i for $i \in I$ is a distinct point on this circle.[10]

Given a scattered placement $s = \{s_1, \ldots, s_n\}$, there is a *neighborhood* $Y_i(s)$ for each candidate $i \in I \equiv \{1, \ldots, n\}$, which is defined to be the maximal arc, containing s_i, but not containing any $s_j \neq s_i$; that is, it is an arc S, containing s_i, but not containing any $s_j \neq s_i$, such that if S' is any arc, containing s_i, but not containing any $s_j \neq s_i$, then $S' \subset S$.[11] As we have defined it, a neighborhood $Y_i(s)$ is a set which is open relative to the circle. The closure of $Y_i(s)$ (relative to the circle) is denoted by $\bar{Y}_i(s)$. Thus, $\bar{Y}_i(s)$ contains $Y_i(s)$ and the boundary points of $Y_i(s)$.

Given a scattered placement $s = \{s_1, \ldots, s_n\}$, for each candidate $i \in I \equiv \{1, \ldots, n\}$, we can define his *neighbors* to be:

$$s_{R(i)} = \arg\min_{s_j \in s\backslash\{s_i\}} a(s_i, s_j) \quad \text{and} \quad s_{L(i)} = \arg\min_{s_j \in s\backslash\{s_i\}} a(s_j, s_i) \qquad \text{(NBS)}$$

These are the candidates with platforms closest to the platform of candidate i, on the right and on the left respectively. Clearly, we have:

$$s_{R(i)} \neq s_i \quad \text{and} \quad s_{L(i)} \neq s_i$$

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[9] When the two arc lengths are not equal, the arc with the larger arc length is called the *major arc*, and the arc with the smaller arc length is called the *minor arc*.

[10] Each s_i can be specified precisely in terms of its Cartesian co-ordinates.

[11] Note that a neighborhood for player i can be a minor arc, and it can also be a major arc, depending on the positions of the other players on the circle.

and the neighbors are *distinct*:

$$s_{R(i)} \neq s_{L(i)}. \tag{DNBS}$$

The statement in (DNBS) holds because if $s_{R(i)} = s_{L(i)}$, then $\bar{Y}_i(s)$ is the circle itself, implying that there are just two distinct platforms on the circle, and this is impossible in a scattered placement with $n \geq 3$.

Using (NBS), it can be shown that:

$$Y_i(s) = arc(s_{L(i)}, s_{R(i)}) \quad \text{and} \quad \bar{Y}_i(s) = arc[s_{L(i)}, s_{R(i)}]. \tag{NBD}$$

Informally, the set $\bar{Y}_i(s)$ contains the platforms of $i's$ "nearest" neighbors, while $Y_i(s)$ does not. (NBD) is a useful characterization of a neighborhood; we make use of this characterization in our analysis below. We define $y_i(s)$ to be the arc length of the arc $Y_i(s)$.

3.1.3 Behavior of voters

Each voter has an ideal platform, v, a point on the circle. The general principle that we wish to use with respect to voting behavior is that each voter votes for the candidate whose platform is "closest" to her ideal platform v. To implement this principle, we need to be explicit about what "closest" means.

However, without specifying a particular distance function, we can say that a *necessary condition* for the voter to vote for platform s_i (of candidate i) is that there be an arc joining v with s_i (either $arc(v, s_i)$ or $arc(s_i, v)$) which contains no other platform s_j (for $j \neq i$). For if there is always some platform s_j in between v and s_i (that is, there is some s_j in $arc(v, s_i)$ *and* there is some s_k in $arc(s_i, v)$), then s_i cannot be closest to v by any reasonable notion of closeness.

A consequence of this is that a voter whose ideal platform v falls outside $\bar{Y}_i(s)$ will definitely *not* vote for candidate i, since $s_{L(i)}$ is in $arc(v, s_i)$ and $s_{R(i)}$ is in $arc(s_i, v)$.

So, the only voters who can possibly vote for candidate i are those whose ideal platform v falls in $\bar{Y}_i(s)$. If $v \in (s_i, s_{R(i)}]$, we postulate that the voter votes for s_i when $a(s_i, v) < a(v, s_{R(i)})$, votes for $s_{R(i)}$ when $a(s_i, v) > a(v, s_{R(i)})$, and votes for either platform with equal probability when $a(s_i, v) = a(v, s_{R(i)})$. A similar rule is postulated if $v \in [s_{L(i)}, s_i)$; that is, the voter votes for s_i when $a(v, s_i) < a[s_{L(i)}, v)$, votes for $s_{L(i)}$ when $a(v, s_i) > a[s_{L(i)}, v)$, and votes for either platform with equal probability when $a(v, s_i) = a[s_{L(i)}, v)$. Finally, if $v = s_i$, then the voter votes for s_i.

Unlike the economics of location of vendors, and customers purchasing from the nearest vendor, where the above postulate would be entirely natural, it is a debatable postulate in electoral politics. The use of the arc length is not itself central to a possible objection to the postulate. It is the fact that a clockwise arc length is compared to an anti-clockwise arc length in deciding on voting behavior. If, for example, a voter always seeks the closest platform, moving in a clockwise direction from his ideal platform, then this postulate of voting behavior would not hold. Because closeness of a platform to another platform is primarily a mental concept, without a convenient physical counterpart, the analogy with spatial location theory need not carry over.

3.1.4 Payoffs of candidates

Given the postulate of voting behavior, and the assumption that the ideal platforms of voters are uniformly distributed on the circle, the payoff $p_i(s)$ of candidate i is one-half the arc length of $Y_i(s)$; that is, $p_i(s) = y_i(s)/2$. It measures the percentage of the votes that candidate i expects to get.

In studying scattered Nash equilibria, it is important to be able to deal with some clustered placements. This is because, for each player, one has to consider *all* unilateral deviations in platforms from her given platform (in the scattered placement). Consider a scattered placement $\{s_1, \ldots, s_n\}$, and focus on player i. Player i can consider a unilateral deviation to a platform s_i'. Now s_i' can coincide with s_k for some $k \neq i$. Thus, after such a deviation, we have a clustered placement, and one must be able to compute the payoff to player i from such a deviation and compare that payoff to her payoff from the scattered placement $\{s_1, \ldots, s_n\}$.[12] To see how this is done, notice that after such a unilateral deviation by player i, we have a clustered placement $s' = \{s_1, ., s_i', .., s_n\}$ and we can define a neighborhood $Y_i(s')$ as above to be the maximal arc, containing s_i', but not containing any $s_j \neq s_i'$. Note that $Y_k(s')$ coincides with $Y_i(s')$. The arc length of $Y_i(s')$ is denoted by $y_i(s')$. The payoff to player i in the new placement s' is defined to be $p_i(s') = y_i(s')/4$, since both i and k have to share the same neighborhood $Y_i(s')$ in the placement s'.

3.2 Characterizing scattered Nash equilibria

Given a scattered placement $s = \{s_1, \ldots, s_n\}$, an *empty stretch* is any arc that does not contain any s_i for $i \in \{1, \ldots, n\}$. The arc length of the empty stretch of maximum arc length is denoted by $x(s)$. We define:

$$y(s) = \min\{y_1(s), \ldots, y_n(s)\}.$$

A characterization of scattered Nash equilibria can be obtained by just comparing $y(s)$ with $x(s)$.

Theorem 1 (A) *Given a scattered placement* $s = \{s_1, \ldots, s_n\}$, *if:*

$$y(s) \geq x(s) \tag{1}$$

then the placement is a Nash equilibrium.
(B) *Given a scattered placement* $s = \{s_1, \ldots, s_n\}$, *if:*

$$y(s) < x(s) \tag{2}$$

then the placement is not a Nash equilibrium.

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[12] Note that the deviation can result in two players (but not more) being at the same location, since the original placement is scattered.

Proof (A) Pick any $j \in I$, and fix it in what follows. By (1), and the definition of $y(s)$, we have:

$$p_j(s) = (y_j(s)/2) \geq (x(s)/2). \tag{3}$$

Suppose j unilaterally changes his platform from s_j to some other point s'_j on the circle. Denote the new placement created by $s' = \{s_1, \ldots, s'_j, \ldots, s_n\}$. We have either (a) s' is a scattered placement, or (b) s' is a clustered placement.

Case (a): If $s'_j \in Y_j(s)$, then the payoff of j does not change. If s'_j is not in $Y_j(s)$, then since s' is a scattered placement, $s'_j \in Y_k(s)$, the neighborhood of some other player $k \in I$ (with $k \neq j$). The neighborhood $Y_k(s)$ consists of two empty stretches, and the new neighborhood of s'_j will have to be one of these two empty stretches (no longer empty after it is occupied by s'_j). Thus, we must have $y_j(s') \leq x(s)$, and so:

$$p_j(s') \leq (x(s)/2). \tag{4}$$

Clearly, (3) and (4) imply that there is no incentive for j to change his platform to s'_j. This establishes that the placement s is a Nash equilibrium in case (a).

