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ORIGINAL PAPER

Inequality measurement with an ordinal and continuous variable

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Abstract
What would be the analogue of the Lorenz quasi-ordering when the variable of interest is continuous and of a purely ordinal nature? We argue that it is possible to derive such a criterion by substituting for the Pigou–Dalton transfer used in the standard inequality literature what we refer to as a Hammond progressive transfer. According to this criterion, one distribution of utilities is considered to be less unequal than another if it is judged better by both the lexicographic extensions of the maximin and the minimax, henceforth referred to as the leximin and the antileximax, respectively. If one imposes in addition that an increase in someone's utility makes the society better off, then one is left with the leximin, while the requirement that society welfare increases as the result of a decrease of one person's utility gives the antileximax criterion. Incidentally, the paper provides an alternative and simple characterisation of the leximin principle widely used in the social choice and welfare literature.

1 Introduction

While inequality is seen as a major concern in most societies and has given rise to a large body of literature, less attention has been paid to the implication of the measurability nature of the variable of interest for its appraisal. So far, the literature

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has mostly focused on the design of indices and criteria for measuring inequality when the attribute distributed among the population's members is (at least) of a cardinal nature. The measurement of income and wealth inequality is a typical example where one even goes on assuming that the variables are measurable on a ratio scale. There are however many other variables that contribute to a person's well-being whose distribution is deemed to be important from a normative point of view, but that, at the same time, cannot be given fully agreed cardinal values. This may be due to a variety of reasons ranging from the fact that, *either* some attributes are by essence not cardinally measurable, *or* there is too much imprecision in the measurement to accord significance to the cardinal values provided, *or* no consensus can be reached about the appropriate cardinal values taken by the attribute.

In this paper, we are interested in the measurement of inequality—and more generally of social welfare—when the information available is purely ordinal. The only restriction—which actually has important consequences—is that the variable of interest is continuous. For convenience, one can think of utility but the analysis carries over for other items like cognitive abilities as measured, for instance, by the PISA scores. The case of categorical data—individuals are allocated into a finite number of ordered categories—raises additional problems and is the subject of a companion paper (see Gravel et al. 2018). Is it legitimate, as some authors do, to import the conventional tools designed for assessing income inequality to this informationally less demanding domain? If it is not, then when can it be reasonably claimed that one distribution is less unequal than another?

In the standard (income) inequality literature, a large consensus prevails for assimilating inequality reduction with the operation consisting of transferring some income from a richer individual to a poorer one in such a way that the beneficiary of the transfer is not made richer than the donor. An important restriction inherent in the definition of a so-called progressive transfer is that the amount taken from the richer individual must equal the amount received by the poorer individual. While it modifies the magnitude of the income differences, a change of scale under cardinal measurability preserves the (weak) inequalities between such differences. This ensures, among other things, that the equality between the amount taken from the richer and that given to the poorer is not affected by a change of scale. This equality does not survive to a change of scale in an ordinal setting even though the relative positions of the individuals are preserved. There is however something that is immune to a change of scale in both the ordinal and the cardinal frameworks: it is the fact that the better-off and the worse-off individuals taking part in the transfer become closer. This captures the most intuitive notion of inequality reduction one could have in mind and it is precisely this idea that was retained by Hammond (1976) in the formulation of his equity principle in a social choice framework (see also Hammond 1979).[1]

Hammond's equity condition requires that, if a person's utility increases while at the same time the utility of a better-off person decreases and if in addition the positions on the utility scale of all individuals are preserved—what we refer to as

[1] While this operation certainly reduces inequality in a two-person society, things are less obvious when the population consists of more than two individuals. There are indeed good reasons to consider that the fact of bringing two individuals closer in terms of their incomes has the effect of moving them farther apart from the other individuals in the society: for more on this, see, e.g., Magdalou and Moyes (2009).

a Hammond progressive transfer—then the society's welfare must improve. We are basically interested in comparing distributions of utilities with a particular focus on inequality, and to this aim we assume a social preference relation defined on the set of the distributions of utilities. Following Hammond's suggestion, we impose that the social preference relation has the property that, if one distribution is obtained from another by means of a Hammond progressive transfer, then it is ranked above the latter. Because there is no particular reason to prefer one social preference relation to another, we follow the dominance approach consisting in requiring unanimity over all those social preference relations obeying the equity condition. This leads us quite naturally to search for a practical means for testing whether one distribution is preferred to another by all the social preference relations that are compatible with the equity condition and—if we further believe that the identities of the utilities' receivers play no role—with the principle of anonymity.

A widely applied criterion for comparing distributions of utilities is the lexicographic extension of the maximin principle which we refer to as the leximin in what follows. The leximin ranks one distribution of utilities above another, if the worst-off person in the first distribution gets a higher utility than that of the worst-off person in the second distribution, or, in the case they enjoy the same utility in both distributions, if the second worst-off person in the first distribution gets a higher utility than what the second worst-off person is given in the second distribution, and so on. The lexicographic extension of the minimax—henceforth the antileximax—is defined analogously but starting the other way round and requiring that the utility of the best-off person, the second best-off person, and so on, is lower in the preferred distribution than in the other. It is remarkable that the simultaneous application of the leximin and the antileximax proves to be the appropriate procedure to check unanimity over all anonymous and equity regarding social preference relations. Adding a preference for more efficiently distributed utilities in addition to anonymity and equity precipitates the leximin, while the judgement that a decrease in someone's utility results in a better situation leads to the antileximax.

As far as the organisation of the paper is concerned, we proceed as follows. We present in Sect. 2 our general framework. We introduce in Sect. 3 our axioms and the different criteria for comparing situations. Section 4 contains the main results and their proofs. We discuss in Sect. 5 the relationships between our approach and other results in the inequality literature, while we provide in Sect. 6 a social choice (re)interpretation of our characterisations of the leximin and antileximax criteria. Finally, Sect. 7 concludes the paper by summarising the results and suggesting avenues for further research.

2 The framework

We are interested in the comparison of distributions of a continuous and ordinal attribute among n individuals, where $n \geqslant 2$. A distribution for a population $N{:=}\{1, 2, \ldots, n\}$ is a vector $\mathbf{u}{:=}(u_1, \ldots, u_n)$, where $u_i \in \mathbb{R}$ may be viewed as the *utility* of individual i, but other interpretations are also possible. We henceforth refer to the vector $\mathbf{u}$ as a *profile* and we indicate by $\mathscr{U}{:=}\mathbb{R}^n$ the set of profiles for the population N. In order to compare profiles, we have recourse to a *social preference* *relation* R on $\mathscr{U}$. We assume that the relation R is *reflexive* and *transitive*, i.e., (i) for all $\mathbf{u} \in \mathscr{U}$, one has $\mathbf{u}\,\mathrm{R}\,\mathbf{u}$ and (ii) for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{U}$, $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$ and $\mathbf{v}\,\mathrm{R}\,\mathbf{w}$ implies that $\mathbf{u}\,\mathrm{R}\,\mathbf{w}$, respectively. On the other hand, we do not impose that R is a complete relation on the set $\mathscr{U}$: the relation R can be an ordering or a quasi-ordering.[2] We indicate respectively by I and P the symmetric and asymmetric components of R defined in the usual way, and we denote by $\mathscr{R}$ the set of social preference relations. An important requirement throughout is that the ranking of the profiles under comparison provided by R is invariant to an increasing transformation of the individuals' utilities. Formally, this amounts to imposing:

Ordinal Scale Invariance (OSI). For all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$ and all increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, we have $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$ if and only if $\varphi(\mathbf{u})\,\mathrm{R}\,\varphi(\mathbf{v})$, where $\varphi(\mathbf{u}){:=}(\varphi(u_1), \ldots, \varphi(u_n))$ and $\varphi(\mathbf{v}){:=}(\varphi(v_1), \ldots, \varphi(v_n))$.

This restriction is reminiscent of the condition of *ordinal level comparability* (OLC) in the social choice literature (see, e.g., Sen 1977). Finally, given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we write: (i) $\mathbf{u} \geq \mathbf{v}$ whenever $u_h \geqslant v_h$, for all $h \in N$; (ii) $\mathbf{u} > \mathbf{v}$ whenever $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$.

3 Axioms and definitions

Given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we say that $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of a *permutation*—or, shortly, that $\mathbf{u}$ is a permutation of $\mathbf{v}$—if there exists a permutation matrix $Q{:=}[q_{ij}]$ such that $\mathbf{u} = \mathbf{v}\,Q$. For later use, we indicate by $\tilde{\mathbf{u}}{:=}(\tilde{u}_1, \ldots, \tilde{u}_n)$ the non-decreasing rearrangement of profile $\mathbf{u}{:=}(u_1, \ldots, u_n)$ defined by $\tilde{u}_1 \leqslant \tilde{u}_2 \leqslant \cdots \leqslant \tilde{u}_n$. We note that, if $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of a permutation, then $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$. Our first condition requires that the ranking of the profiles is not affected by a permutation of the individuals' utilities, which formally amounts to imposing:

Anonymity (A). For all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have $\mathbf{u}\,\mathrm{I}\,\mathbf{v}$ whenever $\mathbf{u}$ is a permutation of $\mathbf{v}$.

It is typically assumed in the economic inequality literature that a transfer of a fixed amount of income from a richer individual to a poorer one that leaves their respective positions unchanged—a so-called progressive transfer—reduces inequality. In an ordinal framework, where the individuals' utilities are defined up to the same increasing transformation, the notion of a progressive transfer makes no sense. Indeed, the equality between the utility gain and the utility loss of the individuals involved in the transfer is likely to be challenged by a change of the scale of measurement. The following transformation, introduced by Hammond (1976), captures the very essence of the notion of inequality reduction without imposing the restriction that the utility gain of the receiver equals the utility loss of the donor. More precisely, given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we say that $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of a *Hammond progressive transfer*, if there exist two individuals $i, j \in N$ such that:

[2] In this respect, our analysis framework is similar to that used by Tungodden (2000) but our approach differs significantly from his.

$$v_i < u_i \leqslant u_j < v_j \text{ and } u_h = v_h, \text{ for all } h \neq i, j. \tag{3.1}$$

Following Hammond, we want that, if one profile is more equal than another, then it is ranked above the latter, hence the following condition:

Hammond's Equity (HE). For all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have $\mathbf{u}\,\mathrm{P}\,\mathbf{v}$ whenever $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of a Hammond progressive transfer.

