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Egalitarianism, utilitarianism, and the Nash bargaining solution

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Abstract

A bargaining solution satisfies *egalitarian–utilitarian monotonicity* (EUM) if the following holds under feasible-set-expansion: a decrease in the value of the Rawlsian (resp. utilitarian) objective is accompanied by an increase in the value of the utilitarian (resp. Rawlsian) objective. A bargaining solution is *welfarist* if it maximizes a symmetric and strictly concave social welfare function. Every 2-person welfarist solution satisfies EUM, but for $n \geq 3$ every n-person welfarist solution violates it. In the presence of other standard axioms, EUM characterizes the Nash solution in the 2-person case, but leads to impossibility in the n-person case.

1 Introduction

Egalitarianism and utilitarianism are key concepts of distributive justice, and the tension between them is the subject of vibrant debates between the supporters of these opposing schools of thought (see, e.g., the book by Fleurbaey et al. 2008). Given this tension, it is natural to look for a "golden path": a criterion that respects, simultaneously, a minimal degree of both utilitarianism and egalitarianism. In the present paper I investigate one possible "golden path" in the context of Nash's (1950) bargaining theory.

In Nash's bargaining model a generic problem is denoted by S, and it consists of non-negative utility vectors from which a single vector needs to be selected.[1] A *bargaining solution* assigns a unique point for every problem, $f(S) \in S$. Let $U(x) \equiv \sum x_i$ and $R(x) \equiv \min_i x_i$. The *utilitarian solution* maximizes U on every problem and the *egalitarian solution* selects for every problem the maximal feasible point whose

[1] Further assumptions on the structure of a problem will be specified in Sect. 2.

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coordinates are equal to one another; under standard assumptions on the structure of the problem, this equal-coordinates point maximizes R.[2]

Typically, $\text{argmax}_{s \in S} U(s) \cap \text{argmax}_{s \in S} R(s) = \emptyset$. If one finds merits in both U and R, then the question arises of how to create a compromise between them. Any solution which is designed to create such a compromise should, I believe, adhere to the following requirement: For all $S \subset T$:

- $R(f(T)) < R(f(S)) \Rightarrow U(f(T)) > U(f(S))$;
  and
- $U(f(T)) < U(f(S)) \Rightarrow R(f(T)) > R(f(S))$.

That is, when opportunities expand, a decrease in the value of the objective-function of one criterion (utilitarianism or egalitarianism) must be accompanied by an increase in the value of the objective-function of the other criterion. Say that a solution satisfies *egalitarian–utilitarian monotonicity* (EUM) if it meets this requirement.

The justification for EUM is that if one recognizes the validity of the positions of both utilitarians and egalitarians, and if no consideration justifies hurting both utilitarians and egalitarians simultaneously, then it would not be desirable to use a solution that, upon an enlargement of the feasible set, makes both constituencies unhappy. If one makes the utilitarians unhappy, there should be a benefit to the egalitarians, and vice versa.

Though EUM's building blocks are utilitarianism and egalitarianism—two *specific* welfarist criteria—in the 2-person case it has the following general aspect: every bargaining solution which is the maximizer of *some* symmetric and strictly concave social welfare function—hereafter, a *welfarist* solution—satisfies EUM. By contrast, if there are more than two players, every welfarist solution violates EUM. Another distinction between the 2-person case and the n-person case is that once EUM is combined with other standard axioms, a characterization of the Nash solution is obtained in the 2-person case, but in the n-person case this combination results in impossibility.[3,4]

The connections among egalitarianism, utilitarianism, and the Nash bargaining solution have been studied ever since the pioneering work of Shapley (1969), who showed that the Nash solution is the only solution that maximizes the utilitarian and Rawlsian objectives simultaneously for an appropriate rescaling of the individual utilities.[5] Mariotti (1999) showed that the Nash solution is characterized by scale covariance and *Suppes-Sen dominance* (SSD). A vector x *Suppes-Sen dominates* (SS-dominates) another vector y if there is a permutation of x such that the permuted vector, call it x', Pareto-dominates y; namely, $x' > y$.[6] SSD requires that the solution

[2] The letter "R," which denotes the minimum operator, stands for "Rawls". In his *Theory of Justice* Rawls (1971) promoted the view that a just society should maximize the well-being of its least well-off member. In the present model, this principle translates to maximizing the utilities-minimum.

