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ORIGINAL PAPER

The weighted-egalitarian Shapley values

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Abstract

We propose a new class of allocation rules for cooperative games with transferable utility (TU-games), weighted-egalitarian Shapley values, where each rule in this class takes into account each player's contributions and heterogeneity among players to determine each player's allocation. We provide an axiomatic foundation for the rules with a given weight profile (i.e., exogenous weights) and the class of rules (i.e., endogenous weights). The axiomatization results illustrate the differences among our class of rules, the Shapley value, the egalitarian Shapley values, and the weighted Shapley values.

1 Introduction

The most eminent allocation rule for cooperative games with transferable utility (TU-games) may be the Shapley value introduced by Shapley (1953b). After the celebrated study of Shapley (1953b), many other axiomatic foundations for the rule were intensively studied.[1] In particular, Young (1985) shows that the Shapley value is the unique efficient rule that satisfies *strong monotonicity* and *symmetry*. These two properties focus on each player's contributions in a game to determine their rewards. More pre-

[1] For recent studies, see Casajus (2011; 2014) and Casajus and Yokote (2017a).

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cisely, (1) strong monotonicity states that each player receives more as his/her marginal contributions weakly increase and (2) symmetry requires that any two players receive the same amount if their contributions are equal. Therefore, the Shapley value can be thought of as an allocation rule completely based on each player's performance.

In actual situations, however, we often use an allocation rule which can assign positive payoff to each player even if he/she cannot contribute for some reason. For example, in the case of wage assignment in a firm, each worker may receive a basic salary in addition to a reward for her contribution. This system may be more secure than a system without a basic salary given the possibility that employees cannot contribute because of, for instance, raising children. Constructing an allocation rule integrating this kind of social equity, which is often referred to as a solidarity principle, with contribution based rule, is one of the main concerns in recent literature of cooperative game theory (Nowak and Radzik 1994; Joosten 1996; Casajus and Huettner 2013, 2014; van den Brink et al. 2013; Joosten 2016). Also, in the same example, the wage may be affected by some index independent of one's contributions, such as seniority, educational background, and entitlements. Moreover, in the case of redistribution of income in a society, each household has a heterogeneous background, such as the number of children or the presence of disability (Abe and Nakada 2017).[2] Modern taxation systems consider such heterogeneities. This observation raises the following question: what allocation rule reconciles performance-based evaluation with a solidarity principle and takes players' heterogeneity into consideration?

To answer this question, this paper considers rules satisfying weaker monotonicity and symmetry. We show that these axioms (elaborated later) characterize the new class of rules, *weighted-egalitarian Shapley values*, where each rule in this class is given as a convex combination of the Shapley value and the weighted division (Béal et al. 2016). Each rule in our class can be interpreted as a redistribution rule, via which a player keeps a part of his income and offers the other part as a tax. After every player's tax is collected, the total tax revenue is distributed among the players, albeit not equally. Depending on a player's weight, the benefits received from the collected taxes may differ. Such weights might be given externally or they might be determined endogenously. In this paper, we study both setups.

First, we characterize the weighted-egalitarian Shapley values where a weight profile is endogenously determined. For monotonicity, we employ weak monotonicity, i.e., each player receives more if his contributions and the worth of the grand coalition increase.[3] This property does not require each player's evaluation to depend only on his contribution but rather allows it to depend also on the worth of the grand coalition to reflect a solidarity principle. For the symmetry property, we consider the following two new axioms: *weak differential marginality for symmetric players* and *ratio invariance for null players*. The former axiom requires the following property. Suppose that there are two null players, i.e., players whose marginal contributions are zero. If the worth of the grand coalition keeps unchanged but these two players make the same

[2] Abe and Nakada (2017) extends a monotonic redistribution rule (Casajus 2015, 2016; Casajus and Yokote 2017b) to exhibit agents' heterogeneity. See Roth (1979), Kalai and Samet (1987), Chun (1988; 1991), and Nowak and Radzik (1995) for other examples.

[3] Weak monotonicity without weights was introduced by van den Brink et al. (2013) to characterize egalitarian Shapley values. We discuss this topic later.

additional contributions, then the two players should receive the same additional payoff. The latter axiom is a minimal fairness requirement for null players, which requires that the payoff ratio between two null players does not vary as long as they are null players. We show that a rule satisfies these axioms, efficiency and nullity if and only if it is a weighted-egalitarian Shapley value.

Second, we suppose that each player's heterogeneity is parametrized by an exogenous weight profile. In this case, how we should integrate the exogenous weights into the two axioms, monotonicity and symmetry, is a problem. The three axioms efficiency*, weak monotonicity*, and ratio invariance for null players* correspond to the axioms in the first case. For the symmetry property, we consider the following two new axioms: *symmetry** and *fair evaluation for contribution*. The former axiom states that any two players receive the same payoff if their contributions and weights are equal. In other words, even though their contributions are equal, we admit that the two players receive different payoffs if their weights are different. The latter axiom states that if a player additionally contributes, then a reward for his additional contributions should be evaluated impartially, regardless of his weight. That is, this axiom requires that we should take each player's contributions and weight into consideration separately. We show that a rule satisfies these axioms if and only if it is a weighted-egalitarian Shapley value with an exogenous weight profile.

Related literature

Our results contribute to the literature regarding axiomatization of variants of the Shapley value to accommodate a solidarity principle and heterogeneity, in particular, the egalitarian Shapley values (Joosten 1996) and the weighted Shapley values (Shapley 1953a). We analyze the axiomatic differences among our rules, the egalitarian Shapley values and the weighted Shapley values.

The egalitarian Shapley values are rules that are convex combinations of the Shapley value and the equal division. That is, each player can obtain some amount of payoff equally and extra amount depending on his/her own contributions. In this sense, this rule can be considered as to combine marginalism and egalitarianism. Since the equal division is a special case of the weighted divisions, our rules subsume this class of rules. We show that the difference between this class and ours stems from different symmetry properties by comparing our result with that of Casajus and Huettner (2014); they characterize their rules by efficiency, weak monotonicity, and symmetry.