While inequality reduction is viewed as a main concern in social evaluation, it is also common to supplement the pursuit of more equality with efficiency considerations. Given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we say that $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of an *increment*, if there exists an individual $i \in N$ such that:

$$u_i > v_i \text{ and } u_h = v_h, \text{ for all } h \neq i. \tag{3.2}$$

Equivalently, we say that $\mathbf{v}$ is obtained from $\mathbf{u}$ by means of a *decrement*. Our next condition is standard and requires that a more efficient profile is always ranked above a less efficient one:

Strong Pareto (SP). For all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have $\mathbf{u}\,\mathrm{P}\,\mathbf{v}$ whenever $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of an increment.

The above condition implicitly assumes that utility is a desirable item: more utility is always preferred to less by any individual and also by the society. Depending on which interpretation of utility we have in mind, it may turn out that less utility might be preferable to more. So is the case when a waste has to be distributed among the population, for it is reasonable to assume that everybody would prefer to have less of it than more of it. In such cases, one is certainly willing to impose the following condition:

Strong AntiPareto (SAP). For all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have $\mathbf{u}\,\mathrm{P}\,\mathbf{v}$ whenever $\mathbf{u}$ is obtained $\mathbf{v}$ by means of a decrement.

For later use, it is convenient to introduce the following sets of social preference relations defined on the set of profiles $\mathscr{U}$:

$$\mathscr{R}_{+}^{\circ} := \{ \mathrm{R} \in \mathscr{R} \mid \text{conditions A and SP hold} \}; \tag{3.3a}$$
$$\mathscr{R}_{-}^{\circ} := \{ \mathrm{R} \in \mathscr{R} \mid \text{conditions A and SAP hold} \}; \tag{3.3b}$$
$$\mathscr{R}^{*} := \{ \mathrm{R} \in \mathscr{R} \mid \text{conditions A and HE hold} \}; \tag{3.3c}$$
$$\mathscr{R}_{+}^{*} := \{ \mathrm{R} \in \mathscr{R} \mid \text{conditions A, HE and SP hold} \}; \text{ and} \tag{3.3d}$$
$$\mathscr{R}_{-}^{*} := \{ \mathrm{R} \in \mathscr{R} \mid \text{conditions A, HE and SAP hold} \}. \tag{3.3e}$$

The following equivalence relation will be needed when we will define the different principles examined in the paper:

$$\forall\, \mathbf{u}, \mathbf{v} \in \mathscr{U}; \;\; \mathbf{u}\,\mathrm{I}^{*}\,\mathbf{v} \Longleftrightarrow \tilde{u}_h = \tilde{v}_h, \; \forall\, h \in N. \tag{3.4}$$

The first principle we consider is due to Suppes (1966) and, although it does not incorporate any concern for equality, we will occasionally refer to it in subsequent discussion and proofs.

SUppes quasi-ordering We say that **u** weakly Suppes dominates **v**, which we write $\mathbf{u}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{v}$, if and only if, *either* $\mathbf{u}\,\mathrm{I}_{\mathrm{SU}}\,\mathbf{v}$, where $\mathrm{I}_{\mathrm{SU}} = \mathrm{I}^*$, *or* $\mathbf{u}\,\mathrm{P}_{\mathrm{SU}}\,\mathbf{v}$ which intends to mean that:

$$\tilde{u}_h \geqslant \tilde{v}_h,\ \forall\, h \in N \text{ and } \exists\, i \in N \mid \tilde{u}_i > \tilde{v}_i\ . \tag{3.5}$$

The second principle is the lexicographic extension of the maximin—known as the leximin ordering—introduced by Sen (1977) that gives priority to the worst-off persons in the society.

LexiMin We say that **u** weakly leximin dominates **v**, which we write $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$, if and only if, *either* $\mathbf{u}\,\mathrm{I}_{\mathrm{LM}}\,\mathbf{v}$, where $\mathrm{I}_{\mathrm{LM}} = \mathrm{I}^*$, *or* $\mathbf{u}\,\mathrm{P}_{\mathrm{LM}}\,\mathbf{v}$ which intends to mean that:

$$\exists\, i \in N \mid \tilde{u}_h = \tilde{v}_h,\ \forall\, h \in \{1, 2, \ldots, i-1\} \text{ and } \tilde{u}_i > \tilde{v}_i\ . \tag{3.6}$$

According to the leximin, profile **u** is considered to be better than profile **v** if the person who is the most disadvantaged in profile **u** gets a higher utility than the most disadvantaged person in profile **v** or, if they both enjoy the same utility, then the second worst-off person in **u** gets a higher utility than her counterpart in **v**, and so on. The third principle mirrors the preceding one by focusing on the situations of the best-off persons.

AntiLeximaX We say that **u** weakly antileximax dominates **v**, which we write $\mathbf{u}\,\mathrm{R}_{\mathrm{ALX}}\,\mathbf{v}$, if and only if, *either* $\mathbf{u}\,\mathrm{I}_{\mathrm{ALX}}\,\mathbf{v}$, where $\mathrm{I}_{\mathrm{ALX}} = \mathrm{I}^*$, *or* $\mathbf{u}\,\mathrm{P}_{\mathrm{ALX}}\,\mathbf{v}$ which intends to mean that:

$$\exists\, j \in N \,\Big|\, \tilde{u}_j < \tilde{v}_j \quad \text{and} \quad \tilde{u}_h = \tilde{v}_h,\ \forall\, h \in \{j+1, j+2, \ldots, n\}\ . \tag{3.7}$$

The antileximax ranks profile **u** above profile **v** as soon as the best-off person in profile **u** gets a lower utility than the best-off person in profile **v** or, if they have the same utility, whenever the second best-off person in **u** gets a lower utility than her counterpart in **v**, and so on. The leximin and the antileximax pay attention exclusively to the worst-off and the best-off persons in the society, respectively. The next and last principle—which appears to be new to the best of our knowledge—can be seen as a compromise between the views expressed by the leximin and the antileximax.

LexiMin–antileximaX We say that **u** weakly leximin–antileximax dominates **v**, which we write $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$, if and only if, *either* $\mathbf{u}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{v}$, where $\mathrm{I}_{\mathrm{LMX}} = \mathrm{I}^*$, *or* $\mathbf{u}\,\mathrm{P}_{\mathrm{LMX}}\,\mathbf{v}$ which intends to mean that:

$$\exists\, i, j \in N\,(i < j) \mid \tilde{u}_i > \tilde{v}_i;\ \tilde{u}_j < \tilde{v}_j;\ \text{and} \tag{3.8a}$$

$$\tilde{u}_h = \tilde{v}_h,\ \forall\, h \in \{1, 2, \ldots, i-1\} \cup \{j+1, j+2, \ldots, n\}. \tag{3.8b}$$

The leximin–antileximax expresses a concern for less unequally distributed utilities in the sense that the utilities are more concentrated in the preferred profile if we leave aside those individuals at both ends of the distributions who are not affected by the choice between the two profiles. It must be emphasised at this stage that the leximin–antileximax pays no attention to the utilities of those individuals who are ranked between i and j. A profile can be ranked above another by the leximin–antileximax while the utilities of these individuals are more unequal in the preferred profile than in the other, something that may be seen a limitation of this criterion. Consider for instance the profiles $\mathbf{u} = (1, 3, 4, 6, 7, 9)$ and $\mathbf{v} = (1, 2, 5, 5, 8, 9)$: clearly, $\mathbf{u}\,\mathrm{P}_{\mathrm{LMX}}\,\mathbf{v}$, while at the same time the utilities of the individuals with ranks 3 and 4 are more equal in $\mathbf{v}$ than in $\mathbf{u}$. Implicit in the definition of the leximin–antileximax is the idea that the reduction of the inequalities between individuals ranked i and j overcome all the possible—and whatever their number—inequalities between the individuals whose ranks lie between i and j. As we will see later on, it is an interesting—and somewhat surprising—property of the leximin–antileximax that the preferred profile can always been derived from the less preferred one by a succession of inequality reducing operations of the kind described above. It follows from the definitions above that these four criteria are nested in the way described below.

Remark 1 For all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have: (i) $\mathbf{u}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{v}$ implies $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$; (ii) $\mathbf{v}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{u}$ implies $\mathbf{u}\,\mathrm{R}_{\mathrm{ALX}}\,\mathbf{v}$; and (iii) $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$ and $\mathbf{u}\,\mathrm{R}_{\mathrm{ALX}}\,\mathbf{v}$ if and only if $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$.

It must be noted that, while the Suppes criterion and the leximin–antileximax provide partial rankings of the feasible profiles, the leximin and the antileximax totally order the elements of $\mathscr{U}$. The following example illustrates the preceding definitions and show to which extent the different principles depart from each other when ranking particular profiles.

Example 3.1 Let $n = 4$ and consider the four following profiles and their corresponding non-decreasing rearrangements:

$$\begin{aligned} \mathbf{u}^{(1)} &= (1, 2, 5, 8); & \tilde{\mathbf{u}}^{(1)} &= (1, 2, 5, 8);\\ \mathbf{u}^{(2)} &= (4, 6, 5, 7); & \tilde{\mathbf{u}}^{(2)} &= (4, 5, 6, 7);\\ \mathbf{u}^{(3)} &= (4, 2, 3, 7); & \tilde{\mathbf{u}}^{(3)} &= (2, 3, 4, 7);\\ \mathbf{u}^{(4)} &= (4, 2, 5, 7); & \tilde{\mathbf{u}}^{(4)} &= (2, 4, 5, 7). \end{aligned}$$

Application of the four above criteria gives the results indicated in Table 1 where the symbol "+1" at the intersection of row i and column j means that profile $\mathbf{u}^{(i)}$ is preferred to profile $\mathbf{u}^{(j)}$, the symbol "−1" that profile $\mathbf{u}^{(j)}$ is preferred to profile $\mathbf{u}^{(i)}$, and the symbol "#" means that they are not comparable.