[3] Whenever I write "n-person case" or "n-person bargaining", I mean $n \geq 3$.

[4] That there are substantial differences between 2-person and n-person bargaining is known ever since the work of Shapley (1969). Recently, Karagözoğlu and Rachmilevitch (2018) showed, in the context of a model different from the one studied here, that it may matter whether the number of bargainers is greater than 4 or not.

[5] Discussions of Shapley's approach can be found in Yaari (1981) and Rachmilevitch (2015).

[6] This notion of dominance is due to Suppes (1966) and Sen (1970).

point not be SS-dominated by any other feasible point. SSD expresses the view that it does not matter which person enjoys what payoff, it only matters what payoffs are being distributed; clearly, it is a weakening of both utilitarianism and egalitarianism. Anbarci and Sun (2011b) showed that Mariotti's result can be improved by weakening SSD to an axiom called *weak SSD*, which requires that there be no feasible point that dominates the solution point both in the SS-sense and in an additional sense, called *equity dominance*. The present paper contributes to this literature by providing a further ethical foundation for the 2-person Nash bargaining solution.

The rest of the paper is organized as follows. Definitions and axioms are presented in Sect. 2. The connection between EUM and welfarism is studied in Sect. 3. In Sect. 4 I show how the combination of EUM with other standard axioms results in a characterization of the Nash solution in the 2-person case, and in impossibility in the n-person case. Both egalitarianism and utilitarianism are based on the idea that utilities of different individuals are comparable; hence, EUM is also based on that assumption. This may seem inconsistent with the Nash solution, which is typically thought of as precluding such comparisons. In Sect. 5 I address this issue and argue that the discrepancy can be settled. There, I also address the issue of symmetry, and highlight the important role it has in my analysis. In Sect. 6 I conclude.

2 Definitions and axioms

A *problem* is a compact and convex set $S \subset \mathbb{R}^n_+$ that contains the origin $\mathbf{0} \equiv (0, \dots, 0)$, contains a vector with all coordinates strictly positive, and is comprehensive. Comprehensiveness means that if $x \in S$ then $y \in S$, for every y that satisfies $\mathbf{0} \leq y \leq x$.[7]

Let $\Delta_r \equiv \{x \in \mathbb{R}^n_+ : \sum x_j \leq r\}$. The bargaining domain in this paper is:

$$\{S : |\text{argmax}_{s \in S} U(S)| = 1\} \cup \{\Delta_r : r > 0\}.$$

That is, the bargaining domain consists of problems for which one of the following applies: (1) the utilitarian point is unique, or (2) the problem is a simplex.

The *weak Pareto frontier of* S is $WP(S) \equiv \{x \in S : y > x \Rightarrow y \notin S\}$ and its *strong Pareto frontier* is $P(S) \equiv \{x \in S : y \gneqq x \Rightarrow y \notin S\}$. The best that player i can hope for in S—his *ideal payoff* in S—is $a_i(S) \equiv \max\{s_i : s \in S\}$. The point $a(S) \equiv (a_1(S), \dots, a_n(S))$ is called the *ideal point* of S.