Case (b): In this case, there is some $k \in I$, with $k \neq j$, such that $s_k = s'_j$. There are two subcases to consider: (I) $Y_j(s)$ is disjoint from $\bar{Y}_k(s)$; (II) $Y_j(s) \cap \bar{Y}_k(s)$ is non-empty. In subcase (I),

$$Y_k(s') = Y_k(s) \tag{5}$$

and so $Y_j(s') = Y_k(s)$.[13] The neighborhood $Y_k(s)$ contains two empty stretches, and the point s_k. Thus, $y_k(s) \leq 2x(s)$, and we get:

$$p_j(s') = y_j(s')/4 = y_k(s)/4 \leq 2x(s)/4 = (x(s)/2). \tag{6}$$

Clearly, (3) and (6) imply that there is no incentive for j to change his platform to s'_j. This establishes that the placement s is a Nash equilibrium in case (b)(I).

In subcase (II), $Y_j(s) \cap \bar{Y}_k(s)$ is non-empty. Let us write:

$$Y_k(s) = arc(s_{L(k)}, s_{R(k)}) \quad \text{and} \quad Y_j(s) = arc(s_{L(j)}, s_{R(j)}). \tag{7}$$

We now claim[14] that:

$$\text{Either (i) } s_j = s_{L(k)} \text{ and } s_{R(j)} = s_k; \quad \text{or (ii) } s_k = s_{L(j)} \text{ and } s_{R(k)} = s_j. \tag{8}$$

---

[13] While the claim (5) is intuitively clear, it takes a bit of work to establish this analytically. It is included in an Appendix.

[14] Claim (8) is intuitively clear, but it takes a bit of work to establish this analytically. It is included in an Appendix.

Taking the claim for granted, we can proceed to complete the proof of case (b)(II) as follows. In case (i) of (8), after candidate j changes his platform from s_j to $s'_j = s_k$, the entire $arc(s_{L(j)}, s_j]$ becomes part of the neighborhood of player j, in addition to $Y_k(s)$. That is,

$$
\begin{aligned}
Y_j(s') &= arc(s_{L(j)}, s_j] \cup Y_k(s) \\
&= arc(s_{L(j)}, s_{L(k)}] \cup arc(s_{L(k)}, s_{R(k)}) \\
&= arc(s_{L(j)}, s_{L(k)}] \cup arc(s_{L(k)}, s_k) \cup arc[s_k, s_{R(k)}) \\
&= arc(s_{L(j)}, s_{L(k)}] \cup arc(s_{L(k)}, s_{R(j)}) \cup arc[s_k, s_{R(k)}) \\
&= arc(s_{L(j)}, s_j] \cup arc(s_j, s_{R(j)}) \cup arc[s_k, s_{R(k)}) \\
&= Y_j(s) \cup arc[s_k, s_{R(k)})
\end{aligned}
\tag{9}
$$

where we have used (8)(i) in lines 2, 4 and 5 of (9). [This is also the new neighborhood $Y_k(s')$ of player k].

Since $arc(s_k, s_{R(k)})$ is one of the two empty stretches of $Y_k(s)$, its arc length is at most $x(s)$, and so the arc length of $arc[s_k, s_{R(k)})$ is also at most $x(s)$. The arc length of $Y_j(s)$ is $y_j(s)$. Thus, the arc length of $Y_j(s')$ is $\leq x(s) + y_j(s)$. Consequently,

$$
p_j(s') \leq [x(s) + y_j(s)]/4 \leq 2y_j(s)/4 = y_j(s)/2
\tag{10}
$$

where the second inequality in (10) follows from (1). Clearly, (3) and (10) imply that there is no incentive for j to change his platform to s'_j. This establishes that the placement s is a Nash equilibrium, when case (i) of claim (8) holds.

If case (ii) of claim (8) holds, the proof is similar to the one just given for case (i) of claim (8). This establishes that the placement s is a Nash equilibrium in case (b)(II), and completes the proof of part (A) of the theorem.

(B) By (2), and the definition of $y(s)$, there is $h \in I$, such that $y_h(s) = y(s) < x(s)$. Then:

$$
p_h(s) < (x(s)/2).
\tag{11}
$$

Now h can relocate from platform s_h to a platform s'_h (say) in an empty stretch with arc length $x(s)$. Then the new neighborhood of h is the empty stretch with arc length $x(s)$ (no longer empty after it is occupied by s'_h). Consequently,

$$
p_h(s') = (x(s)/2).
\tag{12}
$$

Clearly, (11) and (12) imply that h can increase her payoff by a unilateral deviation from platform s_h to platform s'_h. So, the (original) placement s cannot be a Nash equilibrium. This completes the proof of part (B) of the theorem. $\square$

A consequence of Theorem 1 is the following result, which provides a *sufficient condition* for a scattered placement s to be a Nash equilibrium. Note that, to use this criterion, one does not need to calculate $y(s)$.

Corollary 1 *Given a scattered placement $s = \{s_1, \ldots, s_n\}$, if:*

$$x(s) \leq \frac{1}{n-1} \tag{13}$$

then the placement s is a Nash equilibrium.

Proof We claim that:

$$y(s) \geq x(s). \tag{14}$$

Suppose, on the contrary that $y(s) < x(s)$. Then, there is $j \in I$ such that:

$$y_j(s) = y(s) < x(s). \tag{15}$$

Since $x(s)$ is the arc length of the largest empty stretch, we can find a set of points $X(s)$ which is an empty stretch, with arc length $x(s)$. [There could be several such sets; we are picking one and calling it $X(s)$].

Using (13) and (15), we get:

$$x(s) + y_j(s) < 2x(s) \leq \frac{2}{n-1}$$

and so:

$$1 - x(s) - y_j(s) > 1 - \frac{2}{n-1} = \frac{n-3}{n-1} \geq 0. \tag{16}$$

If the neighborhood $Y_j(s)$ overlaps with $X(s)$ (that is $Y_j(s) \cap X(s) \neq \phi$), then $X(s)$ must be a subset of $Y_j(s)$, and consequently $y_j(s) \geq x(s)$, contradicting (9). Thus, $Y_j(s)$ does not overlap with the empty stretch $X(s)$, with arc length $x(s)$.

Note that $Y_j(s)$ consists of two empty stretches (and the point s_j) and neither one overlaps with $X(s)$. If $n = 3$, then there are no other empty stretches and we must have $y_j(s) + x(s) = 1$, but this contradicts (16). Thus, we must have $n > 3$ and there are a total of $(n-3)$ empty stretches remaining (other than $X(s)$ and the two empty stretches in $Y_j(s)$). Let $z(s)$ be the *maximum* of the (arc) lengths of these remaining empty stretches. Consequently,

$$(n-3)z(s) \geq 1 - x(s) - y_j(s). \tag{17}$$

Using (16) in (17), we obtain:

$$z(s) > \frac{1}{n-1}. \tag{18}$$

Clearly, (18) contradicts (13) and establishes our claim (14).
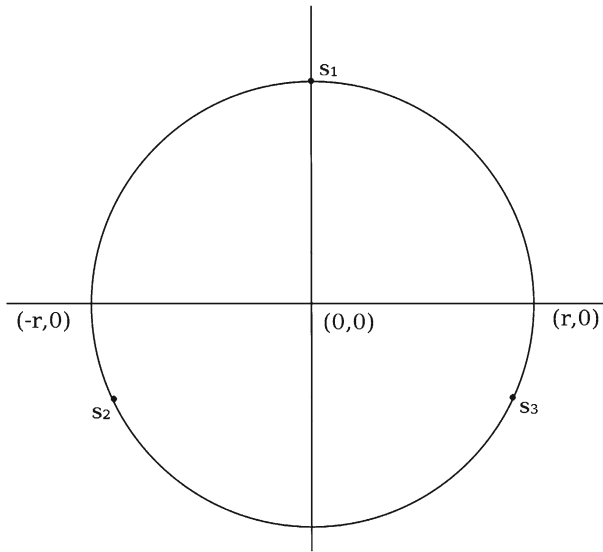
Now, the result follows directly from Theorem 1. $\qquad\qquad\qquad\qquad\square$

Fig. 2 Illustration of Theorem 1(A)

3.2.1 Geometric illustration for $n = 3$

To hone our intuition for the analysis used in Theorem 1, it is useful to consider the special case of $n = 3$.

Considered the scattered placement described in Fig. 2.

It is easy to see no player can do better by shifting platform on the same stretch (strictly) between the other two players. Note, for instance that player 1 will get exactly half the votes on the clockwise stretch traversed from her platform s_1 to the platform s_3 of player 3 and, likewise, half the votes on the anti-clockwise stretch traversed from her platform s_1 to the platform s_2 of player 2. In other words, player 1 gets half the votes on the stretch of rotary between s_2 and s_3 on the side where s_1 is located. Since this makes no mention of where s_1 is located on this stretch, any movement by player 1, where she remains on the same side of the rotary leaves her payoff unchanged. [Analytically, we expressed this above by saying that if s_1 is located anywhere on the $arc(s_2, s_3)$, then her payoff remains the same; this is the arc traversed as one moves from s_2 to s_3 in a clockwise direction].

