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ORIGINAL PAPER

Collective rationality and decisiveness coherence

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Abstract Arrow's impossibility theorem states that if an aggregation rule satisfies unrestricted domain, weak Pareto, independence of irrelevant alternatives, and collective rationality, then there exists a dictator. Among others, Arrow's postulate of collective rationality is controversial. We propose a new axiom for an aggregation rule, decisiveness coherence, which is weaker than collective rationality. It is shown that given the Arrovian axioms other than collective rationality, a dictatorship arises if and only if decisiveness coherence is satisfied. Moreover, we introduce weak versions of decisive coherence and examine these implications.

1 Introduction

In his celebrated study of collective decision making, Arrow (1950, 1951, 1963) showed that there exists a dictator for any aggregation rule satisfying the following postulates:

- Unrestricted domain: the domain of the aggregation rule includes all possible individual orderings;
- Weak Pareto: if every individual in the society prefers an alternative x to another alternative y, then x is socially preferred to y;
- Independence of irrelevant alternatives: the social ranking of any two alternatives, x and y, depends only on the individual ranking of these two alternatives;
- Collective rationality: social preferences must be transitive as well as complete.

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Since Arrow published his impossibility theorem, numerous scholars have criticized its basic postulates.[1] Most notable is the criticism raised by Buchanan (1954, 1975).[2] He questioned Arrow's postulate of collective rationality, which requires consistency of social preferences. In Buchanan's parlance, "[r]ationality or irrationality as an attribute of the social group implies the imputation to the group of an organic existence apart from that of its individual components" [Buchanan (1954, p. 116)]. In response to Buchanan's argument, Arrow claims, "[c]ollective rationality in the social choice mechanism is not …an illegitimate transfer from the individual to society" and that it is important because transitivity guarantees "the independence of the final choice from the path to it" [Arrow (1963, p.120)].

Later, the concept of *path independence* was formalized by Plott (1973). He showed that path independence does not yield full rationality of choice functions.[3] As such, we can construct a path-independent social choice function satisfying non-dictatorship and Arrow's postulates other than collective rationality. This, however, does not imply that the Arrovian impossibility would go away. Indeed, there exists a vetoer for any path-independent Arrovian social choice function (Blair et al. 1976).

The purpose of this study is to re-examine the postulate of collective rationality in Arrow's impossibility theorem. Classic studies in this area, including Sen (1969, 1970), Brown (1975), Blair and Pollak (1979), and Blau (1979), have examined impossibility results under weaker rationality requirements, such as semi-order properties, quasi-transitivity, and acyclicity.[4] We consider a different question—what postulates for an aggregation rule yield Arrow's impossibility theorem?[5] We propose a new postulate, *decisiveness coherence*, which is weaker than collective rationality. Decisiveness coherence requires that

(i) if x is socially at least as good as y, and y is concertedly preferred to z by some decisive coalition, then x is socially better than z; and
(ii) if x is concertedly preferred to y by some decisive coalition, and y is socially at least as good as z, then x is socially better than z.

Here, we say that a coalition concertedly prefers x to y if x is preferred to y for everyone in the coalition. Decisiveness coherence essentially implies that if an alternative is indirectly socially preferred to another alternative (i.e., there is a chain of weak social preference from the former to the latter), then there is no decisive coalition that concertedly prefers the latter to the former. As we will see later, this is associated with a weak form of path independence, which we call *quasi path independence*. Our main result states that, given Arrow's postulates other than collective rationality, there

[1] Suzumura (1983) and Campbell and Kelly (2002) survey various Arrovian impossibility results under modified postulates.

[2] Sen (1995) examines the notion of rationality in social choice through Buchanan's criticism.

[3] Ferejohn and Grether (1977) formulate a weakening of path independence. Bandyopadhyay (1988) provides a version of path independence that guarantees full rationality. Bandyopadhyay (1990) examines its implication for social choice theory.

[4] Cato (2010) provides a systematic approach to proving their impossibility theorems. Sen (1977) provides a classic survey on this subject.

[5] Quesada (2002) and Ozdemir and Sanver (2007) examine domain conditions that yield Arrow's theorem.

exists a dictator if and only if decisiveness coherence is satisfied. That is, decisiveness coherence is a necessary and sufficient condition for the existence of a dictator.

The rest of this paper is organized as follows. In Sect. 2, we introduce basic definitions. Section 3 provides our main theorem. Section 4 introduces weak versions of decisive coherence, and examines their implications. We also extend our approach to a formulation of non-binary social choice. Section 5 concludes the paper.

2 Preliminaries

2.1 Binary relation and preference

The set of alternatives is X with $\#X \geq 3$. Let $R \subseteq X \times X$ be a binary relation on X.[6] The symmetric and asymmetric parts of R are denoted as $I(R)$ and $P(R)$, respectively.

A binary relation may satisfy the following properties:

Completeness: For all $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$.

Transitivity: For all $x, y, z \in X$, $[(x, y) \in R$ and $(y, z) \in R] \Rightarrow (x, z) \in R$.

Quasi-transitivity: For all $x, y, z \in X$, $[(x, y) \in P(R)$ and $(y, z) \in P(R)]$ $\Rightarrow (x, z) \in P(R)$.

Acyclicity: For all $K \in \mathbb{N}$ and $x^0, x^1, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in P(R) \text{ for all } k \in \{1, 2, \ldots, K\} \Rightarrow (x^K, x^0) \notin P(R).$$

Note that transitivity implies quasi-transitivity, and quasi-transitivity implies acyclicity. Let $\mathcal{C}$ be the set of all possible complete binary relations R on X. A complete and transitive binary relation is called an *ordering*. Let $\mathcal{R}$ be the set of all possible orderings R on X.

Now, we introduce operators that are useful to understand the above four properties. The *dual* of a binary relation R is defined by

$$d(R) := \{(x, y) \in X \times X : (y, x) \in R\}.$$

The *composition* of two binary relations R and R' is defined by

$$R \circ R' := \left\{(x, z) \in X \times X : (x, y) \in R \text{ and } (y, z) \in R' \text{ for some } y \in X\right\}.$$

Define a sequence of binary relations $\{R^{(\tau)}\}_{\tau=0}^{\infty}$ by

$$R^{(0)} = R \text{ and } R^{(\tau)} = R^{(\tau-1)} \circ R \text{ for } \tau \in \mathbb{N}.$$

The *transitive closure* of R is defined as follows:

$$tc(R) = \bigcup_{\tau=0}^{\infty} R^{(\tau)}.$$

[6] Cato (2016b) provides a comprehensive argument on fundamental properties of binary relations.

Using these operators, we restate the four properties, as follows.[7]
Completeness: $R \cup d(R) = X \cup X$.
Transitivity: $tc(R) = R$.
Quasi-transitivity: $tc(P(R)) = P(R)$.
Acyclicity: $tc(P(R)) \cap d(P(R)) = \emptyset$.

2.2 Framework of social choice

Let $N = \{1, \ldots, n\}$ be the finite set of individuals. Each individual $i \in N$ has a preference ordering $R_i \in \mathcal{R}$ on X. A preference profile $\mathbf{R} = (R_i)_{i \in N} \in \mathcal{R}^N$ is an n-tuple of individual preference orderings. The restriction of $\mathbf{R}$ on X to the subset Y of X is denoted by $\mathbf{R}|_Y$. The collection $\mathcal{A}$ of profiles, a nonempty subset of $\mathcal{R}^N$, is called the *admissible preference domain*. An *aggregation rule (AR)* is a function $f : \mathcal{A} \to \mathcal{C}$ that maps each profile $\mathbf{R} \in \mathcal{A}$ to a unique social preference $f(\mathbf{R}) \in \mathcal{C}$.