A *solution* is a map f that assigns a unique point for every problem, $f(S) \in S$. The following are important solutions. The *Nash solution* N (Nash 1950) maximizes $\Pi_i s_i$ over $s \in S$ for every S. The *egalitarian solution* E (Kalai 1977) assigns for every S the maximal feasible point of the form $(\lambda, \lambda, \dots, \lambda)$. The *utilitarian solution*, u, assigns for every S the maximizer of $U(s)$ over $s \in S$; on a simplex Δ_r, for which the utility-sum-maximizer is not unique, this solution is defined to select $(\frac{r}{n}, \dots, \frac{r}{n})$. The *Kalai–Smorodinsky solution* KS (Kalai and Smorodinsky 1975) assigns for every S the point $\lambda a(S)$, where λ is the maximum possible. The *i-th dictatorial solution*,

[7] Vector inequalities are as follows: uRv if and only if $u_i R v_i$ for all i, for both $R \in \{\geq, >\}$; $u \gneqq v$ if and only if $u \geq v$ and $u \neq v$. Given a non-empty set $X \subset \mathbb{R}^n_+$, the smallest comprehensive problem containing it is denoted $\text{comp}(X)$.

D^i, assigns for every S the vector whose i-th coordinate is $a_i(S)$ and every other coordinate is zero. The *disagreement solution*, D, assigns for every problem the origin. The *midpoint solution* assigns for each S the point $m(S) \equiv \frac{1}{n}a(S)$.

The following are axioms a solution may satisfy. In their statements, S and T are arbitrary problems. Since the axioms are well-known, I skip a discussion thereof.

Pareto optimality (PO): $f(S) \in P(S)$.

Weak Pareto optimality (WPO): $f(S) \in WP(S)$.

Symmetry (SY): If $S = \pi S$ for every permutation π, then $f_1(S) = \cdots = f_n(S)$.[8]

Suppes-Sen dominance (SSD): There does not exist an $s \in S$ and a permutation π, such that $\pi s > f(S)$.

Scale covariance (S.COV): $f(l \circ S) = l(f(S))$, for every vector of positive linear transformation $l = (l_1, \ldots, l_n)$.

Conflict-freeness (CF): $a(S) \in S \Rightarrow f(S) = a(S)$.

Independence of irrelevant alternatives (IIA): $[f(T) \in S \subset T] \Rightarrow f(S) = f(T)$.

Restricted Independence of irrelevant alternatives (RIIA): $[f(T) \in S \subset T]\&[\exists r \leq 1 \ s.t \ a(S) = ra(T)] \Rightarrow f(S) = f(T)$.

3 Welfarism

A symmetric, strictly concave and non-decreasing function $W \colon \mathbb{R}^n_+ \to \mathbb{R}$ is called a *social welfare function* (SWF).[9] A solution f is *welfarist* if there exists a SWF W such that $f(S) \equiv \text{argmax}_{s \in S} W(s)$. A solution which is not welfarist is called *non-welfarist*.

Proposition 1 *Let $n = 2$. Let f be a welfarist solution. Then f satisfies egalitarian–utilitarian monotonicity.*

Proof Let $n = 2$, let f be a welfarist solution whose underlying SWF is W and let $S \subset T$. Let $A \equiv f(S)$. Suppose first that A is off the 45° line; wlog, to its north–west.

Case 1: $A > \mathbf{0}$. Let I be W's indifference curve that passes through A. Let $B = \pi A$, were π is the non-identity permutation. Note that the line that passes through A and B has slope -1 and it intersects I at A and at B. If $f_1(T) \in (A_1, B_1)$, we are done, since this implies that $R(f(T)) > R(f(S))$. If not, then $U(f(T)) > U(f(S))$.

Case 2: $A_i = 0$ for some i. If $f(T) > \mathbf{0}$, then $R(f(T)) > R(f(S))$. If not, then $U(f(T)) > U(f(S))$.

If A is on the 45°-line then the hyperplane through A with slope -1 separates S and the upper contour set of I; this implies that EUM's requirement must hold as one moves from S to T (Fig. 1). □

[8] Given a permutation π on $\{1, \ldots, n\}$, $\pi S \equiv \{(s_{\pi(1)}, \ldots, s_{\pi(n)}) : s \in S\}$. A problem S that satisfies $S = \pi S$ for every permutation π is called *symmetric*.