The weighted Shapley values are allocation rules based on weighted contributions. Although this rule and ours only share efficiency, the difference is understood as a consequence of the requirement of monotonicity and symmetry by comparing our result with that of Nowak and Radzik (1995); they characterize the class of rules by efficiency, strong monotonicity, mutual dependence and strict positivity, where mutual dependence implies ratio indifference for null players but weak differential marginality for symmetric players and strict positivity implies nullity under efficiency and weak monotonicity.[4] Note that strict positivity, which is formally defined in Sect. 5, is a

[4] For other characterizations of the weighted Shapley values, see Kalai and Samet (1987), Chun (1991), Hart and Mas-Colell (1989), and Yokote (2014).

variation of the positivity axiom discussed by Kalai and Samet (1987) that guarantees every player non-negative payoff for monotonic games. Joosten et al. (1994) consider another weak requirement called social acceptability, which requires that all players obtain non-negative payoff and, in particular, productive players receive more than or as mush as null players in a unanimity game.

Joosten (2016) introduces the egalitarian weighted Shapley values, that is, a class of rules that are convex combinations of a weighted Shapley value and the equal division. The difference between his class and ours lies in how to address a weight profile of players, that is, heterogeneity. In our rules, each player's contributions are evaluated without weights, while the weights determine players' "basic payoffs," namely, the weighted division. In contrast, the egalitarian weighted Shapley values take into account the weights to evaluate each player's contributions, while the "basic payoffs" are given as the equal division. In this sense, Joosten (2016)'s class and ours can be thought of as two different generalizations of the egalitarian Shapley values: the egalitarian Shapley value takes a middle ground between the weighted-egalitarian Shapley value and the egalitarian weighted Shapley values.[5]

The remainder of this paper is organized as follows. In Sect. 2, we provide basic definitions and notation. In Sect. 3, we offer the main characterization of the weighted-egalitarian Shapley values. In Sect. 4, we offer the characterization of the weighted-egalitarian Shapley values in the case of exogenous weight profiles. In Sect. 5, we conclude this paper with some remarks. All proofs are relegated to the appendix.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be the set of players and the function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ denote a characteristic function. A *coalition* of players is defined as a subset of the player set, $S \subseteq N$. Let $|S|$ denote the cardinality of coalition S. A cooperative TU-game is (N, v). Fixing the player set N, we denote by $\mathcal{G}$ the set of all TU-games with the player set N. An allocation rule is denoted by $f : \mathcal{G} \to \mathbb{R}^N$. Player i's *marginal contribution* to coalition $S \subseteq N \backslash \{i\}$ is defined as $v(S \cup \{i\}) - v(S)$. For each $v \in \mathcal{G}$, we say that player $i \in N$ is a *null player* in v if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \backslash \{i\}$. We also say that two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \backslash \{i, j\}$. For any nonempty coalition $T \subseteq N$, the unanimity game $u_T \in \mathcal{G}$ is defined as follows: for any $S \in 2^N$,

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The Shapley value, $Sh(v)$, is given as follows: for any player $i \in N$,

$$Sh_i(v) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(|N| - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

[5] In Sect. 5, we discuss the future direction to unify these two classes.

The Shapley value assigns an average of marginal contributions to each player. Young (1986) shows that the Shapley value satisfies the following properties.

Efficiency, E. For any $v \in \mathcal{G}$, $\sum_{i \in N} f_i(v) = v(N)$.

Strong Monotonicity, M. For any $v, v' \in \mathcal{G}$ and $i \in N$, if $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S)$ for all $S \subseteq N \backslash \{i\}$, then $f_i(v) \geq f_i(v')$.

Symmetry, SYM. For any $v \in \mathcal{G}$ and $i, j \in N$, if i, j are symmetric in v, then we have $f_i(v) = f_j(v)$.[6]

The following theorem shows that the Shapley value is the unique solution satisfying these three properties.

Theorem 1 (Young 1986) *An allocation rule $f : \mathcal{G} \to \mathbb{R}^N$ satisfies (E), (M) and (SYM) if and only if $f(v) = Sh(v)$ for all $v \in \mathcal{G}$.*

Since the Shapley value determines each player's payoff only depending on his/her contributions, it ignores both equity/solidarity and heterogeneity among players, which do not depend on contributions. In the following section, we introduce our class of rules that exhibit these features.

3 Axiomatization of the weighted egalitarian Shapley values

We define $w = (w_i)_{i \in N} \in \mathbb{R}^N_+$ with $\sum_{i \in N} w_i = 1$ as a weight profile and $\mathcal{W}$ as the set of all possible weight profiles.

First, we consider the following weaker version of monotonicity which was introduced by van den Brink et al. (2013).

Weak monotonicity, M$^-$. For each $v, v' \in \mathcal{G}$ with $v(N) \geq v'(N)$, if $v(S) - v(S \backslash \{i\}) \geq v'(S) - v'(S \backslash \{i\})$ for all $S \subseteq N$ with $i \in S$, then $f_i(v) \geq f_i(v')$.

This property states that a player's payoff weakly increases as his marginal contributions and the total value weakly increase. In contrast with (M), this property does not insist that each player's evaluation totally depends on his contributions but rather allows that it can depend on the total value.

The next axiom is a requirement for the treatment of null players.

Ratio invariance for null players, RIN. For any $v, v' \in \mathcal{G}$ and $i, j \in N$ such that i, j are null players in v, v', we have $f_i(v) \cdot f_j(v') = f_i(v') \cdot f_j(v)$.

Ratio invariance for null players requires that as long as some players, say i, j, contribute zero in both games v and v', the ratio of their payoffs, $f_i(v)/f_j(v)$, does not vary.

The following axiom is a requirement for null players about additional contribution.

Weak differential marginality for symmetric players, WDMSP For any $i, j \in N$ and $v, v' \in \mathcal{G}$ such that i, j are null players in v, if i, j are symmetric in v' and $v(N) = v'(N)$, then $f_i(v) - f_i(v') = f_j(v) - f_j(v')$.

Note that players i, j are also symmetric in v. Suppose that there are two null players in v. If the worth of the grand coalition keeps unchanged but these two players make the same additional contributions (from v to v'), then this axiom requires that

[6] This is also known as the *equal treatment property*.

the two players should receive the same additional payoff.[7] That is, the contribution itself is evaluated as the same as under the original symmetry axiom, but the axiom (WDMSP) does not exclude the possibility that the payoffs of two players can differ because of their heterogeneity, that is, $f_i(v) \neq f_j(v)$ and $f_i(v') \neq f_j(v')$ can be allowed.

The following axiom is a harmless feasibility requirement.

Nullity, NY Let $\mathbf{0}$ be the null game. For any $i \in N$, $f_i(\mathbf{0}) = 0$.

Nullity is a weak feasibility condition, which requires that every player receives nothing if every coalition's worth is zero. Nullity is called *triviality* in Chun (1989).

