Soc Choice Welf (2018) 51:415–435
https://doi.org/10.1007/s00355-018-1121-9

ORIGINAL PAPER

Income inequality measurement: a fresh look at two old issues

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Received: 7 June 2016 / Accepted: 15 March 2018 / Published online: 20 March 2018

Abstract The literature on income inequality measurement is split into (1) the ethical approach, from which the Atkinson–Kolm–Sen and Kolm–Pollak classes of indices are derived, and (2) the axiomatic approach, which mainly leads to the generalised entropies. This paper shows how to rationalise, under utilitarianism, the generalised entropies as ethical indices. In this framework a generalised entropy index is consistent with the principle of transfers if and only if the underlying utility function is increasing. This unconventional interpretation explains the strange behaviour of the generalised entropies for some values of the inequality aversion parameter, as identified by Shorrocks (Econometrica 48:613–625, 1980). In that case, the underlying utility function is convex. Then, it provides a solution to escape the so-called Hansson–Sen paradox (Hansson in Foundational problems in the special sciences. Reidel Publishing Company, Dordrecht, 1977; Sen in Personal income distribution. North-Holland, Amsterdam, pp 81–94, 1978) that affects the standard ethical indices and which corresponds to a counterintuitive increase in inequality as a result of a concave transformation of the utility function. A normalised version of the generalised entropies behaves appropriately after such a transformation.

This paper forms part of the research project *RediPref* (Contract ANR-15-CE26-0004) of the French National Agency for Research (ANR) whose financial support is gratefully acknowledged. It has also been supported by *LabEx Entrepreneurship* (Contract ANR-10-LABX-11-01). I want to thank Richard Nock for interesting discussions on the methods used in this paper. I am also indebted to an associate editor and three anonymous referees whose comments have helped me to substantially improve the paper.

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1 Introduction

The literature on income inequality measurement distinguishes two approaches. The ethical approach, initiated by Dalton (1920), aims at assessing inequality as the loss of potential social welfare resulting from the fact that income is unevenly distributed. By contrast, the axiomatic approach aims at characterising measures consistent with properties which seem suitable for inequality indices. This approach usually leads to a unique class of indices, called here the generalised entropies (see Bourguignon 1979; Cowell 1980; Shorrocks 1980, 1984). Unfortunately, neither of these two approaches is free from criticism.

The first limitation of the generalised entropies is the lack of ethical foundations. Indeed, the value given by an index in this class has no particular meaning, and the notion of inequality aversion it captures cannot be tied to the social evaluator's preferences. The second important limitation is the 'strange behaviour' of the generalised entropies, as identified by Shorrocks (1980) for some values of the inequality aversion parameter. As this parameter increases, the index becomes more and more sensitive to progressive income transfers—i.e. transfers from a richer person to a poorer person—at the very top of the distribution, a behaviour which contradicts the principle of weak transfer sensitivity (Kolm 1976; Shorrocks and Foster 1987). A common recommendation for empirical studies is to avoid such parameter values. Nevertheless, what we call in this paper the 'Shorrocks issue' has not been rationalised, to date, through compelling arguments.

The ethical approach—through the so-called Atkinson–Kom–Sen and Kolm–Pollak classes of indices—is also, but for different reasons, controversial. It is here assumed that social welfare can be computed from the individual utilities. The same utility function applies to each individual as a measure of social well-being they receive from their incomes, according to the social evaluator's views. Then concavity of the utility function is supposed to reflect the social evaluator's inequality aversion. Nevertheless, Hansson (1977) and Sen (1978) have identified serious shortcomings for the strategy which aims at defining an inequality index from a social welfare function. The so-called Hansson–Sen paradox states that, for a given income distribution, a concave transformation of the utility function leads to a decrease in inequality of individual welfare, and hence the value given by the inequality index should also decrease. However, and paradoxically enough, the opposite implication is observed: for a given income distribution, the more concave the utility function, the larger the value given by the inequality index.

This paper proposes a new interpretation of the generalised entropies which builds upon an original representation, both formal and geometric. We argue that the function that generates the generalised entropies can be viewed as the antiderivative of a utility function. We first establish that such an interpretation is admissible if we assume that the utility function is a cardinal representation of the social evaluator's preferences. In this framework, inequality aversion is related to the convexity of the antiderivative of the utility function or equivalently to the increasing rate of the utility function. This interpretation, obviously non-standard, provides ethical foundations for the generalised entropies.

The interest of this interpretation is twofold. First, the so-called Shorrocks issue becomes crystal-clear: the strange behaviour of the generalised entropies, for some values of the inequality parameter, results from the convexity of the underlying utility function. We emphasise that the index is here consistent with the Pigou–Dalton principle of transfers as long as the utility function is increasing, whatever the sign of its second derivative. The second interest is to provide a solution to escape the Hansson–Sen paradox. After a simple normalisation—again in order to take into account the cardinal measurability of the utility function—a generalised entropy index decreases after a concave transformation of the utility function.

An association between inequality aversion and the increasing rate of the utility function may seem surprising as it is a common practice in normative economics to link inequality aversion to the concavity of the utility function. We take the stance that this standard way of proceeding may be deemed as a conceptual error. Of course, in a utilitarian framework, an uneven income distribution is suboptimal in terms of production of welfare, and the more concave the utility function, the more intense the suboptimality. Nevertheless, an inequality index that is concerned by the effects of income on welfare should be sensitive to any reduction of the utilities between the individuals. Consequently, it makes more sense to relate, loosely speaking, inequality aversion to the slope of the utility function because for a given income distribution, the more the utility increases, the larger the inequality of welfare is.

The rest of the paper is organised as follows. Section 2 provides a quick presentation of the inequality indices obtained in the ethical and axiomatic approaches. We also discuss the meaning of the 'utility function' which represents the social evaluator's preferences in the ethical approach. We then propose in Sect. 3 a new interpretation of the usual generalised entropies as inequality-of-utility measures, with a geometric representation on the graph of this utility function. We argue in Sect. 4 that this representation can explain the strange behaviour of the generalised entropies for some values of the inequality aversion parameter. Section 5 investigates the fundamental problem pointed out by Hansson (1977) and Sen (1978) that affects the usual ethical inequality indices. We establish that a normalised version of the generalised entropies can escape the paradox. Finally, Sect. 6 concludes the paper.

2 The standard framework of income inequality measurement

2.1 Notation and preliminary definitions

An income distribution is a list $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \in \mathscr{D}$ is the income of individual i, and $\mathscr{D} = [\underline{x}, \overline{x}] \subseteq \mathbb{R}$ is the set of admissible incomes. In this paper we mainly focus, without loss of generality, on equal mean distributions (see Appendix A for the general case), hence distributions drawn from the set $\mathscr{D}^n(\mu) = \{\boldsymbol{x} \in \mathscr{D}^n \,|\, \mu(\boldsymbol{x}) = \mu\}$ where $\mu(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i/n$. Then we let $\mathbf{1}_n = (1, \ldots, 1)$ represent a list where 1 is repeated n-times.

The ethical approach of inequality measurement aims at deriving inequality indices from a social welfare function $W : \mathscr{D}^n \longrightarrow \mathbb{R}$. The equally-distributed-equivalent-income $\Xi(\boldsymbol{x})$ is income which, if received by each individual, results in a distribution socially indifferent to $\boldsymbol{x}$, so that $W(\boldsymbol{x}) = W(\Xi(\boldsymbol{x})\mathbf{1}_n)$. The literature is usually focused on the utilitarian approach, in which social welfare is written as $W_U(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n U(x_i)$ and $\Xi_U(\boldsymbol{x}) = U^{-1}\left(\frac{1}{n}\sum_{i=1}^n U(x_i)\right)$.

Definition 1 For all $\boldsymbol{x} \in \mathscr{D}^n(\mu)$, the class of ethical and utilitarian inequality indices—referred hereafter as ethical indices—can be written as:

$$I_U(\boldsymbol{x}) = \mu - \Xi_U(\boldsymbol{x}), \tag{2.1}$$

for any real-valued (utility) function U, continuously differentiable and concave, defined up to a transformation $U_*(x) = \alpha\, U(x) + \beta$ for all $x \in \mathscr{D}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$.