[9] The functions U and R do not adhere to this definition, as they are not strictly concave. They can be viewed, however, as limit cases: they correspond to the limits $\rho \to 1$ and $\rho \to -\infty$ of $[\sum_i (x_i)^\rho]^{1/\rho}$.

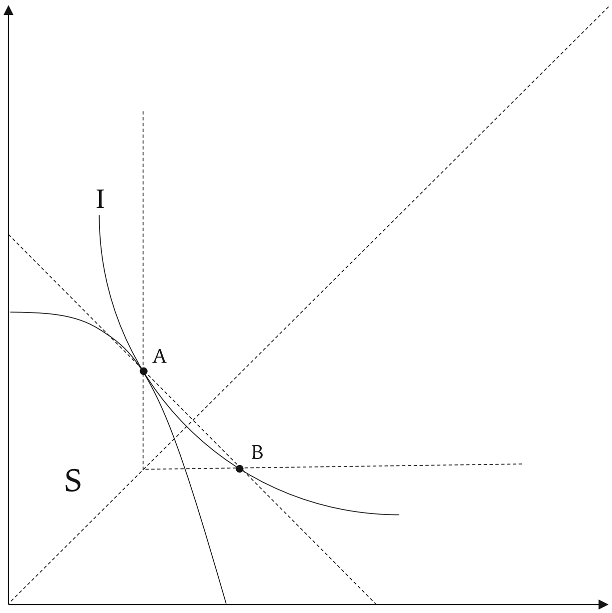

Fig. 1 Illustration of Case 1 in Proposition 1's proof

The converse of Proposition 1 is not true. Moreover, even in the class of solutions that satisfy PO and SY (of which the welfarist solutions are a sub-class), there are non-welfarist EUM-satisfying solutions.[10]

Proposition 2 *Let $n = 2$. Then there exists a non-welfarist solution that satisfies Pareto optimality, symmetry, and egalitarian–utilitarian monotonicity.*

Proof Let $\hat{f}$ be the solution that assigns to each S the maximizer of $(\frac{A(S)}{1+A(S)})ab + (\frac{1}{1+A(S)})(a+b)$ over $(a, b) \in S$, where $A(S)$ is the area of S. Clearly this solution satisfies PO and SY. I will now prove that it satisfies EUM. Let $S \subset T$, let $x = \hat{f}(S)$ and $y = \hat{f}(T)$. Assume by contradiction that $(R(y), U(y)) \lneqq (R(x), U(x))$. Clearly, $y_1 y_2 > x_1 x_2$. Assume, wlog, that $y_1 \geq y_2$.

Suppose first that $x_1 \geq x_2$. In this case $x_2 \geq y_2$ and $y_1 > x_1$. Also, $x_2 = y_2$ is impossible, as it would imply $y_1 + y_2 > x_1 + x_2$. Therefore $x_2 > y_2$. Now, $x_1 + x_2 \geq y_1 + y_2$ implies $(x_1 + x_2)^2 \geq (y_1 + y_2)^2$, or $x_1^2 + 2x_1 x_2 + x_2^2 \geq y_1^2 + 2y_1 y_2 + y_2^2$. Since $y_1 y_2 > x_1 x_2$ it follows that $x_1^2 - 2x_1 x_2 + x_2^2 \geq y_1^2 - 2y_1 y_2 + y_2^2$, or $x_1 - x_2 > y_1 - y_2$. Since $y_1 > x_1$ it follows that $y_2 > x_2$—a contradiction. The case $x_2 \geq x_1$ is treated similarly. Therefore, EUM is satisfied.

Finally, it is easy to see that $\hat{f}$ violates IIA; hence, it is non-welfarist. □

Proposition 1 does not generalize to the n-person case. Moreover, with more than two players there does not exist *any* welfarist solution that satisfies EUM.

[10] PO and SY exclude non-welfarist solutions that satisfy EUM in some trivial way (e.g., D, D^i).