A coalition $M \subseteq N$ is *decisive over* (x, y) *for* f if, for all $\mathbf{R} \in \mathcal{A}$,

$$\left[(x, y) \in P(R_i) \text{ for all } i \in M\right] \Rightarrow (x, y) \in P(f(\mathbf{R})).$$

For each pair $(x, y) \in X \times X$, let $\mathcal{D}_f(x, y)$ denote the set of decisive coalitions over (x, y) for f. A coalition $M \subseteq N$ is *decisive for* f if it is decisive over all pairs (x, y) of distinct alternatives for f. Let $\mathcal{D}_f$ denote the set of decisive coalitions for f. Individual $i \in N$ is called a *dictator for* f if $\{i\} \in \mathcal{D}_f$; individual $i \in N$ is called a *vetoer for* f if, for all $\mathbf{R} \in \mathcal{A}$ and all $x, y \in X$,

$$\left[(x, y) \in P(R_i) \text{ for all } i \in M\right] \Rightarrow (x, y) \in f(\mathbf{R}).$$

Next, we introduce basic axioms on f.

Unrestricted domain: $\mathcal{A} = \mathcal{R}^N$.
Weak Pareto: $N \in \mathcal{D}_f$.
Independence of irrelevant alternatives: For all $x, y \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{A}$, if $\mathbf{R}|_{\{x,y\}} = \mathbf{R}'|_{\{x,y\}}$, then $[(x, y) \in f(\mathbf{R}) \Leftrightarrow (x, y) \in f(\mathbf{R}')]$.

We say that f is *Arrovian* if it satisfies unrestricted domain, weak Pareto, and independence of irrelevant alternatives. In the rest of this paper, we mainly restrict our attention to Arrovian ARs.

Arrow's collective rationality is formulated as follows.
Collective rationality: For all $\mathbf{R} \in \mathcal{A}$, $f(\mathbf{R})$ is transitive.

We require social preference to be complete and, thus, any AR satisfying collective rationality generates an ordering.

[7] See Cato (2016b) for more detail on the operational expressions of the rationality properties.

3 Decisive coherence and Arrow's theorem

3.1 Main theorem

Arrow's impossibility theorem states that, for every Arrovian AR f, there exists a dictator for f if it satisfies collective rationality. In other words, a Paretian AR with unrestricted domain and independence of irrelevant alternatives is dictatorial if it always generates a social preference "ordering".

Social choice is indirectly obtained from f. Given a social preference R, a social choice function can be induced as follows: for each $S \subseteq X$,

$$C_R(S) = \{x \in S : (y, x) \notin P(R) \text{ for all } y \in S\}.$$

A version of path independence, proposed by Plott (1973), is as follows: for all $S, T \subseteq X$,

$$C_R(S \cup T) = C_R(C_R(S) \cup C_R(T)).$$

It is obvious that the induced choice function is path-independent when a social preference R is an ordering. However, the following question arises—is transitivity necessary for path independence? The answer is negative. A possibility theorem is obtained if we weaken transitivity to quasi-transitivity. This implies that there exists a path-independent social choice procedure where weak Pareto, independence of irrelevant alternatives, and non-dictatorship are satisfied.

However, this way of relaxing collective rationality does not allow us to construct a proper democratic aggregation. As demonstrated by Gibbard (2014), Guha (1972), and Mas-Colell and Sonnenschein (1972), there must be a vetoer (weak dictator) for an Arrovian AR that generates a quasi-transitive social preference. Thus, Arrow-type impossibility results must hold under weaker rationality requirements.[8]

Since the seminal work of Sen (1969), many studies have examined possibility and impossibility results under social rationality conditions weaker than collective rationality. On the other hand, we focus on what an additional postulate could contribute to the dictatorship given unrestricted domain, weak Pareto, and independence of irrelevant alternatives. We then propose a new requirement on f.
Decisiveness coherence: For all $\mathbf{R} \in \mathcal{A}$, and for all $M \in \mathcal{D}_f$,

$$\left(f(\mathbf{R}) \circ \left(\bigcap_{i \in M} P(R_i)\right)\right) \cup \left(\left(\bigcap_{i \in M} P(R_i)\right) \circ f(\mathbf{R})\right) \subseteq P(f(\mathbf{R})).$$

Decisiveness coherence has two parts. The first part requires that if $(x, y) \in f(\mathbf{R})$ and there exists $M \in \mathcal{D}_f$ such that $(y, z) \in \bigcap_{i \in M} P(R_i)$, then $(x, z) \in P(f(\mathbf{R}))$; and (ii) if $(y, z) \in f(\mathbf{R})$ and there exists $M \in \mathcal{D}_f$ such that $(x, y) \in \bigcap_{i \in M} P(R_i)$, then $(x, z) \in P(f(\mathbf{R}))$. It is easy to see that decisiveness coherence is weaker than collective rationality.

[8] Blair et al. (1976) provide a comprehensive analysis for path-independent social choice functions.

Now, we consider the intuitive meaning of this axiom using the assumption that $f(\mathbf{R})$ is complete. From the definition of decisiveness and completeness, it follows that

$$(x, y) \in f(\mathbf{R}) \Rightarrow \nexists M \in \mathcal{D}_f : (y, x) \in \bigcap_{i \in M} P(R_i). \tag{1}$$

Then, (1) states that if x is socially at least as good as y, then no decisive coalition concertedly prefers y to x.

Given completeness, decisive coherence can be translated as follows: for all $x, y, z \in X$,

$$(x, y) \in f(\mathbf{R}) \text{ and } (y, z) \in f(\mathbf{R}) \Rightarrow \nexists M \in \mathcal{D}_f : (z, x) \in \bigcap_{i \in M} P(R_i). \tag{2}$$

This means that if x is socially at least as good as y and y is socially at least as good as z, then no decisive coalition concertedly prefers z to x. This can be regarded as an extension of (1). Here, x is said to be *indirectly socially preferred to* z when x is socially at least as good as y and y is socially at least as good as z. This can be extended by employing the transitive closure. If $(x, y) \in tc(R)$, there exist $K \in \mathbb{N}$ and $x^0, \ldots x^K \in X$, such that $x^0 = x$, $(x^0, x^1) \in R$, $(x^1, x^2) \in R, \ldots, (x^{K-1}, x^K) \in R$, and $x^K = y$. That is, we can find a chain of judgments from x to y when $(x, y) \in tc(R)$. Then, an extended form of (2) is as follows:

$$(x, y) \in tc(f(\mathbf{R})) \Rightarrow \nexists M \in \mathcal{D}_f : (y, x) \in \bigcap_{i \in M} P(R_i). \tag{3}$$

We call this requirement *decisive congruence*. This simply means that if x is indirectly socially preferred to y, there is no decisive coalition that concertedly prefers y to x. Although decisive congruence is slightly stronger than (2), it is substantially weaker than collective rationality. As we will see, decisive congruence is equivalent to (2), given the other Arrow axioms. Note that the relationship between (1) and decisive congruence is quite similar to that between the weak axiom of revealed preference and Houthakker's axiom of revealed preference.[9]

There exists a logical gap between decisiveness coherence and collective rationality. Collective rationality is a coherence property for social preferences, which is a restriction on the range of f. On the other hand, decisiveness coherence does not restrict the range of f.

Example 1 Assume that $X = \{x, y, z\}$. Let $\bar{f}$ be an aggregation rule such that $\{(x, y), (y, z), (z, x), (x, x), (y, y), (z, z)\} = \bar{f}(\mathbf{R})$ for any $\mathbf{R} \in \mathcal{A}$.

Under $\bar{f}$, each social preference has a strict preference cycle that is fixed independently of individual preferences. It is obvious that $\bar{f}$ satisfies decisiveness coherence.

[9] See Samuelson (1938) and Houthakker (1950). Suzumura (1983) provides quite general formulations of both axioms of revealed preference.

In general, any AR that yields a fixed social preference satisfies decisiveness coherence.