Alternatively, the axiomatic approach, which consists of identifying suitable properties to be applied to the inequality indices, usually leads to a unique class of indices, called the generalised entropies (see Magdalou and Nock 2011).

Definition 2 For all $\boldsymbol{x} \in \mathscr{D}^n(\mu)$, the class of generalised entropies can be written as:

$$G_\phi(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n \left[\phi(x_i) - \phi(\mu)\right], \tag{2.2}$$

for any real-valued function ϕ, continuously differentiable and strictly convex, defined up to a transformation $\phi_*(x) = \alpha\, \phi(x) + \beta\, x + \gamma$ for all $x \in \mathscr{D}$, where $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$.[1]

2.2 A critical discussion on the ethical approach

A discussion on the meaning of the function U in the ethical approach is required. Of course, it cannot be viewed as a representation of the individuals' real preferences and thus, the use of the term 'utility' may be ambiguous. Following Dalton (1920), this function is sometimes defined in terms of 'individual economic welfare' or 'personal welfare', as it can be assessed by a social evaluator. Alternatively, it can be defined only in terms of the evaluator's view on income inequality:

> 'Note that there is nothing inherently utilitarian in this formulation. The social welfare function is the sum of identical concave transformations of individual incomes, but social welfare is expressed in terms of incomes not utilities. Thus, the concavity of the social welfare function may represent the aversion to inequality of the evaluator rather than the degree of relative risk aversion of a utility function identical across all individuals' (Atkinson and Brandolini 2015, p. 214).

The distinction between the two interpretations is not trivial. First, we recall that the approach initiated by Kolm (1969) and Atkinson (1970) is grounded on the equivalence

[1] Continuous differentiability in Definitions 1 and 2 is actually not necessary, even if satisfied by all the indices in the literature. It is just a simplification for ease of reading thereafter.

between the following two statements, for any distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{D}^n(\mu)$ (Hardy et al. 1934):

(a) $\sum_{i=1}^n U(x_i) \geq \sum_{i=1}^n U(y_i)$ for any strictly concave function U defined on $\mathscr{D}$;
(b) $\boldsymbol{x}$ is obtained from $\boldsymbol{y}$ by means of a finite sequence of permutations and/or progressive transfers, that is, mean-preserving transfers of income from a richer to a poorer person.

This result can be interpreted in several different ways. If one reduces the notion of 'income inequality' as a purely descriptive notion of 'distance in income'—which is unambiguously decreased as the result of progressive transfers—then the concavity of U captures the aversion to income inequality of the measure $\sum_{i=1}^n U(x_i)$, in the sense that the higher the value given by the measure, the lower the income inequality. To go a step further, there is no need at all, in that case, to interpret $\sum_{i=1}^n U(x_i)$ as a social welfare evaluation. Nevertheless, inequality also refers to the normative notion of equity, and the Pigou–Dalton principle of transfers must be seen as an ethical judgment which describes a progressive transfer as a social improvement. Dalton and Pigou take a resolutely utilitarian stance, since it is assumed that a progressive transfer 'must increase the sum of satisfaction' (Pigou 1912, p. 24). This argument justifies the interpretation of statement (a) as a social welfare comparison, and thus endows the concavity of the function U with the ethical views of the principle of transfers.

We emphasise that the normative interpretation of the principle of transfers (statement b) cannot be dissociated to the choice of the social welfare function (statement a). In Kolm–Atkinson's framework, as presented above, a progressive transfer is socially desirable because an uneven income distribution is suboptimal in terms of production of social welfare. Coupled with the fact that the social welfare function is additively separable in the individual situations, it may be difficult not to interpret the concave transformation of individual incomes as a representation of personal welfare, although 'objectively assessed' by a social evaluator, with the property that marginal welfare 'diminishes as income increases' (Dalton 1920, p. 349).

In the same vein, note that Dasgupta et al. (1973) generalise the Atkinson's result to any social welfare function $W = W(U(x_1), U(x_2), \ldots, U(x_n))$ where U is explicitly interpreted as a utility function and such that W is increasing, symmetric, and quasi-concave in individual utilities. The most important argument, for our purpose, is that quasi-concavity is justified in terms of inequality-of-utility aversion: 'Quasi-concavity reflects a tendency to prefer equality; given two *utility distributions* an average of the two must be no worse than the less preferred of the two distributions' (Dasgupta et al. 1973, p. 181).

Aside from the interpretation of the function U, another question is raised by the justification of the quasi-concavity as proposed by Dasgupta et al. (1973): are we only concerned by income inequality, or does the inequality of utility also matter? One may reasonably consider that the income is not, for an individual, an end in itself but a means to achieve personal welfare. In this respect, Dalton's line of argument is perfectly transparent: 'For the economist is primarily interested, not in the distribution of income as such, but in the effects of the distribution of income upon the *distribution* and total amount of economic welfare' (Dalton 1920, p. 431).

The problem is, actually, the ambiguous stance of the Kolm–Atkinson's approach to that question. Indeed, if U is interpreted as a utility function, then the indices I_U are inequality-of-utility averse even if the underlying social welfare function W_U is inequality-of-utility neutral. Let us clarify this argument. If one applies the principle of transfers to the utility distribution instead of the income distribution one observes—if in addition we suppose that U is increasing and concave—that a utility transfer which preserves the sum of the utilities (hence W_U but also Ξ_U) is equivalent to a mean-reducing income transfer. It follows that such a utility transfer—through an underlying income transfer, the only transferable variable—implies a reduction of any index I_U and thus its consistency with the 'principle of utility transfers'. This point can partially explain why this approach has not been much more challenged in the literature.

In this paper we explicitly take the stance—which is of course open to debate—that $U(x_i)$ refers to personal welfare or utility of individual i, and that utility inequality is a matter of concern, as well as income inequality.[2] We also point out that it is impossible to separate the problems of income inequality and utility inequality. Formally, the two variables are inextricably linked, as utility is an increasing and concave transformation of income. 'We have to deal, therefore, not merely with one variable, but with two, or possibly more, between which certain functional relations may be presumed to exist' (Dalton 1920, p. 431). Mean-preserving progressive income transfers can be justified per se through normative arguments, as well as equalising transformations of the utility distribution.

3 An alternative representation of the indices

Even if the ordinal equivalence between indices I_U and G_ϕ (Definitions 1, 2) is often mentioned in the literature, it is rather surprising that the relationship between functions U and ϕ has never been raised nor investigated. However, for the relative and absolute indices, ϕ is nothing but the antiderivative of U ($U_r = \phi_r'$ and $U_a = \phi_a'$, see Appendix A). In this section, we propose a theoretical framework which helps rationalize this interpretation. Notice that the informational content of ϕ and U makes this interpretation possible. Indeed, ϕ is defined up to a transformation $\phi_*(x) = \alpha\,\phi(x) + \beta\,x + \gamma$, where $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$. If $U = \phi'$, then U is defined up to a transformation $U_*(x) = \alpha\,U(x) + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$, as required by the ethical indices: U is a cardinal representation of the social evaluator's preferences.

3.1 Ethical indices and generalised entropies as inequality-of-utility measures

From now on we assume that a social evaluator assesses the social well-being of each individual according to the utility resulting from the income they receive. This section establishes that the generalised entropies can be interpreted as inequality-of-utility measures if ϕ is defined as the antiderivative of U, which is also the case of the

[2] We thus operate in a simplified framework by considering that the situation of an individual is fully described by their income, and that personal welfare is cardinally measurable and interpersonally comparable (see Sen 1976, for a discussion).

standard ethical indices. Let us first define the notion of 'Bregman divergence between two distributions' (see Magdalou and Nock 2011).

Definition 3 For all $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{D}^n(\mu)$, a Bregman divergence can be written as:

$$D_\varphi(\boldsymbol{x} \| \boldsymbol{y}) = \frac{1}{n} \sum_{i=1}^{n} \left[\varphi(x_i) - \varphi(y_i) - (x_i - y_i)\varphi'(y_i)\right], \tag{3.1}$$

for any function φ continuously differentiable and strictly convex.[3]

One remarks that D_φ satisfies 'anonymity', in the sense that a permutation of all the pairs (x_i, y_i) does not modify its value. Because φ is convex, we also have $D_\varphi(\boldsymbol{x} \| \boldsymbol{y}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{D}^n(\mu)$, and $D_\varphi(\boldsymbol{x} \| \boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. Moreover one immediately observes that a generalised entropy index is equivalent to a Bregman divergence between distribution $\boldsymbol{x}$ and the hypothetical situation in which all individuals have mean income μ. Equivalently, $G_\phi(\boldsymbol{x}) = D_\phi(\boldsymbol{x} \| \mu \mathbf{1}_n)$.