Figure 1 provides an illustration of the different criteria above for a population comprising two individuals. The points U and V correspond respectively to the profile $\mathbf{u} = (2, 5)$ and to its permutation $\mathbf{v} = (5, 2)$. For each criteria, the area in light grey (including its boundary) corresponds to the profiles that are considered as least as good as $\mathbf{u}$, while the area in dark grey (including its boundary) represents the profiles that

Table 1 Rankings of profiles $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$ and $\mathbf{u}^{(4)}$

	$\mathbf{u}^{(2)}$	$\mathbf{u}^{(3)}$	$\mathbf{u}^{(4)}$
Suppes			
$\mathbf{u}^{(1)}$	#	#	#
$\mathbf{u}^{(2)}$		+1	+1
$\mathbf{u}^{(3)}$			−1
Leximin			
$\mathbf{u}^{(1)}$	#	#	#
$\mathbf{u}^{(2)}$		+1	+1
$\mathbf{u}^{(3)}$			−1
Antileximax			
$\mathbf{u}^{(1)}$	−1	−1	−1
$\mathbf{u}^{(2)}$		−1	−1
$\mathbf{u}^{(3)}$			+1
Leximin–Antileximax			
$\mathbf{u}^{(1)}$	−1	−1	−1
$\mathbf{u}^{(2)}$		#	#
$\mathbf{u}^{(3)}$			#

are considered as least as bad as $\mathbf{u}$. The white areas represent all the profiles that are not comparable with $\mathbf{u}$ and, thanks to anonymity, also with $\mathbf{v}$. In the particular case where $n = 2$, the sets of the best elements for the leximin and the antileximax are identical to the sets of the best elements for the maximin and the minimax, respectively, hence the impression that the criteria are complete. In the case of the leximin–antileximax, it must be noted that the set of profiles that are better than $\mathbf{u}$ is constituted by the light grey area with the exclusion of its boundary coloured in black and of the points U and V. This corresponds to the intersection of the sets of profiles that are ranked above $\mathbf{u}$ by the leximin and the antileximax. Adding the points U and V gives the set of profiles that are considered to be as least as good as $\mathbf{u}$ by the leximin–antileximax.

4 The main results

Our first result is but a mere restatement of a standard result in the inequality and risk literature and we mostly present it for the sake of completeness.

Theorem 4.1 *For all* $\mathbf{u}, \mathbf{v} \in \mathscr{U}$*, the following three statements are equivalent:*

(a) $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by means of a finite sequence of permutations and/or increments.*
(b) $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$*, for all* $\mathrm{R} \in \mathscr{R}_{+}^{\circ}$.
(c) $\mathbf{u}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{v}$.

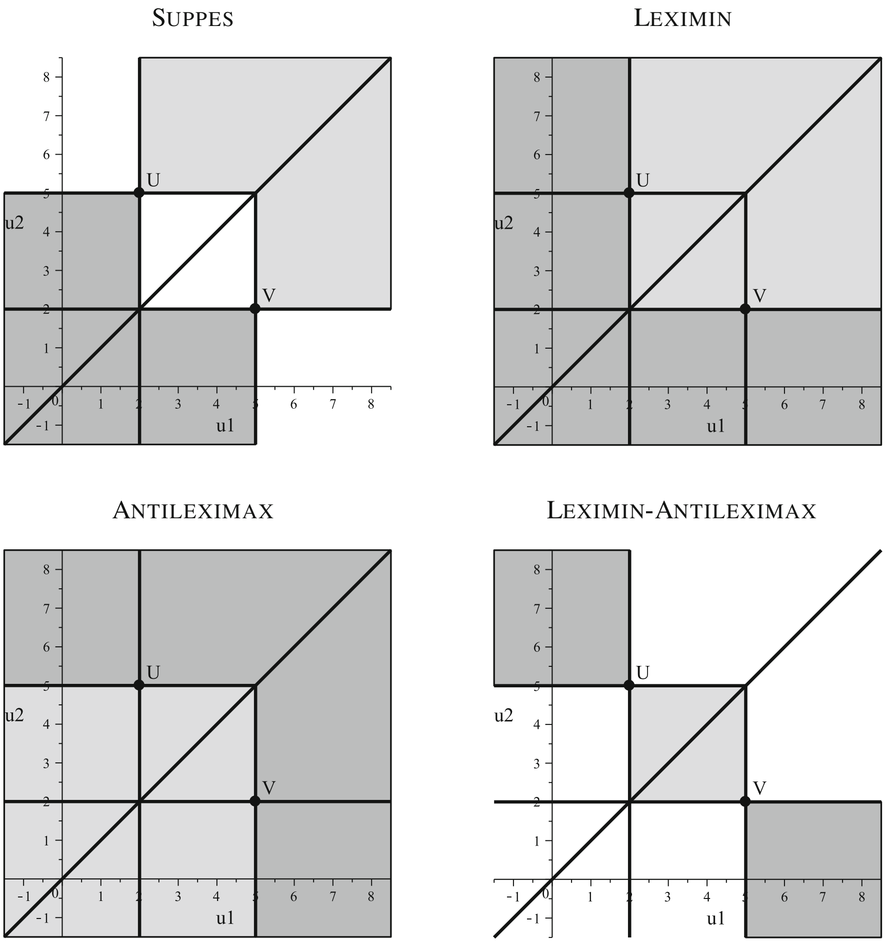

Fig. 1 Dominating and dominated profiles for a two-person society

Proof (a) $\Longrightarrow$ (c). If $\mathbf{u}$ is a permutation of $\mathbf{v}$, then $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$ and $\mathbf{u}\,\mathrm{I}_{\mathrm{SU}}\,\mathbf{v}$. If $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of increments, then it follows from Moyes (2013, Lemma 2.1) that $\tilde{\mathbf{u}} > \tilde{\mathbf{v}}$. To sum up, statement (a) implies that $\mathbf{u}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{v}$.

(c) $\Longrightarrow$ (a). Suppose that $\mathbf{u}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{v}$, in which case there are two possibilities: *either* $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$, *or* $\tilde{\mathbf{u}} > \tilde{\mathbf{v}}$. If $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$, then $\mathbf{u}$ is a permutation of $\mathbf{v}$. If $\tilde{\mathbf{u}} > \tilde{\mathbf{v}}$, then one obtains $\tilde{\mathbf{u}}$ starting from $\tilde{\mathbf{v}}$ by means of at most n increments. In the latter case, one will have recourse to permutations if it happens that $\tilde{\mathbf{u}} \neq \mathbf{u}$ and/or $\tilde{\mathbf{v}} \neq \mathbf{v}$.

(a) $\Longrightarrow$ (b). This follows from the definitions of conditions A and SP.

(b) $\Longrightarrow$ (c). The fact that (a) implies (c) guarantees that the Suppes criterion verifies condition A and SP, hence $\mathrm{R}_{\mathrm{SU}} \in \mathscr{R}_{+}^{\circ}$. Therefore, if statement (b) holds, then so does statement (c). □

According to Theorem 4.1, the Suppes criterion represents the point of view of unanimity among all social preference relations that satisfy anonymity and strong Pareto. It further says that when this unanimous agreement holds, then the profile that is preferred can be derived from the other by making use of permutations and increments only.

The replacement of Strong Pareto by Hammond's Equity gives the following result that may be considered the counterpart of the Hardy-Littlewood-Pólya-theorem in an ordinal framework.

Theorem 4.2 *For all* $\mathbf{u}, \mathbf{v} \in \mathscr{U}$*, the following three statements are equivalent:*

(a) $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by means of a finite sequence of permutations and/or Hammond progressive transfers.*
(b) $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$*, for all* $\mathrm{R} \in \mathscr{R}^*$.
(c) $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$.

Proof (a) $\Longrightarrow$ (c). If $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of permutations, then $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$, and therefore $\mathbf{u}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{v}$. Suppose now that $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of a single Hammond progressive transfer so that (3.1) holds for some $i, j \in N$. Consider the indices g, h, k and ℓ defined by

$$g{:=}\max\big\{t \in \{1,2,\ldots,n\} \,\big|\, \tilde{v}_t \leqslant v_i\big\}, \tag{4.1a}$$

$$h{:=}\max\big\{t \in \{1,2,\ldots,n\} \,\big|\, \tilde{u}_t \leqslant u_i\big\}, \tag{4.1b}$$

$$k{:=}\min\big\{t \in \{1,2,\ldots,n\} \,\big|\, \tilde{u}_t \geqslant u_j\big\}, \text{ and} \tag{4.1c}$$

$$\ell{:=}\min\big\{t \in \{1,2,\ldots,n\} \,\big|\, \tilde{v}_t \geqslant v_j\big\}, \tag{4.1d}$$

that always exist. It follows from (3.1) and the definitions above that

$$\tilde{v}_g = v_i;\;\; \tilde{u}_h = u_i;\;\; \tilde{u}_k = u_j;\;\; \tilde{v}_\ell = v_j; \tag{4.2a}$$

$$\tilde{v}_g < \tilde{u}_h \leqslant \tilde{u}_k < \tilde{v}_\ell;\; \text{and} \tag{4.2b}$$

$$1 \leqslant g \leqslant h < k \leqslant \ell \leqslant n; \tag{4.2c}$$

hence the following situation:

	1	$g-1$	g	$g+1$	$h-1$	h	$h+1$	$k-1$	k	$k+1$	$\ell-1$	ℓ	$\ell+1$	n
						u_i			u_j					
						$\Vert$			$\Vert$					
$\tilde{\mathbf{u}}$:	$\tilde{u}_1 \leqslant$	$\tilde{u}_{g-1} \leqslant$	$\tilde{u}_g \leqslant$	$\tilde{u}_{g+1} \leqslant$	$\tilde{u}_{h-1} \leqslant$	$\tilde{u}_h <$	$\tilde{u}_{h+1} \leqslant$	$\tilde{u}_{k-1} <$	$\tilde{u}_k \leqslant$	$\tilde{u}_{k+1} \leqslant$	$\tilde{u}_{\ell-1} \leqslant$	$\tilde{u}_\ell <$	$\tilde{u}_{\ell+1} \leqslant$	$\tilde{u}_n$
	$\Vert$	$\Vert$											$\Vert$	$\Vert$
$\tilde{\mathbf{v}}$:	$\tilde{v}_1 \leqslant$	$\tilde{v}_{g-1} \leqslant$	$\tilde{v}_g <$	$\tilde{v}_{g+1} \leqslant$	$\tilde{v}_{h-1} \leqslant$	$\tilde{v}_h \leqslant$	$\tilde{v}_{h+1} \leqslant$	$\tilde{v}_{k-1} \leqslant$	$\tilde{v}_k \leqslant$	$\tilde{v}_{k+1} \leqslant$	$\tilde{v}_{\ell-1} <$	$\tilde{v}_\ell \leqslant$	$\tilde{v}_{\ell+1} \leqslant$	$\tilde{v}_n$
			$\Vert$									$\Vert$		
			v_i									v_j		

Inspection of the table makes clear that, for $\mathbf{u}\,\mathrm{P}_{\mathrm{LMX}}\,\mathbf{v}$, one needs to have $\tilde{u}_g > \tilde{v}_g$ and $\tilde{u}_\ell < \tilde{v}_\ell$. At the risk of providing too many details, we distinguish four cases.

CASE 1 $g = h$. Then, we have $\tilde{u}_t = \tilde{v}_t$, for all $t = 1, 2, \ldots, g-1$, and $\tilde{u}_g = u_i > v_i = \tilde{v}_g$.