Now, we introduce the following new class of allocation rules, which we call *weighted-egalitarian Shapley values*:

$$f_i(v) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i v(N) \quad \text{where } \delta \in [0, 1] \text{ and } w \in \mathcal{W}.$$

Note that the allocation rule is specified by two parameters $\delta \in [0, 1]$ and $w \in \mathcal{W}$. For $\delta = 1$, the allocation rule coincides with the Shapley value and distributes the surplus $v(N)$ based only on the players' contributions. For $\delta = 0$, our rule coincides with the weighted devision (Béal et al. 2016). It is clear that rules in this class satisfy all of the axioms. Now, we are ready to offer our main axiomatization result as follows.

Theorem 2 *Suppose that $n \neq 2$. An allocation rule $f : \mathcal{G} \to \mathbb{R}^N$ satisfies (E), (M^-), (RIN), (WDMSP), and (NY) if and only if there exists a $\delta \in [0, 1]$ and a weight profile $w \in \mathcal{W}$ such that $f_i(v) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i v(N)$ for all $v \in \mathcal{G}$.*

Proof See Appendix A. □

For the independence of the axioms, examples are available in Appendix C. Note that, when $n = 2$, there is an allocation rule which satisfies all the axioms, but such a rule is not included in our class. In this sense, uniqueness of the class of rules does not hold when $n = 2$. A counterexample is also available in Appendix C.

4 Axiomatization of the weighted egalitarian Shapley values with an exogenous weight

In this section, we assume that a profile $w = (w_i)_{i \in N} \in \mathcal{W}$ is exogenously given. That is, we consider an allocation rule in the extended domain $f : \mathcal{G} \times \mathcal{W} \to \mathbb{R}^N$.

We first introduce the analogs of the axioms in the previous section.

Efficiency*, **E***. For any $v \in \mathcal{G}$ and $w \in \mathcal{W}$, $\sum_{i \in N} f_i(v, w) = v(N)$.

Weak monotonicity*, **M**$^{-*}$. For any $v, v' \in \mathcal{G}$, $w \in \mathcal{W}$ and $i \in N$, if $v(N) \geq v'(N)$ and $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S)$ for all $S \subseteq N \backslash \{i\}$, then $f_i(v, w) \geq f_i(v', w)$.

Ratio invariance for null players*, **RIN***. For any $v \in \mathcal{G}$, $w \in \mathcal{W}$ and any null players $i, j \in N$, we have $w_i \cdot f_j(v, w) = w_j \cdot f_i(v, w)$.

[7] Note that (WDMSP) is weaker than (SYM). It is also a weaker version of *differential marginality* defined by Casajus (2010, 2011). Formally, differential marginality is defined as follows: for any $i, j \in N$ and $v, v' \in \mathcal{G}$, if $v(S \cup \{i\}) - v(S \cup \{j\}) = v'(S \cup \{i\}) - v'(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, then $f_i(v) - f_j(v) = f_i(v') - f_j(v')$.

Symmetry*, **SYM***. For any $v \in \mathcal{G}$, $w \in \mathcal{W}$ and $i, j \in N$, if i, j are symmetric in v and $w_i = w_j$, then we have $f_i(v, w) = f_j(v, w)$.

Symmetry* requires that two players whose contributions and priorities are the same should receive the same amount. If we fix w to the equal weight $(\frac{1}{n}, \ldots, \frac{1}{n})$, this is equivalent to usual symmetry. Note that we allow different payoffs for two players, even if their contributions are the same, as long as their weights are different.

In addition to the axioms above, we impose the following requirement to take each player's contributions and weights into consideration separately, which supports our motivation to consider heterogeneity.

Fair evaluation for contribution, FEC*. For any $v \in \mathcal{G}$, $w, w' \in \mathcal{W}$ and $i \in N$,

$$f_i(v, w) - f_i(\mu^{v,i}, w) = f_i(v, w') - f_i(\mu^{v,i}, w'),$$

where $\mu^{v,i} = v(N)u_{N\setminus\{i\}}$.

This axiom describes how the payoff of a player changes due to a shift between the two weight profiles. The axiom requires that the payoff difference generated from this shift should be the same as long as productivity of the other players keeps unchanged and this player i is a null player. The following result shows that the rule satisfies these five axioms if and only if it is the weighted-egalitarian Shapley values with an exogenously determined weight profile $w \in \mathcal{W}$.

Theorem 3 *Suppose $n \neq 2$. An allocation rule $f : \mathcal{G} \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E*), (M^{-*}), (RIN*), (SYM*), and (FEC*) if and only if there exists a $\delta \in [0, 1]$ such that $f_i(v, w) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i v(N)$ for all $v \in \mathcal{G}$.*

Proof See Appendix B. □

As elaborated in the appendices, the technical difference between Theorems 2 and 3 lies in the "order" of intermediate claims of the proofs. In the proof of Theorem 3, we first specify the form of the Shapley value, while we first obtain the weighted division in Theorem 2.

Examples for the independence of the axioms and a counterexample for $n = 2$ are given in Appendix C

5 Concluding remarks

In this paper, we propose and axiomatically characterize a new class of allocation rules called weighted-egalitarian Shapley values. This allocation rule integrates equity and heterogeneity with the Shapley value.

As briefly argued in Sect. 1, monotonicity and symmetry distinguish our class of rules from the other rules, such as the egalitarian Shapley values and the weighted Shapley values. Below, we elaborate the differences amongst these classes of rules by comparing the axioms.

Joosten (1996) introduces the class of the egalitarian Shapley values, which are convex combinations of the Shapley value and the equal division:

$$f_i(v) = \delta \cdot Sh_i(v) + (1-\delta) \cdot \frac{v(N)}{n} \quad \text{where } \delta \in [0, 1].$$

Note that this class of rules is a subset of weighted-egalitarian Shapley values (i.e., $w = (\frac{1}{n}, \ldots, \frac{1}{n})$). To clarify the difference between our allocation rules and the egalitarian Shapley values, we consider the characterization of Casajus and Huettner (2014).

Theorem 4 (Casajus and Huettner 2014) *Suppose* $n \neq 2$. *An allocation rule* $f : \mathcal{G} \to \mathbb{R}^N$ *satisfies (E), (M*$^-$*), (SYM) if and only if there exists a* $\delta \in [0, 1]$ *such that* $f_i(v) = \delta \cdot Sh_i(v) + (1-\delta) \cdot \frac{v(N)}{n}$ *for all* $v \in \mathcal{G}$.