Ethical indices are usually described as measures of income loss resulting from inequality. Alternatively the loss of potential social welfare resulting from inequality can be assessed by the difference between the 'mean of the utilities', which is simply written $\mu(U) = W_U(\boldsymbol{x})$, and the 'utility of the mean', $U(\mu)$. From Jensen's inequality and concavity of U, we have $U(\mu) \geq \mu(U)$. The following proposition establishes that an ethical index is actually a Bregman divergence between the distribution of the utilities $\boldsymbol{U} = (U(x_1), U(x_2), \ldots, U(x_n))$ and the hypothetical situation in which all individuals have the mean utility $\mu(U)$. The proofs are relegated in Appendix B.

Proposition 1 *For all* $\boldsymbol{x} \in \mathscr{D}^n(\mu)$ *and all concave utility functions* U, *an ethical index* (Definition 1) *can be written as* $I_U(\boldsymbol{x}) = D_\varphi(\boldsymbol{U} \| \mu(U) \mathbf{1}_n)$, *where* $\varphi(t) = U^{-1}(t)$ *is a convex function.*

Hence an ethical index has an interpretation in terms of utility loss and is thus an inequality-of-utility measure. This remark is in line with the fact that ethical indices are consistent with the 'principle of utility transfers'. An equivalent interpretation is possible for the generalised entropies. Nevertheless, the individual utilities are not compared to the same reference: It is not the mean utility but the utility of the mean income ($\mu(U)$ versus $U(\mu)$).

Proposition 2 *Let* $U = \phi'$. *For all* $\boldsymbol{x} \in \mathscr{D}^n(\mu)$ *and all convex functions* ϕ, *a generalised entropy* (Definition 2) *can be written as* $G_\phi(\boldsymbol{x}) = D_\varphi(U(\mu) \mathbf{1}_n \| \boldsymbol{U})$, *where* $\varphi(t) = t\, U^{-1}(t) - \phi\left(U^{-1}(t)\right)$ *is a convex function.*

Propositions 1 and 2 establish that the ethical indices and the generalised entropies can each be interpreted as measures of loss of potential social welfare resulting from inequality. Nevertheless, there is a fundamental difference. While inequality aversion—through consistency with the principle of transfers—is captured by the

[3] Again, continuous differentiability is usually not required. The symbol $\|$ is used to indicate that a divergence measure is different from a distance measure, as it needs to neither be symmetric nor satisfy the triangle inequality.

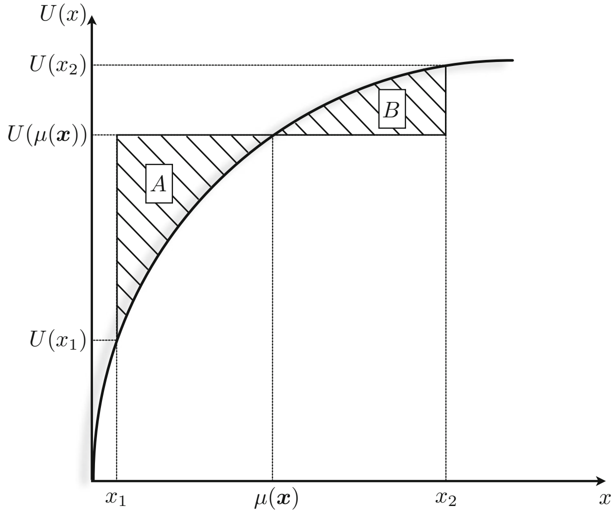

Fig. 1 A representation of an index G_ϕ on the graph of $U = \phi'$

concavity of the utility function in the standard ethical approach, it is here captured by the slope of the utility function $U' = \phi'' > 0$ in the generalised entropies (the convexity of ϕ). This point is quite surprising and, in some sense, disturbing for a reader trained in normative economics, because the association between the concavity of U and the social evaluator's inequality aversion is now well-established. However, it is also well-admitted that this way of defining inequality aversion is not free from criticism, as pointed out by Hansson (1977) and Sen (1978) (see Sect. 5).

3.2 A simple geometric representation of the generalised entropies

Now we provide a geometric representation of G_ϕ when ϕ is interpreted as the antiderivative of the utility function U. It builds on the following result.

Proposition 3 *For all* $\boldsymbol{x} \in \mathscr{D}^n(\mu)$*, a generalised entropy* (Definition 2) *can be written as* $G_\phi(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}\int_{x_i}^{\mu}[U(\mu) - U(x)]\,dx$*, where* $U = \phi'$ *is strictly increasing.*

As an illustration, consider the distribution $\boldsymbol{x} = (x_1, x_2)$ with $x_1 < x_2$. It follows from Proposition 3 that $G_\phi(\boldsymbol{x}) = (A + B)/2$, where $A = \int_{x_1}^{\mu}[U(\mu) - U(x)]\,dx$ and $B = \int_{\mu}^{x_2}[U(x) - U(\mu)]\,dx$. Because A and B are computed as integrals of U, they can be represented as areas on its graph. A representation of A and B, with a strictly concave function U, is provided in Fig. 1. The farther out individual incomes x_1 and x_2 are from the mean, the larger areas A and B are. Figure 1 is thus a simple manner to demonstrate that the convexity of ϕ, or equivalently the fact that its first derivative U is increasing, is sufficient to make the generalised entropies consistent with the principle of transfers. It is also easily recognised that $G_\phi(\boldsymbol{x}) > 0$ as soon as $x_1 \neq x_2$ (areas A and B are strictly positive), and that $G_\phi(\boldsymbol{x}) = 0$ if and only if $x_1 = x_2 = \mu$. We emphasise that U is not necessarily concave, as it just needs to be strictly increasing.

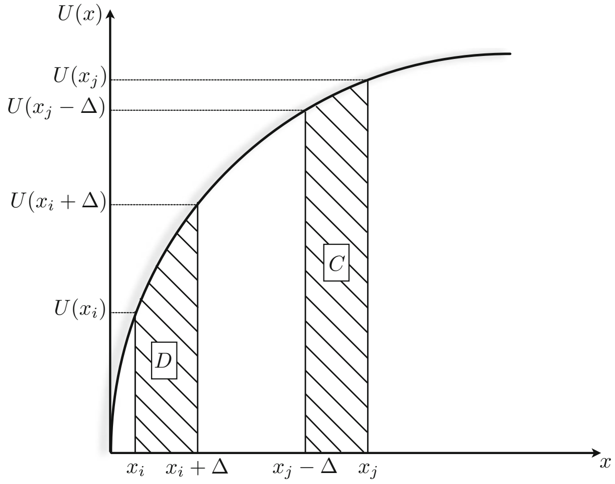

Fig. 2 The impact of a progressive transfer

Because G_ϕ is additively separable according to the individual incomes, this example can be easily generalised for a population consisting of $n > 2$ individuals. In that case, G_ϕ is the mean of areas comparable to A for incomes below the mean and areas comparable to B for incomes above.

By analogy with equivalent notions in other sciences (physics, information theory, …), a generalised entropy can be loosely associated with an 'amount of disorder' generated by income inequality. Because G_ϕ is—in our interpretation—utility based, it can be described as a measure of 'utility change' due to the fact that individuals 1 and 2 have incomes x_1 and x_2 instead of the mean μ. More precisely, area A in Fig. 1, which deals with the situation of individual 1 only, aggregates the utility loss $(U(\mu) - U(x))$ of all potential incomes x, from x_1 to μ. Hence, area A can be viewed as a (surplus-like) measure of utility wasted by individual 1, taking into account the curvature of the utility function over the whole income interval between x_1 and μ. The same interpretation is possible for area B, but as a measure of utility gained by individual 2. Nevertheless, both areas A and B contribute positively to the global measure G_ϕ.