CASE 2 $g < h$. By definition of the indices g and h, we have

$$\begin{array}{ccccccccccccccc} & & \tilde{u}_g & \leqslant & \tilde{u}_{g+1} & \leqslant & \cdots & \leqslant & \tilde{u}_{h-3} & \leqslant & \tilde{u}_{h-2} & \leqslant & \tilde{u}_{h-1} & \leqslant & \tilde{u}_h = u_i \\ & & \| & & \| & & & & \| & & \| & & \| & & \\ v_i = \tilde{v}_g & < & \tilde{v}_{g+1} & \leqslant & \tilde{v}_{g+2} & \leqslant & \cdots & \leqslant & \tilde{v}_{h-2} & \leqslant & \tilde{v}_{h-1} & \leqslant & \tilde{v}_h & & \end{array}$$

from which we deduce that $\tilde{u}_t = \tilde{v}_t$, for all $t = 1, 2, \ldots, g-1$, and $\tilde{u}_g > \tilde{v}_g$.

CASE 3 $k = \ell$. Then, we have $\tilde{u}_\ell = u_j < v_j = \tilde{v}_\ell$ and $\tilde{u}_t = \tilde{v}_t$, for all $t = \ell+1, \ldots, n-1, n$.

CASE 4 $k < \ell$. By definition of the indices k and ℓ, we have

$$\begin{array}{ccccccccccccccc} u_j = \tilde{u}_k & \leqslant & \tilde{u}_{k+1} & \leqslant & \tilde{u}_{k+2} & \leqslant & \cdots & \leqslant & \tilde{u}_{\ell-2} & \leqslant & \tilde{u}_{\ell-1} & \leqslant & \tilde{u}_\ell & & \\ & & \| & & \| & & & & \| & & \| & & \| & & \\ & & \tilde{v}_k & \leqslant & \tilde{v}_{k+1} & \leqslant & \cdots & \leqslant & \tilde{v}_{\ell-3} & \leqslant & \tilde{v}_{\ell-2} & \leqslant & \tilde{v}_{\ell-1} & < & \tilde{v}_\ell = v_j \end{array}$$

from which we deduce that $\tilde{u}_t = \tilde{v}_t$, for all $t = n, n-1, \ldots, \ell+1$, and $\tilde{u}_\ell < \tilde{v}_\ell$.

To sum up, we have $\tilde{u}_t = \tilde{v}_t$, for all $t \in \{1, 2, \ldots, g-1\} \cup \{\ell+1, \ell+2, \ldots, n\}$, $\tilde{u}_g > \tilde{v}_g$ and $\tilde{u}_\ell < \tilde{v}_\ell$, hence $\mathbf{u}\,\mathrm{P}_{\mathrm{LMX}}\,\mathbf{v}$. When more than one Hammond progressive transfer is needed in order to transform $\mathbf{v}$ into $\mathbf{u}$, the result follows from the transitivity of $\mathrm{P}_{\mathrm{LMX}}$. We therefore conclude that, if statement (a) holds, then $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$.

(c) $\Longrightarrow$ (a). Suppose that $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$, in which case there are two possibilities: *either* $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$, *or* $\tilde{\mathbf{u}} \neq \tilde{\mathbf{v}}$. If $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$, then $\mathbf{u}$ is a permutation of $\mathbf{v}$ and we are home. Now, if $\tilde{\mathbf{u}} \neq \tilde{\mathbf{v}}$, then $\mathbf{u}\,\mathrm{P}_{\mathrm{LMX}}\,\mathbf{v}$, hence there exist two indices i and j ($1 \leqslant i < j \leqslant n$) such that:

$$\tilde{u}_t = \tilde{v}_t,\ \forall\, t \in \{1, 2, \ldots, i-1\} \cup \{j+1, j+2, \ldots, n\}; \tag{4.3a}$$
$$\tilde{u}_i > \tilde{v}_i \text{ and } \tilde{u}_j < \tilde{v}_j. \tag{4.3b}$$

Consider now the indices g and h defined by

$$g := \max\left\{t \in \{i, i+1, \ldots, j-2, j-1\} \,\middle|\, \tilde{u}_t > \tilde{v}_t\right\} \text{ and} \tag{4.4a}$$
$$h := \min\left\{t \in \{g+1, g+2, \ldots, j-1, j\} \,\middle|\, \tilde{u}_t < \tilde{v}_t\right\}. \tag{4.4b}$$

Two such indices necessarily exist and it follows from their definitions that $i \leqslant g < h \leqslant j$ and

$$\tilde{u}_t = \tilde{v}_t,\ \forall\, t \in \{g+1, \ldots, h-1\}, \tag{4.5}$$

provided that $g \neq h-1$. The table below summarises the available information derived from the above definitions that is at our disposal for arguing.

	$i-1$		i		$i+1$		$g-1$		g		$g+1$		$h-1$		h		$h+1$		$j-1$		j		$j+1$
$\tilde{\mathbf{u}}:$	$\tilde{u}_{i-1}$	$<$	$\tilde{u}_i$	$\leqslant$	$\tilde{u}_{i+1}$	$\leqslant$	$\tilde{u}_{g-1}$	$\leqslant$	$\tilde{u}_g$	$\leqslant$	$\tilde{u}_{g+1}$	$\leqslant$	$\tilde{u}_{h-1}$	$\leqslant$	$\tilde{u}_h$	$\leqslant$	$\tilde{u}_{h+1}$	$\leqslant$	$\tilde{u}_{j-1}$	$\leqslant$	$\tilde{u}_j$	$<$	$\tilde{u}_{j+1}$
	$=$		$>$		$\geqslant$		$\geqslant$		$>$		$=$		$=$		$<$		$\leqslant$		$\leqslant$		$<$		$=$
$\tilde{\mathbf{v}}:$	$\tilde{v}_{i-1}$	$\leqslant$	$\tilde{v}_i$	$\leqslant$	$\tilde{v}_{i+1}$	$\leqslant$	$\tilde{v}_{g-1}$	$\leqslant$	$\tilde{v}_g$	$<$	$\tilde{v}_{g+1}$	$\leqslant$	$\tilde{v}_{h-1}$	$<$	$\tilde{v}_h$	$\leqslant$	$\tilde{v}_{h+1}$	$\leqslant$	$\tilde{v}_{j-1}$	$\leqslant$	$\tilde{v}_j$	$\leqslant$	$\tilde{v}_{j+1}$

To prove the implication, we first show that it is possible to find a profile $\tilde{\mathbf{w}}:=(\tilde{w}_1, \ldots, \tilde{w}_n)$ with $\tilde{w}_1 \leqslant \tilde{w}_2 \leqslant \cdots \leqslant \tilde{w}_n$ such that (i) $\tilde{\mathbf{w}}$ is obtained from $\tilde{\mathbf{v}}$ by means of a Hammond progressive transfer, (ii) $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{w}$, and (iii) $\tilde{w}_t = \tilde{u}_t$, for at least one $t \in \{g, h\}$. We find convenient to distinguish four cases.

CASE 1 $i < g < h < j$. Choosing $\tilde{\mathbf{w}}$ such that $\tilde{u}_g = \tilde{w}_g > \tilde{v}_g$, $\tilde{u}_h = \tilde{w}_h < \tilde{v}_h$, and $\tilde{w}_t = \tilde{v}_t$, for all $t \neq g, h$, we have the following situation:

	$i-1$		i		$i+1$		$g-1$		g		$g+1$		$h-1$		h		$h+1$		$j-1$		j		$j+1$
$\tilde{\mathbf{u}}:$	$\tilde{u}_{i-1}$	$<$	$\tilde{u}_i$	$\leqslant$	$\tilde{u}_{i+1}$	$\leqslant$	$\tilde{u}_{g-1}$	$\leqslant$	$\tilde{u}_g$	$\leqslant$	$\tilde{u}_{g+1}$	$\leqslant$	$\tilde{u}_{h-1}$	$\leqslant$	$\tilde{u}_h$	$\leqslant$	$\tilde{u}_{h+1}$	$\leqslant$	$\tilde{u}_{j-1}$	$\leqslant$	$\tilde{u}_j$	$<$	$\tilde{u}_{j+1}$
	$=$		$>$		$\geqslant$		$\geqslant$		$=$		$=$		$=$		$=$		$\leqslant$		$\leqslant$		$<$		$=$
$\tilde{\mathbf{w}}:$	$\tilde{w}_{i-1}$	$\leqslant$	$\tilde{w}_i$	$\leqslant$	$\tilde{w}_{i+1}$	$\leqslant$	$\tilde{w}_{g-1}$	$<$	$\tilde{w}_g$	$\leqslant$	$\tilde{w}_{g+1}$	$\leqslant$	$\tilde{w}_{h-1}$	$\leqslant$	$\tilde{w}_h$	$<$	$\tilde{w}_{h+1}$	$\leqslant$	$\tilde{w}_{j-1}$	$\leqslant$	$\tilde{w}_j$	$\leqslant$	$\tilde{w}_{j+1}$
	$=$		$=$		$=$		$=$		$>$		$=$		$=$		$<$		$=$		$=$		$=$		$=$
$\tilde{\mathbf{v}}:$	$\tilde{v}_{i-1}$	$\leqslant$	$\tilde{v}_i$	$\leqslant$	$\tilde{v}_{i+1}$	$\leqslant$	$\tilde{v}_{g-1}$	$\leqslant$	$\tilde{v}_g$	$<$	$\tilde{v}_{g+1}$	$\leqslant$	$\tilde{v}_{h-1}$	$<$	$\tilde{v}_h$	$\leqslant$	$\tilde{v}_{h+1}$	$\leqslant$	$\tilde{v}_{j-1}$	$\leqslant$	$\tilde{v}_j$	$\leqslant$	$\tilde{v}_{j+1}$

By construction $\tilde{\mathbf{w}}$ is obtained from $\tilde{\mathbf{v}}$ by means of a Hammond progressive transfer, hence $\mathbf{w}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{v}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{v}$. Inspection of the table above reveals that $\tilde{\mathbf{w}}$ is non-decreasingly arranged and that $\mathbf{u}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{u}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{w}$. Furthermore $\tilde{u}_g = \tilde{w}_g$ and $\tilde{u}_h = \tilde{w}_h$.