By comparing Theorem 2 with Theorem 4, the difference between these two rules is observed to be the requirement of symmetry. As Table 1 shows, the weighted-egalitarian Shapley values no longer satisfy symmetry, while all egalitarian Shapley values obey symmetry. This difference should be ascribed to the weight which each weighted-egalitarian Shapley value contains. To see what properties the weight makes a solution obey/violate, we briefly introduce another variation of the Shapley value, the weighted Shapley values.

Shapley (1953a) introduces the class of weighted Shapley values $Sh_i^w(v)$, which is a unique linear solution such that for each unanimity game u_T, there is a weight $w \in \mathbb{R}^N_{++}$ with $\sum_{j \in N} w_j = 1$ such that

$$Sh_i^w(u_T) = \begin{cases} \frac{w_i}{\sum_{j \in T} w_j} & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to the Shapley value, the weighted Shapley values satisfy (M). Therefore, this rule can be considered as a performance-based rule. However, the weighted Shapley values allow us to allocate players' surplus based not only on their contributions but also on their weights, that is, heterogeneity is taken into account.

Nowak and Radzik (1995) consider the following axioms.

Mutual dependence, MD. For any two players $i, j \in N$ and $v, v' \in \mathcal{G}$, if i, j are symmetric in v and v', then $f_i(v) f_j(v') = f_i(v') f_j(v)$.

Strict positivity, SP. For any monotonic $v \in \mathcal{G}$ such that there are no null players, we have $f_i(v) > 0$ for all $i \in N$.[8]

They show that these properties, together with (E) and (M), characterize the weighted Shapley values as follows.[9]

Theorem 5 (Nowak and Radzik 1995) *An allocation rule* $f : \mathcal{G} \to \mathbb{R}^N$ *satisfies (E), (M), (MD) and (SP) if and only if there exists a weight* $w \in \mathbb{R}^N_{++}$ *with* $\sum_{j \in N} w_j = 1$ *such that* $f_i(v) = Sh_i^w$ *for all* $v \in \mathcal{G}$.

[8] A game v is monotonic if $v(T) \geq v(S)$ for any $S \subseteq T$.

[9] This is a corollary implied by their more general result. They consider more general weight systems.

Table 1 Axioms and Rules. Abbreviation "w-egSh" means the weighted-egalitarian Shapley values, "egSh" means the egalitarian Shapley values, and "w-Sh" is the weighted Shapley values

	E	M	M$^-$	SYM	WDMSP	RIN	MD	NY	SP
w-egSh	+		+		+	+		+	
egSh	+		+	+					
w-Sh	+	+					+		+

Symbol "+" indicates that the axiom is used for the axiomatization

Note that (MD) implies (RIN) but not (WDMSP). Moreover, under (E) and (M$^-$), (SP) implies (NY).[10] Therefore, considering Theorems 2 and 5, monotonicity and symmetry are the differences between these two rules.

Given that our rule has an axiomatic foundation similar to that of the weighted Shapley values, we may weaken the properties and obtain another class of meaningful rules. For example, there exist $\delta \in [0, 1]$, $w \in \mathcal{W}$ and $z \in \mathbb{R}^N_{++}$ with $\sum_{j \in N} z_j = 1$ such that

$$f_i(v) = \delta \cdot Sh^z_i(v) + (1 - \delta) \cdot w_i v(N).$$

When $w = (1/n, \dots, 1/n)$, as briefly mentioned in Sect. 1, the rules are called the egalitarian weighted Shapley values introduced by Joosten (2016), which is another generalization of the egalitarian Shapley values. Hence, the class above contains both the weighted-egalitarian Shapley values and the egalitarian weighted Shapley values, whose intersection is the egalitarian Shapley values. We conjecture that this general class can be characterized by weak monotonicity and a variant of symmetry.

Appendix A: Proof of Theorem 2

Let $\mathcal{G}^c \subseteq \mathcal{G}$ denote the set of games such that $v(N) = c$. Also, for any player $i \in N$ and $c \in \mathbb{R}$, let $\mathcal{G}^{c,i}$ denote the set of games v such that $v(N) = c$ and i is null player. Note that $\mathcal{G}^{c,i} \subseteq \mathcal{G}^c$ for all $i \in N$ and $c \in \mathbb{R}$. Let $\Delta_i(v) = (v(S \cup \{i\}) - v(S))_{S \subseteq N \setminus \{i\}} \in \mathbb{R}^{2^{(N-1)}}$ be a vector of marginal contributions of i in v. Therefore player $i \in N$ is a null player in v if $\Delta_i(v) = \mathbf{0}$. Let Λ^i be the set of all vectors of marginal contribution of i: $\Lambda^i = \{\Delta_i(v) | v \in \mathcal{G}\}$.

For each $x \in \mathbb{R}^N$, let $m_x \in \mathcal{G}$ be an additive game, $m_x(S) = \sum_{i \in S} x_i$ for all $S \subseteq N$. Let $\mathcal{G}^{add}$ be the set of additive games. Since there is a one-to-one correspondence between $x \in \mathbb{R}^N$ and an additive game m_x, we can identify $\mathcal{G}^{add}$ with $\mathbb{R}^N$. Abe and Nakada (2017) provide the following result, which will be useful later.

Theorem A.1 (Abe and Nakada 2017) *Let $n \neq 2$. $f : \mathcal{G}^{add} \to \mathbb{R}^n$ satisfies (E), (M$^-$), (NY), and (RIN) if and only if there exists some $\delta \in [0, 1]$ and $w \in \mathcal{W}$ such that $f_i(x) = \delta \cdot x_i + (1 - \delta) \cdot w_i \cdot \sum_{l \in N} x_l$ for all $x \in \mathbb{R}^n$ and $i \in N$.*

[10] If $f_i(\mathbf{0}) = \delta < 0$ for all $i \in N$, it contradicts to (E). Hence, supposing that $f_i(\mathbf{0}) = \delta > 0$ for some i, we consider $v \in \mathcal{G}_N$ such that $v(N) = \varepsilon \in (0, \delta)$ and $v(S) = 0$ for $S \neq N$. By (M$^-$), $f_i(v) \geq f_i(\mathbf{0}) = \delta$. Also, by (SP), $f_i(v) > 0$ for all $i \in N$. This contradicts (E) because $v(N) = \varepsilon \in (0, \delta)$.