Proposition 3 *Let $n \geq 3$. Let f be a welfarist solution. Then f violates egalitarian–utilitarian monotonicity.*

To prove Proposition 3, I make use of the following lemma.

Lemma 1 *Let $n \geq 3$. Let $S = conv\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n \alpha_i x_i \leq 1\}$, where $0 < \alpha_1 < \cdots < \alpha_n$. Let $x \in S \cap \mathbb{R}^n_{++}$ be such that $|\{j : x_j = min_i x_i\}| = 1$. Then there exists $y \in S$ such that $U(y) > U(x)$ and $R(y) \geq R(x)$.*

Proof Consider first $n = 3$. Let then $S = \text{conv}\{(a, b, c) \in \mathbb{R}^3_+ : \alpha a + \beta b + \gamma c \leq 1\}$, where $0 < \alpha < \beta < \gamma$, and let $x \in S \cap \mathbb{R}^3_{++}$ be a point whose minimum-value coordinate is unique. Clearly it can be assumed that $x \in P(S)$. Note that the ideal point of S is $a(S) = (\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma})$ and that $x = (\frac{\lambda_1}{\alpha}, \frac{\lambda_2}{\beta}, \frac{1-\lambda_1-\lambda_2}{\gamma})$, where $(\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2) \in \mathbb{R}^3_{++}$ are convex weights.

Case 1: $\min_i x_i = \frac{\lambda_1}{\alpha}$. Let $y(\epsilon) \equiv (\frac{\lambda_1+\epsilon}{\alpha}, \frac{\lambda_2-\epsilon}{\beta}, \frac{1-\lambda_1-\lambda_2}{\gamma})$. For a sufficiently small ϵ, $y(\epsilon) \in S$ and it satisfies $U(y(\epsilon)) > U(x)$ and $R(y(\epsilon)) > R(x)$.

Case 2: $\min_i x_i \neq \frac{\lambda_1}{\alpha}$. Wlog, suppose that $\min_i x_i = \frac{\lambda_2}{\beta}$. In this case, define $y(\epsilon) \equiv (\frac{\lambda_1+\epsilon}{\alpha}, \frac{\lambda_2}{\beta}, \frac{1-\lambda_1-\lambda_2-\epsilon}{\gamma})$. For a sufficiently small ϵ, $y(\epsilon) \in S$ and it satisfies $U(y(\epsilon)) > U(x)$ and $R(y(\epsilon)) = R(x)$.

Now consider $n > 3$. Let $\{j, k, l\}$ be three coordinates one of which is the minimum-value coordinate. Applying the above arguments to these coordinates completes the proof. □

Proof of Proposition 3: Let $n \geq 3$. Assume by contradiction that f is a welfarist solution that satisfies EUM. For such a solution there exists a problem S such that the assumptions of Lemma 1 are satisfied for S and x, where $x = f(S)$. By that lemma, there is a $y \in S$ such that $U(y) > U(x)$ and $R(y) \geq R(x)$. Let $V \equiv \text{comp}(\{y\})$. Clearly, $f(V) = y$. This contradicts EUM, since $V \subset S$. □

4 Characterizations and impossibilities

Continuing the theme from the previous section, the following result describes a further difference between the 2-person case and the n-person case.

Theorem 1 *Let f be a solution that satisfies scale covariance and conflict-freeness. Then:*

- *If $n = 2$ then f satisfies egalitarian–utilitarian monotonicity if and only if it is the Nash solution;*
- *If $n \geq 3$ then f violates egalitarian–utilitarian monotonicity.*

To prove Theorem 1, I make use of the following lemma.