We now propose to illustrate the impact of a progressive transfer on G_ϕ. Precisely, we denote by $\boldsymbol{x}(i, j; \Delta)$ a distribution obtained from $\boldsymbol{x}$ by means of a progressive transfer of income $\Delta > 0$ from individual j to individual i, under the restriction that $x_i + \Delta \leq x_j - \Delta$. One deduces from Proposition 3 that for all distributions $\boldsymbol{x} \in \mathscr{D}^n(\mu)$:

$$G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(i, j; \Delta)) = \frac{1}{n}\left[\int_{x_j-\Delta}^{x_j} U(x)dx - \int_{x_i}^{x_i+\Delta} U(x)dx\right]. \tag{3.2}$$

We have represented $C = \int_{x_j-\Delta}^{x_j} U(x)dx$ and $D = \int_{x_i}^{x_i+\Delta} U(x)dx$ in Fig. 2. It is obvious that $G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(i, j; \Delta)) = C - D > 0$ as soon as U is strictly increasing (or ϕ strictly convex), whatever the sign of the second derivative of U. The interpretation of the first derivative of ϕ as a utility function is used in the next section to clarify the real foundations of what we call in this paper the Shorrocks' issue.

4 A fresh look at the Shorrocks' issue

4.1 The Shorrocks' issue

If we denote again by $\boldsymbol{x}(i, j; \Delta)$ a distribution obtained from distribution $\boldsymbol{x}$ by means of a progressive transfer of an income amount $\Delta > 0$, we know that $G_\phi(\boldsymbol{x}) > G_\phi(\boldsymbol{x}(i, j; \Delta))$ by the strict convexity of ϕ. Here we introduce the notion of weak transfer sensitivity (Shorrocks and Foster 1987), related to 'the position of the transfer' in the distribution. According to this property, when the amount transferred and the income distance between the individuals involved in the transfer is fixed, the lower down the transfer is in the distribution, the more the inequality should be reduced.[4]

Definition 4 (Shorrocks and Foster 1987) An inequality index I is weakly transfer sensitive if and only if, for all $\boldsymbol{x} \in \mathscr{D}^n(\mu)$, we have $I(\boldsymbol{x}) - I(\boldsymbol{x}(i, j; \Delta)) > I(\boldsymbol{x}) - I(\boldsymbol{x}(k, l; \Delta))$ whenever $(x_j - x_i) = (x_l - x_k)$ and $x_i < x_k$.

It can be shown that G_ϕ is weakly transfer sensitive if and only if $\phi''' < 0$. Shorrocks was the first to point out the strange behaviour of indices G_ϕ with $\phi''' \geq 0$ (Shorrocks 1980, p. 623). It is illustrated by the following example, provided by Shorrocks and Foster (1987). Indices G_ϕ with $\phi''' = 0$ have the property that a transfer of £1 between millionaires whose incomes differ by £1 million has the same impact on inequality as a £1 transfer from an individual with an income of £1 million to someone with zero income. Worse still, the first transfer is more inequality-reducing than the second one if $\phi''' > 0$. The interpretation of the first derivative of ϕ as a utility function is a very intuitive way to rationalise the strange behaviour of indices G_ϕ with $\phi''' \geq 0$, what we call in this paper the Shorrocks' issue.

4.2 A new interpretation

If one assumes that $U = \phi'$ is a utility function, concavity ($U'' = \phi''' < 0$) is a necessary and sufficient condition for consistency of G_ϕ with the weak transfer sensitivity property. We have depicted in Fig. 3 the impact of the two progressive transfers $\boldsymbol{x}(i, j; \Delta)$ and $\boldsymbol{x}(k, l; \Delta)$ as described in Definition 4. We actually distinguish two cases. In Fig. 3a, we consider a concave utility function ($U'' = \phi''' < 0$) and in Fig. 3b, we consider a convex one ($U'' = \phi''' > 0$). One can observe in both graphs that $G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(i, j; \Delta)) = C - D$ and $G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(k, l; \Delta)) = E - F$. Without ambiguity, $G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(i, j; \Delta)) > G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(k, l; \Delta))$ when the utility function is concave, whereas inequality is reversed when the function is convex. If the utility is linear, one immediately deduces that $G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(i, j; \Delta)) =$

[4] Actually this property has been introduced by Kolm (1976) and called the principle of diminishing transfers.

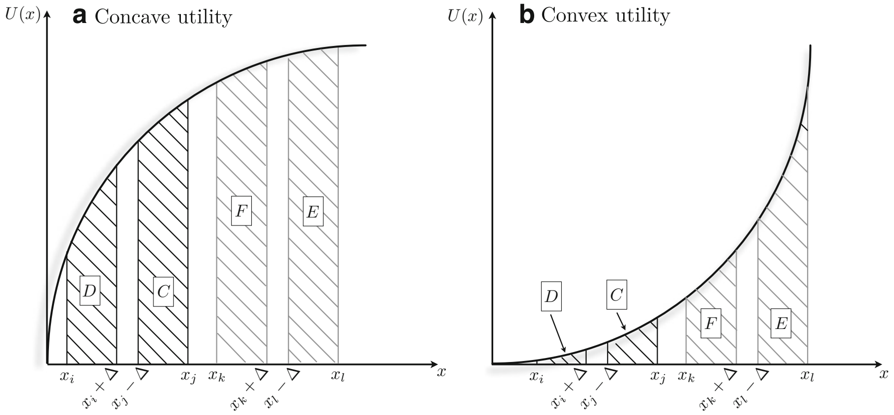

Fig. 3 Position of the transfer and inequality reducing impact

$G_\phi(\boldsymbol{x}) - G_\phi(\boldsymbol{x}(k, l; \Delta))$. In that case, the impact of a transfer is the same whatever its position on the income scale.

We can now discuss the ethical meaning of these different cases. Whatever their positions on the income scale, the two considered transfers have the same inequality reducing impact on incomes: The individuals involved in the transfers are at the same income distance, and the amounts transferred are also the same. Nevertheless, if the inequality measure also appraises the impact in terms of inequality of utility, it seems reasonable to assume that the more the 'utility distance' is reduced between the individuals involved in the transfer, the more the overall inequality is reduced. If the utility function is linear, the decrease in inequality of utility is the same regardless of the position of the transfer on the income scale. If utility is strictly concave, the lower down the progressive transfer is in the distribution, the more the utility inequality is reduced between the individuals involved in the transfer. Thus, if we also assume that the overall inequality measure is additively separable according to the situation of the individuals, the lower down the transfer is in the distribution, the more the value of the inequality index should be reduced. The sensitivity to the position of the transfer is of course reversed with a strictly convex utility function.

The previous description of the generalised entropies, with $U = \phi'$ a utility function, attests that they can be suitably interpreted as measures of income inequality, as well as measures of inequality of utility. This point will be discussed in great detail in the following section.

5 A fresh look at the Hansson–Sen issue

5.1 The Hansson–Sen issue

The Hansson–Sen issue—or paradox—applies to any standard ethical index based on a utilitarian social welfare function W_U, but it can be extended to any index based on a more general but Schur-concave social welfare function (see Sen 1978). At first

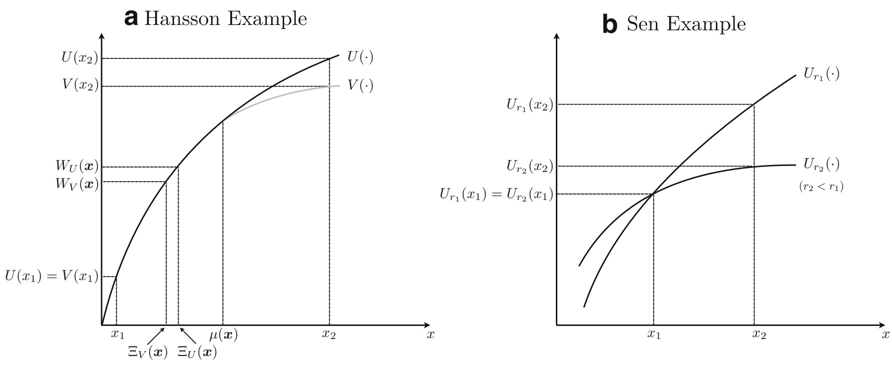

Fig. 4 Hansson–Sen paradox for the standard ethical inequality indices

sight, what is surprising is the discrepancy between, on the one hand, the magnitude of the criticisms raised by Hansson (1977) and Sen (1978) and, on the other hand, the widespread academic support for this approach. Indeed, Sen writes that 'the idea of measuring inequality on the basis of an overall social welfare function is fundamentally misconceived. It leads to a clearcut answer but to a question different from the one that was posed' (Sen 1978, page 428). Hansson even considers that the ethical indices 'are not conceptually linked to inequality at all, even if they accidentally have some relationship to this concept' (Hansson 1977, page 305).