Now, let us consider the following modification of path independence: for all $S, T \subseteq X$,

$$C_{R^*}(S \cup T) \supseteq C_R(C_R(S) \cup C_R(T)), \tag{4}$$

where

$$R^* = \bigcup_{A \in \mathcal{D}_f} \bigcap_{i \in M} P(R_i).$$

Since this is a weak form of path independence, we call it *quasi path independence*. If X is finite, then decisive congruence implies quasi path independence.[10] As mentioned previously, decisive coherence implies decisive congruence under unrestricted domain, weak Pareto, and independence of irrelevant alternatives. Therefore, we can say that a certain form of path independence is satisfied under decisive coherence (given the other Arrow axioms).

Now, we are ready to show our main theorem, which states that decisiveness coherence is a necessary and sufficient condition for the existence of a dictator.

Theorem 1 *For every Arrovian aggregation rule f, there exists a dictator for f if and only if it satisfies decisiveness coherence.*

We show the existence of a dictator in a manner similar to the proof of Arrow's theorem by Sen (1979, 1995). To begin with, we establish a variant of the field expansion lemma.

Lemma 1 (Field Expansion Lemma) *Let f be an Arrovian aggregation rule satisfying decisiveness coherence. For all $x, y \in X$ such that $x \neq y$, if $M \subseteq N$ is decisive over (x, y) for f, then M is decisive for f.*

Proof Let $x, y \in X$ be distinct alternatives. Suppose that $M \subseteq N$ is decisive over (x, y) for f. We have to show that

(i) $M \in \mathcal{D}_f(x, z)$ for all $z \in X \backslash \{x, y\}$;
(ii) $M \in \mathcal{D}_f(z, y)$ for all $z \in X \backslash \{x, y\}$;
(iii) $M \in \mathcal{D}_f(z, w)$ for all $z, w \in X \backslash \{x, y\}$;
(iv) $M \in \mathcal{D}_f(z, x)$ for all $z \in X \backslash \{x, y\}$;

[10] Suppose that f satisfies decisive congruence. By way of contradiction, assume that $x \notin C_{R^*}(S \cup T)$ but $x \in C_R(C_R(S) \cup C_R(T))$ for some $x \in X$. Then, there exists $y \in S \cup T$ such that $(y, x) \in \bigcap_{i \in M} P(R_i)$ for some $M \in \mathcal{D}_f$. By definition, $(y, x) \in P(f(\mathbf{R}))$. Without loss of generality, we can assume that $y \in S$. Since $(y, x) \in P(f(\mathbf{R}))$, it follows that $x \in C_R(T)$ and $y \notin C_R(S)$. Since $y \notin C_R(S)$, finiteness implies that $(z, y) \in tc(P(f(\mathbf{R})))$ for some $z \in C_R(S)$. Since $tc(P(f(\mathbf{R}))) \subseteq tc(f(\mathbf{R}))$, $(z, y) \in tc(f(\mathbf{R}))$. If $(x, z) \in f(\mathbf{R})$, then $(x, y) \in tc(f(\mathbf{R}))$ and $(y, x) \in \bigcap_{i \in M} P(R_i)$. This contradicts (3). Thus, $(z, x) \in P(f(\mathbf{R}))$, by completeness. Therefore, we have $x \notin C_R(C_R(S) \cup C_R(T))$, which is a contradiction. Thus, quasi path independence is satisfied.

(v) $M \in \mathcal{D}_f(y, z)$ for all $z \in X \backslash \{x, y\}$;
(vi) $M \in \mathcal{D}_f(y, x)$.

(i) Take any $z \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (x, y) \in P(R_i) \quad \text{and} \quad (y, z) \in P(R_i),$$
$$\forall i \notin M, (y, z) \in P(R_i).$$

Since M is decisive over (x, y) for f, it follows that $(x, y) \in P(f(\mathbf{R}))$, so that $(x, y) \in f(\mathbf{R})$. From weak Pareto, N is decisive for f, and thus, decisiveness coherence implies that

$$\left[(x, y) \in f(\mathbf{R}) \text{ and } (y, z) \in \bigcap_{i \in N} P(R_i)\right] \Rightarrow (x, z) \in P(f(\mathbf{R})).$$

Since the ranking of x and z is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_f(x, z)$.

(ii) Take any $z \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (x, y) \in P(R_i) \quad \text{and} \quad (z, x) \in P(R_i),$$
$$\forall i \notin M, (z, x) \in P(R_i).$$

Since M is decisive over (x, y) for f, it follows that $(x, y) \in P(f(\mathbf{R}))$, so that $(x, y) \in f(\mathbf{R})$. From weak Pareto, N is decisive for f, and thus, decisiveness coherence implies that

$$\left[(z, x) \in \bigcap_{i \in N} P(R_i) \text{ and } (x, y) \in f(\mathbf{R})\right] \Rightarrow (z, y) \in P(f(\mathbf{R})).$$

Since the ranking of y and z is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_f(z, y)$.

(iii) Take any $z, w \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (z, x) \in P(R_i), (x, y) \in P(R_i), \quad \text{and} \quad (y, w) \in P(R_i),$$
$$\forall i \notin M, (z, x) \in P(R_i) \quad \text{and} \quad (y, w) \in P(R_i).$$

Since M is decisive over (x, y) for f, we have $(x, y) \in P(f(\mathbf{R}))$, so that $(x, y) \in f(\mathbf{R})$. Since N is decisive for f, decisiveness coherence implies that

$$\left[(z, x) \in \bigcap_{i \in N} P(R_i) \text{ and } (x, y) \in f(\mathbf{R})\right] \Rightarrow (z, y) \in P(f(\mathbf{R})).$$

Then, $(z, y) \in f(\mathbf{R})$. Since N is decisive over f, decisiveness coherence implies that

$$\left[(z, y) \in f(\mathbf{R}) \text{ and } (y, w) \in \bigcap_{i \in N} P(R_i)\right] \Rightarrow (z, w) \in P(f(\mathbf{R})).$$

Since the ranking of z and w is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_f(z, w)$.

(iv) Take any $z \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (y, x) \in P(R_i) \quad \text{and} \quad (z, y) \in P(R_i),$$
$$\forall i \notin M, (y, x) \in P(R_i).$$

From (ii), M is decisive over (z, y) for f. Hence, $(z, y) \in P(f(\mathbf{R}))$, so that $(z, y) \in f(\mathbf{R})$. By decisiveness coherence,

$$\left[(z, y) \in f(\mathbf{R}) \text{ and } (y, x) \in \bigcap_{i \in N} P(R_i)\right] \Rightarrow (z, x) \in P(f(\mathbf{R})).$$

Since the ranking of x and z is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_f(z, x)$.

(v) Take any $z \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (y, x) \in P(R_i) \quad \text{and} \quad (x, z) \in P(R_i),$$
$$\forall i \notin M, (y, x) \in P(R_i).$$

From (i), M is decisive over (x, z) for f. Hence, $(x, z) \in P(f(\mathbf{R}))$, so that $(x, z) \in f(\mathbf{R})$. By decisiveness coherence,

$$\left[(y, x) \in \bigcap_{i \in N} P(R_i) \text{ and } (x, z) \in f(\mathbf{R})\right] \Rightarrow (y, z) \in P(f(\mathbf{R})).$$

Since the ranking of y and z is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_f(y, z)$.

(vi) Take any $z \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (z, x) \in P(R_i) \quad \text{and} \quad (y, z) \in P(R_i),$$
$$\forall i \notin M, (z, x) \in P(R_i).$$

From (v), M is decisive over (y, z) for f. Hence, $(y, z) \in P(f(\mathbf{R}))$, so that $(y, z) \in f(\mathbf{R})$. By decisiveness coherence,

$$\left[(y, z) \in f(\mathbf{R}) \text{ and } (z, x) \in \bigcap_{i \in N} P(R_i)\right] \Rightarrow (y, z) \in P(f(\mathbf{R})).$$

Since the ranking of x and z is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_f(y, x)$. □

The following result is a variant of the group contraction lemma.