Now, we offer the proof of Theorem 2. It is clear that the rule satisfies all the axioms. We suppose that a rule $f : \mathcal{G} \to \mathbb{R}^N$ satisfies (E), (M$^-$), (RIN), (WDMSP), and (NY).

Claim 1 For each $i \in N$, there exist functions $\phi_i : \Lambda^i \times \mathbb{R} \to \mathbb{R}$ and $\alpha_i : \mathbb{R} \to \mathbb{R}$ such that $f_i(v) = \phi_i(\Delta_i(v), v(N)) + \alpha_i(v(N))$.

We first take any $c \in \mathbb{R}$. For any $i \in N$ and $v \in \mathcal{G}^c$, we have the following equation: for any $\bar{v} \in \mathcal{G}^c$ such that $\Delta_i(v) = \Delta_i(\bar{v})$,

$$f_i(v) \overset{(\mathrm{M}^-)}{=} f_i(\bar{v}) \ =: \alpha_i(c, \Delta_i(v)). \tag{A.1}$$

Specifically, we denote

$$\alpha_i(c) = \alpha_i(c, \mathbf{0}). \tag{A.2}$$

Moreover, for any $i \in N$ and $v, v' \in \mathcal{G}^c$, we have

$$\begin{aligned} f_i(v) - f_i(v') &\overset{(A.1)}{=} \alpha_i(c, \Delta_i(v)) - \alpha_i(c, \Delta_i(v')) \\ &=: \phi_i(\Delta_i(v), \Delta_i(v'), c). \end{aligned} \tag{A.3}$$

Hence, for any $i \in N$ and $v \in \mathcal{G}^c$, we obtain the following equation: for any $v' \in \mathcal{G}^{c,i}$,

$$\phi_i(\Delta_i(v), \Delta_i(v'), c) \overset{(A.3)}{=} f_i(v) - f_i(v') \overset{(A.1)}{=} f_i(v) - \alpha_i(c). \tag{A.4}$$

Note that $f_i(v) - \alpha_i(c)$ is independent from $v' \in \mathcal{G}^{c,i}$. For any $i \in N$ and $v \in \mathcal{G}^c$ let

$$\phi_i(\Delta_i(v), c) := f_i(v) - \alpha_i(c). \tag{A.5}$$

Hence, for any $i \in N$ and $v \in \mathcal{G}$, we obtain

$$f_i(v) \overset{(A.5)}{=} \phi_i(\Delta_i(v), v(N)) + \alpha_i(v(N)). \tag{A.6}$$

This completes Claim 1.

Claim 2 For any $i \in N$, and $c \in \mathbb{R}$, the function $\phi_i(\cdot, c) : \Lambda^i \to \mathbb{R}$ satisfies (M) within $\mathcal{G}^c$: for any $v, v' \in \mathcal{G}^c$, if $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S)$ for any $S \subset N \backslash \{i\}$, then $\phi(\Delta_i(v), c) \geq \phi(\Delta_i(v'), c)$. Moreover, $\phi_i(\mathbf{0}, c) = 0$.

Let $c = v(N) = v'(N)$. We have

$$\begin{aligned} \phi(\Delta_i(v), c) - \phi(\Delta_i(v'), c) &= \phi(\Delta_i(v), c) + \alpha_i(c) - (\phi(\Delta_i(v'), v'(N)) + \alpha_i(c)) \\ &\overset{\mathrm{C1}}{=} f_i(v) - f_i(v') \\ &\overset{(\mathrm{M}^-)}{\geq} 0. \end{aligned}$$

Moreover, for any $c \in \mathbb{R}$,

$$\phi_i(\mathbf{0}, c) \overset{(A.5),(A.4)}{=} \phi_i(\mathbf{0}, \mathbf{0}, c) \overset{(A.3),(A.1)}{=} \alpha_i(c, \mathbf{0}) - \alpha_i(c, \mathbf{0}) = 0. \tag{A.7}$$

Claim 3 The function ϕ is symmetric: for any $v \in \mathcal{G}$ and $i, j \in N$, if i, j is symmetric in v, then $\phi_i(\Delta_i(v), v(N)) = \phi_j(\Delta_j(v), v(N))$.

For any $i, j \in N$ and $v \in \mathcal{G}$ such that $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N\backslash\{i, j\}$, let $v' = v(N)u_{N\backslash\{i,j\}}$. Then, we have

$$\phi_i(\Delta_i(v), v(N)) \overset{(A.6)}{=} f_i(v) - f_i(v') \overset{(\text{WDMSP})}{=} f_j(v) - f_j(v') \overset{(A.6)}{=} \phi_j(\Delta_j(v), v(N)). \tag{A.8}$$

This completes Claim 3.

Claim 4 The function ϕ satisfies δ-efficiency (δ-E): there is a $\delta \in [0, 1]$ such that $\sum_{i \in N} \phi_i(\Delta_i(v), v(N)) = \delta v(N)$ for any $v \in \mathcal{G}$.

Let $\tilde{f} : \mathcal{G}^{add} \to \mathbb{R}$ be the restriction of f on $\mathcal{G}^{add}$. Then, by Theorem A.1, for each $m_x \in \mathcal{G}^{add}$, we have

$$\begin{aligned} \tilde{f}_i(m_x) &= \phi_i(\Delta_i(m_x), \sum_{l \in N} x_l) + \alpha_i \left(\sum_{l \in N} x_l \right) \\ &= \delta \cdot x_i + (1 - \delta) \cdot w_i \cdot \sum_{l \in N} x_l. \end{aligned} \tag{A.9}$$

for some $\delta \in [0, 1]$ and $w \in \mathcal{W}$. In particular, for $x_i = 0$, (A.7) implies $\alpha_i(\sum_{l \in N} x_l) = (1 - \delta) \cdot w_i \cdot \sum_{l \in N} x_l$. Hence, from Claim 1, it follows that for each $v \in \mathcal{G}$,

$$f_i(v) = \phi_i(\Delta_i(v), v(N)) + (1 - \delta) \cdot w_i \cdot v(N). \tag{A.10}$$

Since f satisfies (E), $\sum_{i \in N} f_i(v) = \sum_{i \in N} \phi_i(\Delta_i(v), v(N)) + (1 - \delta)v(N) = v(N)$, which implies that ϕ satisfies the following property:

$$(\delta - \text{E}) : \sum_{i \in N} \phi_i(\Delta_i(v), v(N)) = \delta v(N).$$

This completes Claim 4.