Hence to check if a scattered placement is a Nash equilibrium all we have to check is if any player is better off by relocating her platform to another empty stretch. For instance, can player 1 do better by changing her platform to the southern stretch of rotary between s_2 and s_3? [Analytically, we expressed this above by analyzing whether player 1 can do better by relocating from platform s_1 to some platform s_1' on the $arc(s_3, s_2)$; this is the arc traversed as one moves from s_3 to s_2 in a clockwise direction]. Clearly, in Fig. 2, the southern stretch of rotary between s_2 and s_3 is less than $(1/2)$, and so player 1 cannot do better by such a relocation of her platform. Given the symmetry of the locations of the three players, the same argument applies to players 2 and 3 as well. So, the placement in Fig. 2 is a Nash equilibrium.
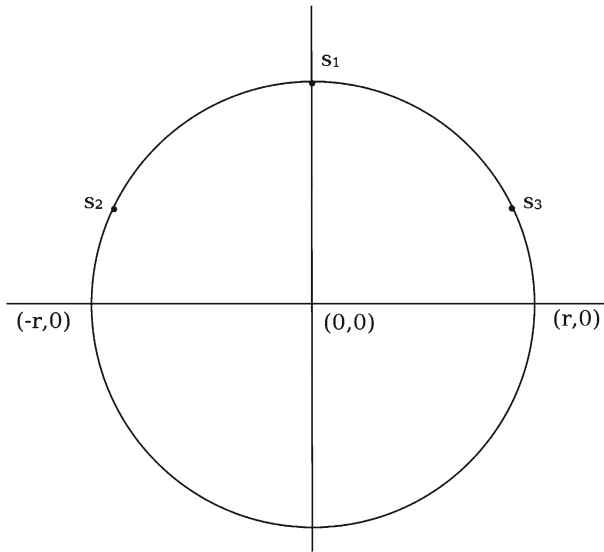
Fig. 3 Illustration of Theorem 1(B)

Now consider the case where the empty stretch $arc(s_3, s_2)$ is more than $(1/2)$. This is illustrated in Fig. 3.

It is obvious that such a placement cannot be Nash, since player 1 can shift her platform from s_1 in the north connector between s_2 and s_3 to s_1' in the south connector between s_2 and s_3.

3.3 Symmetric and asymmetric scattered Nash equilibria

Corollary 1 has the following implication, which might be of interest. In the equally spaced case, the longest empty stretch is $(1/n)$, and therefore such a scattered place-ment is a Nash equilibrium. So, for all placements sufficiently close to the equally spaced case, x is close to $(1/n)$, and therefore less than $[1/(n-1)]$. Thus, all such scattered placements are also Nash equilibria by Corollary 1. That is, there is a con-tinuum of scattered Nash equilibria, clustered around the equally spaced placement.

The fact that the players have no distinctive features ex ante (they can choose platforms anywhere on the circle, and once located the same payoff *rule* applies to all of them) does not, of course, mean that they receive the same or even similar payoffs in a Nash equilibrium. In fact, distinctly asymmetric Nash equilibria can arise in this model. We begin by describing an example with three players, which can be instructive.

3.3.1 The advantage of appearing different

Example 1 With Cartesian co-ordinates, consider the center of the circle to be at $(0, 0)$. There are three players, and the scattered placement s is described as follows. The
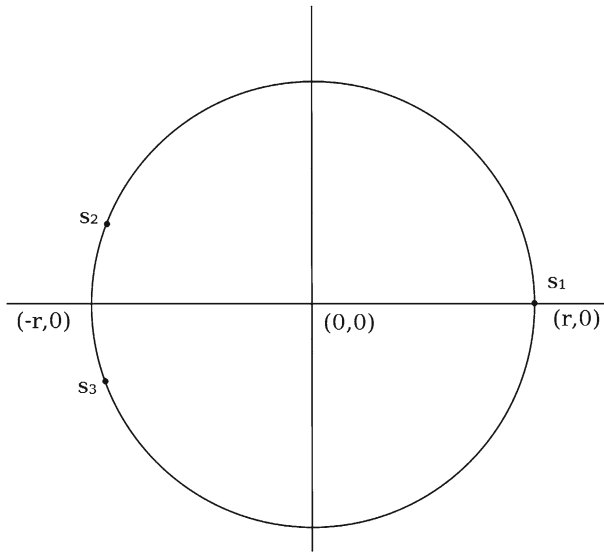
Fig. 4 Scattered Nash equilibrium (Example 1)

platform of player 1 is located at $(r, 0)$, where $r = (1/2\pi)$. The platforms of players 2 and 3 are located near $(-r, 0)$, with the platform for 2 slightly above the axis on the circle, and the platform for 3 symmetrically slightly below the axis on the circle (see Fig. 4). To be precise, the platform of 2 is located at $(-(a^2 - \varepsilon^2)^{1/2}, \varepsilon)$ and the platform of 3 is located at $(-(a^2 - \varepsilon^2)^{1/2}, -\varepsilon)$, where $0 < \varepsilon < a$, and we are to suppose that ε is close to zero. That is, candidates 2 and 3 are very alike in their platforms, while candidate 1 is very different from either one of them.

Note that $y_i(s)$ exceeds $(1/2)$ for all $i \in I = \{1, 2, 3\}$, and $x(s)$ is smaller than $(1/2)$. Thus, s is a Nash equilibrium by Theorem 1. [In fact, Corollary 1 can also be applied directly in this case to reach this conclusion]. Now, notice that as $\varepsilon \to 0$, we have the payoff of player 1 approaching $(1/2)$, as her neighborhood approaches the entire circle, while the payoffs of players 2 and 3 approach $(1/4)$. There is a big advantage from appearing different.

3.3.2 The distribution of payoffs in scattered Nash equilibria

It is a rather remarkable fact that the simple example just discussed illustrates a general result on the *distribution of payoffs* in Nash equilibria in this model. Non-uniqueness of Nash equilibria is the major drawback in its predictive power regarding the outcome of the game. In our next result, instead of focusing on equilibrium locations of platforms, we shift our focus to equilibrium payoffs, and we find that these are subject to rather stringent bounds. Specifically, the Nash equilibrium payoffs of all players must lie in the interval:

$$\Pi = \left[\frac{1}{2(n-1)}, \frac{2}{n+1}\right] \tag{DOP}$$

for an arbitrary number n of players, and for an *arbitrary* Nash equilibrium.

As an implication, in a four-way race, no candidate can get less than one-sixth of the vote in a scattered Nash equilibrium. This makes (1/6) a useful benchmark to decide whether to change one's strategy (platform). That is, it is all right to get less than a quarter of the vote (perform worse than average), but it is not all right to get less than a sixth of the vote.[15]

We now proceed to establish the formula (DOP) using Theorem 1.

Proposition 1 *If a scattered placement $s = \{s_1, \ldots, s_n\}$ is a Nash equilibrium, then*:

$$\frac{1}{2(n-1)} \le p_i(s) \le \frac{2}{n+1} \quad \text{for all } i \in I = \{1, \ldots, n\}. \tag{19}$$

Proof Given the scattered placement $s = \{s_1, \ldots, s_n\}$, recall that the arc length of the empty stretch of maximum arc length is denoted by $x(s)$, and we define:

$$y(s) = \min\{y_1(s), \ldots, y_n(s)\}; \ z(s) = \max\{y_1(s), \ldots, y_n(s)\}.$$

Note that, given the scattered placement $s = \{s_1, \ldots, s_n\}$, there are n empty stretches. Each empty stretch is contained in the neighborhoods of *two* distinct players. Since the arc lengths of the n empty stretches must add up to 1 (the circumference of the circle), the arc lengths of the neighborhoods of the n players must add up to 2; that is:

$$\sum_{i=1}^{n} y_i(s) = 2. \tag{20}$$

There is $k \in I = \{1, \ldots, n\}$ such that $y_k(s) = z(s)$. Since $Y_k(s)$ consists of two empty stretches (and the point s_k), we have:

$$z(s) = y_k(s) \le 2x(s). \tag{21}$$

---

[15] It might be difficult for a candidate to change his platform significantly, given his ideological beliefs. On a more pragmatic note, significant changes in platform by a candidate to suit an election might not be taken seriously by voters, who might rightly view it as an opportunistic move. In such cases, (1/6) would also be a useful benchmark to decide whether to quit a race.