Lemma 2 (Group Contraction Lemma) *Let f be an Arrovian aggregation rule satisfying decisiveness coherence. If $M \subseteq N$ is decisive for f and $\#M > 1$, then some proper subset of M is decisive for f.*

Proof Suppose that $M \subseteq N$ is decisive for f and $\#M > 1$. Let $M^1, M^2 \subseteq M$ be such that $M^1, M^2 \neq \emptyset$, $M^1 \cap M^2 = \emptyset$, and $M^1 \cup M^2 = M$. Take any three distinct alternatives $x, y, z \in X$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M^1, (x, y) \in P(R_i) \quad \text{and} \quad (x, z) \in P(R_i),$$
$$\forall i \in M^2, (y, z) \in P(R_i) \quad \text{and} \quad (x, z) \in P(R_i).$$

Since M is decisive for f, it follows that $(x, z) \in P(f(\mathbf{R}))$.

From completeness of $f(\mathbf{R})$, either $(x, y) \in P(f(\mathbf{R}))$ or $(y, x) \in f(\mathbf{R})$. In the former case, independence of irrelevant alternatives implies that M^1 is decisive over (x, y) for f. Lemma 1 implies that M^1 is decisive for f.

Now consider the latter case. By decisiveness coherence,

$$\left[(y, x) \in f(\mathbf{R}) \text{ and } (x, z) \in \bigcap_{i \in M} P(R_i)\right] \Rightarrow (y, z) \in P(f(\mathbf{R})).$$

Thus, independence of irrelevant alternatives implies that M^2 is decisive over (y, z) for f. Lemma 1 implies that M^2 is decisive for f. The proof is complete. □

Proof of Theorem 1 'Only if.' Let f be an Arrovian AR. Suppose that there exists a dictator $d \in N$ for f. Then, $M \in \mathcal{D}_f$ if and only if $d \in M$. Let $M \in \mathcal{D}_f$ and let

$$(x, y) \in \left(f(\mathbf{R}) \circ \left(\bigcap_{i \in M} P(R_i)\right)\right) \cup \left(\left(\bigcap_{i \in M} P(R_i)\right) \circ f(\mathbf{R})\right).$$

Then,

$$\text{either (i) } (x, y) \in \left(f(\mathbf{R}) \circ \left(\bigcap_{i \in M} P(R_i)\right)\right) \text{ or (ii) } (x, y) \in \left(\left(\bigcap_{i \in M} P(R_i)\right) \circ f(\mathbf{R})\right).$$

Consider case (i). Then, there exists $z \in X$ such that $(x, z) \in f(\mathbf{R})$ and $(z, y) \in \bigcap_{i \in M} P(R_i)$. Since d is a dictator for f, $(x, z) \in R_d$ must hold. Moreover, since $d \in M$, $(z, y) \in \bigcap_{i \in M} P(R_i)$ implies that $(z, y) \in P(R_d)$. Since R_d is an ordering, transitivity implies that $(x, y) \in P(R_d)$. Individual d is a dictator for f, and hence, it follows that $(x, y) \in P(f(\mathbf{R}))$. Analogically, we can prove case (ii).

'If.' By way of contradiction, there exists no dictator for an Arrovian AR f satisfying decisiveness coherence. Weak Pareto implies that $N \in \mathcal{D}_f$. By Lemma 2, either $\{1\} \in \mathcal{D}_f$ or $N \backslash \{1\} \in \mathcal{D}_f$. In the former case, individual 1 is a dictator for f. As such, it follows that $N \backslash \{1\} \in \mathcal{D}_f$. By Lemma 2, either $\{2\} \in \mathcal{D}_f$ or $N \backslash \{1, 2\} \in \mathcal{D}_f$. Since the set of individuals is finite, the repeated applications of Lemma 2 imply that either $\{n-1\} \in \mathcal{D}_f$ or $\{n\} \in \mathcal{D}_f$. That is either individual $n-1$ or n must be a dictator for f. This is a contradiction. □

It may seem that if unrestricted domain, weak Pareto, and independence of irrelevant alternatives are satisfied, then decisiveness coherence implies collective rationality. We now demonstrate, by example, that decisiveness coherence is weaker than collective rationality in the presence of the three axioms.

Example 2 Assume that $N = \{1, 2, 3\}$. Define $\hat{f}$ as follows: for all $x, y \in X$, and for all $\mathbf{R} \in \mathcal{R}^N$,

$$(x, y) \in P(R_1) \Rightarrow (x, y) \in P(\hat{f}(\mathbf{R})),$$
$$(x, y) \in I(R_1) \Rightarrow \left[(x, y) \in \hat{f}(\mathbf{R}) \Leftrightarrow (y, x) \notin P\left(\bigcap_{i \in \{2,3\}} R_i \right) \right].$$

By construction, individual 1 is a dictator for $\hat{f}$. It is obvious that $\hat{f}$ satisfies weak Pareto, independence of irrelevant alternatives, and decisiveness coherence. However, there exists a profile $\mathbf{R} \in \mathcal{R}^N$ such that $\hat{f}(\mathbf{R})$ is not transitive, so that $\hat{f}$ does not satisfy collective rationality. Observe that $\hat{f}$ satisfies collective quasi-rationality and, thus, it generates a quasi-transitive social preference. Hence, it yields a path-independent social choice function.

Now, we show that decisive congruence (3) is equivalent to decisive coherence under unrestricted domain, weak Pareto, and independence of irrelevant alternatives. It suffices to show that if an Arrovian AR satisfies decisive coherence, then it also satisfies quasi path independence (4). As implied by Theorem 1, there exists a dictator $d \in N$. If $(x, y) \in tc(f(\mathbf{R}))$, then there exist $K \in \mathbb{N}$ and $x^0, \ldots x^K \in X$ such that $x^0 = x$, $(x^0, x^1) \in f(\mathbf{R})$, $(x^1, x^2) \in f(\mathbf{R}), \ldots, (x^{K-1}, x^K) \in f(\mathbf{R})$, and $x^K = y$. Given that d is the dictator, we have

$$(x^{k-1}, x^k) \in R_d \quad \text{for all} \quad k \in \{1 \ldots, K\}.$$

Since R_d is transitive, it follows that $(x, y) \in R_d$. Since $d \in M$ for all $M \in \mathcal{D}_f$, there exists no decisive coalition M such that $(y, x) \notin \bigcap_{i \in M} P(R_i)$. Therefore, decisive congruence is satisfied.

Therefore, quasi path independence is satisfied if an Arrovian AR satisfies decisive coherence. However, Plott's path independence may not be satisfied in general. Although $\hat{f}$ in Example 2 is compatible with Plott's path independence, we can find a counter example.

Example 3 Assume that $X = \{x_1, x_2, x_3\}$. Define $\tilde{f}$ as follows: for all $x, y \in X$, and for all $\mathbf{R} \in \mathcal{R}^N$,

$$(x, y) \in P(R_1) \Rightarrow (x, y) \in P(\tilde{f}(\mathbf{R})),$$
$$(x, y) \in I(R_1) \Rightarrow \left[(x, y) \in \tilde{f}(\mathbf{R}) \Leftrightarrow (x, y) \in \bar{R}\right],$$

where $\bar{R} = \{(x_1, x_2), (x_2, x_3), (x_3, x_1), (x_1, x_1), (x_2, x_2), (x_3, x_3)\}$.

By construction, individual 1 is a dictator and $\tilde{f}$ is Arrovian. When $R_1 = X \times X$, a social choice function generated by $\tilde{f}$ is not path-independent.

Examples 2 and 3 show that there is some gap between collective rationality and decisive coherence. However, this observation depends on our domain assumption. As suggested by Example 3, the gap is guaranteed by the possibility that distinct alternatives are indifferent for a dictator. Let us consider the *universal strict-preference domain*: all possible profiles in which every individual has a linear order (no indifference between distinct alternatives). Our theorem is robust in this domain. Thus, if decisive coherence and the other Arrow axioms are satisfied, then there is a dictator, which has no indifference between distinct alternatives. This implies that any social preference must be identical to the dictator's preference and, thus, collective rationality is satisfied.