Lemma 2 *Let f satisfy conflict-freeness and egalitarian–utilitarian monotonicity. Then f satisfies Suppes-Sen dominance.*

Proof Let f satisfy CF and EUM. Assume by contradiction that it violates SSD. Then there is an S such that $x = f(S)$ is SS-dominated by some s^*, where $s^* = \pi s$ for some $s \in S$ and permutation π. That is, $s^* > x$. Note that $U(s^*) = U(s)$ and $R(s^*) = R(s)$. Let $V \equiv \text{comp}(\{s\})$. Note that $V \subset S$. By CF, $f(V) = s$. Since $s^* > x$ it follows that $U(s^*) > U(x)$, hence $U(s) > U(x)$. Therefore, by EUM, it has to be that as one moves from V to S the value of the Rawlsian objective increases, namely $R(x) > R(s)$. Therefore $R(x) > R(s^*)$ and so $x_i = R(x) > R(s^*) = s^*_j$, where i and j are coordinates in which the respective minima are obtained. This implies $x_j \geq s^*_j$, in contradiction to $s^* > x$. □

Proof of Theorem 1: Let f be a solution that satisfies the axioms. By Lemma 2, it satisfies SSD. By Mariotti (1999), since f satisfies SSD and S.COV, $f = N$. By Proposition 3, $n = 2$. □

The axioms in the 2-person part of Theorem 1 are independent: u satisfies all of them but S.COV, KS satisfies all but EUM, and D satisfies all but CF. Nash (1950) characterized N on the basis of WPO, SY, S.COV, and IIA.[11] Since WPO&SY&S.COV imply CF, the 2-person part of Theorem 1 implies that EUM can replace IIA in the 2-person version of Nash's theorem. It should be noted, however, that none of {IIA, EUM} implies the other. As we saw in Proposition 2, EUM does not imply IIA. In the other direction, note that the solution that assigns to each S the unique point of $P(S) \cap \{s \in S : s \geq D^1(S)\}$ satisfies IIA, but not EUM.[12]

The n-person part of Theorem 1 describes an impossibility result. The following theorem stresses this theme further, and shows that in the presence of other plausible axioms and $n \geq 3$, EUM leaves no room for an economically-meaningful solution.

Theorem 2 *Let $n \geq 3$. A solution satisfies egalitarian–utilitarian monotonicity, symmetry, scale covariance, and restricted independence of irrelevant alternatives if and only if it is the disagreement solution.*

Proof It is straightforward that D satisfies the axioms, hence I omit the details. Conversely, let f be a solution that satisfies them. First, I argue that there exists a rectangle R such that $f(R) \neq a(R)$; otherwise, f would satisfy CF, in contradiction to Theorem 1. By S.COV, it can be assumed that R is symmetric. By SY, $f(R) = \theta a(R)$ for some $\theta < 1$. By RIIA and S.COV, $\theta = 0$. Therefore, by S.COV, $f(R') = \mathbf{0}$ for every rectangle R'. Since every problem can be included in a sufficiently large rectangle, EUM implies $f = D$. □

The axioms in Theorem 2 are independent: N satisfies all of them but EUM, any D^i satisfies all of them but SY, E satisfies all of them but S.COV, and m satisfies all of them but RIIA. Theorem 2 does not have a counterpart in the 2-person case: besides D, the 2-person N also satisfies the axioms. Moreover, these are the only such 2-person solutions.

[11] He derived the result for $n = 2$, but the generalization to $n \geq 3$ is straightforward.

[12] For example, this solution assigns comp$\{(1, 1)\}$ the point $(1, 1)$, but assigns comp$\{(1, 1), (1 + \epsilon, 0)\}$ the point $(1 + \epsilon, 0)$, for every $\epsilon > 0$. Hence, it violates EUM (to check that it satisfies IIA is easy).

Theorem 3 *Let $n = 2$. A solution satisfies egalitarian–utilitarian monotonicity, symmetry, scale covariance, and restricted independence of irrelevant alternatives if and only if it is either the Nash solution or the disagreement solution.*[13]

Proof Clearly both N and D satisfy the axioms.[14] Conversely, let f be a solution that satisfies them. If f does not satisfy CF, then the arguments from the proof of Theorem 2 apply, and $f = D$. If f does satisfy CF, then by Theorem 1 $f = N$. □

The axioms in Theorem 3 are independent: KS satisfies all of them but EUM, each D^i satisfies all of them but SY, E satisfies all of them but S.COV, and m satisfies all of them but RIIA.