CASE 2: $i = g < h < j$. Choosing $\tilde{\mathbf{w}}$ such that $\tilde{u}_g > \tilde{w}_g > \tilde{v}_g$, $\tilde{u}_h = \tilde{w}_h < \tilde{v}_h$, and $\tilde{w}_t = \tilde{v}_t$, for all $t \neq g, h$, we have the following situation:

	$g-2$		$g-1$		$g=i$		$g+1$		$g+2$		$h-2$		$h-1$		h		$h+1$		$j-1$		j		$j+1$
$\tilde{\mathbf{u}}:$	$\tilde{u}_{g-2}$	$\leqslant$	$\tilde{u}_{g-1}$	$\leqslant$	$\tilde{u}_g$	$\leqslant$	$\tilde{u}_{g+1}$	$\leqslant$	$\tilde{u}_{g+2}$	$\leqslant$	$\tilde{u}_{h-2}$	$\leqslant$	$\tilde{u}_{h-1}$	$\leqslant$	$\tilde{u}_h$	$\leqslant$	$\tilde{u}_{h+1}$	$\leqslant$	$\tilde{u}_{j-1}$	$\leqslant$	$\tilde{u}_j$	$<$	$\tilde{u}_{j+1}$
	$=$		$=$		$>$		$=$		$=$		$=$		$=$		$=$		$\leqslant$		$\leqslant$		$<$		$=$
$\tilde{\mathbf{w}}:$	$\tilde{w}_{g-2}$	$\leqslant$	$\tilde{w}_{g-1}$	$<$	$\tilde{w}_g$	$\leqslant$	$\tilde{w}_{g+1}$	$\leqslant$	$\tilde{w}_{g+2}$	$\leqslant$	$\tilde{w}_{h-2}$	$\leqslant$	$\tilde{w}_{h-1}$	$\leqslant$	$\tilde{w}_h$	$<$	$\tilde{w}_{h+1}$	$\leqslant$	$\tilde{w}_{j-1}$	$\leqslant$	$\tilde{w}_j$	$\leqslant$	$\tilde{w}_{j+1}$
	$=$		$=$		$>$		$=$		$=$		$=$		$=$		$<$		$=$		$=$		$=$		$=$
$\tilde{\mathbf{v}}:$	$\tilde{v}_{g-2}$	$\leqslant$	$\tilde{v}_{g-1}$	$\leqslant$	$\tilde{v}_g$	$<$	$\tilde{v}_{g+1}$	$\leqslant$	$\tilde{v}_{g+2}$	$\leqslant$	$\tilde{v}_{h-2}$	$\leqslant$	$\tilde{v}_{h-1}$	$<$	$\tilde{v}_h$	$\leqslant$	$\tilde{v}_{h+1}$	$\leqslant$	$\tilde{v}_{j-1}$	$\leqslant$	$\tilde{v}_j$	$\leqslant$	$\tilde{v}_{j+1}$

By construction $\tilde{\mathbf{w}}$ is obtained from $\tilde{\mathbf{v}}$ by means of a Hammond progressive transfer, hence $\mathbf{w}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{v}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{v}$. Inspection of the table above reveals that $\tilde{\mathbf{w}}$ is non-decreasingly arranged, that $\tilde{u}_h = \tilde{w}_h$, and that $\mathbf{u}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{u}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{w}$.

CASE 3: $i < g < h = j$. Choosing $\tilde{\mathbf{w}}$ such that $\tilde{u}_g = \tilde{w}_g > \tilde{v}_g$, $\tilde{u}_h < \tilde{w}_h < \tilde{v}_h$, and $\tilde{w}_t = \tilde{v}_t$, for all $t \neq g, h$, we have the following situation:

	$i-1$		i		$i+1$		$g-1$		g		$g+1$		$h-2$		$h-1$		$h=j$		$h+1$		$h+2$		$n-1$
$\tilde{\mathbf{u}}:$	$\tilde{u}_{i-1}$	$<$	$\tilde{u}_i$	$\leqslant$	$\tilde{u}_{i+1}$	$\leqslant$	$\tilde{u}_{g-1}$	$\leqslant$	$\tilde{u}_g$	$\leqslant$	$\tilde{u}_{g+1}$	$\leqslant$	$\tilde{u}_{h-2}$	$\leqslant$	$\tilde{u}_{h-1}$	$\leqslant$	$\tilde{u}_h$	$\leqslant$	$\tilde{u}_{h+1}$	$\leqslant$	$\tilde{u}_{h+2}$	$\leqslant$	$\tilde{u}_{n-1}$
	$\parallel$		$\vee$		$\vee\!\wedge$		$\vee\!\wedge$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\wedge$		$\parallel$		$\parallel$		$\parallel$
$\tilde{\mathbf{w}}:$	$\tilde{w}_{i-1}$	$\leqslant$	$\tilde{w}_i$	$\leqslant$	$\tilde{w}_{i+1}$	$\leqslant$	$\tilde{w}_{g-1}$	$<$	$\tilde{w}_g$	$\leqslant$	$\tilde{w}_{g+1}$	$\leqslant$	$\tilde{w}_{h-2}$	$\leqslant$	$\tilde{w}_{h-1}$	$\leqslant$	$\tilde{w}_h$	$<$	$\tilde{w}_{h+1}$	$\leqslant$	$\tilde{w}_{h+2}$	$\leqslant$	$\tilde{w}_{n-1}$
	$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\vee$		$\parallel$		$\parallel$		$\parallel$		$\wedge$		$\parallel$		$\parallel$		$\parallel$
$\tilde{\mathbf{v}}:$	$\tilde{v}_{i-1}$	$\leqslant$	$\tilde{v}_i$	$\leqslant$	$\tilde{v}_{i+1}$	$\leqslant$	$\tilde{v}_{g-1}$	$\leqslant$	$\tilde{v}_g$	$<$	$\tilde{v}_{g+1}$	$\leqslant$	$\tilde{v}_{h-2}$	$\leqslant$	$\tilde{v}_{h-1}$	$<$	$\tilde{v}_h$	$\leqslant$	$\tilde{v}_{h+1}$	$\leqslant$	$\tilde{v}_{h+2}$	$\leqslant$	$\tilde{v}_{n-1}$

By construction $\tilde{\mathbf{w}}$ is obtained from $\tilde{\mathbf{v}}$ by means of a Hammond progressive transfer, hence $\mathbf{w}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{v}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{v}$. Inspection of the table above reveals that $\tilde{\mathbf{w}}$ is non-decreasingly arranged, that $\tilde{u}_g = \tilde{w}_g$, and that $\mathbf{u}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{u}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{w}$.

CASE 4: $i = g < h = j$. It is identical to case 1 with the particularity that now we have the simple situation depicted below:

	2		$g-2$		$g-1$		$g=i$		$g+1$		$g+2$		$h-2$		$h-1$		$h=j$		$h+1$		$h+2$		$n-1$
$\tilde{\mathbf{u}}:$	$\tilde{u}_2$	$\leqslant$	$\tilde{u}_{g-2}$	$\leqslant$	$\tilde{u}_{g-1}$	$<$	$\tilde{u}_g$	$\leqslant$	$\tilde{u}_{g+1}$	$\leqslant$	$\tilde{u}_{g+2}$	$\leqslant$	$\tilde{u}_{h-2}$	$\leqslant$	$\tilde{u}_{h-1}$	$\leqslant$	$\tilde{u}_h$	$<$	$\tilde{u}_{h+1}$	$\leqslant$	$\tilde{u}_{h+2}$	$\leqslant$	$\tilde{u}_{n-1}$
	$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\parallel$
$\tilde{\mathbf{w}}:$	$\tilde{w}_2$	$\leqslant$	$\tilde{w}_{g-2}$	$\leqslant$	$\tilde{w}_{g-1}$	$<$	$\tilde{w}_g$	$\leqslant$	$\tilde{w}_{g+1}$	$\leqslant$	$\tilde{w}_{g+2}$	$\leqslant$	$\tilde{w}_{h-2}$	$\leqslant$	$\tilde{w}_{h-1}$	$\leqslant$	$\tilde{w}_h$	$<$	$\tilde{w}_{h+1}$	$\leqslant$	$\tilde{w}_{h+2}$	$\leqslant$	$\tilde{w}_{n-1}$
	$\parallel$		$\parallel$		$\parallel$		$\vee$		$\parallel$		$\parallel$		$\parallel$		$\parallel$		$\wedge$		$\parallel$		$\parallel$		$\parallel$
$\tilde{\mathbf{v}}:$	$\tilde{v}_2$	$\leqslant$	$\tilde{v}_{g-2}$	$\leqslant$	$\tilde{v}_{g-1}$	$\leqslant$	$\tilde{v}_g$	$<$	$\tilde{v}_{g+1}$	$\leqslant$	$\tilde{v}_{g+2}$	$\leqslant$	$\tilde{v}_{h-2}$	$\leqslant$	$\tilde{v}_{h-1}$	$<$	$\tilde{v}_h$	$\leqslant$	$\tilde{v}_{h+1}$	$\leqslant$	$\tilde{v}_{h+2}$	$\leqslant$	$\tilde{v}_{n-1}$

By construction $\tilde{\mathbf{w}}$ is obtained from $\tilde{\mathbf{v}}$ by means of a Hammond progressive transfer, hence $\mathbf{w}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{v}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{v}$. Inspection of the table above reveals that $\tilde{\mathbf{w}}$ is non-decreasingly arranged and $\mathbf{u}\,\mathrm{I}_{\mathrm{LMX}}\,\tilde{\mathbf{u}} = \tilde{\mathbf{w}}\,\mathrm{I}_{\mathrm{LMX}}\,\mathbf{w}$.

Let us denote by $d(\mathbf{u}, \mathbf{v}) := \#\{t \in \{1, 2, \ldots, n\} \mid u_t \neq v_t\}$ the number of distinct components in $\mathbf{u}$ and $\mathbf{v}$. To sum up, we have shown that, if $\tilde{\mathbf{u}}\,\mathrm{P}_{\mathrm{LMX}}\,\tilde{\mathbf{v}}$, then it is possible to find a profile $\tilde{\mathbf{w}}$ such that (i) $\tilde{\mathbf{w}}$ is obtained from $\tilde{\mathbf{v}}$ by means of a Hammond progressive transfer, (ii) $\tilde{\mathbf{u}}\,\mathrm{R}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}$, and (iii) $\tilde{w}_t = \tilde{u}_t$, for at least one $t \in \{g, h\}$, which implies that $d(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \leqslant n - 1$. Repeating the same argument as above, we obtain a sequence of profiles $\{\tilde{\mathbf{w}}^s\}$ such that at each step s one has: (i) $\tilde{\mathbf{w}}^s$ is obtained from $\tilde{\mathbf{w}}^{s-1}$ by means of a Hammond progressive transfer, (ii) $\tilde{\mathbf{u}}\,\mathrm{R}_{\mathrm{LMX}}\,\tilde{\mathbf{w}}^s$, and (iii) $\tilde{w}_t^s = \tilde{u}_t$, for at least one t. Letting $\tilde{\mathbf{w}}^1 \equiv \tilde{\mathbf{w}}$, it follows that

$$0 \leqslant d(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}^s) < d(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}^{s-1}) < \cdots < d(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}^2) < d(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}^1) < d(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \leqslant n, \quad (4.6)$$

for all s. The sequence $\{d(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}^s)\}$ is bounded—from above and below—and strictly decreasing. We therefore conclude that profile $\tilde{\mathbf{u}}$ can be obtained from profile $\tilde{\mathbf{v}}$ by making use of at most $n - 1$ Hammond progressive transfers. Permutations will be used to complete the argument in the case where $\tilde{\mathbf{u}} \neq \mathbf{u}$ and/or $\tilde{\mathbf{v}} \neq \mathbf{v}$.

