Soc Choice Welf (2017) 48:295–326
DOI 10.1007/s00355-016-1004-x

ORIGINAL PAPER

Inequality and isoelastic equivalence scales: restrictions and implications

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Received: 18 November 2014 / Accepted: 1 November 2016 / Published online: 28 November 2016

Abstract Comparisons of well-being across heterogenous households necessitate that households' incomes are adjusted for differences in size and composition: equivalence scales are commonly used to achieve this objective. Equivalence scales with constant elasticity with respect to family size have been argued to provide a good approximation to a large variety of scales (see, e.g., Buhmann et al., Equivalence scales, well-being, inequality and poverty: sensitivity estimates across ten countries using the Luxembourg Income Study (LIS) database. Rev Income Wealth 34:115–142, 1988) and they therefore play a prominent role in empirical work. Focusing on inequality of well-being, we first show that, if one requires that the index of inequality is—in addition to standard properties—invariant to modifications of the relative (marginal) distributions of needs and income across households, then the equivalence scales must be isoelastic. In addition, if all households' members have the same preferences and if households maximise the sum of their members' utilities, then the only preferences consistent with isoelastic scales are of the Cobb–Douglas type.

1 Introduction and motivation

The measurement of economic well-being requires among other things that the households' incomes are adjusted in order to accommodate differences in needs. Equivalence

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scales are designed to accomplish this adjustment by taking into account those household characteristics deemed to affect its members' well-being.[1] Given a reference household type—typically a single adult—the procedure consists in deflating the household's original income by a scale factor that reflects the household's members' needs. The resulting *equivalent income* is a cardinal measure of the household's well-being indicating the income that, if given to a single adult, will allow it to enjoy the same well-being as a typical member of the original household. Traditional characteristics include region, location, age of adults and children but by far the most common factor that affects the household's members' well-being is family size. The equivalence scales currently used in empirical work are extremely varied both in the way in which they are derived and in their values. Different procedures, such as expert opinions, interviews with consumers or econometric studies, are used for determining the values of the equivalence scales (see, e.g., Coulter et al. 1992a). This results in contrasted figures which in turn significantly affect the normative conclusions to be drawn (see, e.g., Whiteford 1985; Coulter et al. 1992b; Figini 1998).

Among the different formulae proposed in the literature, equivalence scales with a constant elasticity with respect to family size play a prominent role (see, for instance, Atkinson et al. 1995; Fleurbaey and Gaulier 2009). Recent OECD publications acknowledge differences in needs by dividing household incomes by the square root of the total number of individuals in the household.[2] According to such equivalence scales, a proportional increase in family size results in a proportional—not necessarily of the same magnitude—increase in the scale.[3] Different reasons may explain why isoelastic equivalence scales have been meeting such success both in empirical and theoretical work. A first reason—advanced for instance by Buhmann et al. (1988)—is that equivalence scales with constant elasticity approximate reasonably well most of the scales currently used in the literature. A second, and more important, reason is that these scales permit one to control for the impact of family size on adjusted income through a single parameter that measures the elasticity of the scale. This, in turn, has nice implications for applied work where one is interested in the changes in the extent of poverty, inequality or welfare implied by modifications of the distribution of household size. For instance, Coulter et al. (1992b) showed that the cardinal value

[1] Adjustments for differences in needs by means of equivalence scales may be considered too specific an approach and an alternative procedure has been proposed by Atkinson and Bourguignon (1987) (see also Bourguignon 1989; Jenkins and Lambert 1993; Bazen and Moyes 2003; Ebert 2010; Moyes 2012; Gravel and Moyes 2012). While this approach has mainly focused on the derivation of quasi-orderings like the sequential Lorenz dominance criterion for making comparisons of living standards across heterogenous populations, it is equally possible to use multidimensional (cardinal) indices (see, e.g., Maasoumi 1999; Ebert 1995; Gravel et al. 2009 among others).

[2] The so-called *square root scale* replaces the former OECD-*modified scale* proposed by Hagenaars et al. (1994) that assigned a value of 1 to the household head, of 0.5 to each additional adult and of 0.3 to each child. While the levels of poverty, inequality or welfare are sensitive the use of one scale rather than another, trends over time and rankings of countries are much less affected (see, e.g., Burniaux et al. 1998).

[3] This makes only sense if the equivalence scales are independent of household income which implicitly amounts to imposing strong restrictions on the preferences of the household's members (see, Blackorby and Donaldson 1993; Blundell and Lewbel 1991). Admittedly, this assumption is debatable and there indeed is ample empirical evidence that it is violated in practice (see, e.g., Donaldson and Pendakur 2004; Koulovatianos et al. 2005a, b).

of a poverty measure can be sensitive to the choice of the size elasticity parameter. Lanjouw and Ravallion (1995) addressed the question of whether large households are poorer than smaller ones and argued that the answer depends critically on the extent of dispersion in family sizes and the size elasticity of the equivalence scale.[4]

In this paper we are interested in the measurement of *inequality of well-being* using the distributions of the households' equivalent incomes. Typically, the equivalent income incorporates two main elements of the household's well-being: *neediness*, which is determined by the household's socio-demographic characteristics, and household *income*. Therefore, the equivalent income can be considered a bidimensional (cardinal) measure of the household's well-being and inequalities of well-being can be traced back to differences in the joint distributions of neediness and household income. Having this in mind, it is likely that one would insist that the *inequality measure of household well-being* reacts appropriately to particular changes in the (unidimensional) distributions of needs and income as well as to the way the (joint) distribution of these two elements varies. It is a standard requirement for a (unidimensional) inequality index that equiproportionate changes in incomes leave inequality unchanged. By the same token, one might want that those modifications of the distributions of neediness and income among households that do not affect their relative marginal distributions have no impact on the inequality of well-being. In particular, this requirement implies that changes in the distribution of demographic variables that do not modify the (relative) distribution of needs across households are inequality-neutral. On the other hand, a modification of the joint distribution of neediness and income that reduce the correlation between these two attributes is generally considered an improvement that contributes to the reduction of the inequality of well-being (see, e.g., Atkinson and Bourguignon 1982; Gravel and Moyes 2012, among others).

The implications for the structure of the inequality measure of household well-being of such changes in the distributions of the households' needs and incomes is something that—to the best of our knowledge—has been largely unexplored in the context of equivalence scales. Building on Ebert and Moyes (2003), we first show that the condition according to which inequality of well-being is reduced by a correlation decreasing transformation of the joint distribution of income and needs places severe constraints on the way households' incomes are adjusted for differing needs. Indeed, the equivalence scale is independent of the household's income and the weight associated to the equivalent income is proportional to the equivalence scale. A direct implication of this way of adjusting households' incomes is that inequality of well-being does not change when the incomes of all households—whatever their sizes—increase or decrease in the same proportion. Still, the fact that equivalence scales do not depend on income leaves open a number of possibilities for taking the households' needs into account. But, if in addition we impose that inequality of well-being is unaffected by proportional changes in the household degree of neediness, then the only possibility left for acknowledging differences in households' needs is to use isoelastic equivalence scales. Here again, it must be noted that this particular adjustment method has strong

[4] Under certain conditions, it is indeed possible to establish the existence of a single critical value of the size elasticity for which the poverty ranking of household-size groups switches (see Lanjouw and Ravallion 1995 for details).

implications for normative assessment: a replication of the population that preserves the relative distribution by size of the households does not modify the inequality of well-being.[5] This condition means that, if—other things being equal—all household sizes double, then inequality of well-being must remain the same. While it is certainly not our intention to suggest that the above conditions (in particular the last one) are normatively appealing, we insist on the fact that— in our model—to have recourse to isoelastic equivalence scales for adjusting incomes for needs or to subscribe to the above conditions are but two sides of the same coin.

For sure, equivalence scales with a constant elasticity are very special and they are expected to imply particularly strong restrictions on the households' patterns of preference. It is therefore doubtful whether such scales can be retrieved from the observation of the households' circumstances and behaviour, or, at the very least, they are expected to imply strong restrictions on the latter. It is important at this stage to distinguish between the preferences of the individuals who constitute the household and the principles that the household's members rely on when deciding how to distribute resources among themselves. Other things equal, the way the household decides about the distribution of its resources is expected to have an impact the distribution of well-being among its members. Various models, ranging from the unitary model (see, e.g., Samuelson 1956) to the collective one (see, e.g., Chiappori 1997) and comprising models rooted in bargaining theory (see, e.g., McElroy and Horney 1981) have been proposed in the literature to explain the household's behaviour.[6] If one is able to observe directly the well-being of each household member or to derive it from some behavioural model, then there is no need to search for equivalence scales: in this case, the distribution of well-being within the household is readily obtained.[7] Therefore, the question addressed in this paper makes little sense when one adopts the collective model of household behaviour or a bargaining model.

In this paper, we adopt an approach much in line with the unitary model of household behaviour where the household's members pool their incomes. Following Ebert and Moyes (2009), we consider an economy where the individuals derive utility from the consumption of two commodities: a private good and a public (to the household's members) good. We assume that all the individuals within the household have the same preferences and that the representation of the common preference ordering is the same for all the household's members. As far as the behaviour of the household is concerned, we posit that the household is utilitarian: it maximises the sum of utilities of its members. In our particular framework, the latter assumption amounts to assuming that the household maximises a symmetric and quasi-concave social welfare function of its members' utilities (see, e.g., Ebert and Moyes 2009). In this particular economy and under these restrictive assumptions, we are able to show that the only preferences

[5] Admittedly, for this requirement to make sense, one has to assimilate neediness with household size: for more on this, see Sect. 2.

[6] We refer the reader to Apps and Rees (2009, Chapter 3) that provides an expositional survey of the different models of household behaviour available in the literature.

[7] The difficulty facing researchers in practice is precisely that most microdata bases provide limited information—usually households' incomes and compositions—that does not allow one to uncover the actual distribution of well-being among the households' members.

that are consistent with isoelastic equivalence scales are of the Cobb–Douglas type. Needless to say, considering a different economy (e.g., introducing labour supply by the household members) or relaxing any of our assumptions (e.g., admitting the possibility that preferences differ or that the household is no longer maximising a social welfare function) would challenge this result.

We introduce in Sect. 2 the framework for our analysis and briefly recall how equivalence scales are derived in this setting. Section 3 is concerned with the measurement of inequality of well-being in the society where a person's well-being depends on the income and size of the household she belongs to. We show there that the conditions that inequality of well-being is reduced by a correlation decreasing transformation of the joint distribution of income and needs and unaffected by proportional changes in both variables—coupled with standard properties of (unidimensional) inequality measures—imply that the equivalence scales must be isoelastic. We investigate in Sect. 4 the implications for the household's members' preferences of this particular way of adjusting household incomes for needs under the assumptions that the individuals have the same utility function and that the household maximises the sum of utilities of its members. Finally, Sect. 5 concludes the paper, while the proofs of our results are collected in Sect. 6.

2 General framework and notation

A *household* consists of a finite group of individuals and we identify it with its *size* n. Typically, household size refers to the number of persons in the household and it therefore takes its values in the set of positive integers. For technical reasons, we assume throughout that household size is a continuous variable ranging from unity to infinity so that $n \in [1, +\infty)$. While it is made for convenience, this assumption suggests that n could be interpreted as an *index of neediness* that depends on both the household's composition and size. This way of proceeding would make it possible to take into account the heterogeneity nature of the households due to the fact that they comprise different types of individuals: think of adults and children both of different ages. It is reasonable to assume that a child contributes less to the household's expenses than an adult and she should therefore count less than an adult. A practical way of acknowledging this is to let $n(m_A, m_C) = m_A + \lambda m_C$ measure household size, where m_A and m_C are respectively the number of adults and children in the household and $\lambda \in [0, 1]$ is a parameter reflecting the importance of a child with respect to an adult (see, e.g., Cutler and Katz 1992). Given an arbitrary real $n \in [1, +\infty)$, it is always possible to find two non-negative integers m°, m^* and a real $\lambda \in [0, 1]$ such that $n = m^\circ + \lambda m^*$. This procedure can be generalised to an arbitrary list of individual types acknowledging for differences in age, health and the like. According to this convention, n is assimilated with the household's size—measured in terms of *equivalent adults* in the household—or equivalently as the household's type. While which interpretation one has in mind is immaterial for our results, we find convenient in what follows to refer to n as the household's size or type.