Claim 5 There is a $\delta \in [0, 1]$ and a $w \in \mathcal{W}$ such that $f_i(x) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i \cdot v(N)$ for any $v \in \mathcal{G}$.

Fixing $c \in \mathbb{R}$, we write $\phi_i^c(v) = \phi_i(\Delta_i(v), c)$ for each $v \in \mathcal{G}^c$. The function $\phi^c(v) : \mathcal{G}^c \to \mathbb{R}^N$ satisfies (δ-E), (M) and (SYM) within $\mathcal{G}^c$ by Claims 2, 3 and 4. Hence, by the same argument of Theorem 1, we have

$$\phi_i^c(v) = \delta Sh_i(v).$$

Since $c \in \mathbb{R}$ is arbitrarily chosen, for any $v \in \mathcal{G}$, we have

$$\phi_i(\Delta_i(v), v(N)) = \phi_i^{v(N)}(v) = \delta Sh_i(v) \tag{A.11}$$

Finally, by (A.10) and (A.11), we obtain $f_i(x) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i \cdot v(N)$, which completes the proof. □

Appendix B: Proof of Theorem 3

We first show that (E*), (M^{-*}), (SYM*) characterizes the egalitarian Shapley value when we fix $w = (\frac{1}{n}, \ldots, \frac{1}{n})$, which can be useful in later.

Lemma 1 *Suppose that $\mathcal{W} = \{(\frac{1}{n}, \ldots, \frac{1}{n})\}$ and $n \neq 2$. Then, an allocation rule $f : \mathcal{G} \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E*), (M^{-*}), (SYM*) if and only if it is an egalitarian-Shapley value.*

Proof This follows from the axioms and arguments in Casajus and Huettner (2014) if $w = (\frac{1}{n}, \ldots, \frac{1}{n})$. □

Now, we offer the proof of Theorem 3. It is clear that the rule satisfies all the axioms. We suppose that a rule $f : \mathcal{G} \to \mathbb{R}^N$ satisfies (E*), (M^{-*}), (RIN*), (SYM*), and (FEC*). Claim 1 can be thought of as an analog of that of Theorem 2. The differences lie in Claims 2, 3 and 4. In this proof, we first specify the form of the Shapley value, while we first specify the weighted division in Theorem 2.

Claim 1 For each $i \in N$, there exists functions $\phi_i(v) : \Lambda^i \times \mathbb{R} \to \mathbb{R}$ and $\alpha_i : \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ such that $f_i(v, w) = \phi_i(\Delta_i(v), v(N)) + \alpha_i(w, v(N))$.

We first take any $c \in \mathbb{R}$. For any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$, we have the following equation: for any $\bar{v} \in \mathcal{G}^c$ such that $\Delta_i(v) = \Delta_i(\bar{v})$,

$$f_i(v, w) \overset{(\mathrm{M}^{-*})}{=} f_i(\bar{v}, w) =: \alpha_i(w, c, \Delta_i(v)). \tag{B.1}$$

Specifically, we denote

$$\alpha_i(w, c) = \alpha_i(w, c, \mathbf{0}). \tag{B.2}$$

By (FEC*), for any $c \in \mathbb{R}$ and $i \in N$, there is a function $\phi_i^c : \mathcal{G}^c \to \mathbb{R}$ such that

$$\begin{aligned} \phi_i^c(v) &= f_i(v, w) - f_i(cu_{N\setminus\{i\}}, w) \\ &\overset{(B.2)}{=} f_i(v, w) - \alpha_i(w, c). \end{aligned}$$

By (B.1), we know that $\phi_i^c(v) = \phi_i^c(\bar{v})$ if $v(N) = \bar{v}(N) = c$ and $\Delta_i(v) = \Delta_i(\bar{v})$. Hence, we can define $\phi_i(\Delta_i(v), c) : \Lambda^i \times \mathbb{R} \to \mathbb{R}$ as $\phi_i(\Delta_i(v), c) =: \phi_i^c(v)$. Therefore, for any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$, we obtain $f_i(v, w) = \phi_i(\Delta_i(v), v(N)) + \alpha_i(w, v(N))$. This completes Claim 1.

Claim 2 For any $v \in \mathcal{G}$, there exists a $\delta \in [0, 1]$ and $d_i^{v(N)} \in \mathbb{R}$ such that $\phi_i(\Delta_i(v), v(N)) = \delta Sh_i(v) + d_i^{v(N)}$.

Let $w^* = (1/n, \ldots, 1/n) \in \mathcal{W}$, i.e., the equal weight. For any $c \in \mathbb{R}$ and any $v \in \mathcal{G}^c$, by Claim 1, we have

$$f_i(v, w^*) = \phi_i(\Delta_i(v), v(N)) + \alpha_i(w^*, v(N)), \tag{B.3}$$

and, by Lemma 1, there exists $\delta \in [0, 1]$ such that

$$f_i(v, w^*) = \delta Sh_i(v) + (1-\delta)\frac{1}{n}c. \tag{B.4}$$

Note that δ does not depend on $c \in \mathbb{R}$. For any $v' \in \mathcal{G}^{c,i}$, we have

$$\begin{aligned}\phi_i(\Delta_i(v'), c) + \alpha_i(w^*, c) &\overset{(B.3)}{=} f_i(v', w^*) \overset{(B.4)}{=} \delta Sh_i(v') + (1-\delta)\frac{1}{n}c \\ &= (1-\delta)\frac{1}{n}c.\end{aligned} \tag{B.5}$$

Note that player i is a null player in game $v' \in \mathcal{G}^{c,i}$. Hence, for any $v', v'' \in \mathcal{G}^{c,i}$, we have $\phi_i(\Delta_i(v'), c) + \alpha_i(w^*, c) \overset{(B.5)}{=} (1-\delta)\frac{1}{n}c \overset{(B.5)}{=} \phi_i(\Delta_i(v''), c) + \alpha_i(w^*, c)$ and, so, denote $d_i^c := \phi_i(\Delta_i(v'), c) = \phi_i(\Delta_i(v''), c)$. We obtain

$$\alpha_i(w^*, c) \overset{(B.5), d_i^c}{=} (1-\delta)\frac{1}{n}c - d_i^c. \tag{B.6}$$

Therefore, for every $v \in \mathcal{G}^c$, we must have

$$\phi_i(\Delta_i(v), c) \overset{(B.3)(B.4)(B.6)}{=} \delta Sh_i(v) + d_i^c. \tag{B.7}$$

Since $c \in \mathbb{R}$ is arbitrary chosen, we obtain $\phi_i(\Delta_i(v'), v(N)) = \delta Sh_i(v) + d_i^{v(N)}$ for all $v \in \mathcal{G}$.