3.2 Discussion: serial dictatorship

Some authors have characterized the serial dictatorship in the Arrovian framework (Fishburn 1975; Gevers 1979; Cato 2013a, b; Man and Takayama 2013; Takayama and Yokotani 2017). An AR f is called *serially dictatorial* if there is a linear order $\leq_L$ on N such that

$$(x, y) \in P((f(\mathbf{R})) \Leftrightarrow [\exists k \in N : (x, y) \in I(R_i) \text{ for all } i <_L k \text{ and } (x, y) \in P(R_k)];$$
$$(x, y) \in I((f(\mathbf{R})) \Leftrightarrow [(x, y) \in I(R_i) \text{ for all } i \in N].$$

Clearly, any serially dictatorial AR satisfies all the Arrow axioms: universal domain, weak Pareto, independence of irrelevant alternatives, and collective rationality. However, this is not the only rule that satisfies these axioms.[11]

A key axiom for characterizing the serial dictatorship is *strong Pareto*, which requires that if x is at least as good as y for all individuals and x is better than y for some individual, then x is socially better than y (more formally, $P(\bigcap_{i \in N} R_i) \subseteq P(f(\mathbf{R}))$ for all $\mathbf{R} \in \mathcal{R}^N$). If f satisfies universal domain, strong Pareto, independence of irrelevant alternatives, and collective rationality, then the serial dictatorship arises (Fishburn 1975). Example 2 demonstrates that collective rationality is also crucial for this characterization, because the AR in this example satisfies strong Pareto, but is not serially dictatorial.

[11] For example, the following rule is not serially dictatorial, but satisfies all axioms: $f(\mathbf{R}) = R_1$ for all profiles.

The point is that the collection of decisive coalitions is not only a matter in the serial dictatorship. Consider the serial dictatorship where $i \leq_L j \Leftrightarrow i \leq j$. That is, individual 1 is the first dictator, and if the alternatives are indifferent for 1, then individual 2 can decide the ranking, and so on. Under this AR, the collection of decisive coalitions is as follows:

$$\mathcal{D}_f = \{A \subseteq N : 1 \in A\}.$$

That is, $\mathcal{D}_f$ says nothing about the case where the alternatives are indifferent for individual 1. Since decisive coherence cares only about decisive coalitions, the serial structure cannot be derived, even with universal domain, strong Pareto, and independence of irrelevant alternatives.

The key property of the serial dictatorship is that a certain power structure arises, given the first dictator's indifference. The concept of *conditional decisiveness* is introduced by Cato (2013a, 2013b) in order to capture the power structure behind the serial dictatorship.[12] Given $M \subseteq N$, a coalition $M' \subseteq N \setminus M$ is *M-conditionally decisive over* (x, y) *for* f if, for all $\mathbf{R} \in \mathcal{A}$,

$$\Big[(x, y) \in I(R_i) \text{ for all } i \in M \text{ and } (x, y) \in P(R_i) \text{ for all } i \in M'\Big] \Rightarrow (x, y) \in P(f(\mathbf{R})).$$

Let $\mathcal{D}_f(M)$ denote the collection of M-conditionally decisive coalitions for f. By definition, M is decisive for f if and only if it is $\emptyset$-conditionally decisive for f. Cato (2013a, 2013b) shows that $\mathcal{D}_f(M)$ satisfies certain mathematical conditions: more specifically, it forms an ultrafilter. This concept is incorporated by extending the approach developed in this section.[13]However, there is no need to consider this extension. This is because any type of serial dictatorship satisfies collective rationality: given universal domain, strong Pareto, and independence of irrelevant alternatives, collective rationality is a necessary and sufficient condition for the serial dictatorship. That is, transitivity is unnecessary for the dictatorship, but is necessary for the serial dictatorship.

12 Takayama and Yokotani (2017) carefully examine the structure of the set of conditionally decisive coalitions.

13 Consider the following axiom, which is an extension of decisive coherence:
Conditional decisiveness coherence: For all $\mathbf{R} \in \mathcal{A}$, all $M \subseteq N$, and all $M' \in \mathcal{D}_f(M)$,

$$\left(f(\mathbf{R}) \circ \left(\left(\bigcap_{i \in M} I(R_i)\right) \cap \left(\bigcap_{i \in M'} P(R_i)\right)\right)\right) \cup \left(\left(\left(\bigcap_{i \in M} I(R_i)\right) \cap \left(\bigcap_{i \in M'} P(R_i)\right)\right) \circ f(\mathbf{R})\right) \subseteq P(f(\mathbf{R})).$$

4 Extensions

4.1 Relaxing decisive coherence

Now, we relax decisive coherence in order to capture oligarchic structures. Gibbard (2014) shows that any Arrovian AR has an oligarchy under quasi-transitivity. A coalition $M \subseteq N$ is *semi-decisive for* f if, for all $(x, y) \in X \times X$, and for all $\mathbf{R} \in \mathcal{A}$,

$$\big[(x, y) \in P(R_i) \text{ for all } i \in M\big] \Rightarrow (x, y) \in f(\mathbf{R}).$$

Note that a semi-decisive coalition has a *veto* power. A coalition $M \subseteq N$ is an *oligarchy for* f if it is decisive for f, and for all $i \in M$, $\{i\}$ is semi-decisive for f. For an oligarchical AR, there exists a decisive coalition M, in which every member has a veto power. An oligarchy M is the smallest decisive coalition, in the sense that

$$M \subseteq M' \text{ for all decisive coalitions } M'.$$

If it is not the smallest decisive coalition, then $M \nsubseteq M'$ for some decisive coalition M'. Taking $v \in M \backslash M'$, consider a profile $\mathbf{R} \in \mathcal{R}^N$ such that

$$(x, y) \in \bigcap_{i \in M'} P(R_i) \quad \text{and} \quad (y, x) \in P(R_v).$$

Since M is decisive, x is socially better than y. Since v has a veto power (i.e., $\{v\}$ is semi-decisive), y is socially at least as good as y. This is a contradiction. Thus, an oligarchy is always the smallest decisive coalition.

However, the smallest decisive coalition is not necessarily an oligarchy.

Example 4 Assume that $X = \{x_1, x_2, x_3\}$. Define $\tilde{f}$ as follows: for all $x, y \in X$, and for all $\mathbf{R} \in \mathcal{R}^N$,

$$(x, y) \in \bigcap_{i \in N} P(R_i) \Rightarrow (x, y) \in P(\tilde{f}(\mathbf{R})),$$
$$(x, y) \notin \bigcap_{i \in N} P(R_i) \Rightarrow \big[(x, y) \in \tilde{f}(\mathbf{R}) \Leftrightarrow (x, y) \in \bar{R}\big],$$

where $\bar{R} = \{(x_1, x_2), (x_2, x_3), (x_3, x_1), (x_1, x_1), (x_2, x_2), (x_3, x_3)\}$. Here, N is the smallest decisive coalition. Since there exists no vetoer, there is no oligarchy.

Then, we can distinguish two points:

(i) there exists a smallest decisive coalition M;
(ii) every individual in M has a veto power.

As long as quasi-transitivity is imposed, there is no gap between (i) and (ii). Our goal is to relax decisive coherence to capture (i) and (ii) separately. This means that we consider alternative weakenings of quasi-transitivity.