There are two commodities: a *private good* and a *public*—to the household's members—*good*. The quantity of the private good consumed by member i in the household is denoted by x_i, while Z represents the quantity of the public good purchased by the household. The prices of the private and public goods are indicated by $p > 0$ and $q > 0$, respectively. In order to allow for the possibility of congestion, we denote by

$$G = \phi(Z, \theta, n) := \frac{Z}{\psi(\theta, n)} \tag{2.1}$$

the *effective consumption of the public good* by any member of a household of type n. The parameter $\theta \in [0, 1]$ is a measure of the *degree of publicness* within the household, where $\theta = 0$ indicates the pure public good case while $\theta = 1$ corresponds to the private good case. The *congestion function* $\psi(\theta, n)$ has the following properties:

$$\psi(\theta, n) \text{ is differentiable in } \theta \text{ and } n, \quad \forall\, \theta \in [0, 1], \ \forall\, n \in [1, +\infty); \tag{2.2a}$$

$$\psi(\theta, n) \text{ is increasing in } \theta, \quad \forall\, n \in (1, +\infty); \tag{2.2b}$$

$$\psi(\theta, n) \text{ is increasing in } n, \quad \forall\, \theta \in [0, 1); \tag{2.2c}$$

$$\psi(0, n) = \psi(\theta, 1) = 1, \ \forall\, \theta \in [0, 1], \quad \forall\, n \in [1, +\infty); \tag{2.2d}$$

$$\psi(1, n) = n, \quad \forall\, n \in [1, +\infty). \tag{2.2e}$$

This allows for a wide range of specifications where the congestion function can follow different patterns with respect to n (see, e.g., Edwards 1990; Reiter and Weichenrieder 1999 for details). We denote by $\boldsymbol{\Psi}$ the set of congestion functions satisfying conditions (2.2a)–(2.2e). A particular instance of such congestion functions, that will be shown to be of particular interest later on, is given by $\psi(\theta, n) = n^{\theta}$ (see Borcherding and Deacon 1972). The possibility of crowding is best exemplified in the case where the household consists of students sharing an apartment. While such goods as lightning and heating are—at least in principle—public, other goods like access to the television, the newspaper(s), the telephone or the washing machine are partially excludable. Admittedly, congestion also happens in the family even though its extent is more limited.

We assume that all individuals have the same preferences and we indicate by $U(x, G)$ the utility derived by an individual consuming x units of the private good and G units of the public good, where the *individual utility function* U is assumed to be differentiable, monotone and strictly concave.[8] Different principles can be used by the household for allocating resources among its members and the way the household decides about this distribution has been shown to have important consequences (see, e.g., Ebert and Moyes 2009). We assume here that the individuals who constitute the household agree to allocate their resources between private and public consumption so that the household's welfare is maximised. This guarantees that the household members do not pursue their own self-interest but rather act in such a way as to provide all of them with the greatest well-being.

[8] In general, the fact that all individuals in the society have the same preferences does not imply that they have the same utility function. Since, in our model, individuals are all alike, it is natural to assume that they have also the same (cardinal) utility function.

The cooperative model and the assumption that individuals are identical may be considered reasonable approximations of the households' behaviour in modern societies where the household consists of a single family. For cultural and historical reasons, members of the same family are expected to share some common values, and cooperation among its members is the founding element of the family. One may therefore abstract from differences between the individuals who constitute the household—like adults and children—because they subscribe to the same common objectives and values. Things are quite different when the household is constituted by different families who join forces to achieve particular economic objectives. It follows from our assumptions that at the optimum all the household's members will get the same amount of the private good.

Formally, the household maximises the sum of the utilities derived by the equivalent adults who compose it.[9] The problem of a household of type n can be written as

$$\max \ n\,U(x, G) \ \text{ s.t. } \ pnx + (\psi(\theta, n)\,q)\,G \leqslant y, \qquad \text{P(H)}$$

where x is the consumption of the private good equal for all equivalent adults in the household, $y > 0$ is the household's total income, and $\psi(\theta, n)\,q$ can be interpreted as the price of one unit of effective consumption of the public good. Denoting as $X(p, q, y; \psi, \theta, n)$ and $G(p, q, y; \psi, \theta, n)$ the unique solution to problem P(H) and upon substitution into the utility function, we get the *representative indirect utility function*

$$V(p, q, y; \psi, \theta, n) := U(X(p, q, y; \psi, \theta, n), G(p, q, y; \psi, \theta, n)). \qquad (2.3)$$

Letting $u = V(p, q, y; \psi, \theta, n)$ and upon inverting, we obtain the *household expenditure function* $y = C(p, q, u; \psi, \theta, n)$, that indicates the minimum household income that guarantees that each of its members will reach the utility level u.

We follow the standard practice which involves choosing the household comprising a single individual as the reference household type ($n = 1$). Then, the *equivalent income function* $E(p, q, y; \psi, \theta, n)$ is implicitly defined by

$$V(p, q, y; \psi, \theta, n) = V(p, q, E(p, q, y; \psi, \theta, n); \psi, \theta, 1), \qquad (2.4)$$

which upon inverting gives

$$E(p, q, y; \psi, \theta, n) = C(p, q, V(p, q, y; \psi, \theta, n); \psi, \theta, 1). \qquad (2.5)$$

The *equivalent income* $E(p, q, y; \psi, \theta, n)$ represents the income that has to be given to a household of type $n = 1$ in order that its member enjoys the same utility as any member of household n with income y, given the prices p and q, the congestion function ψ, and the degree of publicness θ. It is quite standard in the literature to

[9] This is consistent with the maximisation of a symmetric, monotone non-decreasing and quasi-concave *social welfare function* $F(U(x_1, G), \ldots, U(x_n, G))$, where n is the number of persons in the household (see Bourguignon 1989; Blackorby and Donaldson 1993).

consider equivalence scales rather than the more general concept of the equivalent income function. The introduction of equivalence scales requires extra assumptions to be made about the relationship between the scales and the equivalent income or the household expenditure function. Starting with the household expenditure function $C(p, q, u; \psi, \theta, n)$, standard (relative) *equivalence scales* are defined by

$$M(p, q, u; \psi, \theta, n) = \frac{C(p, q, u; \psi, \theta, n)}{C(p, q, u; \psi, \theta, 1)}. \tag{2.6}$$

The scale $M(p, q, u; \psi, \theta, n)$ measures the extra cost for a household of size n of providing each of its members with the utility level u relative to that of the single person household. Equivalence scales can also be defined by using the equivalent income function rather than the expenditure function. Upon substituting $u = V(p, q, y; \psi, \theta, n)$ into (2.6), we obtain

$$\begin{aligned} m(p, q, y; \psi, \theta, n) &:= M(p, q, V(p, q, y; \psi, \theta, n); \psi, \theta, n) \\ &= \frac{y}{C(p, q, V(p, q, y; \psi, \theta, n); \psi, \theta, 1)} \\ &= \frac{y}{E(p, q, y; \psi, \theta, n)}, \end{aligned} \tag{2.7}$$

or equivalently

$$E(p, q, y; \psi, \theta, n) = \frac{y}{m(p, q, y; \psi, \theta, n)}. \tag{2.8}$$

In this case, the equivalent income is obtained by deflating household income by a scale factor, that depends on household size and, in principle, also on household income. So far, it is difficult to infer anything concerning the way the equivalent income and the equivalence scale react to modifications of the household's income and size because not much is known about the properties of the representative indirect utility function.

Consider now an individual with utility function U who has to allocate an income y/n between the consumption of the private good and the public good whose prices are p and $\psi(\theta, n)\, q/n$, respectively. The individual's optimisation problem is given by

$$\max \; U(x, G) \quad \text{s.t.} \quad px + \frac{\psi(\theta, n)\, q}{n} G \leqslant \frac{y}{n}, \tag{P(I)}$$

the solution to which is indicated by $\bar{X}(p, \psi(\theta, n)\, q/n, y/n)$ and $\bar{G}(p, \psi(\theta, n)\, q/n, y/n)$. Upon substitution and insertion into the individual utility function, we get the standard *individual indirect utility function*

$$\bar{V}\left(p, \frac{\psi(\theta, n)\, q}{n}, \frac{y}{n}\right) := U\left(\bar{X}\left(p, \frac{\psi(\theta, n)\, q}{n}, \frac{y}{n}\right), \bar{G}\left(p, \frac{\psi(\theta, n)\, q}{n}, \frac{y}{n}\right)\right), \tag{2.9}$$

which, upon inversion, gives the *individual expenditure function* $\bar{C}(p, \psi(\theta, n)\, q/n, u)$. We deduce from the definitions of the household expenditure and the individual expenditure functions that

$$y \equiv C(p, q, u; \psi, \theta, n) = n\,\bar{C}\left(p, \frac{\psi(\theta, n)\, q}{n}, u\right) \equiv n\,\frac{y}{n}. \tag{2.10}$$

Upon substituting into (2.6) and, since $q \geqslant (\psi(\theta, n)/n)\, q$, we finally get

$$M(p, q, u; \psi, \theta, n) = \frac{n\,\bar{C}\left(p, \frac{\psi(\theta,n)\, q}{n}, u\right)}{\bar{C}(p, q, u)} \leqslant n. \tag{2.11}$$

It follows from Ebert and Moyes (2009) that an arbitrary small amount of publicness is a necessary and sufficient condition for $M(p, q, u; \psi, \theta, n) < n$, whenever $n > 1$. This also implies that $E(p, q, y; \psi, \theta, n) < y$, for $n > 1$, in which case we say that we have (positive) *economies of size*, and, from now, we assume on that $\theta \in [0, 1)$. On the other hand, it must be noted that the assumptions we have made concerning the utility function do not permit us to tell how the equivalent income varies with household size.

3 Inequality of well-being and isoelastic scales

We consider *populations* comprising H households ($H \geqslant 2$), where each household is described by two attributes: its *income* and its *size*. A *heterogenous distribution*—or for short a *situation*—is a partitioned vector $(\mathbf{x}; \mathbf{n}) := (x_1, \ldots, x_H; n_1, \ldots, n_H)$, where $x_h > 0$ and $n_h \in [1, +\infty)$ are respectively the income and the size—equivalently, the degree of neediness or type—of household h. The set of situations for a population comprising H households is indicated by $\mathscr{S}_H$. While the household population size H is fixed throughout, we insist on the fact that the distribution of households' types $\mathbf{n} := (n_1, n_2, \ldots, n_H)$ may vary. The assumption of a fixed population size is not restrictive as long as the normative criteria one appeals to for comparing situations obey the *principle of population*.[10] Assuming that prices p and q, as well as the degree of publicness θ, and the congestion function ψ are fixed, we drop these from the formulae in this section in order to simplify the notation.