5 Further remarks

5.1 Interpersonal comparisons

The starting point of my analysis was (the tension between) egalitarianism and utilitarianism—two concepts that assume interpersonal utility comparisons. The Nash solution, on the other hand, satisfies S.COV, an axiom which is often understood as precluding such comparisons. This discrepancy can be settled by viewing S.COV as precluding interpersonal comparisons only in practice, not in principle.[15]

5.2 Symmetry

Throughout the analysis I have assumed symmetry; in particular, EUM is defined on the basis of the symmetric versions of egalitarianism and utilitarianism, and a social welfare function, per the definition of Sect. 3, is symmetric. In principle, symmetry can be relaxed, as follows.

Let $n = 2$. Given $p \in (0, 1)$, let $U^p(x) \equiv px_1 + (1 - p)x_2$ and $R^p(x) \equiv \min\{(1 - p)x_1, px_2\}$.[16] The parameter p expresses the importance of individual 1 over that of individual 2. EUM is then generalized as follows: a 2-person solution f satisfies p-EUM if whenever the feasible set expands, a decrease of U^p's value (resp. R^p's value) is accompanied by an increase of R^p's value (resp. U^p's value). A 2-person solution f is a *general welfarist solution* if $f(S) \equiv \text{argmax}_{s \in S} W(s)$, where $W: \mathbb{R}^2_+ \to \mathbb{R}$ is a strictly concave and non-decreasing function (that may or may not be symmetric). The following question presents itself: does every general welfarist 2-person solution satisfy p-EUM for some p? The result below answers this question in the negative.

[13] Theorem 3 has a similar flavor to a result of Roth (1979), who showed that when WPO is deleted from Nash's (1950) axiom-list, a joint characterization of N and D obtains. Other papers that provide WPO-free axiomatizations of N include Lensberg and Thomson (1988) and Anbarci and Sun (2011a).

[14] As I mentioned in the proof of the previous theorem, it is straightforward that D satisfies the axioms. That N satisfies EUM follows from Proposition 1, since N is welfarist; that is satisfies the other axioms follows from Nash (1950).

[15] For more on this idea, see Mariotti (1999).

[16] $U = U^{\frac{1}{2}}$ and $R = R^{\frac{1}{2}}$.

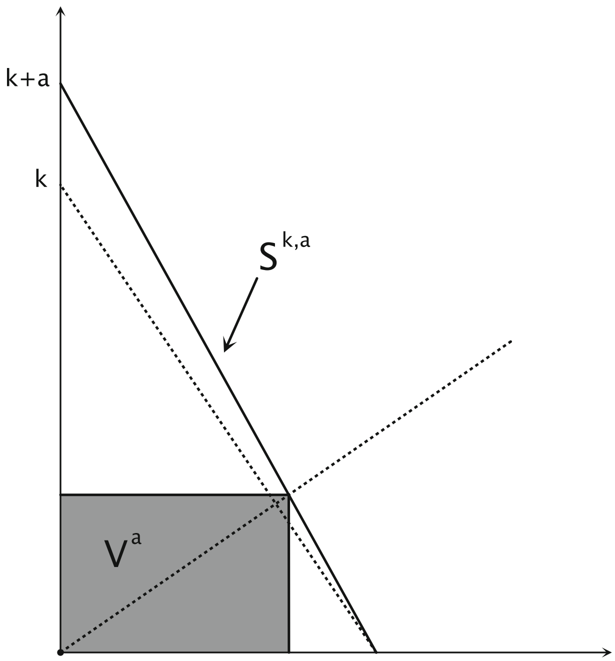

Fig. 2 Illustration of Proposition 4's proof

Proposition 4 *There exists a general welfarist 2-person solution that does not satisfy p-EUM for any* $p \in (0, 1)$.