(a) $\Longrightarrow$ (b). This follows from the definitions of conditions A and HE.

(b) $\Longrightarrow$ (c). The fact that (a) implies (c) guarantees that the leximin–antileximax criterion verifies conditions A and HE, hence $\mathrm{R}_{\mathrm{LMX}} \in \mathscr{R}^*$. Therefore, if statement (b) holds, then so does statement (c). □

According to Theorem 4.2, the only way to make sure that one profile is ranked above another by all the social preference relations that satisfy anonymity and Hammond equity is to subject the two profiles to the verdicts of the leximin and of the antileximax. More precisely, one profile will be preferred to another by unanimity among all these social preference relations if and only if both criteria agree. When this happens, Theorem 4.2 goes on saying that the dominating profile can always be derived from the dominated profile by making use of only permutations and Hammond progressive transfers.

So far we have focused on inequality and we have therefore prevented ourselves from taking into account any consideration for efficiency. Adding strong Pareto to anonymity and Hammond equity precipitates the leximin as the following result demonstrates.

Theorem 4.3 *For all* $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, *the following three statements are equivalent:*

(a) $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by means of a finite sequence of permutations, Hammond progressive transfers and/or increments.*
(b) $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$, *for all* $\mathrm{R} \in \mathscr{R}^*_+$.
(c) $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$.

Proof (a) $\Longrightarrow$ (c). We know from Theorem 4.1 that, if $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of permutations and/or increments, then $\mathbf{u}\,\mathrm{R}_{\mathrm{SU}}\,\mathbf{v}$, and it follows from Remark 1 that $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$. Similarly, Theorem 4.2 tells us that $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$ whenever $\mathbf{u}$ results from $\mathbf{v}$ through a finite sequence of Hammond progressive transfers, and Remark 1 again ensures that $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$.

(c) $\Longrightarrow$ (a). Suppose that $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$, in which case there are two possibilities. If $\mathbf{u}\,\mathrm{I}_{\mathrm{LM}}\,\mathbf{v}$, then $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$ and $\mathbf{u}$ can be obtained from $\mathbf{v}$ by means of permutations only. Now, if $\mathbf{u}\,\mathrm{P}_{\mathrm{LM}}\,\mathbf{v}$, then there exists $i \in N$ such that $\tilde{u}_h = \tilde{v}_h$, for all $h \in \{1, 2, \ldots, i-1\}$, and $\tilde{u}_i > \tilde{v}_i$, and there are two cases to consider.

Case 1: $\tilde{u}_h \geqslant \tilde{v}_h$, for all $h \in \{i+1, i+2, \ldots, n\}$. Then $\mathbf{u}\,\mathrm{P}_{\mathrm{SU}}\,\mathbf{v}$ and it follows from Theorem 4.1 that $\mathbf{u}$ can be obtained from $\mathbf{v}$ by means of a finite sequence of permutations and/or increments.

Case 2: $\tilde{u}_h < \tilde{v}_h$, for some $h \in \{i+1, \ldots, n\}$. Consider the index j defined as follows:

$$j := \max\,\{h \in \{i+1, \ldots, n-1, n\} \mid \tilde{u}_h < \tilde{v}_h\}\,. \tag{4.7}$$

Then, we have $\tilde{u}_h = \tilde{v}_h$, for all $h \in \{1, 2, \ldots, i-1\} \cup \{j+1, j+2, \ldots, n\}$, $\tilde{u}_i > \tilde{v}_i$ and $\tilde{u}_j < \tilde{v}_j$. Therefore $\mathbf{u}\,\mathrm{P}_{\mathrm{LMX}}\,\mathbf{v}$, and it follows from Theorem 4.2 that $\mathbf{u}$ can be obtained from $\mathbf{v}$ by means of a finite sequence of permutations and/or Hammond progressive transfers.

(a) $\Longrightarrow$ (b). This follows from the definitions of conditions A, HE and SP.

(b) $\Longrightarrow$ (c). The fact that (a) implies (c) guarantees that the leximin verifies conditions A, HE and SP, hence $\mathrm{R}_{\mathrm{LM}} \in \mathscr{R}_{+}^{*}$. We therefore conclude that, if statement (b) holds, then so does statement (c). □

The search for unanimity of point of views among the social preference relations that satisfy anonymity, Hammond equity and strong Pareto leads to the leximin. In addition, Theorem 4.3 confirms that only permutations, Hammond progressive transfers and increments are needed to convert the dominated profile into the dominating one.

Not surprisingly, substitution of strong antiPareto for strong Pareto, in conjunction with anonymity and Hammond equity, gives the next result which emphasises the decisive role played by the antileximax in the search for unanimity over the corresponding class of social preference relations.

Theorem 4.4 *For all* $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, *the following three statements are equivalent:*

(a) $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by means of a finite sequence of permutations, Hammond progressive transfers and/or decrements.*
(b) $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$, *for all* $\mathrm{R} \in \mathscr{R}_{-}^{*}$.
(c) $\mathbf{u}\,\mathrm{R}_{\mathrm{ALX}}\,\mathbf{v}$.

Proof It is a straightforward adaptation of the arguments used when proving Theorem 4.3 above and it is therefore omitted. □

According to Theorem 4.4, the antileximax is the right criterion to use in order to check that one profile is ranked above another by all the social preference relations that satisfy anonymity, Hammond equity and strong antiPareto. If the antileximax succeeds to decide between two profiles, then Theorem 4.4 guarantees that the dominating profile can always be derived from the dominated profile by making use of only permutations, Hammond progressive transfers and decrements.

While Theorems 4.1, 4.2, 4.3 and 4.4 possess the same structure, close attention reveals a notable difference between Theorems 4.1 and 4.2 and Theorems 4.3 and 4.4 that may look surprising at first sight. It originates in statement (c) of the different theorems that introduces the Suppes quasi-ordering, the leximin–antileximax, the leximin, and the antileximax. Because the leximin and the antileximax are orderings—they are reflexive, transitive and complete—it follows that the classes of social preferences considered in statement (b) of Theorems 4.3 and 4.4, respectively, contains only one element, contrary to what happens in the case of Theorems 4.1 and 4.2. We will elaborate on this particular point in the next section.

5 Relationship with other results in the inequality literature

Comparison with the Lorenz approach Theorems 4.1, 4.2, 4.3 and 4.4 may be considered the natural adaptations of well-known results in the inequality and welfare literature when the variable that is distributed is continuous and ordinal. The Suppes quasi-ordering is actually identical to the first stochastic dominance criterion—or, equivalently, the quantile dominance criterion—extensively used in the risk and inequality literature (see, e.g., Fishburn and Vickson 1978). In this respect, Theorem 4.1 is but a restatement of well-known equivalences with the difference that we had no recourse to representations of the social preference ordering, be they of the expected or non-expected utility types (see, e.g., Levy 1998 for a survey of the literature). Actually, we even do not require the social preference to be an ordering and only impose that it is a reflexive and transitive relation.

The leximin–antileximax leaves aside efficiency considerations and focuses exclusively on the idea of inequality reduction: in an ordinal framework, it can therefore be seen as the counterpart of the Lorenz criterion, also known as *majorisation* in the mathematical literature (see Marshall and Olkin 1979). Like the Lorenz criterion, the leximin–antileximax considers that bringing two individuals closer on the utility scale without reversing their respective positions reduces inequality. But, contrary to the Lorenz criterion, it does not impose the additional constraint that gains and losses balance, which makes no sense in an ordinal framework. Theorem 4.2 may therefore be considered the ordinal version of the celebrated Hardy–Littlewood–Pólya theorem that provides the normative foundation of the Lorenz criterion by uncovering its connections with progressive transfers and the unanimous agreement among all utilitarians endowed with a concave utility function. It clarifies the way Hammond progressive transfers, unanimity over the class of relations that obey the Hammond equity condition and the leximin–antileximax criterion are related.

While the Lorenz criterion is exclusively concerned with inequality, the generalised Lorenz criterion introduced by Shorrocks (1983) allows one to compare distributions from a welfare point of view. More precisely, the generalised Lorenz criterion—also referred to as *weak submajorisation* (see again Marshall and Olkin 1979)—insists that increments and progressive transfers give rise to a social welfare improvement. The leximin plays a similar role in an ordinal framework by substituting Hammond progressive transfers for standard progressive transfers. On the other hand, it is not uncommon that one has to compare situations where an undesirable item is distributed among a population of individuals. For instance, one may think of a decision-maker who has to allocate a waste and who considers that (i) the smaller is the quantity that a person receives, the better off she is, and (ii) the more equally the waste is shared among the population—everybody contributes—the better off the society is. In such a case, *weak supermajorisation* seems to be an appropriate criterion for comparing the alternative distributions or, at least, to rule out those distributions that are deemed unacceptable with no ambiguity. Theorem 4.4 provides good reasons for appealing to the antileximax when such comparisons are made in an ordinal framework.

A common critique addressed to the Lorenz criterion—as well as to its various extensions—is that it fails to provide a complete ranking of the distributions to be compared. While a somewhat similar critique could be addressed to the leximin–antileximax, it must be emphasised that the leximin and the antileximax are exempt from this deficiency: they are able to rank *all* the profiles under consideration. This does not occur in the standard framework where the variable of interest is of a cardinal nature: like the Lorenz criterion, weak submajorisation and weak supermajorisation may not rank all distributions. Because the leximin–antileximax is by definition the intersection of two orderings—the leximin and the antileximax—it fails to be complete: for more on the relation between orderings and quasi-orderings, see Donaldson and Weymark (1998).[3] What is to some extent surprising is the ability of the leximin and the antileximax to totally order the profiles under comparison.[4]

Finally, while we argued that the leximin, antileximax and leximin–antileximax are the right criteria to use in an ordinal framework when one is concerned with inequality reduction, we do not see any objection to appeal to these criteria in richer informational settings. But it must be insisted on the fact that, by doing so, one implicitly adopts a somewhat extreme view of inequality reduction according to which bringing two individuals' utilities closer is always worth from a social point of view, whatever magnitude of the gains and losses in utility. In particular, the inequality judgement made by the Lorenz criterion is compatible with that captured by the leximin–antileximax: if one distribution is ranked above another by the Lorenz criterion, then it will be the same with the leximin–antileximax. In other words, in the standard income inequality setting, domination in terms of the leximin–antileximax is a necessary condition for Lorenz domination.[5] Similarly, the fact that one profile is ranked above another by the leximin is necessary for it to generalised Lorenz dominate the other profile.