For comparisons of well-being across households to be meaningful, it is necessary to correct household incomes for differences in needs and this adjustment involves two *a priori* distinct components. On the one hand, the household's income is converted into an *equivalent income* which is the income needed by a household of type $n = 1$ (the reference household type) in order to achieve the same level of well-being as that attained by the original household. On the other hand, this equivalent income is attached a *weight* that is assumed to depend exclusively on household size. Formally, we associate to the situation $(\mathbf{x}; \mathbf{n}) := (x_1, \ldots, x_H; n_1, \ldots, n_H) \in \mathscr{S}_H$ the *adjusted*

[10] According to this principle, a replication of a situation leaves welfare, poverty, inequality, and the like unaffected.

income distribution

$$(E(\mathbf{x};\mathbf{n}) \mid w(\mathbf{n})) := (E(x_1;n_1),\ldots,E(x_H;n_H) \mid w(n_1),\ldots,w(n_H)), \quad (3.1)$$

where $E(x_h, n_h)$ and $w(n_h) > 0$ are respectively the equivalent income and the weight assigned to household h (see, e.g., Ebert and Moyes 2003). We assume throughout that the *equivalent income function* E has the following properties:

$$E(y;n) \text{ is differentiable in } y \text{ and } n, \quad \forall\, y > 0,\ \forall\, n \in [1,+\infty); \quad (3.2a)$$

$$E(y;n) \text{ is increasing in } y, \quad \forall\, y > 0,\ \forall\, n \in [1,+\infty); \quad (3.2b)$$

$$E(y;n) \text{ is decreasing in } n, \quad \forall\, y > 0,\ \forall\, n \in [1,+\infty);\ \text{and} \quad (3.2c)$$

$$\lim_{y\to 0} E(y;n) = 0, \quad \forall\, y > 0,\ \forall\, n \in [1,+\infty). \quad (3.2d)$$

The first two properties are inherited from the definition of the equivalent income and from the properties of the common utility function. The last property follows from taking the limit of (2.8) and from the fact that $m(y,n) \geqslant 1$, for all $n \in [1,+\infty)$ and all $y > 0$. To the extent that the equivalent income is considered an appropriate measure of the well-being of the household's members, it makes sense to assume that it is decreasing with the household's size: other things equal, the smaller is the size of the household, the greater is the well-being of any of its members.[11] On the other hand, the choice of the *weighting function* w is arbitrary and we only impose it to satisfy the two following conditions:

$$w(n) \text{ is differentiable in } n, \quad \forall\, n \in [1,+\infty) \text{ and} \quad (3.3a)$$

$$w(n) \text{ is non-decreasing in } n, \quad \forall\, n \in [1,+\infty). \quad (3.3b)$$

An *adjustment method* is a couple $(E \mid w)$ and, for later use, we denote by $\mathscr{A}$ the set of admissible adjustment methods, i.e., those $(E \mid w)$ satisfying conditions (3.2) and (3.3). The weighting function allows for different possibilities among which are the standard one consisting in weighting the equivalent income by the number of persons in the household, and the one that gives each household the same weight irrespective of its size and composition. The way in which the households' equivalent incomes are weighted is not innocuous and it has been shown to have important consequences for normative evaluation.[12] Let us indicate by $(\mathbf{s} \mid \mathbf{w}) := (s_1,\ldots,s_H \mid w_1,\ldots,w_H)$ a typical (unidimensional) *income distribution*, where s_h and w_h are respectively the income and the weight attached to household h, and by

[11] The standards properties of the utility function do not guarantee that this property be satisfied unless one imposes additional restrictions on the utility function that still need to be identified. For instance, the quasi-linear utility function $U(x,G) := x + 2\ln(1+G)$ does not generate an equivalent income that decreases with household size.

[12] This was first recognised by Glewwe (1991), who showed that a regressive transfer of income between two households might decrease the inequality of well-being when the equivalent incomes are weighted by the household sizes.

$$\mathscr{D}_H := \{(\mathbf{s} \mid \mathbf{w}) := (s_1, \ldots, s_H \mid w_1, \ldots, w_H) \mid s_h > 0 \text{ and } w_h > 0, \forall h\} \quad (3.4)$$

the set of such distributions. We note that, by definition, $(E(\mathbf{x}; \mathbf{n}) \mid w(\mathbf{n})) \in \mathscr{D}_H$, for all $(\mathbf{x}; \mathbf{n}) \in \mathscr{S}_H$, and we further use $\mu(\mathbf{s} \mid \mathbf{w})$ to represent the (arithmetic) *mean* of the income distribution $(\mathbf{s} \mid \mathbf{w}) \in \mathscr{D}_H$.

We are interested in comparing situations from the point of view of inequality of well-being for populations of households whose members face different circumstances. To this end, we introduce an *index of inequality of well-being* $I : \mathscr{S}_H \to \mathbb{R}$ such that $I(\mathbf{x}; \mathbf{n})$ measures the extent of inequality in situation $(\mathbf{x}; \mathbf{n})$ with the property that

$$I(\mathbf{x}; \mathbf{n}) = J(E(\mathbf{x}; \mathbf{n}) \mid w(\mathbf{n})), \quad \forall\ (\mathbf{x}; \mathbf{n}) \in \mathscr{S}_H, \quad (3.5)$$

where $J : \mathscr{D}_H \to \mathbb{R}$ is a (unidimensional) *inequality index*. According to definition (3.5), the assessment of the inequality of well-being is a two-stage process, where the two dimensions of the households' heterogeneity are first aggregated into a single measure (the equivalent income) and where the distributions of these equivalent incomes—appropriately weighted—are then compared by means of a standard inequality index. This two-stage process in the measurement of inequality for heterogenous populations is rather natural when one uses equivalence scales to adjust households' incomes for differences in needs and it has been widely used in empirical work.

The preceding discussion makes clear that the extent of inequality of well-being depends both on the index J and on the adjustment method $(E \mid w)$ one chooses. For instance, J can be the Gini index, a member of the Atkinson–Kolm–Sen (AKS) family of indices or a member of the generalised entropy family. There is no need for our purpose to choose a particular unidimensional inequality index: it suffices that J verifies four natural (in the inequality literature sense) conditions. The first condition is satisfied by most unidimensional inequality indices and it simply requires that the index takes the value zero only when all incomes are equal.

Normalisation (UN) For all $(\mathbf{s} \mid \mathbf{w}) \in \mathscr{D}_H$, we have $J(\mathbf{s} \mid \mathbf{w}) = 0$, if and only if $s_1 = s_2 = \cdots = s_H$.

The second condition is also standard and it states that inequality is not changed when incomes increase or decrease proportionally.

Scale invariance (USI) For all $(\mathbf{s}^* \mid \mathbf{w}), (\mathbf{s}^\circ \mid \mathbf{w}) \in \mathscr{D}_H$ and all $\nu > 0$, we have $J(\mathbf{s}^* \mid \mathbf{w}) = J(\mathbf{s}^\circ \mid \mathbf{w})$ whenever $\mathbf{s}^* = \nu\, \mathbf{s}^\circ$.

The next condition is but an adaptation in our framework of the *principle of population* due to Dalton (1920), according to which a replication of the population leaves inequality unchanged.

Distributional invariance (UDI) For all $(\mathbf{s} \mid \mathbf{w}^*), (\mathbf{s} \mid \mathbf{w}^\circ) \in \mathscr{D}_H$ and all $\lambda > 0$, we have $J(\mathbf{s} \mid \mathbf{w}^*) = J(\mathbf{s} \mid \mathbf{w}^\circ)$ whenever $\mathbf{w}^* = \lambda\, \mathbf{w}^\circ$.

According to condition UDI, inequality does not change when all the weights are increased or decreased in the same proportion. If the weights attached to the equivalent incomes are equal to the numbers of households having these incomes, then condition UDI reduces to the *principle of population* applied to households.[13] Before we turn to the fourth condition, we need to introduce a piece of additional notation. Let us indicate by $F(\cdot\,; (\mathbf{s}\,|\,\mathbf{w}))$ the *cumulative distribution function* of $(\mathbf{s}\,|\,\mathbf{w})$ and by $F^{-1}(\cdot\,; (\mathbf{s}\,|\,\mathbf{w}))$ its *inverse* obtained by letting $F^{-1}(0; (\mathbf{s}\,|\,\mathbf{w})) := \min\{s_1, s_2, \ldots, s_H\}$ and

$$F^{-1}(p; (\mathbf{s}\,|\,\mathbf{w})) := Inf\,\{z \in (-\infty, +\infty) \mid F(z; (\mathbf{s}\,|\,\mathbf{w})) \geq p\}, \tag{3.6}$$

for $\forall\, p \in (0, 1]$ (see Gastwirth 1971). The *Lorenz curve* of the income distribution $(\mathbf{s}\,|\,\mathbf{w}) \in \mathscr{D}_H$—denoted as $L(p; (\mathbf{s}\,|\,\mathbf{w}))$—is then defined by

$$L(p; (\mathbf{s}\,|\,\mathbf{w})) := \int_0^p F^{-1}(q; (\mathbf{s}\,|\,\mathbf{w}))\,dq, \quad \forall\; p \in [0, 1]. \tag{3.7}$$

Then, we will say that *distribution* $(\mathbf{s}^*\,|\,\mathbf{w}^*)$ *Lorenz dominates distribution* $(\mathbf{s}^\circ\,|\,\mathbf{w}^\circ)$, which we write $(\mathbf{s}^*\,|\,\mathbf{w}^*) \geq_L (\mathbf{s}^\circ\,|\,\mathbf{w}^\circ)$, if

$$L(p; (\mathbf{s}^*\,|\,\mathbf{w}^*)) \geqslant L(p; (\mathbf{s}^\circ\,|\,\mathbf{w}^\circ)),\ \forall\; p \in (0, 1),\ \text{and}\quad \mu(\mathbf{s}^*\,|\,\mathbf{w}^*) = \mu(\mathbf{s}^\circ|\mathbf{w}^\circ). \tag{3.8}$$

The fourth condition captures the very idea of inequality reduction and it requires that inequality does not increase when incomes are more equally distributed in the sense that the Lorenz curve moves upwards.

Lorenz consistency (ULC) For all $(\mathbf{s}^*\,|\,\mathbf{w}^*), (\mathbf{s}^\circ\,|\,\mathbf{w}^\circ) \in \mathscr{D}_H$, we have $J(\mathbf{s}^*\,|\,\mathbf{w}^*) \leqslant J(\mathbf{s}^\circ\,|\,\mathbf{w}^\circ)$ whenever $(\mathbf{s}^*\,|\,\mathbf{w}^*) \geq_L (\mathbf{s}^\circ\,|\,\mathbf{w}^\circ)$.

The above condition is a different way of stating the *principle of transfers* according to which inequality decreases as the result of a progressive transfer.[14]

The class of indices defined by (3.5) and such that the inequality index J possesses the four properties above is presumably large and there is no reason *a priori* to believe that all such indices may be considered appropriate measures of the inequality of well-being. We propose below three conditions that will allow us to put more structure on the index I and, as a consequence, to narrow down the set of admissible indices for measuring inequality of well-being. As we already insisted in the Introduction, this does not mean that all the conditions we review below are ethically acceptable, nor that they exhaust all the possibilities. But, as we will see in a while, they appear to be necessary and sufficient to generate isoelastic equivalence scales. The first condition is merely a restatement of the standard *scale invariance* property in the unidimensional

[13] Actually, condition UDI is stronger than the *principle of population* to the extent that the weights are not necessarily equal to the numbers of households who have that particular income: for instance, weights may be used to improve the representativeness of the sample data.

[14] We refer the interested reader to Ebert and Moyes (2002) for a proof of this assertion in the particular framework considered here.

setting and it demands that proportional changes in the households' incomes have no impact on the inequality of well-being in the population.