We then introduce two weak versions of decisiveness coherence.
Weak decisiveness coherence$^\diamond$: For all $\mathbf{R} \in \mathcal{A}$, and for all $M, M' \in \mathcal{D}_f$,

$$\left(\bigcap_{i \in M} P(R_i)\right) \circ \left(\bigcap_{i \in M'} P(R_i)\right) \subseteq P(f(\mathbf{R})).$$

Weak decisiveness coherence*: For all $\mathbf{R} \in \mathcal{A}$, and for all $M \in \mathcal{D}_f$,

$$\left(P(f(\mathbf{R})) \circ \left(\bigcap_{i \in M} P(R_i)\right)\right) \cup \left(\left(\bigcap_{i \in M} P(R_i)\right) \circ P(f(\mathbf{R}))\right) \subseteq P(f(\mathbf{R})).$$

According to weak decisiveness coherence$^\diamond$, if there exist $M, M' \in \mathcal{D}_f$ such that $(x, y) \in \bigcap_{i \in M} P(R_i)$ and $(y, z) \in \bigcap_{i \in M'} P(R_i)$, then $(x, z) \in P(f(\mathbf{R}))$. According to weak decisiveness coherence*, (i) if $(x, y) \in P(f(\mathbf{R}))$ and there exists $M \in \mathcal{D}_f$ such that $(y, z) \in \bigcap_{i \in M} P(R_i)$, then $(x, z) \in P(f(\mathbf{R}))$; and (ii) if $(y, z) \in P(f(\mathbf{R}))$ and there exists $M \in \mathcal{D}_f$ such that $(x, y) \in \bigcap_{i \in M} P(R_i)$, then $(x, z) \in P(f(\mathbf{R}))$. Weak decisiveness coherence* implies weak decisiveness coherence$^\diamond$, but not vice versa. Note that the two axioms are weaker than quasi-transitivity.

One interpretation of these axioms is associated with quasi path independence (4). Suppose that weak decisiveness coherence* is not satisfied. Then, there exist $x, y, z \in X$ and $\mathbf{R} \in \mathcal{A}$ such that $(x, y) \in P(f(\mathbf{R}))$, $(y, z) \in \bigcap_{i \in M} P(R_i)$ for some $M \in \mathcal{D}_f$, and $(x, z) \notin P(f(\mathbf{R}))$. Then, we have

$$z \notin C_{R^*}(\{x, y, z\}) \quad \text{and} \quad z \in C_R(C_R(\{x, y\}) \cup C_R(\{z\})).$$

Thus, (4) is not satisfied. This means that weak decisiveness coherence* is a necessary condition for (4). Since weak decisiveness coherence$^\diamond$ is weaker than weak decisiveness coherence*, weak decisiveness coherence$^\diamond$ is necessary for (4). There is another form of path independence that is associated with weak decisiveness coherence$^\diamond$: for all $S, T \subseteq X$,

$$C_{R^*}(S \cup T) \supseteq C_R(C_{R^*}(S) \cup C_{R^*}(T)), \tag{5}$$

where $R^* = \bigcup_{A \in \mathcal{D}_f} \bigcap_{i \in M} P(R_i)$. Weak decisiveness coherence$^\diamond$ is necessary for (5).[14]

The following theorem shows the relationship between weak decisiveness coherence$^\diamond$ and the existence of the unique smallest decisive coalition.

Theorem 2 *For every Arrovian aggregation rule f, there exists the unique smallest decisive coalition for f if and only if it satisfies weak decisiveness coherence$^\diamond$.*

Lemma 3 *Let f be an Arrovian aggregation rule satisfying weak decisiveness coherence$^\diamond$. For all $M, M' \subseteq N$, if M and M' are decisive for f, then $M \cap M'$ is decisive for f.*

[14] Weak decisiveness coherence* is not necessary for (5).

Proof Suppose that M and M' are decisive for f. We prove that $M \cap M'$ is decisive for f. Take any $x, y \in X$. Let $z \in X \backslash \{x, y\}$, and let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\begin{aligned} &\forall i \in M \cap M', \{(x, z), (z, y), (x, y)\} \subseteq P(R_i), \\ &\forall i \in M \backslash M', (x, z) \in P(R_i), \\ &\forall i \in M' \backslash M, (z, y) \in P(R_i). \end{aligned}$$

Weak decisiveness coherence$^{\diamond}$ implies that

$$\left[(x, z) \in \bigcap_{i \in M} P(R_i) \text{ and } (z, y) \in \bigcap_{i \in M'} P(R_i)\right] \Rightarrow (x, y) \in P(f(\mathbf{R})).$$

Since the ranking of x and y is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \cap M' \in \mathcal{D}_f(x, y)$. Note that two alternatives x, y are arbitrarily chosen. Hence, $M \cap M' \in \mathcal{D}_f$. □

Proof of Theorem 2. 'Only if.' Let f be an Arrovian AR. Suppose that there exists the unique smallest decisive coalition M^* for f. Let $M, M' \in \mathcal{D}_f$ and let

$$(x, y) \in \left(\bigcap_{i \in M} P(R_i)\right) \circ \left(\bigcap_{i \in M'} P(R_i)\right).$$

Then, there exists $z \in X$ such that $(x, z) \in \bigcap_{i \in M} P(R_i)$ and $(z, y) \in \bigcap_{i \in M'} P(R_i)$. Note that $M^* \subseteq M$ and $M^* \subseteq M'$. Therefore, $(x, z) \in \bigcap_{i \in M^*} P(R_i)$ and $(z, y) \in \bigcap_{i \in M^*} P(R_i)$. Since each individual preferences is transitive,

$$(x, y) \in \bigcap_{i \in M^*} P(R_i).$$

Since M^* is decisive for f, it follows that $(x, y) \in P(f(\mathbf{R}))$.

'If.' Since $N \in \mathcal{D}_f$, there exists a smallest element of $\mathcal{D}_f$. We now show that it must be unique. By way of contradiction, suppose that there exists two smallest elements M, M' of $\mathcal{D}_f$ such that $M \neq M'$. By Lemma 3, $M \cap M' \in \mathcal{D}_f$. This contradicts to the smallestness of M and M'. □

If we impose weak decisiveness coherence*, then there exists an oligarchy for f. Conversely, if there exists an oligarchy for f, then weak decisiveness coherence* must be satisfied. We then have the following theorem.

Theorem 3 *For every Arrovian aggregation rule f, there exists an oligarchy for f if and only if it satisfies weak decisiveness coherence*.*

Proof ‘Only if.’ Let f be an Arrovian AR. Suppose that there exists an oligarchy M^* for f. Let $M \in \mathcal{D}_f$ and let

$$(x, y) \in \Big(P(f(\mathbf{R})) \circ \big(\bigcap_{i \in M} P(R_i)\big)\Big) \cup \Big(\big(\bigcap_{i \in M} P(R_i)\big) \circ P(f(\mathbf{R}))\Big).$$

Then,

$$\begin{aligned} \text{either (i) } (x, y) &\in \left(P(f(\mathbf{R})) \circ \left(\bigcap_{i \in M} P(R_i)\right)\right) \text{ or} \\ \text{(ii) } (x, y) &\in \left(\left(\bigcap_{i \in M} P(R_i)\right) \circ P(f(\mathbf{R}))\right). \end{aligned}$$

Consider case (i). As such, there exists $z \in X$ such that $(x, z) \in P(f(\mathbf{R}))$ and $(z, y) \in \bigcap_{i \in M} P(R_i)$. Since every member i in M^* has a veto power, it follows that $(x, z) \in R_i$ for all $i \in M^*$. Moreover, $M^* \subseteq M$, and thus, $(z, y) \in P(R_i)$ for all $i \in M^*$. Since each individual ordering is transitive, we have $(x, y) \in P(R_i)$ for all $i \in M^*$. Then, M^* is decisive for f, and thus, $(x, y) \in P(f(\mathbf{R}))$. Analogically, we can prove case (ii).