It is now well-accepted that income inequality must be analysed through its impact on social welfare and that an inequality index should capture the social evaluator's views on the notion of inequality itself. Whereas most of the usual properties deal with the effects of changes in the income distribution—such that the principle of transfers—Hansson (1977) focuses on changes of the utility function:

> 'My proposal is the following: whenever the curve relating welfare to income flattens out so that from some point on the new curve is wholly below the old one, but still increasing and concave, then the measure of inequality shall decrease for the same distribution of nominal income. This *principle of reduced welfare from excess income* is related to the principle of transfers in the following way: when the curve changes the welfare of some of those who are better off decreases and even if no actual transfer takes place, there is a general shrinking of difference' (Hansson 1977, page 306).

Figure 4a is an illustration provided by the author to show that indices I_U violate this property. He considers a society consisting of two individuals with incomes x_1 and x_2, and social welfare is assessed from two utility functions, U and V. Function V is equal to U up to the mean μ, and more concave afterwards. One immediately observes that the distance from μ to $\Xi_V(\boldsymbol{x})$ is greater than the distance from μ to $\Xi_U(\boldsymbol{x})$. Hence I_U increases after such a concave transformation of U, whereas the principle of reduced welfare from excess income suggests that it should decrease. This is the problem identified by Hansson.

When discussing Hansson's criticism, Sen (1978) considers another example, in some sense more general. The author rightly argues that, as the utility function becomes more and more concave, the inequality index increases but, in the same way, the gap between the personal welfare values becomes smaller. The decrease in inequality of personal welfare values is in complete contradiction with the index assessment.[5] The author illustrates the situation with Fig. 4b (for relative indices), by normalising the utility functions such that $U(x_1) = V(x_1) = u$, with V more concave than U. Such a normalisation is possible through an affine transformation of the utility function, without affecting the value of the inequality index. Sen (1978) also considers the impact of a progressive transfer on the index. As the utility function becomes more concave, the impact of a transfer on the index is larger and larger, whereas the impact in terms of inequality of personal welfare goes on decreasing.

Nevertheless, the notion of difference in welfare should be accurately defined, which is not the case above. Indeed the utility function is defined up to an affine transformation and it does not make sense to compare differences $(U(x_2) - U(x_1))$ and $(V(x_2) - V(x_1))$ as these quantities are not invariant to such transformations. We propose to appeal to the following classical result in the theory of decision under risk, to clarify the paradox.

Remark 1 (see Pratt 1964, Theorem 1) The following three statements are equivalent:

(a) $V\left(U^{-1}(t)\right)$ is a strictly concave function of t.
(b) $I_V(\boldsymbol{x}) > I_U(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathscr{D}^n(\mu)$.
(c) $\frac{V(y)-V(x)}{V(w)-V(v)} < \frac{U(y)-U(x)}{U(w)-U(v)}$ for all $v, w, x, y \in \mathscr{D}$ with $v < w \leq x < y$.

Statement (a) defines the notion 'V more concave than U'. Hansson–Sen paradox is a consequence of (c), observing that the ratios are invariant to affine transformations of the utilities. Hansson's criticism can be interpreted as follows, with (c) equivalently written as:

$$\frac{V(y) - V(x)}{V'(w)} \leq \frac{U(y) - U(x)}{U'(w)}, \quad \text{for all } w, x, y \in \mathscr{D} \quad \text{with} \quad w \leq x < y. \tag{5.1}$$

Consider a distribution $\boldsymbol{x} = (x_1, x_2)$, let $w = x = x_1$ and $y = x_2$, and normalise the utility functions—by affine transformations—such that $U'(x_1) = V'(x_1)$. Equation (5.1) implies that the difference in welfare, relative to the marginal utility of the poorer individual which is assumed constant whatever the utility function, is lower with utility V than with U. As a contradiction, statement (b) states that the inequality is higher with I_V than with I_U. For the Sen example, two points of the utility function must be normalised. Recalling that the minimum possible income is $\underline{x}$, let $U(\underline{x}) = V(\underline{x})$ and $U(x_1) = V(x_1)$. Now from statement (c), let $v = \underline{x}$, $w = x = x_1$ and $y = x_2$. In that case, we reach the same conclusion as in the Hansson example,

[5] From an empirical perspective, one can argue that it is not meaningful to investigate the impact of a utility transformation on the index as most of the applications assume a fixed utility function (and thus a fixed level of inequality aversion). Nevertheless, some papers adopt a dual approach by considering that each country, for instance, is characterised by its own degree of inequality aversion (see Lambert et al. 2003). In that case, inequality comparisons require taking account of these different views on inequality aversion, which makes empirically relevant the Hansson–Sen paradox.

but the difference in welfare is here relative to the supplement of utility from the minimum the poorer individual beneficiates, which is assumed constant whatever the utility function.

Even if one assumes that only income inequality matters—and not also utility inequality, as assumed in this paper—the Hansson–Sen criticism cannot be entirely ruled out. We recall that the Kolm–Atkinson's ethical approach draws 'on the parallel with the formally similar problem of measuring risk in the theory of decision-making' (Atkinson 1970, p. 244). The Atkinson index is merely the analogue of the proportional risk premium (Pratt 1964). Strictly speaking, by applying this analogy, we emphasise that the (proportional) risk premium is not, in itself, a direct measure of risk as it can be approximated by the product of the (proportional) Arrow–Pratt risk aversion coefficient and an 'objective' risk measure—not affected by preferences—which is the variance. Coming back to our context, one can admit that the more concave the utility is, the more the social evaluator is inequality averse. Nevertheless, the statement that also adds 'the more concave the utility is, the more the income inequality is increased' is at least ambiguous and, at worst, a misinterpretation. As established in the following section, the generalised entropies are not affected by the Hansson–Sen criticism if they are properly normalised.

5.2 Generalised entropies as a solution

The ethical indices I_U have an intuitive interpretation as the 'cost of inequality' in terms of loss of total income. At the opposite, an interpretation in terms of 'measure of inequality' may be viewed as misconceived if we take the stance that U is a function reflecting personal welfare and that inequality-of-utility matters, as assumed in this paper. As illustrated by the so-called Hansson–Sen paradox, these notions are different: It is possible to observe a decrease in the inequality of utility and, at the same time, an increase in the income loss resulting from inequality. In this respect, we argue that an interpretation of the concavity of U as an indicator of inequality aversion is not appropriate. It makes more sense to relate, loosely speaking, inequality aversion to the slope of the utility function (U'). Indeed, for a given income distribution, the more the utility increases, the larger the inequality of welfare is. This is actually the idea—which will become more precise hereafter—behind the interpretation of the generalised entropies as non-standard ethical inequality indices.