Using (20) and (21), and Theorem 1, we get:

$$
\begin{aligned}
2 &= z(s) + \sum_{i \neq k} y_i(s) \\
&\geq z(s) + (n-1)x(s) \\
&\geq z(s) + (n-1)\left[z(s)/2\right] \\
&= z(s)\left[1 + \frac{n-1}{2}\right] \\
&= z(s)\left[\frac{n+1}{2}\right]
\end{aligned}
\tag{22}
$$

where we used (20) in line 1, Theorem 1 in line 2 and (21) in line 3. This yields:

$$
z(s) \leq \left[\frac{4}{n+1}\right]
$$

so that:

$$
p_i(s) \leq p_k(s) = \left[z(s)/2\right] \leq \left[\frac{2}{n+1}\right] \quad \text{for all } i \in I.
\tag{23}
$$

This establishes the right-hand inequality in (19).

We proceed now to establish the left-hand inequality in (19). Note that by using (21) and Theorem 1, we have:

$$
z(s) = y_k(s) \leq 2x(s) \leq 2y(s).
\tag{24}
$$

There is $j \in I = \{1, \ldots, n\}$ such that $y_j(s) = y(s)$. The neighborhood $Y_j(s)$ consists of two empty stretches (and the point s_j). Call these empty stretches $Y_{j1}(s)$ and $Y_{j2}(s)$. The corresponding arc lengths can be written as $[y_j(s)/2]+\varepsilon$ and $[y_j(s)/2]-\varepsilon$, where $-[y_j(s)/2] < \varepsilon < [y_j(s)/2]$. The empty stretch $Y_{j1}(s)$ must be shared with another player $r \neq j$; that is, $Y_{j1}(s)$ must be one of the two empty stretches of $Y_r(s)$, the neighborhood of player r. Similarly, empty stretch $Y_{j2}(s)$ must be shared with another player $t \neq j, r$; that is, $Y_{j2}(s)$ must be one of the two empty stretches of $Y_t(s)$, the neighborhood of player t. Using (20), we can now write:

$$
\begin{aligned}
2 &= y_j(s) + y_r(s) + y_t(s) + \sum_{i \neq j, r, t} y_i(s) \\
&\leq y_j(s) + y_r(s) + y_t(s) + (n-3)z(s) \\
&\leq y_j(s) + \{[y_j(s)/2] + \varepsilon + x(s)\} + \{[y_j(s)/2] - \varepsilon + x(s)\} + (n-3)z(s) \\
&= 2y_j(s) + 2x(s) + (n-3)z(s) \\
&\leq 2y(s) + 2y(s) + (n-3)z(s) \\
&\leq 2y(s) + 2y(s) + 2(n-3)y(s)
\end{aligned}
\tag{25}
$$

where we have used Theorem 1 in line 5 and (24) in line (6) of (25). Thus, we obtain:

$$y(s) \geq \frac{2}{2n-2} = \frac{1}{n-1}. \tag{26}$$

Using (6), we have:

$$p_i(s) \geq p_j(s) = [y(s)/2] \geq \left[\frac{1}{2(n-1)} \right] \quad \text{for all } i \in I$$

which establishes the left-hand inequality in (19). □

Remark In the context of Example 1 with $n = 3$, we see that the lower bound of 25% in (19), and the upper bound of 50% in (19) are approached as we let $\varepsilon \to 0$.

4 Clustered Nash equilibria

To analyze clustered Nash equilibria, it is useful to have a more general set-up of the framework of Sect. 3, which will include all placements, including both scattered and clustered placements.

Consider as before that there are $n \geq 3$ players. A placement will be described by (m, s, t) where $m \leq n$ is a positive integer, $s = \{s_1, \ldots, s_m\}$ an m-tuple of distinct points[16] on the circle (with circumference equal to 1), and $t = \{t_1, \ldots, t_m\}$ an m−tuple of positive integers, with $t_1 + \cdots + t_m = n$. In the placement $c = (m, s, t)$, there are m locations on the circle which represent distinct platforms. For $i \in J = \{1, \ldots, m\}$, there are t_i candidates who share the platform s_i. A scattered placement is a special case, where $m = n$, and $t_i = 1$ for each $i \in J$. Concepts related to this framework can be defined as in Sect. 3.1, but with a few differences which justify our going over these concepts again.

Given a placement $c = (m, s, t)$, with $m \geq 2$, there is a *neighborhood* $Y_i(c)$ for each platform s_i, where $i \in J \equiv \{1, \ldots, m\}$, which is defined to be the maximal arc, containing s_i, but not containing any $s_j \neq s_i$.

Note that the subscript i in $Y_i(c)$ keeps track of the platform s_i, and not a particular candidate who has announced platform s_i. All candidates sharing the same platform s_i have the same neighborhood $Y_i(c)$. Note also the qualification $m \geq 2$ in the above definition. If $m = 1$, then all platforms are identical, and the neighborhood for each candidate announcing the only platform is the entire circle.

As in Sect. 3.1, a neighborhood is an arc, and we define $y_i(c)$ to be the arc length of this arc. As we have defined it, when $m \geq 2$, a neighborhood $Y_i(c)$ is a set which is open relative to the circle. The closure of $Y_i(c)$ (relative to the circle) is denoted by $\bar{Y}_i(c)$. Thus, $\bar{Y}_i(c)$ contains $Y_i(c)$ and the boundary points of $Y_i(c)$.

---

[16] Each s_i can be specified precisely in terms of its Cartesian co-ordinates.

Given a placement $c = (m, s, t)$, with $m \geq 2$, for each platform s_i (with $i \in J$), we can define the *neighboring* platforms by:

$$s_{R(i)} = \arg \min_{s_j \in s \setminus \{s_i\}} a(s_i, s_j) \quad \text{and} \quad s_{L(i)} = \arg \min_{s_j \in s \setminus \{s_i\}} a(s_j, s_i). \qquad \text{(NBS)}$$

Clearly, we have:

$$s_{R(i)} \neq s_i \quad \text{and} \quad s_{L(i)} \neq s_i.$$

Unlike in Sect. 3.1, it is possible to have $s_{R(i)} = s_{L(i)}$. [In this case, there are exactly two distinct platforms].

Using (NBS), it can be shown that when $m \geq 2$,

$$Y_i(c) = arc(s_{L(i)}, s_{R(i)}) \quad \text{and} \quad \bar{Y}_i(c) = arc[s_{L(i)}, s_{R(i)}]. \qquad \text{(NBD)}$$

When $m \geq 2$, the payoff $p_i(c)$ of each candidate who has announced the platform s_i (with $i \in J$) is:

$$p_i(c) = \frac{y_i(c)}{2t_i}. \qquad \text{(PO)}$$

When $m = 1$, there is exactly one platform and the payoff $p(c)$ of each candidate is:

$$p(c) = \frac{1}{n}. \qquad \text{(PO1)}$$

4.1 An example of clustered Nash equilibrium

We start by providing an example of a clustered Nash equilibrium. This can be viewed as a limiting version of Example 1.

Example 2 With Cartesian co-ordinates, consider the center of the circle to be at $(0, 0)$. There are three players, and the placement $c = (m, s, t)$ is described as follows. The platform of player 1 is $s_1 = (r, 0)$. Players 2 and 3 have the same platform, specified by $s_2 = (-r, 0)$. Thus, we have $m = 2$, $s = \{s_1, s_2\}$ and $t = \{t_1, t_2\} = \{1, 2\}$. This placement is illustrated in Fig. 5.

Here we have $y_1(c) = 1$, $y_2(c) = 1$, and so $p_1(c) = (1/2)$, while the payoff of each player who have announced the same platform s_2 is $p_2(c) = (1/4)$. We can check that c is a Nash equilibrium by using the following elementary analysis.

A unilateral deviation by any player will create a new placement $c' = (m', s', t')$, where the platforms of the other two players remain the same as in the placement c. If player 1 changes his platform to $s_1' \neq s_2$, his neighborhood remains the same as in c, and his payoff remains $(1/2)$. If $s_1' = s_2$, then his neighborhood remains the same as in c, but his payoff becomes $(1/3)$. So, clearly, player 1 has no incentive to deviate from s_1.
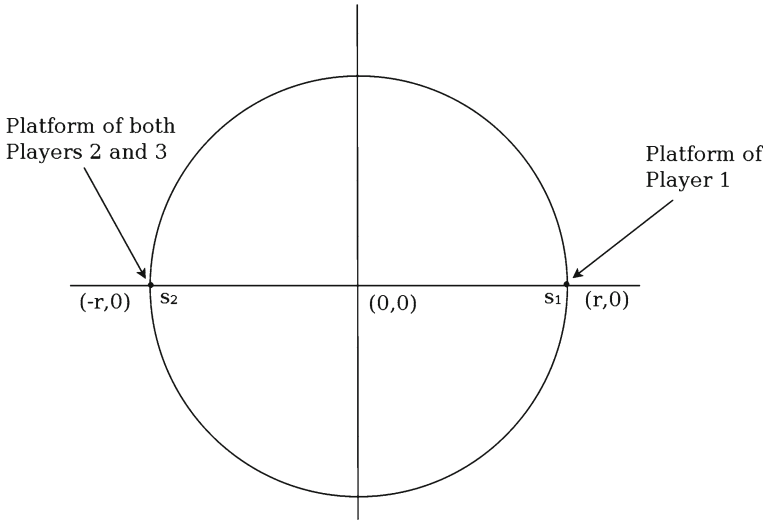
Fig. 5 Clustered Nash equilibrium (Example 2)

Consider next a unilateral deviation by player 2 from her original platform of s_2 to a new platform $s_2' \neq s_2$. If $s_2' \neq s_1$, her new neighborhood is the semi-circle, and so her new payoff is $(1/4)$. If, on the other hand, $s_2' = s_1$, then her neighborhood is the full circle, and her payoff is again $(1/4)$. So, clearly, player 2 has no incentive to deviate from s_2. Similarly, player 3 has no incentive to deviate from s_2.