Claim 3 $\alpha_i(w, v(N)) = (1-\delta) \cdot w_i v(N) - d_i^{v(N)}$ for each $w \in \mathcal{W}$.

Consider any $w \in \mathcal{W}$ and player $k^* \in N$ such that $k^* \in \text{argmin}_{i \in N, w_i > 0} w_i$. Note that k^* is well-defined because $\sum_{i \in N} w_i = 1$ and $w_i \geq 0$ for any $i \in N$. By Claim 2, for any player $i \neq k^*$ and any $c \in \mathbb{R}$,

$$f_k(cu_{\{i\}}, w) = \begin{cases} \delta c + d_k^c + \alpha_k(w, c) & \text{if } k = i, \\ d_k^c + \psi_k^c(w) & \text{otherwise.} \end{cases}$$

Hence, we have

$$\sum_{k \in N} (\alpha_k(w, c) + d_k^c) \overset{(\text{E}^*)}{=} (1-\delta)c. \tag{B.8}$$

Moreover, for any $i \neq k^*$, j $(j \neq i,\ j \neq k^*)$ and, by considering a game $cu_{\{j\}}$, we have

$$\alpha_i(w, c) + d_i^c \overset{(\text{RIN}^*)}{=} \frac{w_i}{w_{k^*}}(\alpha_{k*}(w, c) + d_{k^*}^c), \tag{B.9}$$

because i and k^* are null players in $cu_{\{j\}}$. Therefore, for any $i \in N$, we have

$$\begin{aligned}
&\alpha_i(w,c) + d_i^c - (1-\delta)w_i c \\
&\overset{(B.9)}{=} w_i \cdot \left[\frac{1}{w_{k^*}}(\alpha_{k*}(w,c) + d_{k^*}^c) - (1-\delta)c\right] \\
&\overset{(B.8)}{=} w_i \cdot \left[\frac{1}{w_{k^*}}(\alpha_{k*}(w,c) + d_{k^*}^c) - \sum_{k\in N}(\alpha_k(w,c) + d_k^c)\right] \\
&\overset{(B.9)}{=} \frac{w_i}{w_{k^*}} \cdot \left[(\alpha_{k*}(w,c) + d_{k^*}^c) - \sum_{k\in N} w_k(\alpha_{k*}(w,c) + d_{k^*}^c)\right] \\
&\overset{\sum_k w_k=1}{=} \frac{w_i}{w_{k^*}} \cdot \left[(\alpha_{k*}(w,c) + d_{k^*}^c) - (\alpha_{k*}(w,c) + d_{k^*}^c)\right] \\
&= 0.
\end{aligned}$$

Since $c \in \mathbb{R}$ is arbitrary chosen, we obtain $\alpha_i(w, v(N)) = (1-\delta)\cdot w_i v(N) - d_i^{v(N)}$ for all $v \in \mathcal{G}$.

Claim 4 For any $v \in \mathcal{G}$ and $w \in \mathcal{W}$, there exists a $\delta \in [0,1]$ such that $f_i(v,w) = \delta \cdot Sh_i(v) + (1-\delta)\cdot w_i v(N)$.

For any $v \in \mathcal{G}$ and $w \in \mathcal{W}$, we have

$$\begin{aligned}
f_i(v,w) &\overset{\text{C1}}{=} \phi_i(\Delta_i(v), v(N)) + \alpha_i(w, v(N)) \\
&\overset{\text{C2}}{=} \delta Sh_i(v) + d_i^{v(N)} + \alpha_i(w, v(N)) \\
&\overset{\text{C3}}{=} \delta Sh_i(v) + d_i^{v(N)} + (1-\delta)\cdot w_i v(N) - d_i^{v(N)} \\
&= \delta Sh_i(v) + (1-\delta)\cdot w_i v(N).
\end{aligned}$$

This completes the proof. □

Appendix C: Independence of axioms and a counterexample for *n* = 2

Independence of axioms for Theorem 2

The independence of the axioms is shown in the examples listed below.

Example C.1 Consider the following function: for any $i \in N$ and $v \in \mathcal{G}$,

$$f_i^{\mathrm{E}}(v) = 0.$$

This function satisfies all axioms except (E).

Example C.2 Consider the following function: for any $i \in N$ and $v \in \mathcal{G}$,

$$f_i^{\mathrm{M}^-}(v) = 2Sh_i(v) - \frac{v(N)}{n}.$$

This function satisfies all axioms except (M^-).

Example C.3 Consider the following function: for any $i \in N$ and $v \in \mathcal{G}$,

$$f_i^{\mathrm{RIN}}(v) = \delta Sh_i + (1-\delta)\frac{i + v(N)^2}{\bar{N} + n(v(N))^2}v(N),$$

where $\bar{N} = \sum_{i \in N} i = \frac{n(n-1)}{2}$ and i is the natural number representing player i. This rule satisfies (E), (M^-), (WMDSP) and (NY) but not (RIN). To check (M^-), let $h_i(a) = \frac{i+a^2}{\bar{N}+a^2}a = \frac{ia+a^3}{\bar{N}+a^2}$. Then, we have $\frac{dh_i(a)}{da} = \frac{na^4+(3\bar{N}-ni)a^2+i\bar{N}}{(\bar{N}+a^2)^2} > 0$ for all $a \in \mathbb{R}$ because $na^4 + i\bar{N} > 0$ for all i and $3\bar{N} - ni \geq \frac{n(n-3)}{2} \geq 0$.

Example C.4 Consider the following function: for any $i \in N$ and $v \in \mathcal{G}$,

$$f_i^{\mathrm{WMDSP}}(v) = v(P_i^\sigma \cup \{i\}) - v(P_i^\sigma)$$

where P_i^σ is the set of predecessors of i in σ. This function satisfies all the axioms except (WMDSP).

Example C.5 Consider the following function: for any $i \in N$ and $v \in \mathcal{G}$,

$$f_i^{\mathrm{NY}}(v) = \begin{cases} Sh_1 + 10 & \text{if } \ i = 1, \\ Sh_i - \frac{10}{n-1} & \text{if } \ i \neq 1. \end{cases}$$

This rule satisfies all the axioms except (NY).

Independence of axioms for Theorem 3

The independence of the axioms is shown in the examples listed below.