By assuming that U is a strictly increasing function representing the social evaluator's preferences, let G_ϕ be defined with $\phi' = U$. It is worth emphasising that the consistency with the Pigou–Dalton principle of transfers is here ensured by $U' > 0$, not by $U'' < 0$. We also recall that indices G_ϕ are defined up to a transformation of ϕ by $\phi_*(x) = \alpha\,\phi(x) + \beta\,x + \gamma$ for all $x \in \mathscr{D}$, with $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$ or, equivalently, by an affine transformation of the utility function, $U_*(x) = \alpha\,U(x) + \beta$. In order to escape the Hansson–Sen criticism, G_ϕ should satisfy the following property: The more concave the utility function, the lower the value given by the index. Even if an affine transformation of the utility function does not modify the ranking of the distributions provided by the index, a problem is that its value is not invariant to such a transformation ($G_{\phi_*} = \alpha\,G_\phi$ when $\phi_*(x) = \alpha\,\phi(x) + \beta\,x + \gamma$). Hence, it does not

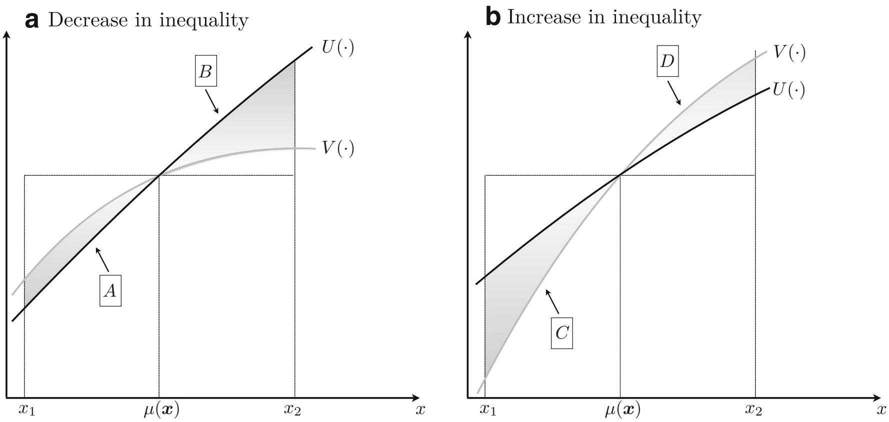

Fig. 5 Generalised entropy and concave transformation of the utility function

make sense to directly compare, without normalisation, the values of two different indices based on a more or less concave utility function, as illustrated in Fig. 5.

This figure is based on the geometric interpretation of G_ϕ as provided in Sect. 3.2. We consider a distribution $\boldsymbol{x} = (x_1, x_2)$ and two utility functions $U = \phi'$ and $V = \psi'$ such that V is strictly more concave than U. In Fig. 5a, one observes that $G_\psi(\boldsymbol{x}) < G_\phi(\boldsymbol{x})$. Indeed, $G_\phi(\boldsymbol{x}) = G_\psi(\boldsymbol{x}) + (A + B)$, where areas A and B are positive. In that case, a concave transformation of the utility function reduces the inequality. In Fig. 5b, the outcome is precisely the opposite: $G_\psi(\boldsymbol{x}) > G_\phi(\boldsymbol{x})$, observing that $G_\phi(\boldsymbol{x}) = G_\psi(\boldsymbol{x}) - (C + D)$ with positive areas C and D. The difference between these two situations can be easily understood: The function V is increasing less than U in Fig. 5a, and increasing more in Fig. 5b.

A simple normalisation can solve the problem. The requirements are: (1) the index becomes invariant to any affine transformation of the utility function, (2) the index remains consistent with the interesting properties of the unnormalised index (principle of transfers, principle of diminishing transfers…) and (3) the index escapes the Hansson–Sen criticism. Several normalisations are actually possible. We propose a normalisation based on the following Lemma.

Lemma 1 *Consider two strictly increasing utility functions U and V defined on $\mathscr{D}$ with values in $\mathbb{R}$ such that $V\left(U^{-1}(t)\right)$ is a strictly concave function of t. Moreover assume that there exist $x, y \in \mathscr{D}$ with $x < y$ such that $V'(x) \leq U'(x)$ and $V(y) = U(y)$. It follows that $V(z) > U(z)$ for all $z \in [x, y)$, and that $V(z) < U(z)$ for all $z > y$.*

This Lemma establishes sufficient conditions to be in a situation like in Fig. 5a, where a decrease in inequality follows a concave transformation of the utility function. The fixed-point y in Lemma 1 can be chosen such as $y = \mu$, the mean income. Then, in order to have V above U up to μ and below afterwards, a sufficient condition is to have $V'(x) \leq U'(x)$ where x is the lowest considered income in distribution $\boldsymbol{x}$. A solution to make this income independent from $\boldsymbol{x}$ is to let $x = \underline{x}$, the lower bound of the set of admissible incomes $\mathscr{D} = [\underline{x}, \overline{x}]$. We then suggest letting $V'(\underline{x}) = U'(\underline{x})$. To that end, we propose the following normalisation of the utility function: Let $U_*(x) = \alpha\, U(x) + \beta$ with α and β such that $U_*{}'(\underline{x}) = 1$ and $U_*(\mu) = c$ where c is a constant. Moreover we know that $U_* = \phi_*{}'$ so that $\phi_*(x) = \alpha\, \phi(x) + \beta\, x + \gamma$, and thus $G_{\phi_*} = \alpha\, G_\phi$. It follows that, in our case, $\alpha = 1/U'(\underline{x})$.

Inequality aversion can be associated with the slope of the utility function but, regarding the cardinal nature of U, this slope can be meaningfully interpreted only in relative terms (see statement (c) in Remark 1). Here, we propose to choose as reference the slope of U at the minimal income. We finally obtain the following class of normalised indices.

Definition 5 Consider G_ϕ as presented in Definition 2. For all $\boldsymbol{x} \in \mathscr{D}^n(\mu)$, the class of the normalised and generalised entropies can be written as:

$$I_\phi(\boldsymbol{x}) = \frac{1}{U'(\underline{x})} G_\phi(\boldsymbol{x}), \tag{5.2}$$

where $U = \phi'$ is the utility function representing the social evaluator's preferences. The lower bound $\underline{x}$ of the set of admissible incomes $\mathscr{D} = [\underline{x}, \overline{x}]$ can be replaced by the lowest considered income in the distributions under comparison.[6]

This class of inequality indices inherits all the properties of indices G_ϕ. A new property is the invariance of I_ϕ to any transformation of ϕ by $\phi_*(x) = \alpha\, \phi(x) + \beta\, x + \gamma$ or, equivalently, by an affine transformation of the utility function ($I_\phi = I_{\phi_*}$). Finally, as established by Proposition 4 below, this class of indices escapes the Hansson–Sen criticism: The more concave the utility function, the lower the value given by the index. Notice that it makes sense now to compare the values of two different indices, respectively based on a more or less concave utility function, because they are invariant to affine transformations of the utility functions.

Proposition 4 *Let $V = \psi'$ and $U = \phi'$. Statement* (a) *below implies statement* (b)*, but the converse is false:*

(a) $V\left(U^{-1}(t)\right)$ *is a strictly concave function of t.*
(b) $I_\psi(\boldsymbol{x}) < I_\phi(\boldsymbol{x})$ *for all $\boldsymbol{x} \in \mathscr{D}^n(\mu)$.*

The fact that (b) does not imply (a) is not surprising: In contrast to the standard ethical indices I_U, inequality aversion for the generalised entropies is not directly related to the concavity of the utility function but to its slope. Hence a concave transformation of the utility function is sufficient to imply a decrease of the normalised indices I_ϕ, but it is not necessary.

[6] This definition immediately extends to the general case of unequal-mean distributions. We illustrate here the normalisation for the relative generalised entropies G_r (see Appendix A), where admissible incomes are the strictly positive real numbers, recalling that $G_r(\boldsymbol{x}) = G_{\phi_r}(\hat{\boldsymbol{x}})$ with $\hat{\boldsymbol{x}} = \boldsymbol{x}/\mu(\boldsymbol{x})$ and ϕ_r as defined in (A.3). Consider for instance that the distributions under comparison are $\boldsymbol{x} = (1, 3)$ and $\boldsymbol{y} = (2, 18)$, such that $\hat{\boldsymbol{x}} = (0.5, 1.5)$ and $\hat{\boldsymbol{y}} = (0.2, 1.8)$. The normalisation is here concerned by the lowest observable reduced income, namely 0.2. Hence one can choose the normalised index $G_{\phi_r}/U'(0.2)$. For the absolute generalised indices, the same reasoning applies to the centred incomes.

6 Discussion

The aim of this paper is to revisit the well-known class of inequality indices called the generalised entropies, through a new interpretation: We argue that the generating function ϕ can be interpreted as the antiderivative of the utility function representing social evaluator's preferences. In this framework, these indices are consistent with the Pigou–Dalton principle of transfers if and only if the antiderivative of the utility function is convex, or equivalently if and only if the utility function is increasing. Then, they satisfy the principle of diminishing transfers if and only if the utility function is concave. The originality of this approach is to separate the standard notion of diminishing marginal utility to the degree of inequality aversion of the social evaluator.