4.2 General properties of Nash equilibria

We now proceed to establish four general properties of Nash equilibria. Given that we already have a complete characterization of scattered Nash equilibria in Sect. 3, the focus of three of these properties is on clustered Nash equilibria.

First, in any clustered Nash equilibrium, there cannot be more than two candidates announcing the same platform. That is, for any Nash equilibrium placement $c = (m, s, t)$, we must have $t_i \in \{1, 2\}$ for all $i \in J$. This implies that $m \geq 2$ (since we have $n \geq 3$).

Second, if $c = (m, s, t)$ is a clustered Nash equilibrium, and there are two candidates who have announced the same platform s_j, then s_j must be precisely at the mid-point of its neighborhood $Y_j(c)$.

Third, the analogue of part (B) of Theorem 1 is valid for all Nash equilibria. To describe it, let us define two key concepts. Given a placement $c = (m, s, t)$, an *empty stretch* is any arc that does not contain any s_i for $i \in J$. The arc length of an empty stretch of maximum arc length (among all empty stretches) is denoted by $x(c)$. We define for $m \geq 2$,

$$y(c) = \min \left\{ \frac{y_1(c)}{t_1}, \ldots, \frac{y_m(c)}{t_m} \right\}.$$

If $c = (m, s, t)$ is any Nash equilibrium, then it must be the case that $y(c) \geq x(c)$.

Fourth, if $c = (m, s, t)$ is any clustered Nash equilibrium, then it must be the case that $y(c) = x(c)$.

Proposition 2 *If a placement $c = (m, s, t)$ is a Nash equilibrium, then the following properties must hold*:

  (i) $t_i \in \{1, 2\}$ *for all $i \in J$, and $m \geq 2$.*
 (ii) *For each location s_i with $t_i = 2$, s_i must be at the mid point of its neighborhood $Y_i(c)$.*
(iii) $y(c) \geq x(c)$.
(iv) *If the placement c is clustered, then $y(c) = x(c)$; further, for each location s_i with $t_i = 2$, each of the two empty stretches of $Y_i(c)$ has arc length $x(c)$.*

Proof (i) Let $c = (m, s, t)$ be a Nash equilibrium placement. Suppose there is some platform s_j which has been announced by $k \geq 3$ candidates.

First consider the case where $k = n$; that is, all players have announced the same platform. Then the neighborhood of each player is the entire circle, and the payoff to each player is $(1/n)$. If one player deviates to another platform she will still have the entire circle as her neighborhood, but her payoff will be $(1/2)$. Since $n \geq 3$, a deviation is worthwhile. Hence c is not a Nash equilibrium.

Now suppose $3 \leq k < n$, so that $m \geq 2$. The neighborhood of each player who has announced platform s_j is $Y_j(c)$, and the payoff of each player who has announced platform s_j is $p_j(c) = y_j(c)/2t_j = y_j(c)/2k$. The neighborhood $Y_j(c)$ consists of two empty stretches (and the point s_j). Call these empty stretches $Y_{j1}(c)$ and $Y_{j2}(c)$. Comparing the arc lengths of these empty stretches, pick an empty stretch whose arc length is not smaller than the arc length of the other empty stretch, and call it $Y_{j1}(c)$ (without loss of generality).

If one of the k players who has announced platform s_j deviates to a platform $s_j' \in Y_{j1}(c)$, then this player's neighborhood becomes $Y_{j1}(c)$ and so this player's payoff becomes half the arc length of $Y_{j1}(c)$, which will therefore be greater than or equal to $[y_j(c)/4]$. Since $k \geq 3$,

$$\frac{y_j(c)}{4} > \frac{y_j(c)}{2k}.$$

Hence, the player benefits by deviating from s_j to s_j'. So c cannot be a Nash equilibrium.

This establishes that $t_i \in \{1, 2\}$ for all $i \in J$. If $m = 1$, then there would be at least three players who announce the same platform, since $n \geq 3$. But this is ruled out by the property just established. Thus, we must have $m \geq 2$.

(ii) Let $c = (m, s, t)$ be a Nash equilibrium placement. Suppose there is some platform s_j which has been announced by $t_j = 2$ players. The neighborhood of each player who has announced platform s_j is $Y_j(c)$, and the payoff of each player who has announced platform s_j is $p_j(c) = y_j(c)/2t_j = y_j(c)/4$. The neighborhood $Y_j(c)$ consists of two empty stretches (and the point s_j). Call these empty stretches $Y_{j1}(c)$ and $Y_{j2}(c)$. Suppose the arc lengths of these two empty stretches are *not* the same. Pick the empty stretch whose arc length is the larger of the two, and call it $Y_{j1}(c)$ (without loss of generality).

If one of the 2 players who has announced platform s_j deviates to a location $s'_j \in Y_{j1}(c)$, then this player's neighborhood becomes $Y_{j1}(c)$ and so this player's payoff becomes half the arc length of $Y_{j1}(c)$, which will therefore be greater than $[y_j(c)/4]$. Hence, the player benefits by deviating from s_j to s'_j. So c cannot be a Nash equilibrium. This establishes that the arc lengths of $Y_{j1}(c)$ and $Y_{j2}(c)$ must be the same. That is, s_j must be at the mid point of its neighborhood $Y_j(c)$.

(iii) Let $c = (m, s, t)$ be a Nash equilibrium placement. Suppose $y(c) < x(c)$. By the definition of $y(c)$, there is some platform s_h (with $h \in J$) such that $y_h(c)/t_h = y(c) < x(c)$. Then the payoff of each player who has announced platform s_h is:

$$p_h(c) = y_h(c)/2t_h < (x(c)/2).$$

Now a player (who has announced platform s_h) can relocate from s_h to a platform s'_h in an empty stretch with arc length $x(c)$. Then the new neighborhood of this player is the empty stretch with arc length $x(c)$ (no longer empty after it is occupied by this player). Consequently, the new payoff to this player is $(x(c)/2)$. Hence, this player benefits by deviating from s_j to s'_j. So, the placement c cannot be a Nash equilibrium. This contradiction establishes that $y(c) \geq x(c)$.

(iv) Let $c = (m, s, t)$ be a Nash equilibrium placement, where the placement is clustered. Then, there is some platform s_h (with $h \in J$) which has been announced by more than one player. By (i), there are two players who have announced platform s_j. Thus, by definition of $y(c)$, we have:

$$\frac{y_j(c)}{2} \geq y(c) \geq x(c) \tag{27}$$

where the second inequality in (27) follows from the result (iii).

The neighborhood of each player who has announced platform s_j is $Y_j(c)$, and $Y_j(c)$ consists of two empty stretches (and the point s_j).Thus,

$$y_j(c) \leq 2x(c). \tag{28}$$

Combining (27) and (28), we obtain:

$$\frac{y_j(c)}{2} \geq y(c) \geq x(c) \geq \frac{y_j(c)}{2} \tag{29}$$

so that equality must hold in place of *each* inequality in (29). Thus, $y(c) = x(c)$ must hold. Further, $y_j(c) = 2x(c)$, so by result (ii), each of the two (equal) empty stretches of $Y_j(c)$ constitutes a largest empty stretch in the placement c. $\qquad\square$

Remark Result (i) in Proposition 2 sheds interesting light on the median voter theorem which says that, when the available platforms are points on a straight line, two candidates will invariably choose the same platform (the median). What Example 2 and Proposition 2(i) show is that in an electoral democracy with voter preference on a circle of available platforms, two candidates may choose the same political agenda, but there will never be three candidates doing so.

4.3 A property of clustered Nash equilibria

Results (i), (ii) and (iv) of Proposition 2 place rather strong restrictions on *clustered* Nash equilibria. If $n = 3$ and $c = (m, s, t)$ is a clustered Nash equilibrium, then there must be precisely two platforms. One candidate would announce one platform; call this s_1. The other two candidates would both announce a distinct platform, s_2. Given the platform s_2, property (ii) of Proposition 2 demands that it be at the mid-point of its neighborhood. This implies that s_1 must be diametrically opposite to s_2 on the circle. In other words, Example 2 is essentially the *only* example of a clustered Nash equilibrium when $n = 3$. In particular, the distribution of payoffs in any clustered Nash equilibrium must be:

$$\Omega = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right\}.$$

More generally, for arbitrary $n \geq 3$, we note the following property for clustered Nash equilibria, which should be compared to Corollary 1 for scattered Nash equilibria. It is a useful property which is helpful in studying the distribution of payoffs in arbitrary Nash equilibria, which we take up in the next subsection.