Income scale invariance (BISI) For all $(\mathbf{x}^*; \mathbf{n}), (\mathbf{x}^\circ; \mathbf{n}) \in \mathscr{S}_H$ and all $\nu > 0$, we have $I(\mathbf{x}^*; \mathbf{n}) = I(\mathbf{x}^\circ; \mathbf{n})$ whenever $\mathbf{x}^* = \nu\mathbf{x}^\circ$.

Similarly, our second condition insists on the fact that demographic changes that result in proportional shifts in the distribution of household types have no impact on the inequality of well-being.

Neediness scale invariance (BNSI) For all $(\mathbf{x}; \mathbf{n}^*), (\mathbf{x}; \mathbf{n}^\circ) \in \mathscr{S}_H$ and all $\lambda > 0$, we have $I(\mathbf{x}; \mathbf{n}^*) = I(\mathbf{x}; \mathbf{n}^\circ)$ whenever $\mathbf{n}^* = \lambda\mathbf{n}^\circ$.

While the standard *scale invariance* property (USI) for unidimensional inequality indices is mirrored by condition BISI in our bidimensional setting, things are totally different as far as conditions BNSI and UDI are concerned. Indeed, according to UDI, a replication of the population of individuals leaves inequality of income unchanged whereas, in the case of BNSI, it is a proportional change in the distribution of household types that—other things equal—is deemed to have no impact of the inequality of well-being. Thus, there is no guarantee that a replication of the population of individuals translates into a proportional shift of the distribution of household types. On the other hand, the latter transformation can be seen as a particular replication of the population of individuals.

Formally, conditions BISI and BNSI impose that changes that do not modify the (relative) inequality of the marginal distributions of household income and household size have no impact on the inequality of well-being.[15] At this stage, and in the absence of additional restrictions placed on the adjustment method, there is no guarantee that these two conditions be satisfied. Nor it is clear what is the interest of requiring that the unidimensional inequality index J is Lorenz consistent. Indeed, the latter property has no particular meaning in the present context, unless one is able to relate the *shifts of the Lorenz curves* of the adjusted income distributions to *particular modifications* of the joint distributions of household income and neediness that represent uncontroversial reductions of inequality of well-being. In this respect, the next transformation (see, e.g., Ebert 2000; Gravel and Moyes 2012), which fully exploits the bidimensionality of a situation, constitutes in our model a natural generalisation of the notion of a (unidimensional) progressive transfer. Given two situations $(\mathbf{x}^*; \mathbf{n}^*), (\mathbf{x}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$, we will say that $(\mathbf{x}^*; \mathbf{n}^*)$ *is obtained from* $(\mathbf{x}^\circ; \mathbf{n}^\circ)$ *by means of a between-type progressive transfer*, if there exists $\Delta > 0$ and two households i, j such that

$$x_i^\circ < x_i^* \leqslant x_j^* < x_j^\circ;\ \ n_j^\circ = n_j^* < n_i^* = n_i^\circ; \tag{3.9a}$$

$$x_i^* - x_i^\circ = x_j^\circ - x_j^* = \Delta;\ \text{and} \tag{3.9b}$$

$$(x_h^*, n_h^*) = (x_h^\circ, n_h^\circ),\ \ \forall\, h \neq i, j. \tag{3.9c}$$

[15] Combining these two conditions, we obtain the kind of invariance property considered in the standard multidimensional inequality literature, where different scalings are used for different attributes (see, e.g., Tsui 1995).

Like a progressive transfer, a between-type progressive transfer consists in taking some income from a richer household for giving it to a poorer household in such a way that the donor is not made poorer than the receiver. The difference is that the household who is poor in income is also the one who has greater needs while the rich in income household has less needs. A between-type progressive transfer fully acknowledges the multidimensional nature of inequality—household i is more deprived than household j in both income and needs—but only the first attribute is used for reducing inequality.[16] Indeed, given the nature of the need variable, it does not make sense to redistribute needs from needy to less needy households and we therefore prevent ourselves from so doing. Hence, by definition, a between-type progressive transfer does not modify the distribution of household types: if $(\mathbf{x}^\circ; \mathbf{n}^\circ)$ is converted into $(\mathbf{x}^*; \mathbf{n}^*)$ by means of a between-type progressive transfer, then $\mathbf{n}^* = \mathbf{n}^\circ$. If we subscribe to the judgement that a between-type progressive transfer constitutes an unambiguous inequality-reducing operation in our particular setting, then we surely want to impose the following condition:

Weak equity (BWE) For all $(\mathbf{x}^*; \mathbf{n}^*), (\mathbf{x}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$, we have $I(\mathbf{x}^*; \mathbf{n}^*) \leqslant I(\mathbf{x}^\circ; \mathbf{n}^\circ)$ whenever $(\mathbf{x}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{x}^\circ; \mathbf{n}^\circ)$ by means of a between-type progressive transfer.

Although the condition that inequality of well-being is reduced by a between-type progressive transfer appears at first sight to be a mild requirement, it has important consequences for the structure of the inequality index I as the next result demonstrates.

Proposition 3.1 *Let I be as defined by (3.5) and assume that J satisfies condition* ULC. *Then:*

$$E(y; n) = \frac{y}{K(n)}, \quad \forall\, y > 0,\ \forall\, n \in [1, +\infty),\ \textit{and} \tag{3.10a}$$

$$w(n) = \eta K(n), \quad \forall\, n \in [1, +\infty)\ \textit{and for some}\ \eta > 0, \tag{3.10b}$$

are necessary and sufficient for condition BWE *to be fulfilled.*

According to Proposition 3.1, the imposition of condition BWE restricts the way the adjusted income distributions are derived in two respects. Firstly, the equivalence scale must be independent of household income: in other terms, the equivalence scale satisfies the *relative equivalence scale exactness* condition of Blackorby and Donaldson (1993), or equivalently the *independence of base level* condition due to Lewbel (1989) (see also Blundell and Lewbel 1991). Secondly, the weights associated to the households' equivalent incomes must be proportional to the corresponding equivalence scales. It follows that the equivalent income function and the weighting function cannot be chosen independently from each other if we want inequality of well-being—

[16] To this extent, a between-type progressive transfer is a particular case of the more general transformation introduced by Kolm (1977) who requires that transfers take place in all attributes. It must also be stressed that Kolm (1977) imposes no restrictions on the respective positions—with respect to the different attributes—of the households involved in this generalised transfer. In particular, it is not necessary that one household be richer than another in all attributes for the transfer to make sense.

as measured by an index defined by (3.5)—to decrease as the result of a between-type progressive transfer.[17]

On the other hand, Proposition 3.1 does not give any indication about the form of the size dependency of the equivalence scale and there are a number of possibilities consistent with the restrictions expressed by (3.10). Upon substituting (3.10a) and (3.10b) into (3.5), we obtain:

$$I(\mathbf{x}; \mathbf{n}) = J\left(\frac{x_1}{K(n_1)}, \ldots, \frac{x_H}{K(n_H)} \,\middle|\, \eta K(n_1), \ldots, \eta K(n_H)\right). \tag{3.11}$$

Clearly, all indices I as defined by (3.11) satisfy BISI provided that the unidimensional inequality index J is *scale invariant*. On the other hand, *scale invariance* (USI)—even combined with *distributional invariance* (UDI)—does not guarantee that an index of inequality of well-being I of the form (3.11) will obey condition BNSI. However, BWE in conjunction with the four standard properties UN, USI, UDI and ULC ensures that the index I satisfies BNSI. The following result summarises the above discussion:

Proposition 3.2 *Let I be as defined by* (3.5) *and assume that J satisfies conditions* UN, USI, UDI *and* ULC *and I satisfies condition* BWE. *Then:*

$$K(n) = \xi\, n^{\varepsilon} \ (\xi, \varepsilon > 0), \quad \textit{for all } n \in [1, +\infty), \tag{3.12}$$

is necessary and sufficient for conditions BNSI *and* BISI *to be fulfilled.*

While the imposition of BNSI in a heterogenous setting is admittedly open to debate, we insist on the fact that it is this condition that—in conjunction with condition BWE and standard properties of unidimensional inequality indices— precipitates isoelastic equivalence scales. We admit that, among the three properties of the index of inequality of well-being that we have considered, condition BNSI is with no doubt the most controversial one. However, if we subscribe to isoelastic equivalence scales and to the two-stage procedure we have followed for measuring inequality of well-being, then we have to accept condition BNSI. Or, if we feel unable to accept condition BNSI, then we have to abandon isoelastic equivalence scales: within the model considered here, there is no way to escape this dilemma.

4 Recovering the household's members' common preferences

We have shown that under certain conditions relative to the way one conceives of inequality of well-being, equivalence scales with a constant elasticity constitute the only possibility for adjusting household income for needs. Isoelastic scales are very specific and they are likely to imply strong restrictions both on the behaviour of the household and on its members' preferences. Taking for granted that the household maximises the sum of utilities of its members who, in addition, are assumed

[17] This result is reminiscent of Ebert and Moyes (2003) who obtained similar restrictions on the adjustment method but using a slightly different approach.

to be all alike, we are interested here in the implications of such scales for the common—to all the household's members—preference ordering. Upon substituting $\xi =: \tilde{g}(p, q, y; \psi, \theta, 1)$ and $\varepsilon =: \tilde{f}(p, q, y; \psi, \theta, 1)$ into (3.12), we get

$$m(p, q, y; \psi, \theta, n) = \tilde{g}(p, q, y; \psi, \theta, 1)\, n^{\tilde{f}(p,q,y;\psi,\theta,1)}, \tag{4.1}$$

which holds for all $\psi \in \boldsymbol{\Psi}$, $n \geqslant 1$, $\theta \in [0, 1)$, $y > 0$ and all $(p, q) \gg (0, 0)$. Appealing next to (2.7), we obtain a similar condition when the equivalence scale refers to the household representative member's utility. More precisely, letting

$$g(p, q, u; \psi, \theta, 1) := \tilde{g}(p, q, C(p, q, u; \psi, \theta, 1); \psi, \theta, 1) \text{ and} \tag{4.2a}$$

$$f(p, q, u; \psi, \theta, 1) := \tilde{f}(p, q, C(p, q, u; \psi, \theta, 1); \psi, \theta, 1), \tag{4.2b}$$

we obtain the utility-based equivalence scale

$$M(p, q, u; \psi, \theta, n) = g(p, q, u; \psi, \theta, 1)\, n^{f(p,q,u;\psi,\theta,1)}, \tag{4.3}$$

for all $\psi \in \boldsymbol{\Psi}$, $n \geqslant 1$, $\theta \in [0, 1)$, $u \in \mathbb{R}$ and all $(p, q) \gg (0, 0)$.