This interpretation clarifies the strange behaviour of the generalised entropies, as identified by Shorrocks (1980) for some values of the inequality aversion parameter, and what we call the Shorrocks issue: It appears with convex utility functions. In that case, the higher a progressive transfer is on the income scale, the more the inequality of personal welfare decreases between the individuals involved in the transfer, and thus the more the inequality as measured by the inequality index decreases. This is the reason why, in that case, the principle of diminishing transfers is violated.

This interpretation also provides a solution to escape the Hansson–Sen paradox, which affects ethical inequality indices. This paradox states that a concave transformation of the utility function leads to a decrease in inequality of personal welfare, and thus presumes that it should also imply an inequality reduction as measured by the indices, whereas the opposite implication is observed. We establish that a normalised version of the generalised entropies can escape the paradox. A normalisation is necessary because a generalised entropy index is not invariant to an affine transformation of the utility function. We propose to divide the index by the marginal utility at the lower bound of the set of admissible incomes, or at the lowest income as observed in the distributions under comparison. Nevertheless, the value given by such an index cannot be directly interpreted, which is probably the price to pay to escape the paradox.

A. Inequality indices with unequal mean distributions

When no restriction is placed on the mean income, indices are obtained from Definitions 1 and 2 by normalising the income distributions. We denote by $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ the reduced distribution of $\boldsymbol{x} \in \mathscr{D}^n$ where $\hat{x}_i = x_i/\mu(\boldsymbol{x})$, and by $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ the centered distribution where $\tilde{x}_i = x_i - \mu(\boldsymbol{x})$. By using Definition 1, the so-called Atkinson–Kolm–Sen class of relative ethical indices (Kolm 1969; Atkinson 1970; Sen 1973) is defined by $I_r(\boldsymbol{x}) = I_{U_r}(\hat{\boldsymbol{x}})$, where $r < 2$ and:

$$U_r(x) = \begin{cases} \frac{1}{r-1}\, x^{r-1}, \text{ if } & r \neq 1, \\ \ln x + 1, \text{ if } & r = 1. \end{cases} \tag{A.1}$$

It follows that $I_r(\boldsymbol{x}) = 1 - \Xi_{U_r}(\boldsymbol{x})/\mu(\boldsymbol{x})$. Equivalently, the Kolm–Pollak class of absolute ethical indices (Kolm 1969; Pollak 1971) is defined by $I_a(\boldsymbol{x}) = I_{U_a}(\tilde{\boldsymbol{x}})$, where $a > 0$ and:

$$U_a(x) = \begin{cases} -\frac{1}{a}\, e^{-a\,x}, & \text{if} \quad a \neq 0, \\ x, & \text{if} \quad a = 0. \end{cases} \tag{A.2}$$

Hence $I_a(\boldsymbol{x}) = \mu(\boldsymbol{x}) - \Xi_{U_a}(\boldsymbol{x})$. A relative (resp. absolute) ethical index measures the relative (resp. absolute) income loss resulting from inequality. We also point out that, by definition, the utility functions U_r and U_a are strictly increasing and concave.

By applying Definition 2, the class of the relative generalised entropies (Bourguignon 1979; Cowell 1980; Shorrocks 1980, 1984) can be written as $G_r(\boldsymbol{x}) = G_{\phi_r}(\hat{\boldsymbol{x}})$ with $r \in \mathbb{R}$ and:

$$\phi_r(x) = \begin{cases} \frac{1}{r(r-1)}\, x^r, & \text{if} \quad r \neq 0, 1, \\ x\, \ln x, & \text{if} \quad r = 1, \\ -\ln x, & \text{if} \quad r = 0. \end{cases} \tag{A.3}$$

Finally the class of the absolute generalised entropies (Bosmans and Cowell 2010), can be written as $G_a(\boldsymbol{x}) = G_{\phi_a}(\tilde{\boldsymbol{x}})$ with $a \in \mathbb{R}$ and:

$$\phi_a(x) = \begin{cases} \frac{1}{a^2}\, e^{-ax}, & \text{if} \quad a \neq 0, \\ \frac{1}{2}x^2, & \text{if} \quad a = 0. \end{cases} \tag{A.4}$$

Whatever the values of r and a, ϕ_r and ϕ_a are always strictly convex.

The Atkinson–Kolm–Sen (resp. Kolm–Pollak) class is ordinally equivalent to a part of the relative (resp. absolute) generalised entropies class. Indeed $I_r = 1 - \left[(r-1)(r-2)G_{r-1} + 1\right]^{1/(r-1)}$ if $r \neq 1$ and $I_1 = 1 - \exp(-G_0)$, so that I_r is an increasing monotonic transformation of G_{r-1} as soon as $r < 2$. Equivalently $I_a = \ln(a^2 G_a + 1)/a$, so that I_a is an increasing monotonic transformation of G_a when $a > 0$. This relationship is well-established in the literature. In this paper we provide a new relationship, more transparent, through an original interpretation of the generalised entropies.

B. Proofs

Proof of Proposition 1 By definition, $I_U(\boldsymbol{x}) = \mu - U^{-1}\left(\frac{1}{n}\sum_{i=1}^n U(x_i)\right)$. Hence we can write $I_U(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n \left[U^{-1}\left(U(x_i)\right) - U^{-1}\left(\mu(U)\right)\right]$. Recalling that $\sum_{i=1}^n \left[U(x_i) - \mu(U)\right] = 0$, one obtains the desired result. □

Proof of Proposition 2 We know that $G_\phi(\boldsymbol{x}) = D_\phi(\boldsymbol{x} \| \mu \mathbf{1}_n)$ for any $\boldsymbol{x} \in \mathscr{D}^n(\mu)$. The proof is based on the Legendre duality which characterises Bregman divergences (see Boissonnat et al. 2010). By definition, $\phi : \mathscr{D} \to \mathbb{R}$. The conjugate function of ϕ, denoted by ϕ^*, is defined as follows:

$$\phi^*(t) = \sup_{x \in \mathscr{D}} \left(tx - \phi(x)\right). \tag{B.1}$$

Recalling that ϕ is continuously differentiable and strictly convex, it follows that we also have $\phi^*(t) = t\,(\phi')^{-1}(t) - \phi\left((\phi')^{-1}(t)\right)$. Moreover, it can be shown that $(\phi')^{-1} = (\phi^*)'$ and $(\phi^*)^{-1} = \phi'$. By letting $F = D_{\phi^*}(\phi'(\mu), \ldots, \phi'(\mu) \| \phi'(x_1), \ldots, \phi'(x_n))$ we first notice that, by definition:

$$F = \frac{1}{n} \sum_{i=1}^{n} \left[(\phi^*)(\phi'(\mu)) - (\phi^*)(\phi'(x_i)) - (\phi'(\mu) - \phi'(x_i))(\phi^*)'(\phi'(x_i)) \right]. \quad \text{(B.2)}$$

Moreover $(\phi^*)(\phi'(x)) = \phi'(x)x - \phi(x)$, and $(\phi'(\mu) - \phi'(x))(\phi^*)'(\phi'(x)) = (\phi'(\mu) - \phi'(x))x$ by applying the property $(\phi^*)^{-1} = \phi'$. It follows that:

$$F = \frac{1}{n} \sum_{i=1}^{n} \left[\phi'(\mu)\mu - \phi(\mu) - \phi'(x_i)x_i + \phi(x_i) - (\phi'(\mu) - \phi'(x_i))x_i \right]. \quad \text{(B.3)}$$

After simplification, one obtains $F = \frac{1}{n} \sum_{i=1}^{n} [\phi(x_i) - \phi(\mu)] = D_\phi(\boldsymbol{x} \| \mu \mathbf{1}_n)$. If now we define U as $U = \phi'$, one immediately observes that $\phi^* = \varphi$ with φ as stated in the proposition. Hence we also have from the definition of F that $F = D_\varphi(U(\mu)\mathbf{1}_n \| \boldsymbol{U})$, which completes the proof. □