Corollary 2 *If a placement $c = (m, s, t)$ is a clustered Nash equilibrium, then*:

$$x(c) \geq \frac{1}{n-1}. \tag{30}$$

Proof Given the placement $c = (m, s, t)$, there are m empty stretches. Since the arc lengths of the m empty stretches must add up to 1 (the circumference of the circle), we must have:

$$mx(c) \geq 1. \tag{31}$$

Since the placement $c = (m, s, t)$ is a clustered Nash equilibrium, there is at least one platform s_k, which is announced by multiple candidates; by Proposition 2(i), we know in fact that there are precisely two candidates who announce the platform s_k. The remaining $(n-2)$ candidates announce the remaining $(m-1)$ platforms. In order that each of the remaining $(m-1)$ platforms is announced by at least one candidate each, we must have:

$$n - 2 \geq m - 1$$

and consequently $n - 1 \geq m$. Using this in (31), we obtain (30). □

4.4 Distribution of payoffs in Nash equilibria

We now present a generalization of Proposition 1, extending that result to all Nash equilibria. It will be noted that in our result, the bounds on the payoffs are determined

in terms of n, the number of players (candidates), and independent of m, the number of platforms announced in a Nash equilibrium. Thus, one needs no information about the equilibrium placements to make use of the bounds.

Theorem 2 *If a placement* $c = (m, s, t)$ *is a Nash equilibrium, then*:

$$\frac{1}{2(n-1)} \leq p_i(c) \leq \frac{2}{n+1} \quad \text{for all } i \in J = \{1, \ldots, m\}. \tag{DOP'}$$

Proof Given the placement $c = (m, s, t)$, recall that the arc length of the empty stretch of maximum arc length is denoted by $x(c)$, and we define:

$$y(c) = \min\left\{\frac{y_1(c)}{t_1}, \ldots, \frac{y_m(c)}{t_m}\right\}; \quad z(c) = \max\left\{\frac{y_1(c)}{t_1}, \ldots, \frac{y_m(c)}{t_m}\right\}.$$

Note that, given the placement $c = (m, s, t)$, there are m empty stretches. Each empty stretch is contained in the neighborhoods of *two* distinct platforms. Since the arc lengths of the m empty stretches must add up to 1 (the circumference of the circle), the arc lengths of the neighborhoods of the m platforms must add up to 2; that is:

$$\sum_{i=1}^{m} y_i(c) = 2. \tag{32}$$

There is $k \in J = \{1, \ldots, m\}$ such that $(y_k(c)/t_k) = z(c)$. By Proposition 2(i), there are two possibilities (a) $t_k = 2$; (b) $t_k = 1$. In case (a), we know by Proposition 2(iv) that $y_k(c) = 2x(c)$. Thus, we have:

$$z(c) = y_k(c)/t_k = 2x(c)/t_k = x(c) \tag{33}$$

and so by definition of $z(c)$,

$$y_i(c)/t_i \leq z(c) = x(c) \quad \text{for all } i \in J. \tag{34}$$

But, we also know from the definition of $y(c)$ and Proposition 2(iii) that:

$$y_i(c)/t_i \geq y(c) = x(c) \quad \text{for all } i \in J. \tag{35}$$

Thus, we have:

$$y_i(c)/t_i = y(c) = z(c) = x(c) \quad \text{for all } i \in J. \tag{36}$$

Using (36) in (32), we obtain:

$$2 = \sum_{i=1}^{m} y_i(c) = \sum_{i=1}^{m} t_i \left[\frac{y_i(c)}{t_i}\right] = \sum_{i=1}^{m} t_i x(c) = nx(c). \tag{37}$$

Thus, (34) and (37) yield:

$$p_i(c) = y_i(c)/2t_i \leq x(c)/2 = (1/2)\left[\frac{2}{n}\right] = \left[\frac{1}{n}\right] \quad \text{for all } i \in J.$$

Since $n \geq 3$, we have $(1/n) < [2/(n+1)]$, and so the right-hand inequality in (DOP') holds in case (a).

In case (b), we have $t_k = 1$, and so:

$$z(c) = y_k(c)/t_k = y_k(c) \leq 2x(c). \tag{38}$$

Using (32) and (38), and Proposition 2(iii), we get:

$$
\begin{aligned}
2 &= y_k(c) + \sum_{i \neq k} y_i(c) \\
&\geq z(c) + \sum_{i \neq k} t_i\, y(c) \\
&\geq z(c) + \sum_{i \neq k} t_i\, x(c) \\
&\geq z(c) + \sum_{i \neq k} t_i\, \{z(c)/2\} \\
&= z(c)\left[1 + \sum_{i \neq k} (t_i/2)\right] \\
&= z(c)\left[\{1 - (t_k/2)\} + \sum_{i=1}^{m} (t_i/2)\right] \\
&= z(c)\left[(1/2) + (n/2)\right] \tag{39}
\end{aligned}
$$

where we used (32) in line 1, Proposition 2(iii) in line 3 and (38) in line 4. This yields:

$$z(c) \leq \left[\frac{4}{n+1}\right]$$

so that:

$$p_i(c) = y_i(c)/2t_i \leq z(c)/2 \leq \left[\frac{2}{n+1}\right] \quad \text{for all } i \in J.$$

This establishes the right-hand inequality in (DOP') in case (b).

We proceed now to establish the left-hand inequality in (DOP'). We divide up our analysis into two cases (a) $c = (m, s, t)$ is a scattered Nash equilibrium; and (b) $c = (m, s, t)$ is a clustered Nash equilibrium. In case (a), the result follows directly

from Proposition 1. We now analyze case (b). In case (b), we note that by Corollary 2,

$$x(c) \geq \frac{1}{n-1}. \tag{40}$$

Thus, for all platforms s_i (with $i \in J$) we must have:

$$p_i(c) = y_i(c)/2t_i \geq y(c)/2 \geq x(c)/2 \geq \left[\frac{1}{2(n-1)}\right]$$

where the second inequality follows from Proposition 2(iii), and the last inequality from (40). This establishes the left-hand inequality in (DOP') in case (b). □

4.5 Best bounds in the distribution of payoffs

In the context of Example 2 with $n = 3$, we see that the lower bound of $[1/2(n-1)] = 25\%$ in (DOP'), and the upper bound of $[2/(n+1)] = 50\%$ in (DOP') are attained. This shows that the bounds in (DOP') are the best possible for $n = 3$.

This raises the question of whether, in general, the bounds given in Theorem 2 (for the distribution of payoffs in an arbitrary Nash equilibrium) are the best possible. In this subsection, we provide answers to this question separately for the lower bound in (DOP') and the upper bound in (DOP').

4.5.1 Lower bound in the distribution of payoffs

We present an example to show that the lower bound of the interval, given in both Proposition 1 and Theorem 2, is the "best possible" in the sense that given any $n \geq 3$, it is possible to construct a scattered Nash equilibrium with n players, such that the payoff of some player in that equilibrium is exactly equal to $[1/2(n-1)]$.

Example 3 Let $n \geq 3$ be given. Divide the circumference of the circle into $(n-1)$ equal parts. This will generate $(n-1)$ points on the circle, equally spaced.

Place the platforms $\{s_1, \ldots, s_{n-1}\}$ at these $(n-1)$ points. To make the instruction more precise, place s_1 at any one of the $(n-1)$ points. Place s_2 at the next available point (of the remaining $n-2$ points) as you go clockwise around the circle, starting from s_1. Continue similarly with $s_3, \ldots, s_{n-1}$. Finally, place the platform s_n at the mid point of the arc $arc(s_{n-1}, s_1)$. For an illustration of this procedure for $n = 5$, see Fig. 6.

The largest empty stretch is:

$$x(s) = \frac{1}{n-1}. \tag{41}$$

The neighborhood with the smallest arc length is:
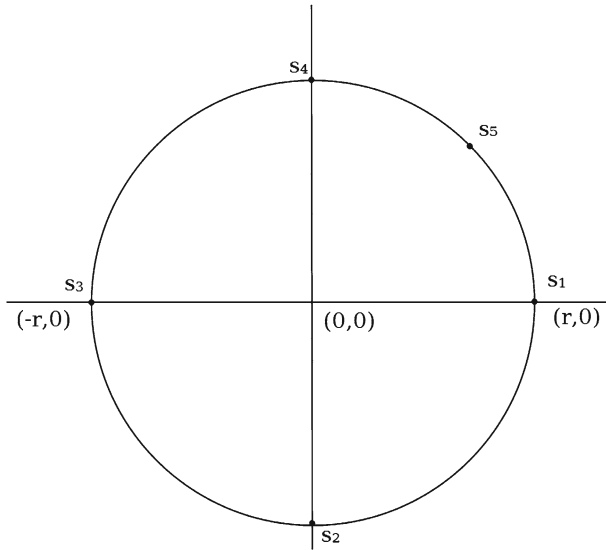
$$y(s) = y_n(s) = \frac{1}{n-1}. \tag{42}$$

Fig. 6 Lower bound of DOP (Example 3)

Thus, by Theorem 1, the placement $s = \{s_1, \ldots, s_n\}$ described above is a Nash equilibrium.