By definition, the equivalence scale $m(p, q, y; \psi, \theta, n)$ is isoelastic if (4.1) holds, for all $\psi \in \boldsymbol{\Psi}$, all $n \in [1, +\infty)$, all $\theta \in [0, 1)$, all $y > 0$, and all $(p, q) \gg (0, 0)$. To see what this actually means, consider the condition according to which the relative change in the equivalence scale due to a proportional increase of household size is independent of the latter. Formally, this condition amounts to requiring that:

$$\frac{m(p, q, \tilde{y}; \psi, \theta, \lambda\tilde{n})}{m(p, q, \tilde{y}; \psi, \theta, \tilde{n})} = \frac{m(p, q, \hat{y}; \psi, \theta, \lambda\hat{n})}{m(p, q, \hat{y}; \psi, \theta, \hat{n})}, \tag{4.4}$$

for all $\tilde{n}, \hat{n} \geqslant 1$, all $\tilde{y}, \hat{y} > 0$, all $\lambda > 1$ and all $(p, q; \psi, \theta)$, or equivalently that

$$m(p, q, y; \psi, \theta, \lambda n) = \varphi(p, q, y; \psi, \theta, \lambda)\, m(p, q, y; \psi, \theta, n), \tag{4.5}$$

for all $n \geqslant 1$, all $y > 0$, all $\lambda > 1$ and all $(p, q; \psi, \theta)$. This is a functional equation the solution of which (see Aczel (1966, Chapter 3)) is precisely (4.1). Hence, isoelastic scales have the property that a proportional increase in household size translates into a proportional—but not necessarily of the same magnitude— increase in the scale. Upon substitution of (4.1) into (2.8), we obtain the equivalent income:

$$E(p, q, y; \psi, \theta, n) = \frac{y}{\tilde{g}(p, q, y; \psi, \theta, 1)}\, n^{-\tilde{f}(p,q,y;\psi,\theta,1)}, \tag{4.6}$$

for all $n \geqslant 1$, all $\lambda > 1$ and all $(p, q, y; \psi, \theta)$. The equivalent income function inherits the properties of the equivalence scale: it is isoelastic with respect household size. Whatever the household income, the relative increase of the equivalent income needed for compensating a proportional increase of the household size is independent of its size. To make things clearer, suppose that there are four types of households: singles, couples with no children, couples with one child, and couples with two children.

Assume further, as we have been doing implicitly throughout the paper, that children and adults are counted as identical persons in the household. Consider a population consisting of two households i and j, and let $(\mathbf{x}^\circ; \mathbf{n}^\circ) = (x_i^\circ, x_j^\circ; n_i^\circ, n_j^\circ) = (s, t; 1, 2)$ and $(\mathbf{x}^*; \mathbf{n}^*) = (x_i^*, x_j^*; n_i^*, n_j^*) = (s, t; 2, 4)$, be two alternative situations for this society. The fact that the equivalent income function is isoelastic with respect to household size ensures that:

$$\frac{E(p, q, t; \psi, \theta, 4)}{E(p, q, s; \psi, \theta, 2)} = \frac{E(p, q, t; \psi, \theta, 2)}{E(p, q, s; \psi, \theta, 1)}. \tag{4.7}$$

According to (4.7), a doubling of households i and j's sizes preserves the initial ratio of households i and j's equivalent incomes. This is true whatever the values of the households i and j's incomes and it generalises to arbitrary household sizes and proportional changes of these sizes. Equation (4.7) can be considered a partial homotheticity condition of the equivalent income function with respect to household size.[18]

Taking for granted that isoelastic equivalence scales are deemed to be ethically relevant, the question is to know whether there exist preferences that, in our *particular* model, would generate such scales. Consider an individual who has Cobb-Douglas preferences represented by the utility function $U(x, G) = x^c G^{1-c}$, where $0 < c < 1$. Letting $n = 1$, the solutions to problem P(I) are $x = \bar{X}(p, q, y) = cy/p$ and $G = \bar{G}(p, q, y) = (1-c)y/q$. Upon substitution into $U(x, G)$, we obtain the individual indirect utility function

$$\bar{V}(p, q, y) = \left(\frac{c}{p}\right)^c \left(\frac{1-c}{q}\right)^{1-c} y, \tag{4.8}$$

which upon inverting gives the individual expenditure function

$$\bar{C}(p, q, u) = \left(\frac{p}{c}\right)^c \left(\frac{q}{1-c}\right)^{1-c} u. \tag{4.9}$$

Substituting $\psi(\theta, n)q/n$ for q into (4.9) and using (2.10), we get the household expenditure function

$$C\,(p, q, u; \psi, \theta, n) = n\,\bar{C}\left(p, \frac{\psi(\theta, n)\,q}{n}, u\right) = n\,\left(\frac{p}{c}\right)^c \left(\frac{\psi(\theta, n)\,q}{(1-c)\,n}\right)^{1-c} u, \tag{4.10}$$

[18] In the case of two variables, standard homotheticity requires that $f(\lambda\mathbf{u})/f(\mathbf{u}) = f(\lambda\mathbf{v})/f(\mathbf{v})$, for all $\mathbf{u} := (u_1, u_2), \mathbf{v} := (v_1, v_2) \in \mathbb{R}^2_{++}$ and all $\lambda > 1$. On the other hand, (partial) homotheticity in the first variable would impose that $f(\lambda u_1, u_2)/f(u_1, u_2) = f(\lambda v_1, v_2)/f(v_1, v_2)$, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{++}$ and all $\lambda > 1$.

which can be rewritten as

$$C\,(p,q,u;\psi,\theta,n) = n\left(\frac{\psi(\theta,n)}{n}\right)^{1-c}\left(\frac{p}{c}\right)^{c}\left(\frac{q}{1-c}\right)^{1-c}u. \tag{4.11}$$

We derive the household (relative) equivalence scale

$$M\,(p,q,u;\psi,\theta,n) = n\left(\frac{\psi(\theta,n)}{n}\right)^{1-c}, \tag{4.12}$$

which is independent of the household representative utility. If, in addition,

$$\psi(\theta,n) = n^{\rho(\theta)}, \tag{4.13}$$

then we get

$$M\,(p,q,u;\psi,\theta,n) = n^{c+(1-c)\,\rho(\theta)}, \tag{4.14}$$

and the equivalence scale is isoelastic with respect to household size. To sum up: (1) if preferences are Cobb–Douglas and (2) if the congestion function is isoelastic, then the (relative) equivalence scale is isoelastic. It is fair to acknowledge that a related result has been obtained by Fleurbaey and Gaulier (2009) in the case of a pure household public good which amounts to setting $\theta = 0$ in (4.13) above. However, we insist on the fact that, in the general case where one allows for the possibility of congestion and where $\psi(\theta,n) \neq n^{\rho(\theta)}$, Cobb-Douglas preferences do not guarantee that the equivalence scale will be isoelastic. This can be seen by letting $c = 0.50$ and choosing

$$\psi(\theta,n) = \frac{n^{\theta^2} + n^{\theta}}{2}, \tag{4.15}$$

that satisfies conditions (2.2a)–(2.2e). Consider again a population consisting of two households i and j, and choose $(\mathbf{x}^\circ;\mathbf{n}^\circ) = (x_i^\circ, x_j^\circ; n_i^\circ, n_j^\circ) = (3,6;1,2)$ and $(\mathbf{x}^*;\mathbf{n}^*) = (x_i^*, x_j^*; n_i^*, n_j^*) = (3,6;3,6)$ be two alternative situations for this society. Letting $p = q = 1$ and $\theta = 0.50$, we get

$$\frac{m(p,q,6;\psi,\theta,6)}{m(p,q,3;\psi,\theta,3)} = 2.1508 > 2.1382 = \frac{m(p,q,6;\psi,\theta,2)}{m(p,q,3;\psi,\theta,1)}, \tag{4.16}$$

which violates condition (4.4). Hence, the presumption that, in the presence of congestion, Cobb–Douglas preferences alone do not guarantee that the equivalence scales will be isoelastic.

The discussion above makes clear that, provided that the congestion function has the particular shape given by (4.12), Cobb–Douglas preferences give rise to equivalence scales with constant elasticity with respect to household size. The interesting thing is that, in our model, Cobb–Douglas preferences and an isoelastic congestion function are also necessary for the equivalence scales to be isoelastic, as the following result indicates.

Proposition 4.1 *Assume that* $\psi \in \boldsymbol{\Psi}$ *and* $\theta \in [0,1)$. *Then,* $m\,(p,q,y;\psi,\theta,n)$ *satisfies condition* (4.1)*—equivalently,* $M\,(p,q,u;\psi,\theta,n)$ *satisfies condition* (4.3)*—if and only if:*

(a) *There exists* $\rho:[0,1)\rightarrow[0,1)$ *such that* $\psi(\theta,n)=n^{\rho(\theta)}$, *and*

(b) $U(x,G)=x^{c}G^{1-c}$, *for some* $c\in(0,1)$.

It follows from Cobb–Douglas preferences that the consumption of the private good by any member of the household is linear and increasing in household income and similarly for the consumption public good. On the other hand, the consumption of both goods by any member of the household is a decreasing and convex function of household size, and public consumption decreases at a slower pace than does private consumption whenever $c > 1 - c$.

To the extent that Cobb–Douglas preferences are a particular case of homothetic preferences, one might wonder whether the homotheticity of preferences would have not been sufficient to generate isoelastic scales in the presence of a congestion function of the form (4.13). Actually, this is not true as it can be seen by considering the CES family of utility functions:

$$U(x,G)=\left(a\,x^{c}+b\,G^{c}\right)^{\frac{1}{c}}, \quad \text{where } a,b>0 \text{ and } c>0. \tag{4.17}$$

Indeed, assuming a pure household public good ($\theta = 0$) and letting $a = b = 1.00$, $c = 0.50$ and $p = q = 1$, we obtain

$$m\,(p,q,y;\psi,\theta,n)=\frac{2n}{n+1}, \quad \text{for all } n\geqslant 1 \text{ and all } y>0, \tag{4.18}$$

which is clearly not isoelastic. Therefore, CES preferences—and more generally homothetic preferences—do not guarantee that the equivalence scales will always be isoelastic. On the other hand, we note that the equivalence scale does not depend on the household income as it is the case for the Cobb-Douglas utility function. This ceases to be the case when the utility function is not homothetic as it can be seen by choosing the utility function $U(x,G) := x + 2\ln(1+G)$ for instance. Then, when $p = q = 1$ and $\theta = 0.80$, one can check that $m(p,q,y;\psi,\theta,n)$ increases with y, whatever $n > 1$.

5 Concluding remarks

In this paper, we have taken for granted that, for comparisons of well-being across populations of heterogenous households to be meaningful, one has to make proper adjustments of household incomes to acknowledge differences in needs. We have, to a large extent, followed the current practice in most empirical studies that consists in comparing the distributions of the households' equivalent incomes by means of standard (unidimensional) measures of welfare, poverty or inequality. On the *normative* side, we have shown that, if we want (1) that inequality decreases as the result of a natural extension of the notion of a progressive transfer, and (2) that it is unaffected by proportional changes in both household incomes and sizes, then the scales have to be isoelastic (Propositions 3.1 and 3.2). On the *positive* side, we have shown that, if (1) all individuals have identical preferences and (2) households maximise the sum of their members' utilities, then Cobb–Douglas preferences are necessary and sufficient for such equivalence scales to arise, provided that the congestion function has a specific form (Proposition 4.1). While it is generally claimed that isoelastic scales provide reasonable approximations of the scales currently used in practice, Proposition 4.1 uncovers the restrictions one has to place implicitly on the household's members preferences as well as on the way the intra-allocation of resources is determined by the household in order to generate such scales. It is interesting to note that isoelastic scales can also be recovered from personal judgements within the subjective approach model of Kapteyn and Praag (1976).

By definition, the equivalence scale depends both on the household welfare—measured by the utility of a representative member—and on the prices of the private and public goods. Experimental evidence suggests that equivalence scales are not independent of household income—or equivalently of household welfare—as shown, for instance, by Koulovatianos et al. (2005a), Koulovatianos et al. (2005a, b). This is supported by econometric studies that indicate that the scales values vary significantly with the income of the household (see, e.g., Donaldson and Pendakur 2004). This is in total contradiction with our results which insist on the fact that—among other things—a necessary condition for the inequality of well-being to be reduced by a transfer of income from a richer and smaller household to a poorer and larger household is that the equivalence scale $m(p, q, y; \psi, \theta, \lambda n)$ is independent of household income y (Proposition 3.1) or, equivalently, that $M(p, q, u; \psi, \theta, n)$ is independent of the household representative utility u. It follows that isoelastic equivalence scales satisfy the condition of *relative equivalence scale exactness* (Blackorby and Donaldson 1993) or the condition of *independence of base level* (Lewbel 1989; Blundell and Lewbel 1991). Proposition 4.1 indicates in addition that, if households behave cooperatively and individuals have the same Cobb–Douglas preferences, then the equivalence scales are also independent of prices.