‘If’. By Theorem 2, there exists the unique smallest decisive set M^* for f. We now prove that M^* is an oligarchy for f. It suffices to show that every member of M^* has a veto power, i.e., for all $i \in M^*$, for all $\mathbf{R} \in \mathcal{R}^N$, and for all $x, y \in X$,

$$\big[(x, y) \in P(R_i) \Rightarrow (x, y) \in f(\mathbf{R})\big].$$

By way of contradiction, suppose that there exist $i^* \in M^*$, $\mathbf{R} \in \mathcal{R}^N$, and $x, y \in X$ such that $(x, y) \in P(R_{i^*})$ and $(x, y) \notin f(\mathbf{R})$. By completeness, we have $(y, x) \in P(f(\mathbf{R}))$. Define $M^1, M^2, M^3 \subseteq N$ by

$$\begin{aligned} M^1 &:= \{i \in N : (x, y) \in P(R_i)\}, \\ M^2 &:= \{i \in N : (x, y) \in I(R_i)\}, \\ M^3 &:= \{i \in N : (y, x) \in P(R_i)\}. \end{aligned}$$

Note that $i^* \in M^1$. Let $\mathbf{R}' \in \mathcal{R}^N$ be such that

$$\begin{aligned} \forall i \in M^*, (x, z) \in P(R'_i), \\ \mathbf{R}'_{\{x,y\}} = \mathbf{R}_{\{x,y\}}. \end{aligned}$$

By independence of irrelevant alternatives, $(y, x) \in P(f(\mathbf{R}'))$. By weak decisiveness coherence, it follows that $(y, z) \in P(f(\mathbf{R}))$. Note that

$$(x, z) \in P(R'_i) \text{ for all } i \in M^2 \cup M^3.$$

Since the ranking of x and z is not specified for individuals outside of $M^2 \cup M^3$, independence of irrelevant alternatives implies that $M^2 \cup M^3 \in \mathcal{D}_f(x, z)$. The field expansion lemma holds under weak decisiveness coherence*. Hence, $M^2 \cup M^3 \in \mathcal{D}_f$. However, $M^2 \cup M^3$ is a proper subset of M^*. This contradicts to the supposition that M^* is the unique smallest decisive set for f. □

In the preceding analysis, we investigated a dictator or an oligarchy. Next, we study a coherence property that yields the existence of a non-empty intersection of all decisive coalitions for f, which is called a *collegium*. Formally, a coalition M^* is a collegium for f if

$$M^* \neq \emptyset \text{ and } M^* = \bigcap_{M \in \mathcal{D}_f} M.$$

A collegium is not necessarily the smallest decisive coalition. For example, consider the following collection of decisive coalitions:

$$\mathcal{D}_f = \big\{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\big\}.$$

In this case, there exists a collegium that is not the smallest decisive coalition.

Brown (1975) and Banks (1995) show that if an acyclical AR f satisfies unrestricted domain and weak Pareto, there exists a collegium for f whenever the number of alternatives is sufficiently large. We consider a necessary and sufficient condition of coherency for the existence of a collegium.

A good starting point is the existence of the smallest decisive coalition. If the smallest decisive coalition exists, the following always holds: for all $K \in \mathbb{N}$, and for all $M^1, \ldots, M^K \in \mathcal{D}_f$,

$$\left(\left(\bigcap_{i \in M^1} P(R_i)\right) \circ \cdots \circ \left(\bigcap_{i \in M^K} P(R_i)\right)\right) \subseteq P(f(\mathbf{R})). \tag{6}$$

This is a strong form of weak decisiveness coherence$^\diamond$. This strong form is equivalent to weak decisiveness coherence$^\diamond$ for any Arrovian AR. A possible weak version of (6) is as follows: for all $K \in \mathbb{N}$, and for all $M^1, \ldots, M^K \in \mathcal{D}_f$,

$$\left(\left(\bigcap_{i \in M^1} P(R_i)\right) \circ \cdots \circ \left(\bigcap_{i \in M^K} P(R_i)\right)\right) \subseteq f(\mathbf{R}). \tag{7}$$

We can show that this is sufficient for the existence of a collegium, but this is not necessary. A further weakening is needed. Note that, given completeness, (7) is rephrased as follows: for all $K \in \mathbb{N}$, and for all $M^1, \ldots, M^K \in \mathcal{D}_f$,

$$d\big(P(f(\mathbf{R}))\big) \cap \left(\left(\bigcap_{i \in M^1} P(R_i)\right) \circ \cdots \circ \left(\bigcap_{i \in M^K} P(R_i)\right)\right) = \emptyset.$$

This leads us to the following condition, which is a weak version of the acyclicity of social preferences.

Decisiveness acyclicity: For all $\mathbf{R} \in \mathcal{A}$, for all $K \in \mathbb{N}$, and for all $M^0, M^1, \ldots, M^K \in \mathcal{D}_f$,

$$d\left(\bigcap_{i \in M^0} P(R_i)\right) \cap \left(\left(\bigcap_{i \in M^1} P(R_i)\right) \circ \cdots \circ \left(\bigcap_{i \in M^K} P(R_i)\right)\right) = \emptyset.$$

Given $K \in \mathbb{N}$ and $x^0, x^1, \ldots, x^K \in X$, this axiom requires that no decisive coalition concertedly prefers x^K to x^0 whenever $M^k \in \mathcal{D}_f$ concertedly prefers x^{k-1} to x^k for all $k \in \{1, \ldots, K\}$. As implied in the derivation, decisiveness acyclicity is weaker than weak decisiveness coherence$^\diamond$ for any Arrovian AR.

We now offer the theorem. Note that independence of irrelevant alternatives is not imposed in this theorem.

Theorem 4 *Suppose that $\#X \geq \#N$ and an aggregation rule f satisfies unrestricted domain and weak Pareto. Then, there exists a collegium for f if and only if it satisfies decisiveness acyclicity.*

Proof 'Only if'. Suppose that there exists a collegium for f. Let $M^0, M^1, \ldots, M^K \in \mathcal{D}_f$. By way of contradiction, assume that

$$d\left(\bigcap_{i \in M^0} P(R_i)\right) \cap \left(\left(\bigcap_{i \in M^1} P(R_i)\right) \circ \cdots \circ \left(\bigcap_{i \in M^K} P(R_i)\right)\right) \neq \emptyset.$$

Then, there exists $x, y \in X$ such that

$$(x, y) \in \left(\bigcap_{i \in M^1} P(R_i)\right) \circ \cdots \circ \left(\bigcap_{i \in M^K} P(R_i)\right),$$

and

$$(y, x) \in \bigcap_{i \in M^0} P(R_i).$$

Since there exists a collegium for f, there exists $i^* \in N$ such that $i^* \in \bigcap_{k=\{0,\ldots,K\}} M^k$. This implies that $(x, y) \in P(R_{i^*})$ and $(y, x) \in P(R_{i^*})$. This is a contradiction.

'If'. Suppose, on the contrary, that f satisfies decisiveness acyclicity and there exists no collegium for f. Then, for all $i \in N$, there exists M^i such that M^i is decisive for f and $i \notin M^i$. By definition of decisiveness, $N \setminus \{i\}$ is decisive for f because $M^i \subseteq N \setminus \{i\}$.

Choose distinct $x^1, \dots, x^n \in X$, and let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$
\begin{aligned}
&\{(x^1, x^2), (x^2, x^3), \dots, (x^{n-1}, x^n)\} \subseteq P(R_1),\\
&\{(x^2, x^3), (x^3, x^4), \dots (x^n, x^1)\} \subseteq P(R_2),\\
&\{(x^3, x^4), (x^4, x^5), \dots, (x^1, x^2)\} \subseteq P(R_3),\\
&\quad \vdots\\
&\{(x^n, x^1), (x^1, x^2), \dots, (x^{n-2}, x^{n-1})\} \subseteq P(R_n).
\end{aligned}
$$

Since $\#X \geq \#N$ and unrestricted domain is satisfied, this combination and the profile are available. Note that

$$
\begin{aligned}
(x^n, x^1) &\in \bigcap_{i \in N \setminus \{1\}} P(R_i),\\
j = 2, \dots, n : (x^{j-1}, x^j) &\in \bigcap_{i \in N \setminus \{j\}} P(R_i).
\end{aligned}
$$

Then, it follows that

$$
(x^1, x^n) \in \left(\bigcap_{i \in N \setminus \{2\}} P(R_i) \circ \cdots \circ \bigcap_{i \in N \setminus \{n\}} P(R_i) \right).
$$

Note that

$$
(x^n, x^1) \in \bigcap_{i \in N \setminus \{1\}} P(R_i) \Leftrightarrow (x^1, x^n) \in d \left(\bigcap_{i \in N \setminus \{1\}} P(R_i) \right).
$$

This contradicts to decisiveness acyclicity. □

4.2 Non-binary social choice

The original work of Arrow (1951, 1963) examines the problem of social choice in terms of social preferences. In the 1970s and 1980s, many authors investigated the problem of collective decision making in terms of a social choice function (Schwartz 1970; Suzumura 1976; Campbell 1976; Parks 1976; Suzumura 1976; Grether and Plott 1982; Matsumoto 1985; Richelson 1977, 1978; Sen 1993). A non-binary formulation of social choice is useful to capture social decision procedures.[15] In this section, we demonstrate that our results in the previous section are robust under this formulation.