Using (42), the payoff of player n in this Nash equilibrium is:

$$p_n(s) = \frac{y_n(s)}{2} = \frac{1}{2(n-1)} \tag{43}$$

which is precisely the lower bound of Π in (DOP').

Remark The lower bound in (DOP') is also attained when $n = 2$, since in that case, the payoff of each player in any Nash equilibrium is equal to $(1/2)$.

4.5.2 Upper bound in the distribution of payoffs

We now focus on the question whether the upper bound of the interval, given in both Proposition 1 and Theorem 2, is the "best possible". When $n \geq 3$, and n is odd, we show that the upper bound in (DOP') is the best possible in the following sense. Given $n \geq 3$ and n odd, and any positive number β smaller than this upper bound:

$$\beta < \frac{2}{n+1} \tag{44}$$

we can construct an example of a scattered Nash equilibrium, such that the payoff of some player exceeds β in that equilibrium. Thus, $[2/(n+1)]$ is not just an upper bound, it is the least upper bound of the payoff of the players in a Nash equilibrium, for $n \geq 3$ and n odd.

Unlike the result for the lower bound, we do not claim that the bound $[2/(n+1)]$ is actually attained in a Nash equilibrium for $n \geq 3$ with n odd. This is principally because we work with scattered Nash equilibrium, for which there is a complete characterization provided by Theorem 1. The partial characterization result (in Proposition 2) of all Nash equilibria (scattered and clustered) does not provide us with a *sufficient* condition to check whether a proposed placement is a Nash equilibrium. It is certainly *possible* that the least upper bound $[2/(n+1)]$ is actually attained in a clustered Nash equilibrium, but we have not pursued this issue. [For $n = 3$, this least upper bound is attained in a clustered Nash equilibrium, as we checked in Example 2].

Example 4 (The result for n odd) Let $n \geq 3$ be given, with n odd. Then $(n+1)$ is even, and $N = (n+1)/2$ is an integer, greater than or equal to 2. Let β be any positive number satisfying (44). Denote:

$$\varepsilon \equiv \frac{2}{n+1} - \beta \in \left(0, \frac{2}{n+1}\right). \tag{45}$$

Divide the circumference of the circle into N equal parts. This will generate N points on the circle, equally spaced.

Place the platform s_1 at any one of the N points. Place s_3 at the next available point (of the remaining $N - 1$ points) as you go clockwise around the circle, starting from s_1. Continue similarly with $s_5, \ldots, s_n$. Notice that only the odd numbered platforms have been located at these N points. For example, with $n = 5$, we will have $N = 3$, and this procedure will place platforms s_1, s_3, and s_5 at the $N = 3$ available points, moving clockwise from s_1 to s_3 and then clockwise from s_3 to s_5. Observe that:

$$arc(s_1, s_3) = \cdots = arc(s_{n-2}, s_n) = arc(s_n, s_1) = \frac{1}{N} = \frac{2}{n+1}. \tag{46}$$

For an illustration of this procedure for $n = 5$, see Fig. 7.

We now proceed to place the even numbered platforms as follows. Place s_2 in $arc(s_1, s_3)$ so that $arc(s_2, s_3) = \varepsilon$. Since:

$$\varepsilon < \frac{2}{n+1} = \frac{1}{N} = arc(s_1, s_3)$$

this is certainly possible. Similarly, place s_k in $arc(s_{k-1}, s_{k+1})$ with $arc(s_k, s_{k+1}) = \varepsilon$ for all $k \in \{2, 4, \ldots, n-1\}$. See the figure again for an illustration of this procedure for $n = 5$.

It is now easy to check that:

$$y_j(s) = (1/N) \quad \text{for all } j \in \{2, 3, \ldots, n-1\}. \tag{47}$$

Further, we have:

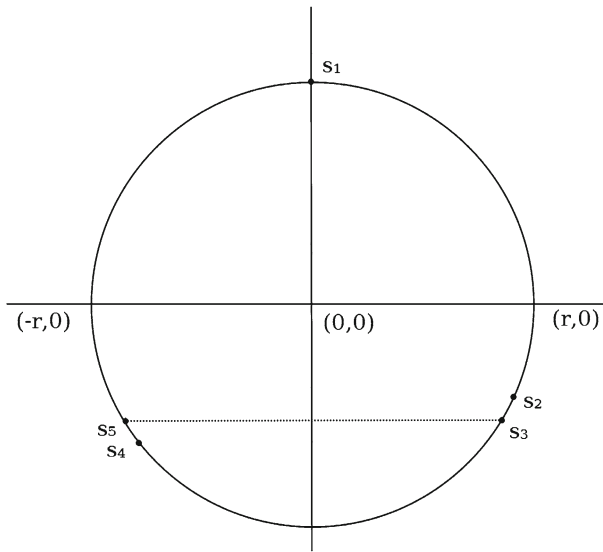$$y_n(s) = (1/N) + \varepsilon; \quad y_1(s) = (2/N) - \varepsilon. \tag{48}$$

Fig. 7 Upper bound of DOP (Example 4)

Using (45), (47) and (48), the neighborhood with the smallest arc length is:

$$y(s) = \frac{1}{N}. \tag{49}$$

The largest empty stretch is:

$$x(s) = a(s_n, s_1) = \frac{1}{N}. \tag{50}$$

Thus, by Theorem 1, the placement $s = \{s_1, \ldots, s_n\}$ described above is a Nash equilibrium.

Using (48), the payoff of player 1 in this Nash equilibrium is:

$$p_1(s) = \frac{y_1(s)}{2} = \frac{1}{N} - \frac{\varepsilon}{2} = \frac{2}{n+1} - \frac{\varepsilon}{2} > \frac{2}{n+1} - \varepsilon = \beta. \tag{51}$$

This shows that $[2/(n+1)]$ is the least upper bound of the payoff of the players in a Nash equilibrium, for $n \geq 3$ and n odd.

Example 5 (The result for n even) For $n \geq 3$ and n even, the upper bound in (DOP') is not necessarily a least upper bound of the payoff of the players in a Nash equilibrium. For instance, with $n = 4$, it can be shown that an upper bound of the payoff of the players in any scattered Nash equilibrium is equal to:

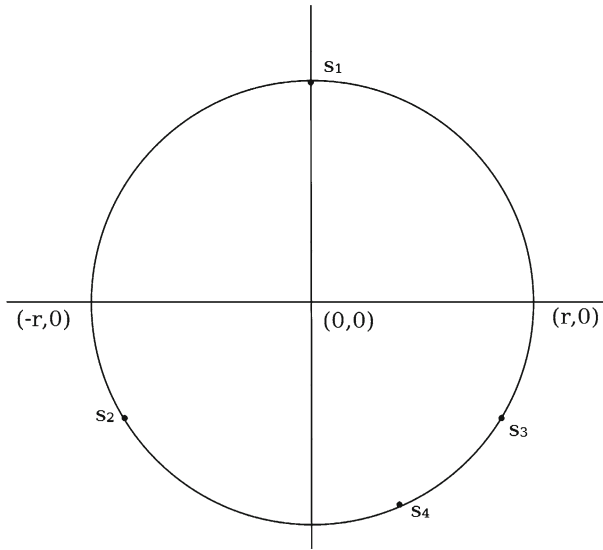$$\frac{1}{3} = \frac{2}{n+2} < \frac{2}{n+1} = \frac{2}{5}. \tag{52}$$

Fig. 8 Upper bound of DOP for n = 4 (Example 5)

To see this, we proceed as follows. With $n = 4$, let $s = \{s_1, s_2, s_3, s_4\}$ be an arbitrary scattered Nash equilibrium. Recall that the arc length of the empty stretch of maximum arc length is denoted by $x(s)$, and we define:

$$y(s) = \min\{y_1(s), \ldots, y_4(s)\}; \quad z(s) = \max\{y_1(s), \ldots, y_4(s)\}.$$

There is $k \in I = \{1, \ldots, 4\}$ such that $y_k(s) = z(s)$. Note that by Theorem 1, we have:

$$z(s) = y_k(s) \leq 2x(s) \leq 2y(s). \tag{53}$$

Without loss of generality, we can take $k = 1$. The neighbourhood $Y_1(s)$ defines s_j and s_q with $s_j \neq s_q$ such that:

$$Y_1(s) = arc(s_j, s_q). \tag{54}$$

Again without loss of generality, we can take $j = 2$ and $q = 3$. It then follows that s_4 belongs to the arc $arc(s_q, s_j) = arc(s_3, s_2)$. Thus,

$$Y_4(s) = arc(s_3, s_2). \tag{55}$$

See Fig. 8 for an illustration of the locations s_1, s_2, s_3, and s_4.