Crucial for our result are the assumptions that the household members behave in a *cooperative* way and that they are *identical* in all respects. Suppose the household members do not cooperate—every household member decides in isolation the amount she is willing to contribute to the household public good—and that preferences are identical and Cobb-Douglas. Then, the (relative) equivalence scale is

$$M(p, q, u; \psi, \theta, n) = (1 + c(n-1))\psi(\theta, n)^{1-c} \tag{5.1}$$

(see Ebert and Moyes 2009), which, assuming that the congestion function is isoelastic, reduces to

$$M(p, q, u; \psi, \theta, n) = (1-c)n^{(1-c)\theta} + cn^{(1-c)\theta+1}. \tag{5.2}$$

Thus, as far as our first assumption is concerned, we note that in the non-cooperative case Cobb-Douglas preferences no longer lead to isoelastic scales. However, for sufficiently small values of c, isoelastic scales provide a reasonable approximation of the resulting scales. But, the main limitation of our approach is certainly the strong assumption that individuals are identical in all respect. This is at variance with the real world where households typically consist of individuals of different types— for instance adults and children—who are likely to have distinct preferences. Allowing for such a heterogeneity into our model and investigating its implications for the structure of the equivalence scales is certainly the next step to be taken.

Finally, although our results are concerned with the measurement of inequality in a context where the income-receiving units are heterogenous, we would like to emphasise that they apply equally to the measurement of welfare and poverty provided (1) that the corresponding measures can be factorised in the same way as is our measure of inequality of well-being and (2) that conditions BWE, BISI and BNSI be suitably adapted. One the one hand, most empirical studies appeal to the two-stage procedure we have used in this paper for assessing social welfare and poverty. On the other hand, it seems reasonable to require that poverty decreases and social welfare increases as the result of a between-type progressive transfer. Therefore, if one subscribes to the two above views, then one would end up with the adjustment method defined by (3.10a) and (3.10b): the equivalence scale is independent of household income and the weight associated to the equivalent income is proportional to the scale. This still leaves some room for those who do not accept condition BNSI even though the range of remaining admissible equivalence scales is limited.

6 Proofs

Proof of Proposition 3.1 To the extent that the proof builds to a large extent on arguments used by the authors in Ebert and Moyes (2003), we only sketch the general line of reasoning. Consider the following condition that refers to the impact of a between-type progressive transfer (BTPT) on the distributions of equivalent incomes. □

Weak living standard equity (WLSE) Assume that $(E \mid w) \in \mathscr{A}$. Then, for all $(\mathbf{x}^*; \mathbf{n}^*), (\mathbf{x}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$, we have

$$(E(\mathbf{x}^*; \mathbf{n}^*) \mid w(\mathbf{n}^*)) \geq_L (E(\mathbf{x}^\circ; \mathbf{n}^\circ) \mid w(\mathbf{n}^\circ)), \tag{6.1}$$

whenever $(\mathbf{x}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{x}^\circ; \mathbf{n}^\circ)$ by means of a between-type progressive transfer.

Note that both conditions BWE and WLSE emphasise the impact of a between-type progressive transfer. Whereas the first condition is concerned with the inequality of household well-being, the second condition focuses on the distribution of the adjusted incomes. Given the structure of our index of inequality of well-being, we expect that these two conditions are related as Fig. 1 suggests, and in fact, they are as we shall see.

We first show that, whenever J fulfills condition ULC, a necessary and sufficient condition for I to satisfy BWE is that the adjustment method $(E \mid w)$ obeys condition WLSE. It is obvious that, if J and $(E \mid w)$ satisfy respectively ULC and WLSE, then condition BWE is fulfilled by I. To prove the converse, let $(\mathbf{x}^*; \mathbf{n}^*), (\mathbf{x}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$ be

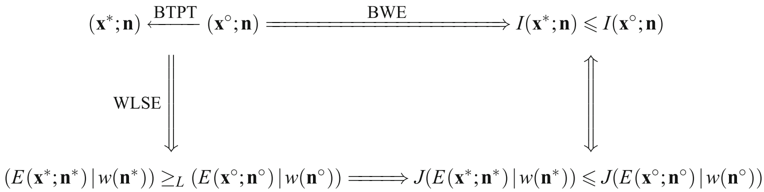

Fig. 1 Logic of proof of Proposition 3.1

two situations such that $(\mathbf{x}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{x}^\circ; \mathbf{n}^\circ)$ by means of a between-type progressive transfer. Assuming that I satisfies condition BWE, we have:

$$I(\mathbf{x}^*; \mathbf{n}^*) = J(E(\mathbf{x}^*; \mathbf{n}^*)\,|\,w(\mathbf{n}^*)) \leqslant J(E(\mathbf{x}^\circ; \mathbf{n}^\circ)\,|\,w(\mathbf{n}^\circ)) = I(\mathbf{x}^\circ; \mathbf{n}^\circ), \quad (6.2)$$

that holds for all J that are Lorenz-consistent and it follows from Proposition 3.1 in Ebert and Moyes (2002) that

$$(E(\mathbf{x}^*; \mathbf{n}^*)\,|\,w(\mathbf{n}^*)) \geq_L (E(\mathbf{x}^\circ; \mathbf{n}^\circ)\,|\,w(\mathbf{n}^\circ)). \quad (6.3)$$

This reasoning applies whatever between-type progressive transfer is considered and we therefore conclude that J satisfies condition WLSE.

It remains for us to establish that condition WLSE holds if and only if the adjustment method is the one defined by (3.10a) and (3.10b). More precisely, we have to prove the following result:

Proposition 6.1 *The adjustment method* $(E\,|\,w)$ *satisfies condition* WLSE *if and only if*

$$E(y; n) = \frac{y}{K(n)}, \quad \forall\ y > 0,\ \forall\ n \in [1, +\infty),\ \textit{and} \quad (6.4a)$$

$$w(n) = \eta K(n), \quad \forall\ n \in [1, +\infty)\ \textit{and for some}\ \eta > 0. \quad (6.4b)$$

The proof of this result is pretty close to that of Proposition 4.2 in Ebert and Moyes (2003) and we refer the interested reader to Ebert and Moyes (2016) that provides the full details. □

Proof of Proposition 3.2 As we already noted, it follows from Proposition 3.1 that condition BISI is automatically satisfied. Hence, we only have to show that isoelastic scales are necessary and sufficient for BISI to hold. □

Necessity Assume that the unidimensional inequality index J satisfies UN, USI, UDI and ULC, and that the index of inequality of well-being I satisfies condition BWE. We have to show that, if the index I fulfills condition BNSI, then the equivalence scale is isoelastic. Choosing first $(\mathbf{x}^\circ; \mathbf{n}^\circ) = (K(n_1), \ldots, K(n_H); n_1, \ldots, n_H)$ and making use of (3.5), we obtain

$$I(\mathbf{x}^\circ; \mathbf{n}^\circ) = J\left(E(x_1^\circ; n_1^\circ), \ldots, E(x_H^\circ; n_H^\circ)\ \middle|\ w(n_1^\circ), \ldots, w(n_H^\circ)\right) \quad (6.5)$$

$$= J\left(\frac{K(n_1)}{K(n_1)}, \ldots, \frac{K(n_H)}{K(n_H)} \,\middle|\, \eta\, K(n_1), \ldots, \eta\, K(n_H)\right) \qquad \text{by Prop. 3.1}$$
$$= J\,(1, \ldots, 1 \mid \eta\, K(n_1), \ldots, \eta\, K(n_H))$$
$$= J\,(1, \ldots, 1 \mid K(n_1), \ldots, K(n_H)) \qquad \text{by UDI}$$
$$= 0. \qquad \text{by UN}$$

Choosing next $(\mathbf{x}^*; \mathbf{n}^*) = (K(n_1), \ldots, K(n_H); \lambda n_1, \ldots, \lambda n_H)$ with $\lambda > 0$, we obtain by a similar reasoning

$$I(\mathbf{x}^*; \mathbf{n}^*) = J\left(E(x_1^*; n_1^*), \ldots, E(x_H^*; n_H^*) \,\middle|\, w(n_1^*), \ldots, w(n_H^*)\right) \tag{6.6}$$
$$= J\left(\frac{K(n_1)}{K(\lambda n_1)}, \ldots, \frac{K(n_H)}{K(\lambda n_H)} \,\middle|\, \eta\, K(\lambda n_1), \ldots, \eta\, K(\lambda n_H)\right) \quad \text{by Prop. 3.1}$$
$$= J\left(\frac{K(n_1)}{K(\lambda n_1)}, \ldots, \frac{K(n_H)}{K(\lambda n_H)} \,\middle|\, K(\lambda n_1), \ldots, K(\lambda n_H)\right). \qquad \text{by UDI}$$

Invoking BNSI, we have $I(\mathbf{x}^*; \mathbf{n}^*) = I(\mathbf{x}^\circ; \mathbf{n}^\circ)$, which, upon using (6.5) and (6.6), implies that

$$J\left(\frac{K(n_1)}{K(\lambda n_1)}, \ldots, \frac{K(n_H)}{K(\lambda n_H)} \,\middle|\, K(\lambda n_1), \ldots, K(\lambda n_H)\right) = 0, \tag{6.7}$$

which holds true whatever the equivalence scale function K, the distribution of household size $\mathbf{n} := (n_1, \ldots, n_H)$ and $\lambda > 0$. This implies in turn that

$$\frac{K(\lambda n_1)}{K(n_1)} = \frac{K(\lambda n_2)}{K(n_2)} = \cdots = \frac{K(\lambda n_H)}{K(n_H)}. \tag{6.8}$$

Because (6.8) holds true for all $\mathbf{n} = (n_1, \ldots, n_H) \in [1, +\infty)^H$ and all $\lambda > 0$, we deduce that

$$K(\lambda n) = \phi(\lambda)\, K(n), \;\; \forall\, \lambda > 0, \quad \forall\, n \in [1, +\infty). \tag{6.9}$$

This is a functional equation the solution of which (see Aczel 1966, Chapter 2) is precisely (3.12).