Example C.6 Consider the following function: for any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$,

$$f_i^{\mathrm{E}^*}(v, w) = 0.$$

Then, the function satisfies all axioms except (E^*).

Example C.7 Consider the following function: for any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$,

$$f_i^{\mathrm{M}^{-*}}(v, w) = 2Sh_i(v) - w_i v(N).$$

Then, the function satisfies all axioms except (M^{-*}).

Example C.8 Consider the following function: for any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$,

$$f_i^{\mathrm{RIN}^*}(v, w) = \delta \cdot \frac{v(N)}{|N|} + (1-\delta) \cdot w_i v(N).$$

The function satisfies all axioms except (RIN^*).

Example C.9 Consider the following function: for any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$,

$$f_i^{\mathrm{SYM}^*}(v, w) = \delta \cdot Sh_i^z(v) + (1-\delta) \cdot w_i v(N),$$

where $Sh_i^z(v)$ is the weighted Shapley value for a given weight $z \in R_{++}^N$. Since anonymity is defined over $\mathcal{G}$ and $\mathcal{W}$, the function satisfies all axioms except (SYM*).

Example C.10 Consider the following function: for any $i \in N$, $v \in \mathcal{G}$ and $w \in \mathcal{W}$,

$$f_i^{\mathrm{FEC}^*}(v, w) = w_{\min} \cdot Sh_i(v) + (1 - w_{\min}) w_i v(N),$$

where $w_{\min} = \min_{j \in N} w_j$. This function satisfies all axioms except (FEC*).

A counterexample to Theorems 2 and 3 for $n = 2$

Theorems 2 and 3 fail for $n = 2$. Consider the following allocation rule $f^{\heartsuit}$ on $N = \{1, 2\}$:

$$(f_1^{\heartsuit}(v, w), f_2^{\heartsuit}(v, w)) = \begin{cases} (Sh_1(v), Sh_2(v)), & Sh_1(v) \geq 0 \text{ and } Sh_2(v) \geq 0, \\ (0, v(N)), & Sh_1(v) < 0 \text{ and } Sh_2(v) > 0 \wedge v(N) \geq 0, \\ (v(N), 0), & Sh_1(v) < 0 \text{ and } Sh_2(v) > 0 \wedge v(N) < 0, \\ (Sh_1(v), Sh_2(v)), & Sh_1(v) \leq 0 \text{ and } Sh_2(v) \leq 0, \\ (0, v(N)), & Sh_1(v) > 0 \text{ and } Sh_2(v) < 0 \wedge v(N) \leq 0, \\ (v(N), 0), & Sh_1(v) > 0 \text{ and } Sh_2(v) < 0 \wedge v(N) > 0, \end{cases}$$

for any $v \in \mathcal{G}$ and $w \in \mathcal{W}$.

Note that this function does not depend on w. It is clear that $f^{\heartsuit}$ satisfies (E*) and (M^{-*}). It satisfies (SYM*) because if the players 1 and 2 are symmetric in the sense of marginal contribution and have the same weight, they receive $(Sh_1(v), Sh_2(v))$. It satisfies (RIN*) because if the players 1 and 2 are null players, the game v is the null game: $v(12) = v(1) = v(2) = 0$. Since $f^{\heartsuit}$ does not depend on w, it clearly satisfies (FEC*). The same argument applies to Theorem 2.

References

Abe T, Nakada S (2017) Monotonic redistribution: reconciling performance-based allocation and weighted division. Int Game Theory Rev 19:175022

Béal S, Casajus A, Huettner F, Rémila E, Solal P (2016) Characterizations of weighted and equal division values. Theory Decis 80:649–667

van den Brink R, Funaki Y, Ju Y (2013) Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. Soc Choice Welf 40:693–714

Casajus A (2010) Another characterization of the Owen value without the additivity axiom. Theory Decis 69:523–536

Casajus A (2011) Differential marginality, van den Brink fairness, and the Shapley value. Theory Decis 71:163–174

Casajus A (2014) The Shapley value without efficiency and additivity. Math Soc Sci 68:1–4

Casajus A (2015) Monotonic redistribution of performance-based allocations: a case for proportional taxation. Theor Econ 10:887–892

Casajus A (2016) Differentially monotonic redistribution of income. Econ Lett 141:112–115
Casajus A, Huettner F (2013) Null players, solidarity, and the egalitarian Shapley values. J Math Econ 49:58–61
Casajus A, Huettner F (2014) Weakly monotonic solutions for cooperative games. J Econ Theory 154:162–172
Casajus A, Yokote K (2017a) Weak differential marginality and the Shapley value. J Econ Theory 167:274–284
Casajus A, Yokote K (2017b) Weak differential monotonicity, flat tax, and basic income. Econ Lett 151:100–103
Chun Y (1988) The proportional solution for rights problems. Math Soc Sci 15:231–246
Chun Y (1989) A new axiomatization of the Shapley value. Games Econ Behav 1:119–130
Chun Y (1991) On the symmetric and weighted Shapley values. Int J Game Theory 20:183–190
Hart S, Mas-Colell A (1989) Potential, value, and consistency. Econometrica 57:589–614
Joosten R, Peters H, Thuijsman F (1994) Socially acceptable values for transferable utility games. Report M94-03, Maastricht University
Joosten R (1996) Dynamics, equilibria and values. Dissertation, Maastricht University
Joosten R (2016) More on linear-potential values and extending the 'Shapley family' for TU-games, presented in Stony Brook, see the conference website http://www.gtcenter.org/?page=Archive/2016/ConfTalks.html
Kalai E, Samet D (1987) On weighted Shapley values. Int J Game Theory 16:205–222
Nowak A, Radzik T (1994) A solidarity value for n-person transferable utility games. Int J Game Theory 23:43–48
Nowak A, Radzik T (1995) On axiomatizations of the weighted Shapley values. Games Econ Behav 8:389–405
Roth A (1979) Proportional solutions to the bargaining problem. Econometrica 47:775–778
Shapley L (1953a) Additive and non-additive set functions. PhD Thesis, Department of Mathematics, Princeton University
Shapley L (1953b) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games II (Annals of Mathematics Studies 28). Princeton University Press, Princeton, pp 307–317
Shapley L, Shubik M (1969) On the core of an economic system with externalities. Am Econ Rev 59:678–684
Young P (1985) Monotonic solutions of cooperative games. Int J Game Theory 14:65–72
Yokote K (2015) Weak addition invariance and axiomatization of the weighted Shapley value. Int J Game Theory 44:275–293