Inclusion quasi-ordering and unanimity From now on, we assume without loss of generality that the profiles under consideration are non-decreasingly arranged. A recurrent limitation of a progressive transfer—be it a Hammond transfer or a standard one—is the fact that, while it reduces inequality between the individuals involved, the transfer may generate an increase in the inequalities between each of these two individuals and the rest of the population.[6] Consider for instance the profiles $\mathbf{u}^{(5)} = (1, 2, 5, 8, 9)$ and $\mathbf{u}^{(6)} = (1, 4, 5, 7, 9)$, where $\mathbf{u}^{(6)}$ is obtained from $\mathbf{u}^{(5)}$ by means of a Hammond progressive transfer involving individuals 2 and 4. While $u_2^{(5)} < u_2^{(6)} < u_4^{(6)} < u_4^{(5)}$, we observe that $u_1^{(6)} = u_1^{(5)} < u_2^{(5)} < u_2^{(6)}$ and $u_4^{(6)} < u_4^{(5)} < u_5^{(5)} = u_5^{(6)}$.[7] Whereas individuals 2 and 4 came closer to each other in terms of utility, individuals of 1 and 2, as well as individuals of 4 and 5, moved away from each other. Thus, the reduction in inequality between individuals 2 and 4 attributable to the Hammond progressive transfer is achieved at the cost of an increase in inequality for the pairs of individuals $\{1, 2\}$ and $\{4, 5\}$.

The implicit concept of inequality mentioned above refers to the notion of *pairwise inclusion* that suggests a slightly more general way of appraising inequality in an

[3] Actually, the same holds true for the Lorenz criterion that is the intersection of weak supermajorisation—or, equivalently, generalised Lorenz dominance—and weak submajorisation.

[4] The adjunction of strong Pareto to the anonymity and Hammond equity conditions suffices for rendering the social preference relation complete. This has to be contrasted to what happens in the standard framework: imposing that the social welfare function is monotone increasing in addition to being Schur-concave does not make the corresponding dominance criterion complete.

[5] This is readily inferred from the fact that, in a cardinal framework, a progressive transfer is but a particular case of a Hammond progressive transfer.

[6] Questionnaire studies have highlighted the fact that a large proportion of respondents do not subscribe to the view according to which a progressive transfer reduces inequality (see for instance Amiel and Cowell 1999). In this respect, a notion like that of a uniform on the right progressive transfer proposed by Magdalou and Moyes (2009), which is closely related to the reduction of deprivation, is more likely to fit the views expressed by the interviewed public.

[7] It must be noted however that, for all the other pairs of individuals $\{i, j\}$ such that $i < j$, we have, *either* $u_i^{(5)} < u_i^{(6)} < u_j^{(6)} < u_j^{(5)}$, *or* $u_i^{(5)} = u_i^{(6)} < u_j^{(6)} = u_j^{(5)}$.

Table 2 Pairwise inclusion ranking in the case of profiles $\mathbf{u}^{(5)}$ and $\mathbf{u}^{(6)}$

	2	3	4	5
1	$(1,4) \prec (1,2)$	$(1,5) \sim (1,5)$	$(1,7) \succ (1,8)$	$(1,9) \sim (1,9)$
2		$(4,5) \succ (2,5)$	$(4,7) \succ (2,8)$	$(4,9) \succ (2,9)$
3			$(5,7) \succ (5,8)$	$(5,9) \sim (5,9)$
4				$(7,9) \prec (8,9)$

ordinal framework [see Kolm (1999, Sect. 3.2)]. Let us say that the profile $\mathbf{u}$ is obtained from the profile $\mathbf{v}$ by means of a *pairwise inclusion* if there exist two individuals i and j such that

$$[u_i, u_j] \subseteq [v_i, v_j] \quad \text{and} \quad u_h = v_h, \text{ for all } h \neq i, j. \tag{5.1}$$

We find convenient to write $(u_i, u_j) \succsim (v_i, v_j)$ when $[u_i, u_j] \subseteq [v_i, v_j]$ and we denote respectively by $\succ$ and $\sim$ the asymmetric and symmetric components of $\succsim$. The inclusion is *semi-strict* if (5.1) holds and, *either* $u_i \neq v_i$, *or* $u_j \neq v_j$. If $u_i \neq v_i$ and $u_j \neq v_j$, then the inclusion is *strict* and it reduces to a Hammond progressive transfer. Table 2 indicates the corresponding ranking of the couples $(u_i^{(5)}, u_j^{(5)})$ and $(u_i^{(6)}, u_j^{(6)})$ for each pair of individuals $\{i, j\}$, and confirms that a Hammond progressive transfer fails to reduce inequality over all pairs of individuals.

An obvious solution in order to avoid such a situation would be to declare that inequality decreases if and only if *all* pairwise inequalities are simultaneously reduced. The following quasi-ordering, that is due to Kolm (1997, Chapter III-C), precisely aims at capturing this basic idea. Given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we say that $\mathbf{u}$ *unanimously pairwise inclusion dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_{\text{UPI}} \mathbf{v}$, if:

$$[u_i, u_j] \subseteq [v_i, v_j], \quad \text{for all } i, j \in N \ (i \neq j). \tag{5.2}$$

In order to decide whether $\mathbf{u}$ dominates $\mathbf{v}$ according to unanimous pairwise inclusion, one has to perform $n(n-1)/2$ pairwise comparisons, something that rapidly involves a huge number of computations in practice. Interestingly, Kolm (1997) claims that unanimous pairwise inclusion dominance is equivalent to what he called bi-truncation dominance that actually only requires n pairwise comparisons to be made. Given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we say that $\mathbf{u}$ *bi-truncation dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_{\text{BTR}} \mathbf{v}$, if there exist two individuals i, j and two reals a, b such that $v_i \leqslant a < b \leqslant v_j$ and:

$$u_h = a, \quad \text{for } h = 1, 2, \ldots, i; \tag{5.3a}$$

$$u_h = v_h, \quad \text{for } h = i+1, \ldots, j-1; \text{ and} \tag{5.3b}$$

$$u_h = b, \quad \text{for } h = j, j+1, \ldots, n. \tag{5.3c}$$

It is almost immediate that $\mathbf{u} \geq_{\text{BTR}} \mathbf{v}$ implies that $\mathbf{u} \geq_{\text{UPI}} \mathbf{v}$. Conversely, one can show that, if $\mathbf{u} \geq_{\text{UPI}} \mathbf{v}$, then one has

$$v_1 \leqslant u_1 = a \leqslant u_2 = v_2 \leqslant \cdots \leqslant v_{n-1} = u_{n-1} \leqslant b = u_n \leqslant v_n, \tag{5.4}$$

which implies that (5.4) holds with $i = 1$ and $j = n$, hence $\mathbf{u} \geq_{\text{BTR}} \mathbf{v}$. One might be tempted to apply the same technique to Hammond progressive transfers and say that the profile $\mathbf{u}$ is *unanimously Hammond dominates* the profile $\mathbf{v}$, which we write $\mathbf{u} \geq_{\text{UH}} \mathbf{v}$, if $v_i < u_i \leqslant u_j < v_j$, for all $i \neq j$. For instance, in the case where $n = 3$, the above definition requires that the following inequalities are verified:

$$\{1, 2\}: \quad v_1 < u_1 \leqslant u_2 < v_2; \tag{5.5a}$$

$$\{1, 3\}: \quad v_1 < u_1 \leqslant u_3 < v_3; \text{ and} \tag{5.5b}$$

$$\{2, 3\}: \quad v_2 < u_2 \leqslant u_3 < v_3. \tag{5.5c}$$

Clearly, conditions (5.5a) and (5.5c) cannot be met simultaneously, which shows that the unanimous Hammond dominance criterion fails to exist whenever $n \geqslant 3$.

Contrary to the notion of a Hammond progressive transfer, the increment and the decrement need not be matched in a pairwise inclusion. According to the latter, all increments are considered inequality reducing transformations as long as there exists some individual whose initial endowment is not smaller than that of the individual who benefited from the increment. A similar remark applies *mutatis mutandis* in the case of decrements. Note that, in the approach of Allison and Foster (2004), increments and decrements reduce inequality subject to the proviso that (i) increments take place below the median, (ii) decrements occur above the median, and (iii) neither of them jump over the median. Therefore, the implicit notion of *inequality reducing transformation* considered by these authors is a particular case of the pairwise inclusion of Kolm. A possibility in order to disentangle inequality reduction and efficiency improvements in the pairwise inclusion approach would be to say that $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of an increment if $u_i = v_i + \Delta > v_i$ for some i and there exists no j such that $u_i \leqslant v_j$. According to this definition, the only admissible increment is such that $u_i = v_i + \Delta > v_n$, hence $\Delta > v_n - v_i$: all other increases in person i' utility would result in pairwise inclusions.