Sufficiency Assume that condition (3.12) holds and consider an arbitrary situation $(\mathbf{x}; \mathbf{n}) := (x_1, \ldots, x_H; n_1, \ldots, n_H)$. Choosing any $\lambda > 0$ and making use of (3.5) again, we obtain

$$I(\mathbf{x}; \lambda\, \mathbf{n}) = J\,(E(x_1; \lambda n_1), \ldots, E(x_H; \lambda n_H) \mid w(\lambda n_1), \ldots, w(\lambda n_H)) \tag{6.10}$$
$$= J\left(\frac{x_1}{K(\lambda n_1)}, \ldots, \frac{x_H}{K(\lambda n_H)} \,\middle|\, \eta\, K(\lambda n_1), \ldots, \eta\, K(\lambda n_H)\right) \qquad \text{by Prop. 3.1}$$

$$= J\left(\frac{x_1}{\xi(\lambda n_1)^{\varepsilon}}, \ldots, \frac{x_H}{\xi(\lambda n_1)^{\varepsilon}} \,\middle|\, \eta\,\xi(\lambda n_1)^{\varepsilon}, \ldots, \eta\,\xi(\lambda n_H)^{\varepsilon}\right) \quad \text{by assumption}$$
$$= J\left(\frac{1}{\lambda^{\varepsilon}}\frac{x_1}{\xi n_1^{\varepsilon}}, \ldots, \frac{1}{\lambda^{\varepsilon}}\frac{x_H}{\xi n_H^{\varepsilon}} \,\middle|\, \eta\,\lambda^{\varepsilon}(\xi n_1^{\varepsilon}), \ldots, \eta\,\lambda^{\varepsilon}(\xi n_H^{\varepsilon})\right)$$
$$= J\left(\frac{1}{\lambda^{\varepsilon}}\frac{x_1}{\xi n_1^{\varepsilon}}, \ldots, \frac{1}{\lambda^{\varepsilon}}\frac{x_H}{\xi n_H^{\varepsilon}} \,\middle|\, \eta\,\xi n_1^{\varepsilon}, \ldots, \eta\,\xi n_H^{\varepsilon}\right) \quad \text{by UDI}$$
$$= J\left(\frac{x_1}{\xi n_1^{\varepsilon}}, \ldots, \frac{x_H}{\xi n_H^{\varepsilon}} \,\middle|\, \eta\,\xi n_1^{\varepsilon}, \ldots, \eta\,\xi n_H^{\varepsilon}\right) \quad \text{by USI}$$
$$= J\left(\frac{x_1}{K(n_1)}, \ldots, \frac{x_H}{K(n_H)} \,\middle|\, \eta\,K(n_1), \ldots, \eta\,K(n_H)\right) \quad \text{by assumption}$$
$$= J\left(E(x_1; n_1), \ldots, E(x_H; n_H) \mid w(n_1), \ldots, w(n_H)\right) \quad \text{by Prop. 3.1}$$
$$= I(\mathbf{x}; \mathbf{n}), \quad \text{by (3.5)}$$

hence condition BNSI is verified. □

Before we proceed to the proof of Proposition 4.1, we find it convenient to introduce the following intermediate result that will be used repeatedly.

Lemma 6.1 *Let $D := [1, c)$, whenever $c > 1$, and $D := (d, 1]$, whenever $0 \leqslant d < 1$. Then, there exist a non-constant function h and a function k such that*

$$h(ab) = h(a) + k(a)\,h(b), \quad \forall\ a, b, ab \in D, \tag{6.11}$$

if and only if

$$\text{either } h(a) = \alpha \ln a \text{ and } k(a) = 1, \quad \forall\ a \in D, \tag{6.12a}$$
$$\text{or } h(a) = \alpha\left[a^{\eta} - 1\right] \text{ and } k(a) = a^{\eta}\ (\lambda \neq 0), \quad \forall\ a \in D. \tag{6.12b}$$

Furthermore, the functions h and k defined in (6.12) satisfy (6.11), for all $a, b, ab > 0$.

Proof of Lemma 6.1 The proof is analogous to that of Aczel (1984, Lemma 3), but we provide it for the sake of completeness. □

Step 1. It is obvious that the functions h and k defined in (6.12a) and (6.12b) satisfy (6.11). We therefore confine ourselves to the proof of necessity and we distinguish two cases.
Case 1: $k(a) = 1$, for all $a \in D$. Then (6.11) reduces to

$$h(ab) = h(a) + h(b), \quad \forall\ a, b, ab \in D, \tag{6.13}$$

the solution of which (see Aczel (1966, Chapter 2)) is

$$h(a) = \alpha \ln a, \quad \forall\ a \in D. \tag{6.14}$$

Case 2: $k(a^\circ) \neq 1$, for some $a^\circ \in D$. Making use of the commutativity of h on D, we have

$$h(ab) = h(a) + k(a)\,h(b) \quad \text{and} \tag{6.15a}$$
$$h(ba) = h(b) + k(b)\,h(a), \tag{6.15b}$$

for all $a, b \in D$, from which we deduce that

$$h(a)\,[\,k(b) - 1\,] = h(b)\,[\,k(a) - 1\,], \tag{6.16}$$

for all $a, b \in D$. In particular, choosing $b := a^\circ$, we obtain

$$h(a) = \alpha\,[\,k(a) - 1\,], \tag{6.17}$$

where

$$\alpha := \frac{h(a^\circ)}{k(a^\circ) - 1}. \tag{6.18}$$

The fact that h is non-constant by assumption guarantees that there exists $\alpha \neq 0$ and we can therefore insert (6.17) into (6.11) to get

$$\alpha\,[\,k(ab) - 1\,] = \alpha\,[\,k(a) - 1\,] + k(a)\,\alpha\,[\,k(b) - 1\,], \tag{6.19}$$

which reduces to

$$k(ab) = k(a)\,k(b), \quad \forall\ a, b \in D, \tag{6.20}$$

which is a functional equation whose solution (see again Aczel (1966, Chapter 2)) is

$$k(a) = a^\eta, \quad \forall\ a \in D. \tag{6.21}$$

Upon substitution into (6.17), we obtain

$$h(a) = \alpha\,\left[\,a^\eta - 1\,\right], \quad \forall\ a \in D. \tag{6.22}$$

Step 2. So far the solutions we have obtained are restricted to the domain $D \subset \mathbb{R}_{++}$ and we have to show that they are still valid for the entire domain $\mathbb{R}_{++}$. For Case 1, this follows from Aczel (1984, Theorem A) and by setting $k(a) = 1$, for all $a \in \mathbb{R}_{++}$. For Case 2, observe that, for all $a \in \mathbb{R}_{++}$, there exists an integer n such that $\sqrt[n]{a} \in D$. Therefore, we define

$$\overline{k}(a) := k(\sqrt[n]{a})^n, \quad \forall\ a \in \mathbb{R}_{++} \text{ and} \tag{6.23a}$$
$$\overline{h}(a) := \alpha\,[\,\overline{k}(a) - 1\,], \quad \forall\ a \in \mathbb{R}_{++}. \tag{6.23b}$$

Direct computation shows that the functions $\overline{h}$ and $\overline{k}$ defined above satisfy (6.11), for all $a \in \mathbb{R}_{++}$. □

Proof of Proposition 4.1 We have already established that conditions (a) and (b) are sufficient for the (relative) equivalence scale to be isoelastic and we therefore confine ourselves to showing that they are also necessary. While the function $\psi \in \boldsymbol{\Psi}$ and the parameter $\theta \in [0, 1)$ are kept fixed throughout the proof, we maintain them for clarity even though this makes notation more complicated. Suppose that $M(p, q, u; \psi, \theta, n)$ is isoelastic with respect to household size n, in which case there exist continuous functions $g(p, q, u; \psi, \theta, 1)$ and $f(p, q, u; \psi, \theta, 1)$ such that

$$M(p, q, u; \psi, \theta, n) = g(p, q, u; \psi, \theta, 1)\, n^{f(p,q,u;\psi,\theta,1)}, \tag{6.24}$$

for all $n \geqslant 1$ and all (p, q, u). The proof is relatively long and it involves four successive steps. In Step 1, we derive the implications of our assumptions for the expenditure function $\bar{C}(p, \psi(\theta, n)\, q/n, u)$. We obtain a complex functional equation which is simplified in Step 2, and whose solutions are derived in Step 3. Finally, the consequences for the expenditure function are examined in Step 4, and it turns out that preferences must be Cobb–Douglas and the congestion function isoelastic. □

Step 1. Making use of (2.10) and (6.24), we obtain the functional equation

$$C(p, q, u; \psi, \theta, n) = C(p, q, u; \psi, \theta, 1)\, g(p, q, u; \psi, \theta, 1)\, n^{f(p,q,u;\psi,\theta,1)}, \tag{6.25}$$

for all $n \geqslant 1$ and all (p, q, u). Setting $n = 1$ in (6.25) implies that

$$g(p, q, u; \psi, \theta, 1) = 1, \quad \forall\, (p, q, u), \tag{6.26}$$

which upon substituting into (6.25) and making use of (2.10) and (2.2d) gives

$$n\, \bar{C}\left(p, \frac{\psi(\theta, n)\, q}{n}, u\right) = \bar{C}(p, q, u)\, n^{f(p,q,u;\psi,\theta,1)}, \quad \forall n \geqslant 1, \ \forall\, (p, q, u). \tag{6.27}$$

Substituting $(1, q/p)$ for (p, q) in (6.27) and dividing both sides by n, we obtain

$$\bar{C}\left(1, \frac{\psi(\theta, n)}{n} \frac{q}{p}, u\right) = \bar{C}\left(1, \frac{q}{p}, u\right) n^{f\left(1, \frac{q}{p}, u; \psi, \theta, 1\right) - 1}, \quad \forall n \geqslant 1, \ \forall\, (p, q, u). \tag{6.28}$$

Letting $\xi(\theta, n) := n/\psi(\theta, n)$, equation (6.28) becomes

$$\bar{C}\left(1, \frac{1}{\xi(\theta, n)} \frac{q}{p}, u\right) = \bar{C}\left(1, \frac{q}{p}, u\right) n^{f\left(1, \frac{q}{p}, u; \psi, \theta, 1\right) - 1}, \quad \forall n \geqslant 1, \ \forall\, (p, q, u). \tag{6.29}$$

Since $\xi(\theta, n)$ is strictly monotone in n, it has an inverse $\varphi(\theta, r)$ defined by

$$\xi(\theta, \varphi(\theta, r)) = r, \quad \forall r \in S(\theta) := \{s \mid \exists n \in [1, +\infty) : \xi(\theta, n) = s\}, \tag{6.30}$$

where

$$S(\theta) \subseteq (0, 1] \text{ if } \xi(\theta, n) \text{ is decreasing in } n, \text{ and} \tag{6.31a}$$
$$S(\theta) \subseteq [1, +\infty) \text{ if } \xi(\theta, n) \text{ is increasing in } n. \tag{6.31b}$$

For later use, we find it convenient to introduce the set

$$T(\theta) := \left\{ s \,\middle|\, \exists r \in S(\theta) : s = \frac{1}{r} \right\}, \tag{6.32}$$

where

$$T(\theta) \subseteq (0, 1] \text{ if } \xi(\theta, n) \text{ is decreasing in } n, \text{ and} \tag{6.33a}$$
$$T(\theta) \subseteq [1, +\infty) \text{ if } \xi(\theta, n) \text{ is increasing in } n. \tag{6.33b}$$

Replacing n by $\varphi(\theta, r)$ in (6.29), we obtain

$$\bar{C}\left(1, \frac{q}{p}\frac{1}{r}, u\right) = \bar{C}\left(1, \frac{q}{p}, u\right) \varphi(\theta, r)^{f\left(1, \frac{q}{p}, u; \psi, \theta, 1\right) - 1}, \quad \forall r \in S(\theta), \ \forall (p, q, u). \tag{6.34}$$

Step 2. Now define

$$h(a, u) := \ln \bar{C}(1, a, u), \quad (a = q/p) \tag{6.35a}$$
$$k(\theta; a, u) := f(1, a, u; \psi, \theta, 1) - 1, \tag{6.35b}$$
$$\ell(\theta; b) := \ln\, \varphi(\theta, 1/b). \quad (b = 1/r) \tag{6.35c}$$

For later use, we note that the function $h(a, u)$ inherits the properties of the (individual) expenditure function: in particular, it is increasing in a. Similarly, $\ell(\theta; b)$ is strictly monotonic in b since the inverse $\varphi(\theta, r)$ of $\xi(\theta, n)$ is strictly monotonic in r. Then (6.34) can be rewritten as

$$h\left(\frac{q}{p}\frac{1}{r}, u\right) = h\left(\frac{q}{p}, u\right) + k\left(\theta; \frac{q}{p}, u\right) \ell\left(\theta; \frac{1}{r}\right), \quad \forall r \in S(\theta), \ \forall (p, q, u), \tag{6.36}$$

which is equivalent to the following functional equation:

$$h(ab, u) = h(a, u) + k(\theta; a, u)\, \ell(\theta; b), \ \forall b \in T(\theta), \ \forall a > 0, \ \forall u. \tag{6.37}$$