Proof Consider the general welfarist solution corresponding to $W(x_1, x_2) = x_1 + ln(1+x_2)$. I argue that it does not satisfy p-EUM for any p. To see this, fix a $p \in (0, 1)$. Let l be the line whose equation is $x_2 = (\frac{1-p}{p})x_1$. Given a constant $k > 0$, let l^k be the line whose equation is $x_2 = -(\frac{p}{1-p})x_1 + k$. Note that l^k is orthogonal to l and $l^k_+ \equiv l^k \cap \mathbb{R}^2_+ = \text{conv}\{(k(\frac{1-p}{p}), 0), (0, k)\}$. Let x^k be the unique point of $l \cap l^k$.

Let $S^{k,a}$ be the problem whose Pareto frontier is $\text{conv}\{(k(\frac{1-p}{p}), 0), (0, k + a)\}$. Thus, for $a = 0$ the frontier is given by l^k_+. Momentarily I will consider positive, but arbitrarily small a.

Note that, in any problem, f selects a point which is either on the horizontal axis or in the relative interior of the frontier; when f selects a point in the relative interior of $S^{k,0}$'s frontier, the associated tangency condition is $1+x_2 = \frac{p}{1-p}$, therefore $x_2 = \frac{2p-1}{1-p}$ and $x_1 = \frac{k(1-p)+1-2p}{p}$.

Let $x \equiv f(S^{k,0})$. I argue that for a large enough k, $x_1 > x^k_1$. This is clear if $x = (x_1, 0)$; I will now prove that this is also true when x is given by the aforementioned tangency condition. That is, the claim is:

$$\frac{k(1-p) + 1 - 2p}{p} > k\left[\frac{p(1-p)}{p^2 + (1-p)^2}\right].$$

Multiplying both sides by $\frac{p}{1-p}$ gives:

$$k + \frac{1-2p}{p} > k\left[\frac{p^2}{p^2+(1-p)^2}\right]. \tag{1}$$

It is clear that (1) holds for all sufficiently large k's. Fix such a k and consider $S^{k,a}$, where $a > 0$ is a small number. Let x^a be the unique point of $P(S^{k,a}) \cap l$. Let $V^a \equiv \text{comp}(\{x^a\})$. Note that $f(V^a) = x^a$, that $V^a \subset S^{k,a}$, and that $f(S^{k,a}) \sim x$. In the move $V^a \mapsto S^{k,a}$ both the U^p-value and the R^p-value of the solution point decrease. Hence, f does not satisfy p-EUM (Fig. 2). □

6 Conclusion

I have presented a new axiom, egalitarian–utilitarian monotonicity (EUM), that offers a compromise between utilitarianism and egalitarianism in the Nash bargaining framework. In the 2-person case EUM is a weakening of (a particular sense of) welfarism, but with more than two players it is inconsistent with welfarism. With more than two players it is impossible to reconcile EUM with other standard requirements, whereas in the 2-person case the combination of EUM and other standard axioms pins down the Nash solution.

The following table summarizes the solutions I have considered and the various axioms;[17] for each solution and axiom, it is indicated whether the axiom is satisfied by the 2-person version of the solution. For $n \geq 3$ the same table applies, except that EUM is violated by N.

Solution\axiom	EUM	PO	WPO	SY	S.COV	IIA	RIIA	SSD
N	√	√	√	√	√	√	√	√
KS	–	√	√	√	√	–	√	–
E	√	–	√	√	–	√	√	√
u	√	√	√	√	–	√	√	√
D^i	√	–	√	–	√	√	√	–
D	√	–	–	√	√	√	√	–
m	√	–	–	√	√	–	–	–

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[17] The only axiom which is omitted from the table is CF. The column associated with it is identical to the one of PO.

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