Variable population size The criteria discussed in the paper can be easily adapted to the general case where the population size is allowed to vary through the introduction of the *principle of population* (PP) of Dalton (1920). Let us say that distribution $\mathbf{u}:=(u_1, \ldots, u_n)$ is a *replication* of distribution $\mathbf{v}:=(v_1, \ldots, v_m)$ if there exists $q \geqslant 2$ such that $\mathbf{u} = (\mathbf{v}; \ldots; \mathbf{v}) \in \mathbb{R}^{qm}$. A social preference relation R defined over $\bigcup_{n=2}^{\infty} \mathbb{R}^n$ verifies the principle of population if $\mathbf{u}\,\mathrm{I}\,\mathbf{v}$ whenever $\mathbf{u}$ is a replication of $\mathbf{v}$. Then, given two profiles $\mathbf{u}, \mathbf{v} \in \bigcup_{n=2}^{\infty} \mathbb{R}^n$, we would say that $\mathbf{u}$ (weakly) leximin–antileximax dominates $\mathbf{v}$, which we write $\mathbf{u}\,\mathrm{R}^*_{\text{LMX}}\,\mathbf{v}$, if and only if, *either* $U(p) = V(p)$, for all $p \in [0, 1]$, *or*

$$\exists\, 0 < s < t < 1 \mid U(s) > V(s); \;\; U(t) < V(t); \text{ and} \tag{5.6a}$$

$$U(p) = V(p),\ \forall\, p \in [0, s) \cup (t, 1], \tag{5.6b}$$

where $U(p)$ and $V(p)$ are the *quantile* functions corresponding to $\mathbf{u}$ and $\mathbf{v}$ defined in the usual way [see, e.g., Moyes (1999, Sect. 2.2]. We indicate by $\mathrm{I}^*_{\mathrm{LMX}}$ and $\mathrm{P}^*_{\mathrm{LMX}}$ the symmetric and asymmetric components of $\mathrm{R}^*_{\mathrm{LMX}}$. To illustrate things, consider the profiles $\mathbf{u} = (1, 2, 3, 4)$ and $\mathbf{v} = (1, b, 4)$. Indicate by $\tilde{\mathbf{u}} = (1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4)$ and $\tilde{\mathbf{v}} = (1, 1, 1, 1, b, b, b, b, 4, 4, 4, 4)$ the 3-fold and 4-fold replicates of $\mathbf{u}$ and $\mathbf{v}$, respectively. Then, we have

$$\tilde{u}_4 > \tilde{v}_4;\ \ \tilde{u}_9 < \tilde{v}_9 \text{ and } \tilde{u}_i = \tilde{v}_i,\ \forall\, i \in \{1, 2, 3\} \cup \{10, 11, 12\}, \tag{5.7}$$

or equivalently

$$U\left(\tfrac{1}{4}\right) > V\left(\tfrac{1}{4}\right);\ \ U\left(\tfrac{3}{4}\right) < V\left(\tfrac{3}{4}\right) \text{ and } U(p) = V(p),\ \forall\, p \in \left[0, \tfrac{1}{4}\right) \cup \left(\tfrac{3}{4}, 1\right]. \tag{5.8}$$

We therefore conclude that $\mathbf{u}\ \mathrm{I}^*_{\mathrm{LMX}}\ \tilde{\mathbf{u}}\ \mathrm{P}^*_{\mathrm{LMX}}\ \tilde{\mathbf{v}}\ \mathrm{I}^*_{\mathrm{LMX}}\ \mathbf{v}$ as long as $1 < b < 4$. One adapts in a similar way the definitions of the Suppes, leximin and antileximax criteria to accommodate the possibility that the populations are of different sizes.

6 Some implications for social choice

Invoking the *welfarism theorem* makes clear the connection between our Theorems , 4.2, 4.3 and 4.4 and standard results in the social choice literature. The welfarism theorem allows one to translate the ordering of the social states into an ordering of the utility profiles associated with these social states provided that the social welfare ordering (SWO) satisfies the conditions of *universal domain* (UD), *independence with respect to irrelevant alternatives* (IIA), and *Pareto indifference* (PI) [see, e.g., Bossert and Weymark (2004, Theorem 4.2)]. Assuming *ordinal level comparability* (OLC) and UD, it has been shown that, if a social welfare ordering satisfies IIA, A and SP, then it is *positional dictatorship* (see Gevers 1979; Roberts 1980). On the one hand, this result is somewhat related to our Theorem 4.1, where it is said that profile $\mathbf{u}$ is preferred to profile $\mathbf{v}$ by the Suppes criterion if and only if *all* positional dictators agree to rank $\mathbf{u}$ above $\mathbf{v}$. On the other hand, no use is made of conditions OLC, UD and IIA in Theorem 4.1, nor is the social preference relation even assumed to be an ordering. Adding HE to the list of conditions listed above precipitates the leximin (see Hammond 1976). In this respect, Theorem 4.3 can be seen as providing an alternative characterisation of the leximin, albeit in a more specific framework than that commonly used in the social choice theory. More precisely, we have the following result, where we remind the reader that the elements of $\mathscr{R}$ are reflexive, transitive but not necessarily complete:

Theorem 6.1 *The social preference relation* $\mathrm{R} \in \mathscr{R}$ *satisfies conditions* A, SP *and* HE *if and only if it is the leximin.*

Proof That the leximin satisfies conditions A, SP and HE is readily inferred from Theorem 4.3. Indeed, it is said there that, given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, if $\mathbf{u}$ is obtained from **v** by means of a finite sequence of permutations, Hammond progressive transfers and/or increments, then **u** is preferred to **v** by the leximin. In other words, the leximin satisfies conditions A, SP and HE, hence $\mathrm{R}_{\mathrm{LM}} \in \mathscr{R}^*_+$. To establish that conditions A, SP and HE imply the leximin amounts to showing that $\mathscr{R}^*_+$ contains only the leximin. Note that, because the leximin is a complete relation, given two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have *either* $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$, *or* $\mathbf{v}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{u}$. Suppose that there exists an element $\mathrm{R} \in \mathscr{R}^*_+$ distinct from the leximin. Then, there are two profiles $\mathbf{u}, \mathbf{v} \in \mathscr{U}$ such that $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$ and $\neg\,[\,\mathbf{u}\,\mathrm{R}\,\mathbf{v}\,]$. But this is clearly impossible, for invoking Theorem 4.3 again, we know that, if $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$, then **u** is obtained from **v** by means of a finite sequence of permutations, Hammond progressive transfers and/or increments. Since $\mathrm{R} \in \mathscr{R}^*_+$, it must be that $\mathbf{u}\,\mathrm{R}\,\mathbf{v}$, which contradicts our assumption. □

Making use of Theorem 4.4, one can prove along a similar reasoning that the antileximax is the only relation satisfying conditions A, SAP and HE.

Theorem 6.2 *The social preference relation* $\mathrm{R} \in \mathscr{R}$ *satisfies conditions* A, SAP *and* HE *if and only if it is the antileximax.*

Contrary to what happens with Theorems 4.3 and 4.4, it must be noted that it is not possible to derive a characterisation of the leximin–antileximax starting with Theorem 4.2. The difficulty originates in the fact that the leximin–antileximax is not the only relation satisfying conditions A and HE. Then, it is possible to find a social preference in $\mathrm{R}^* \in \mathscr{R}^*$ distinct from $\mathrm{R}_{\mathrm{LMX}}$ such that $\mathbf{u}\,\mathrm{P}^*\,\mathbf{v}$ and $\neg\left[\,\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}\,\right]$, for some $\mathbf{u}, \mathbf{v} \in \mathscr{U}$. For instance, both the leximin and the antileximax are consistent with the leximin–antileximax. Indeed, since $\mathscr{R}^*_+ \subset \mathscr{R}^*$ and $\mathscr{R}^*_- \subset \mathscr{R}^*$, it follows that, for all $\mathbf{u}, \mathbf{v} \in \mathscr{U}$, we have $\mathbf{u}\,\mathrm{R}_{\mathrm{LM}}\,\mathbf{v}$ and $\mathbf{u}\,\mathrm{R}_{\mathrm{ALX}}\,\mathbf{v}$ whenever $\mathbf{u}\,\mathrm{R}_{\mathrm{LMX}}\,\mathbf{v}$ (see Remark 1). Choose $\mathbf{u} = (a, b, b, \ldots, b, d, d)$ and $\mathbf{v} = (a, a, c, \ldots, c, c, d)$, where $a < b < c < d$. Then, we get $\mathbf{u}\,\mathrm{P}_{\mathrm{LM}}\,\mathbf{v}$, $\mathbf{v}\,\mathrm{P}_{\mathrm{ALX}}\,\mathbf{u}$, while **u** and **v** cannot be ranked by the leximin–antileximax.

7 Concluding remarks

Suppose that we are interested in the comparisons of distributions of a continuous and ordinal attribute with a particular concern for inequality consideration. In such a context, the principle of transfers, according to which inequality decreases as the result of a progressive transfer, is meaningless because it is not invariant to arbitrary changes in the scale of measurement. The fact that one distribution is obtained from another by means of a progressive transfer does not imply that it will be possible to perform the same operation after these distributions have been subjected to the same increasing transformations of the individuals' utilities. The difficulty originates in that the equality of distances between the utility of the beneficiary and that of the recipient of the transfer does not survive to such transformations. Abandoning the equality restriction, while still insisting fact that individuals must be brought closer without affecting their relative positions, leads us to what we refer to as Hammond progressive transfers. The corresponding Hammond equity principle constitutes therefore in an ordinal setting the natural analogue of the principle of transfers.

We have shown in the paper that (i) imposing that the social preference relation satisfies Hammond equity and anonymity, and (ii) requiring unanimity over all such social preferences inexorably lead to what we have called the leximin–antileximax. In other words, one profile is judged to be better than another if the leximin and the antileximax agree to rank the first profile above the second. Adding strong Pareto to Hammond equity and anonymity precipitates the leximin, while the adjunction of strong antiPareto results in the antileximax. The leximin–antileximax and leximin can be considered the analogues in an ordinal framework of the majorisation and weak submajorisation criteria better known as the Lorenz and generalised Lorenz quasi-orderings, respectively, in the inequality and welfare literature. Similarly, the antileximax criterion is the adaptation of weak supermajorisation when the available information is of an ordinal nature.

An important difference between the cardinal and ordinal frameworks when accounting for inequality is the ability of the resulting criteria to discriminate among the distributions. It is well-known—and this is sometimes considered a weakness—that the Lorenz criterion, as well as its extensions, does not permit one to completely order the distributions under comparison. While a similar critique—though somehow weaker—may be addressed to the leximin–antileximax criterion, it must be emphasised that the leximin and the antileximax are able to rank *all* the distributions under comparison. Thus, the addition of strong Pareto—similarly, of strong antiPareto—to Hammond equity et anonymity allows one to break all the incomparabilities that may result from the application of the leximin–antileximax criterion. This does not occur in the standard framework where the introduction of a concern for efficiency in addition to the principle of transfers is unable to resolve all the cases of incomparabilities that arise from the use of the Lorenz criterion.

The social preference relations discussed in the paper—the Suppes quasi-ordering, the leximin, the antileximax, and the leximin–antileximax—are unidimensional criteria. This is not a problem in a social choice setting as long as a person's utility incorporates everything is deemed to be relevant for her well-being. Following persuasive arguments from Sen, this welfarist position has been criticised on the grounds that it fails to take into account some important dimensions of a person's well-being (see, e.g., Sen 1985) and it has been suggested that a multidimensional approach to inequality might be preferable. A much neglected difficulty in the multidimensional approach is the recognition that the components of a person's well-being—one may think of income, health, or cognitive ability, among other things—involve measurement scales of different nature ranging from cardinal to ordinal scales when they are not categorical. Extending the approach followed in the paper to the comparison of distributions of attributes involving different measurement scales is certainly the way to go in the future.

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