The neighborhood $Y_1(s)$ consists of two empty stretches (and the point s_1), $arc(s_2, s_1)$ and $arc(s_1, s_3)$. The corresponding arc lengths can be written as $[y_1(s)/2] + \varepsilon$ and $[y_1(s)/2] - \varepsilon$ respectively, where $- [y_1(s)/2] < \varepsilon < [y_1(s)/2]$. Similarly, the

neighborhood $Y_4(s)$ consists of two empty stretches (and the point s_4), $arc(s_3, s_4)$ and $arc(s_4, s_2)$. The corresponding arc lengths can be written as $[y_4(s)/2] + \eta$ and $[y_4(s)/2] - \eta$ respectively, where $-[y_4(s)/2] < \eta < [y_4(s)/2]$. Note that ε and η are being allowed to be either negative or positive or zero. [In Fig. 8, $\varepsilon = 0$ while $\eta < 0$].

With this notation we have:

$$y_2(s) = a(s_4, s_2) + a(s_2, s_1) = [y_4(s)/2] - \eta + [y_1(s)/2] + \varepsilon \qquad (56)$$

and:

$$y_3(s) = a(s_1, s_3) + a(s_3, s_4) = [y_1(s)/2] - \varepsilon + [y_4(s)/2] + \eta. \qquad (57)$$

Using (56) and (57), we can now write:

$$
\begin{aligned}
2 &= y_1(s) + y_2(s) + y_3(s) + y_4(s) \\
&= y_1(s) + \{[y_4(s)/2] - \eta + [y_1(s)/2] + \varepsilon\} + \\
&\quad \{[y_1(s)/2] - \varepsilon + [y_4(s)/2] + \eta\} + y_4(s) \\
&= 2y_1(s) + 2y_4(s) \\
&\geq 2z(s) + 2y(s) \\
&\geq 2z(s) + z(s) \\
&= 3z(s) \qquad (58)
\end{aligned}
$$

where we have used (53) in the last but one line of (58). Thus, we obtain:

$$z(s) \leq \frac{2}{3} \qquad (59)$$

and so:

$$p_i(s) = \frac{y_i(s)}{2} \leq \frac{z(s)}{2} \leq \frac{1}{3} \quad \text{for all } i \in I = \{1, 2, 3, 4\}. \qquad (60)$$

5 Concluding remarks

This paper was an exercise in abstract geometric reasoning. When drawing on it for more applied models it will be important to add relevant complications. In applying it to retail trading and location choice or brand proliferation it will be important to introduce the option to vary prices. When applying to electoral politics we need to account for the fact that candidates typically come with a prior record of ideology and a sharp shift in this can cause voter dissonance. In all these models there may also be a case for considering sequential moves which would convert this into an extensive-form game. It is hoped that the simple model presented here can be of value for such extensions.

6 Appendix

6.1 Proof of claim (5) in Sect. 3

Since $k \neq j$, the definition of s' implies that $s'_k = s_k \in Y_k(s)$. If there is some s'_r in $Y_k(s)$ with $s'_r \neq s'_k$, then $r \neq k, j$, and so by definition of s', $s'_r = s_r$ must hold. In this case, we have s_r in $Y_k(s)$ with $s_r \neq s_k$, a contradiction to the definition of $Y_k(s)$. This establishes that there is no s'_r in $Y_k(s)$ with $s'_r \neq s'_k$. Since $Y_k(s)$ is a connected set containing s'_k, and there is no s'_r in $Y_k(s)$ with $s'_r \neq s'_k$, we must have $Y_k(s) \subset Y_k(s')$, and consequently $\bar{Y}_k(s) \subset \bar{Y}_k(s')$ as well.

Using (NBD), let us write $Y_k(s) = arc(s_{L(k)}, s_{R(k)}) \equiv arc(s_p, s_q)$. Note that $s_{L(k)} \neq s_k$, and $s_{R(k)} \neq s_k$, and $s_{L(k)} \neq s_{R(k)}$. Now, if $s_{L(k)} = s_j$, then since $s_{L(k)} \in \bar{Y}_k(s)$ and $s_j \in Y_j(s)$, we would have $\bar{Y}_k(s) \cap Y_j(s) \neq \phi$, a contradiction to the definition of case (b)(I). Thus, we must have $s_{L(k)} \neq s_j$, and similarly $s_{R(k)} \neq s_j$. By definition of s' it follows that $s'_p = s_p$ and $s'_q = s_q$.

Both s_p and s_q belong to $\bar{Y}_k(s)$, and since $\bar{Y}_k(s) \subset \bar{Y}_k(s')$, they belong to $\bar{Y}_k(s')$. However, they cannot be interior points of $\bar{Y}_k(s')$; that is, they cannot belong to $Y_k(s')$. To see this, note that $s_p \neq s_k$, $s_p = s'_p$ and $s_k = s'_k$, and so $s'_p \neq s'_k$. Then, by definition of $Y_k(s')$, we infer that s'_p cannot be in $Y_k(s')$; that is, s_p cannot be in $Y_k(s')$. A similar reasoning establishes that s_q cannot be an interior point of $\bar{Y}_k(s')$. Thus, both are boundary points of $\bar{Y}_k(s')$. This means that $\bar{Y}_k(s')$ is either the set $arc[s_p, s_q]$ or the set $arc[s_q, s_p]$. Since $Y_k(s) = arc(s_p, s_q)$, and $\bar{Y}_k(s')$ has to contain the set $\bar{Y}_k(s) = arc[s_p, s_q]$, we conclude that $\bar{Y}_k(s')$ must be the set $arc[s_p, s_q]$. Thus, $Y_k(s')$ must be the set $arc(s_p, s_q)$, and so $Y_k(s) = Y_k(s')$.

6.2 Proof of claim (8) in Sect. 3

We break up our analysis into two cases: (A1) $s_{L(j)} = s_{R(k)}$ and (A2) $s_{L(j)} \neq s_{R(k)}$.

Case (A1): In this case, s_j belongs to $arc(s_{R(k)}, s_{L(k)}]$. If s_j actually belongs to $arc(s_{R(k)}, s_{L(k)})$, then $Y_j(s) = arc(s_{R(k)}, s_{L(k)})$, which is disjoint from $\bar{Y}_k(s)$, contradicting the fact that $Y_j(s) \cap \bar{Y}_k(s) \neq \phi$ in subcase (b)(II). Thus, $s_j = s_{L(k)}$, and so $s_k = s_{R(j)}$, so that (8)(i) in claim (8) holds.

Case (A2): We subdivide our analysis into the following two possibilities (A2a) $s_{L(j)} \in \bar{Y}_k(s)$, (A2b) $s_{L(j)} \notin \bar{Y}_k(s)$.

In subcase (A2a), $s_{L(j)}$ must be equal to $s_{L(k)}$ or s_k or $s_{R(k)}$. The last possibility is ruled out since in (A2) we have $s_{L(j)} \neq s_{R(k)}$. If $s_{L(j)} = s_{L(k)}$, then s_j must be equal to s_k, which is ruled out since s is a scattered placement (recall that $k \neq j$ in case (b)). Thus, we must have $s_{L(j)} = s_k$ and consequently $s_j = s_{R(k)}$. That is, (8)(ii) in claim (8) holds.

In subcase (A2b), $s_{L(j)} \in arc(s_{R(k)}, s_{L(k)})$, and consequently $s_{R(j)} \notin arc(s_{R(k)}, s_{L(k)})$. This is because if $s_{R(j)} \in arc(s_{R(k)}, s_{L(k)})$, then $Y_j(s)$ is entirely contained in $arc(s_{R(k)}, s_{L(k)})$, which is disjoint from $\bar{Y}_k(s)$, contradicting the fact that $Y_j(s) \cap \bar{Y}_k(s) \neq \phi$ in subcase (b)(II).

Since $s_{R(j)} \notin arc(s_{R(k)}, s_{L(k)})$, we must have $s_{R(j)} \in \bar{Y}_k(s) \equiv arc[s_{L(k)}, s_{R(k)}]$. Thus, $s_{R(j)}$ must be equal to $s_{L(k)}$ or s_k or $s_{R(k)}$. The first possibility is ruled out since $s_{R(j)} = s_{L(k)}$ would imply that $Y_j(s) \cap \bar{Y}_k(s) = \phi$, a contradiction. The third possibility is ruled out because $s_{R(j)} = s_{R(k)}$ would imply that s_j must be equal to s_k, which would contradict the fact that s is a scattered placement (recall that $k \neq j$ in case (b)). Thus, we must have $s_{R(j)} = s_k$ and consequently $s_j = s_{L(k)}$, so that (8)(i) in claim (8) holds.

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