Proof of Proposition 3 Let $\boldsymbol{x} \in \mathscr{D}^n(\mu)$. We know that, by definition, $G_\phi(\boldsymbol{x}) = D_\phi(\boldsymbol{x} \| \mu \mathbf{1}_n) = \frac{1}{n} \sum_{i=1}^{n} \left[\phi(x_i) - \phi(\mu) - (x_i - \mu)\phi'(\mu) \right]$. By denoting $U = \phi'$ and using the fundamental theorem of calculus, one has $\phi(x_i) - \phi(\mu) = -\int_{x_i}^{\mu} U(x)dx$ and $-(x_i - \mu)\phi'(\mu) = \int_{x_i}^{\mu} U(\mu)dx$, from which the result is deduced. The function ϕ is strictly convex, hence U is strictly increasing. □

Proof of Lemma 1 Notice first that:

$$\frac{d}{dt} V\left(U^{-1}(t)\right) = \frac{V'\left(U^{-1}(t)\right)}{U'\left(U^{-1}(t)\right)}. \quad \text{(B.4)}$$

If $V\left(U^{-1}(t)\right)$ is a strictly concave function of t then, for all $s, t \in \mathbb{R}$ with $s < t$, we have:

$$\frac{V'\left(U^{-1}(s)\right)}{U'\left(U^{-1}(s)\right)} > \frac{V'\left(U^{-1}(t)\right)}{U'\left(U^{-1}(t)\right)}. \quad \text{(B.5)}$$

Let $s = U(x)$. After substitution in (B.5), and assuming that $(0 <)\ V'(x) \leq U'(x)$, it follows that:

$$\frac{V'\left(U^{-1}(t)\right)}{U'\left(U^{-1}(t)\right)} < 1, \quad \forall t > U(x). \quad \text{(B.6)}$$

By letting $t = U(z)$ with $z > x$, one obtains $V'(z) < U'(z)$ for all $z > x$. If moreover there exists $y \in \mathscr{D}$ such that $x < y$ and $V(y) = U(y)$, one immediately obtains the desired result. □

Proof of Proposition 4 We first demonstrate that (a) implies (b). Let $V = \psi'$ and $U = \phi'$, and choose any distribution $\boldsymbol{x} \in \mathscr{D}^n(\mu)$. One first remarks that:

$$I_\phi = G_{\phi_\star}, \text{ where } \phi_*(x) = \frac{1}{U'(\underline{x})}\phi(x) + \beta x + \gamma, \text{ for any } \beta, \gamma \in \mathbb{R}. \quad \text{(B.7)}$$

It follows that $\phi_*''(\underline{x}) = U_*'(\underline{x}) = 1$. By choosing $\beta = c - U(\mu)/U'(\underline{x})$ and $\gamma = 0$ it follows that, in addition, $U_*(\mu) = c$. Equivalently we have:

$$I_\psi = G_{\psi_\star}, \text{ where } \psi_*(x) = \frac{1}{V'(\underline{x})}\psi(x) + \delta\, x + \eta, \text{ for any } \delta, \eta \in \mathbb{R}. \tag{B.8}$$

Hence $\psi_*''(\underline{x}) = V_*'(\underline{x}) = 1$ and, by choosing $\delta = c - V(\mu)/V'(\underline{x})$ with $\eta = 0$, we have $V_*(\mu) = c$. By using Proposition 3, we know that:

$$G_{\phi_\star}(\boldsymbol{x}) = \sum_{i\,:\,x_i \le \mu} \int_{x_i}^{\mu} [U_*(\mu) - U_*(x)]\, dx + \sum_{i\,:\,x_i > \mu} \int_{\mu}^{x_i} [U_*(x) - U_*(\mu)]\, dx, \tag{B.9}$$

and:

$$G_{\psi_\star}(\boldsymbol{x}) = \sum_{i\,:\,x_i \le \mu} \int_{x_i}^{\mu} [V_*(\mu) - V_*(x)]\, dx + \sum_{i\,:\,x_i > \mu} \int_{\mu}^{x_i} [V_*(x) - V_*(\mu)]\, dx. \tag{B.10}$$

Because $U_*(\mu) = V_*(\mu) = c$, by substracting (B.10) from (B.9), one obtains:

$$\begin{aligned} G_{\phi_\star}(\boldsymbol{x}) - G_{\psi_\star}(\boldsymbol{x}) = & \sum_{i\,:\,x_i \le \mu} \int_{x_i}^{\mu} [V_*(x) - U_*(x)]\, dx \\ & + \sum_{i\,:\,x_i > \mu} \int_{\mu}^{x_i} [U_*(x) - V_*(x)]\, dx, \end{aligned} \tag{B.11}$$

From Lemma 1, recalling that $U_*'(\underline{x}) = V_*'(\underline{x})$ and $U_*(\mu) = V_*(\mu)$, we deduce that $V_*(x) > U_*(x)$ for all $x < \mu$ and $V_*(x) < U_*(x)$ for all $x > \mu$. From (B.11) we have $G_{\phi_\star}(\boldsymbol{x}) - G_{\psi_\star}(\boldsymbol{x}) > 0$, or equivalently $I_\psi(\boldsymbol{x}) < I_\phi(\boldsymbol{x})$. Hence statement (a) implies statement (b). To prove that the converse implication is false, consider two increasing functions U and V such that $U'(\underline{x}) = V'(\underline{x})$, $U(\mu) = V(\mu)$, $V(x) > U(x)$ for all $x \in [\underline{x}, \mu)$, $V(x) < U(x)$ for all $x \in (\mu, \overline{x}]$, and such that U is concave. It follows that $I_\psi(\boldsymbol{x}) < I_\phi(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathscr{D}$, where $V = \psi'$ and $U = \phi'$ (see Fig. 1). Since V is not necessarily concave it cannot be, in all cases, a concave transformation of U (which is already concave). □

References

Atkinson A (1970) On the measurement of inequality. J Econ Theory 2:244–263

Atkinson A, Brandolini A (2015) Unveiling the ethics behind inequality measurement: Dalton's contribution to economics. Econ J 125:209–234

Boissonnat J, Nielsen F, Nock R (2010) Bregman Voronoi diagrams. Discrete Comput Geom 44:281–307

Bosmans K, Cowell F (2010) The class of absolute decomposable inequality measures. Econ Lett 109:154–156

Bourguignon F (1979) Decomposable income inequality measures. Econometrica 47:901–920

Cowell F (1980) On the structure of additive inequality measures. Rev Econ Stud 47:521–531

Dalton H (1920) The measurement of the inequality of incomes. Econ J 30:348–361
Dasgupta P, Sen A, Starrett D (1973) Notes on the measurement of inequality. J Econ Theory 6:180–187
Hansson B (1977) The measurement of social inequality. In: Butts R, Hintikka J (eds) Foundational problems in the special sciences. Reidel Publishing Company, Dordrecht
Hardy G, Littlewood J, Polya G (1934) Inequalities. Cambridge University Press, London
Kolm S (1976) Unequal inequalities II. J Econ Theory 13:82–111
Kolm S-C (1969) The optimal production of social justice. In: Margolis J, Guitton H (eds) Public economics. Macmillan, London, pp 145–200
Lambert P, Millimet D, Slottje D (2003) Inequality aversion and the natural rate of subjective inequality. J Public Econ 87:1061–1090
Magdalou B, Nock R (2011) Income distributions and decomposable divergence measures. J Econ Theory 146:2440–2454
Pigou A (1912) Wealth and welfare. Maxmillan, London
Pollak R (1971) Additive utility functions and linear Engel curves. Rev Econ Stud 38:401–414
Pratt J (1964) The class of additively decomposable inequality measures. Econometrica 32:122–136
Sen A (1973) On economic inequality. Clarendon Press, Oxford
Sen A (1976) Welfare inequalities and Rawlsian axiomatics. Theor Decis 7:243–262
Sen A (1978) Ethical measurement of inequality: some difficulties. In: Krelle W, Shorrocks A (eds) Personal income distribution. North-Holland, Amsterdam, pp 81–94
Shorrocks A (1980) The class of additively decomposable inequality measures. Econometrica 48:613–625
Shorrocks A (1984) Inequality decomposition by population subgroups. Econometrica 52:1369–1385
Shorrocks A, Foster J (1987) Transfer sensitive inequality measures. Rev Econ Stud 54:485–497