Setting $a = 1$ in the preceding equation, we get

$$h(b, u) = h(1, u) + k(\theta; 1, u)\, \ell(\theta; b), \quad \forall\, b \in T(\theta), \ \forall\, u. \tag{6.38}$$

Because $h(\,\cdot\,, u)$ is increasing, it must be the case that $k(\theta; 1, u) \neq 0$, and we deduce from (6.38) that

$$\ell(\theta; b) = \frac{h(b, u) - h(1, u)}{k(\theta; 1, u)}, \quad \forall\, b \in T(\theta), \ \forall\, u. \tag{6.39}$$

Substituting into (6.37) and subtracting $h(1, u)$ from both sides, we obtain

$$h(ab, u) - h(1, u) = h(a, u) - h(1, u) + \frac{k(\theta; a, u)}{k(\theta; 1, u)}\, [\, h(b, u) - h(1, u)], \tag{6.40}$$

for all $b \in T(\theta)$, all $a > 0$, and all u. Now define

$$\hat{h}(a, u) := h(a, u) - h(1, u), \tag{6.41a}$$

$$\hat{k}(\theta; a, u) := k(\theta; a, u)/k(\theta; 1, u), \tag{6.41b}$$

and substitute into (6.40) to get

$$\hat{h}(ab, u) = \hat{h}(a, u) + \hat{k}(\theta; a, u)\, \hat{h}(b, u), \ \ \forall\, a > 0, \ \forall\, b \in T(\theta), \quad \forall\, u, \tag{6.42}$$

which implies that $\hat{k}(\theta; a, u)$ is independent of θ, hence $\hat{k}(\theta; a, u) = \tilde{k}(a, u)$.
Step 3. Given $\theta \in [0, 1)$ and $u \in \mathbb{R}$, we want to solve the functional equation

$$\hat{h}(ab, u) = \hat{h}(a, u) + \tilde{k}(a, u)\, \hat{h}(b, u), \quad \text{where } a, b, ab \in T(\theta). \tag{6.43}$$

Invoking Lemma 6.1 and letting

$$\beta(u) := h(1, u) \quad \text{and} \quad \gamma(\theta; u) := k(\theta; 1, u), \tag{6.44}$$

there are two cases to be considered.
Case 1: $\hat{h}(a, u) = \alpha(u) \ln a$ and $\tilde{k}(a, u) = 1$, for all $a \in \mathbb{R}_{++}$. Making use of (6.41a), (6.41b), (6.44), and upon substituting into (6.35a), (6.35b) and (6.35c), we obtain

$$h(a, u) = \hat{h}(a, u) + h(1, u) = \alpha(u)\ \ln a + \beta(u), \quad \forall\, a \in \mathbb{R}_{++}; \tag{6.45a}$$

$$k(\theta; a, u) = \tilde{k}(a, u)\, k(\theta; 1, u) = \gamma(\theta; u), \quad \forall\, a \in \mathbb{R}_{++}; \tag{6.45b}$$

$$\ell(\theta; a) = \frac{h(a, u) - h(1, u)}{k(\theta; 1, u)} = \frac{\alpha(u)\ \ln a}{\gamma(\theta; u)}, \quad \forall\, a \in \mathbb{R}_{++}. \tag{6.45c}$$

Case 2: $\hat{h}(a, u) = \alpha[a^{\eta} - 1]$ and $\tilde{k}(a, u) = a^{\eta}$ ($\eta \neq 0$), for all $a \in \mathbb{R}_{++}$. Making use again of (6.41a), (6.41b), (6.44), and upon substituting into (6.35a), (6.35b) and (6.35c), we obtain

$$h(a, u) = \hat{h}(a, u) + h(1, u) = \alpha(u)\,[\,a^{\eta(u)} - 1\,] + \beta(u), \quad \forall\, a \in \mathbb{R}_{++}; \tag{6.46a}$$

$$k(\theta; a, u) = \tilde{k}(a, u)\, k(\theta; 1, u) = \gamma(\theta; u), \quad \forall\, a \in \mathbb{R}_{++}; \tag{6.46b}$$

$$\ell(\theta; a) = \frac{h(a, u) - h(1, u)}{k(\theta; 1, u)} = \alpha(u)\,[\,a^{\eta(u)} - 1\,], \quad \forall\, a \in \mathbb{R}_{++}. \tag{6.46c}$$

Step 4. Now, we examine the implications for the expenditure function and the congestion function of the two solutions we have obtained above.
Case 1: $h(t, u) = \alpha(u)\,\ln t + \beta(u)$. Then, we have

$$e^{h(t,u)} = e^{\alpha(u)\,\ln t + \beta(u)} = e^{\beta(u)}\, e^{\ln t^{\alpha(u)}} = \delta(u)\, t^{\alpha(u)}, \tag{6.47}$$

where $\delta(u) := e^{\beta(u)}$, for all $t > 0$ and all u. Substituting into (6.35a) and acknowledging the linear homogeneity in prices of the expenditure function, we get

$$\bar{C}(p, q, u) = p\,\bar{C}\left(1, \frac{q}{p}, u\right) = p\, e^{h\left(\frac{q}{p}, u\right)} = \delta(u)\, p^{1-\alpha(u)}\, q^{\alpha(u)}, \tag{6.48}$$

for all $(p, q) \gg (0, 0)$ and all u. The monotonicity of the expenditure function in prices implies that $1 - \alpha(u) > 0$ and $\alpha(u) > 0$, hence $0 < \alpha(u) < 1$, for all u. By definition, the individual expenditure function is strictly increasing in u, which implies that[19]

$$\frac{\partial\, \bar{C}(p, q, u)}{\partial\, u} = p^{1-\alpha(u)}\, q^{\alpha(u)} \left[\delta'(u) + \delta(u)\, \alpha'(u)\, \ln\left(\frac{q}{p}\right)\right] > 0, \tag{6.49}$$

where $\alpha'(u)$ is the derivative of $\alpha(u)$ with respect to u. Since $\ln(q/p) \in (-\infty, +\infty)$, it is necessary for (6.49) to hold that $\alpha'(u) = 0$, hence $\alpha(u)$ is independent of u and $\alpha(u) = \varepsilon \in (0, 1)$, for all u. Furthermore, we have

$$\ell(\theta; s) = \ln s^{\alpha(u)/\gamma(\theta;u)} = \ln s^{\varepsilon/\gamma(\theta;u)}, \quad \forall\, s > 0,\ \forall\, u, \tag{6.50}$$

and we conclude that $\gamma(\theta; u)$ is independent of u, hence $\gamma(\theta; u) = \zeta(\theta)$. By definition

$$\ell(\theta; s) := \ln\, \varphi(\theta, 1/s) = \ln s^{\varepsilon/\zeta(\theta)}, \quad \forall\, s > 0, \tag{6.51}$$

which implies that

$$\varphi(\theta, 1/s) = s^{\varepsilon/\zeta(\theta)}, \quad \forall\, s > 0, \tag{6.52}$$

[19] Since by assumption the utility function is differentiable, so are the indirect utility function and its inverse.

or equivalently

$$\varphi(\theta, r) = (1/r)^{\varepsilon/\zeta(\theta)} = r^{-\varepsilon/\zeta(\theta)}, \quad \forall\, r > 0. \tag{6.53}$$

Using the fact that by definition $\varphi(\theta, r) = n$, we obtain

$$\xi(\theta, n) = \xi(\theta, \varphi(\theta, r)) = r = n^{-\zeta(\theta)/\varepsilon}, \tag{6.54}$$

and finally

$$\psi(\theta, n) = \frac{n}{\xi(\theta, n)} = \frac{n}{n^{-\zeta(\theta)/\varepsilon}} = n^{\rho(\theta)}, \tag{6.55}$$

where $\rho(\theta) := 1 + \zeta(\theta)/\varepsilon$, and it is an admissible congestion function.
Case 2: $h(t, u) = \alpha(u)\,[\,t^{\eta(u)} - 1\,] + \beta(u)$. Then, we have

$$e^{h(t,u)} = e^{\alpha(u)\,[\,t^{\eta(u)}-1\,]+\beta(u)} = e^{\beta(u)}\, e^{\alpha(u)\,[\,t^{\eta(u)}-1\,]} =: \delta(u)\, e^{\alpha(u)\,[\,t^{\eta(u)}-1\,]}, \tag{6.56}$$

for all $t > 0$ and all u. Substituting into (6.35a) and using the linear homogeneity in prices of the expenditure function, we obtain

$$\bar{C}(p, q, u) = p\,\bar{C}\left(1, \frac{q}{p}, u\right) = p\, e^{h\left(\frac{q}{p}, u\right)} = \delta(u)\, p\, e^{\alpha(u)\left[\left(\frac{q}{p}\right)^{\eta(u)}-1\right]}, \tag{6.57}$$

for all $(p, q) \gg 0$ and all u. By definition, the expenditure function $\bar{C}(p, q, u)$ must be increasing in prices. On the one hand, we must have

$$\frac{\partial\, \bar{C}(p, q, u)}{\partial\, q} = \delta(u)\, p\, e^{\alpha(u)\left[\left(\frac{q}{p}\right)^{\eta(u)}-1\right]}\, \alpha(u)\, \frac{\eta(u)}{p}\, \left(\frac{q}{p}\right)^{\eta(u)-1} > 0, \tag{6.58}$$

which simplifies to

$$\frac{\partial\, \bar{C}(p, q, u)}{\partial\, q} = \delta(u)\, e^{\alpha(u)\left[\left(\frac{q}{p}\right)^{\eta(u)}-1\right]}\, \alpha(u)\, \eta(u)\, \left(\frac{q}{p}\right)^{\eta(u)-1} > 0. \tag{6.59}$$

Since $\delta(u) > 0$, it is necessary for (6.59) to hold that $\alpha(u)\,\eta(u) > 0$. On the other hand, it must be the case that

$$\frac{\partial\, \bar{C}(p, q, u)}{\partial\, p} = \delta(u)\, e^{\alpha(u)\left[\left(\frac{q}{p}\right)^{\eta(u)}-1\right]}\left[1 - \alpha(u)\, \eta(u)\, \left(\frac{q}{p}\right)^{\eta(u)}\right] > 0. \tag{6.60}$$

Since $\delta(u) > 0$, it is necessary for (6.60) to hold that the term within squared brackets is positive. However, depending on the values of q/p and $\eta(u) - 1$, this term can be positive, negative, or zero, and we therefore conclude that Case 2 is impossible.

To sum up, we have shown that

$$\bar{C}(p,q,u) = \delta(u)\, p^{1-\varepsilon}\, q^{\varepsilon}\ (0 < \varepsilon < 1), \quad \forall\ (p,q) \gg (0,0), \quad \forall\ u, \text{ and} \tag{6.61a}$$

$$\psi(\theta, n) = n^{\rho(\theta)}, \quad \forall\ n \in [1, +\infty), \tag{6.61b}$$

where $\theta \in [0, 1)$, and the proof is complete. □

Acknowledgements This paper forms part of the research project Heterogeneity and Well-Being Inequality (Contract No. HEWI/ANR-07-FRAL-020) of the ANR-DFG programme whose financial support is gratefully acknowledged. We are indebted to Stephen Bazen, the associate editor and two anonymous referees for very useful comments and suggestions when preparing this version. Needless to say, none of the persons mentioned above should be held responsible for remaining deficiencies.

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