Let $\mathcal{X}$ denote the power set of X excluding the empty set. A *social choice function* is a mapping $C : \mathcal{X} \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \mathcal{X}$. Let $\mathcal{S}$ be the set of all

[15] Sen (1977, 1993) discusses the meanings and significance of this approach.

social choice functions. A *functional collective choice rule* is a function $F : \mathcal{A} \to \mathcal{S}$ that maps each profile $\mathbf{R} \in \mathcal{A}$ to a social choice function $C(\cdot) \in \mathcal{S}$. A coalition $M \subseteq N$ is *decisive for* F if for all $x, y \in X$, and for all $\mathbf{R} \in \mathcal{A}$,

$$\big[(x, y) \in P(R_i) \text{ for all } i \in M\big] \Rightarrow \{x\} = C(\{x, y\}),$$

where $C(\cdot) = F(\mathbf{R})$. Let $\mathcal{D}_F$ denote the set of decisive coalitions for F. Individual $i \in N$ is a *pair-choice dictator for* F if $\{i\} \in \mathcal{D}_F$.

Arrovian conditions are translated as follows.

Pair-choice weak Pareto: For all $\mathbf{R} \in \mathcal{A}$, and for all $x, y \in X$, if $(x, y) \in \bigcap_{i \in N} P(R_i)$, then $\{x\} = C(\{x, y\})$, where $C(\cdot) = F(\mathbf{R})$.

Pair-choice independence: For all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{A}$, if $\forall i \in N : [(x, y) \in R_i \Leftrightarrow (x, y) \in R'_i]$, then $C(\{x, y\}) = C'(\{x, y\})$, where $C(\cdot) = F(\mathbf{R})$ and $C'(\cdot) = F(\mathbf{R}')$.

We say that F is *Arrovian* if it satisfies unrestricted domain, pair-choice weak Pareto, and pair-choice independence.

In order to translate decisiveness coherence, we introduce an auxiliary concept. Given a social choice function $C(\cdot)$, the *base relation* $\bar{R}_C$ can be defined as follows: for all $x, y \in X$,

$$(x, y) \in \bar{R}_C \Leftrightarrow x \in C(\{x, y\}).$$

A choice-functional version of decisiveness coherence is formulated as follows.

Base decisiveness coherence: For all $\mathbf{R} \in \mathcal{A}$, and for all $M \in \mathcal{D}_F$,

$$\left(\bar{R}_C \circ \left(\bigcap_{i \in M} P(R_i)\right)\right) \cup \left(\left(\bigcap_{i \in M} P(R_i)\right) \circ \bar{R}_C\right) \subseteq P(\bar{R}_C),$$

where $\bar{R}_C$ is the base relation induced from $C(\cdot) = F(\mathbf{R})$.

A choice-functional version of the dictator theorem is as follows.

Theorem 5 *For every Arrovian functional collective choice rule F, there exists a pair-choice dictator for F if and only if it satisfies base decisiveness coherence.*

The proof is essentially the same as that of Theorem 1. First, we prove the field expansion lemma and then show the group contraction lemma. Here, we mention only the former.

Lemma 4 *Let F be an Arrovian functional collective choice rule satisfying base decisiveness coherence. For all $x, y \in X$ such that $x \neq y$, if $M \subseteq N$ is decisive over (x, y) for F, then M is decisive for F.*

Sketch of Proof. Let $x, y \in X$ be distinct alternatives. Suppose that $M \subseteq N$ is decisive over (x, y) for F. Here, we prove that $M \in \mathcal{D}_F(x, z)$ for all $z \in X \backslash \{x, y\}$. Take any $z \in X \backslash \{x, y\}$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\forall i \in M, (x, y) \in P(R_i) \text{ and } (y, z) \in P(R_i),$$
$$\forall i \notin M, (y, z) \in P(R_i).$$

Since M is decisive over (x, y) for F, it follows that $(x, y) \in P(\bar{R}_C)$. From pair-choice weak Pareto, decisiveness coherence implies that

$$\left[(x, y) \in \bar{R}_C \text{ and } (y, z) \in \bigcap_{i \in N} P(R_i)\right] \Rightarrow (x, z) \in P(\bar{R}_C).$$

Since the ranking of x and z is not specified for individuals outside of M, independence of irrelevant alternatives implies that $M \in \mathcal{D}_F(x, z)$. □

All things to do for proving Theorem 5 is replacing $f(\mathbf{R})$ with $\bar{R}_C$ in the proof of Theorem 1. We omit the rest of the proof. Also, Theorems 2–4 can be translated into choice functional versions.

5 Concluding remarks

Collective rationality is a crucial postulate for Arrow's impossibility theorem. This paper introduced the concept of decisiveness coherence, which is weaker than collective rationality. Decisiveness coherence can be interpreted as a requirement of social rationality. Assume that (i) an alternative x is socially at least as good as another alternative y, and (ii) y is socially preferred to z. Situation (ii) holds under either of the following two cases: (ii-a) x is preferred to y for every member in some decisive coalition, or (ii-b) otherwise. Decisiveness coherence restricts social preference in the former case, but requires nothing in the latter case. Therefore, it is "conditional" collective rationality in the sense that it has the same role as collective rationality when social preference is associated with a decisive coalition.

For any Arrovian AR, decisiveness coherence is a necessary and sufficient condition for the existence of a dictator. Three weak versions of the axiom were also introduced in order to clarify the necessary and sufficient conditions for the existence of an oligarchy, the smallest decisive coalition, and a collegium, respectively. We clarified how these axioms are related to path-independent collective decision-making.

We believe that our theorems and new axioms are useful for the further development of social choice theory and related subjects. Here, we mention three possible applications and extensions of our approach. The first extension is to consider an infinite population. In this study, we assume the set of individuals to be finite. It is known that Arrow's theorem does not hold in a society with an infinite population: there exists a non-dictatorial AR that satisfies unrestricted domain, weak Pareto, independence of irrelevant alternatives, and collective rationality (Fishburn 1970). Thus, our results are no longer robust under the infinite-population assumption. However, decisive coherence can work effectively under the infinite-population assumption, and can be utilized to examine the general structure of the collection of decisive coalitions, especially, by following the approaches developed by Kirman and Sondermann (1972) and Hansson (1976).

The second extension is to weaken weak Pareto or the independence of irrelevant alternatives, which many existing studies do (Wilson 1972; Baigent 1987; Denicolò 1998; Cato 2012, 2014, 2016, b). The structure of decisive coalitions is affected by relaxing these axioms. We can examine the implications of decisive coherence or its extensions in combination with such relaxed Arrovian axioms. The third extension is to consider an economic environment. We assumed unrestricted domain in this study. In economic environments, individual preferences are restricted to be monotonic and continuous. This implies that the domain is restricted in a certain way under such environments. A significant number of studies examine Arrovian ARs under such domains because economic environments are important to derive the direct implication of Arrow's theorem for welfare economics.[16] Applying decisive coherence to these domains can further show the significance of collective rationality in the theory of social choice.

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[16] See, for example, Kalai et al. (1979), Border (1983) and Fleurbaey et al. (2005), and Tadenuma (2005) Le Breton and Weymark (2011) provide a comprehensive survey of Arrovian results in economic domains.

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