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# A note on extended stable sets

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**Abstract** We study abstract decision problems by introducing an extended dominance relation with respect to a set of alternatives. This extension is in between the traditional dominance relation as formulated by Von Neumann and Morgenstern (Theory of games and economic behavior, Princeton University Press, Princeton, 1944) and the transitive closure of it. Subsequently, stable sets are defined and studied for this extended relation. We formulate a characterization of stable sets for this relation and an existence theorem. Finally, we discuss its relation with Von Neumann–Morgenstern stable sets and generalized stable sets.

## 1 Introduction

We study a variation of stable sets for abstract decision problems. An abstract decision problem is represented by a pair $(X, R)$, where $X$ is a set of alternatives and $R$ is a dominance relation on $X$. Here, we read $xRy$ as $x$ dominates $y$ for any $x, y$ belonging to $X$. Moreover, we limit our discussion to a nonempty finite $X$ and asymmetric $R$. The concept of stable set is offered in Von Neumann and Morgenstern (1944) and interpreted in terms of accepted standards of behaviour based on two stability conditions. A subset $V$ of $X$ is called a stable set of $(X, R)$ if it possesses two

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properties: (i) no alternative in $V$ is dominated by another alternative in $V$(called *internal stability*); (ii) any alternative outside $V$ is dominated by some alternative inside $V$(called *external stability*). However, these two stabilities fail to guarantee the existence of stable sets for all abstract decision problems. For instance, stable set will not exist in case of $x_1 R x_2, x_2 R x_3, \ldots, x_{6-1} R x_7, x_7 R x_1$. As for the existence of stable sets, see Lucas (1992) for an excellent review of the state of knowledge at that time.

To solve the existence problem of stable sets, several generalizations and modifications of the notion of stable sets have been proposed in the literature. These extensions can be roughly divided into two categories. In the first category, the focus is on redefining dominance relation. Examples are generalized stable sets in Van Deemen (1991) and socially stable sets in Delver and Monsuur (2001). As for generalized stable sets, both internal stability and external stability are defined in terms of the transitive closure of $R$ while as for socially stable sets, internal stability is defined by the asymmetric part of transitive closure of $R$ with the restriction on this solution, but external stability is defined by $R$. The focus of the second category is on the reconstruction of criteria of stabilities. Examples are absorbing set in Kalai et al. (1976) and generalized optimal choice set in Schwartz (1986), admissible set in Kalai and Schmeidler (1977), and $m$-stable set in Peris and Subiza (2013). All these modifications in this category satisfy a weakened internal stability[1] and an alternative notion of external stability.[2] The relations between all these generalizations are summarized in Inarra et al. (2005), Nicolas (2009) and Peris and Subiza (2013).

Here, we approach stable sets of abstract decision problems by introducing a new dominance relation (denoted by $R^{\omega}$) that is an extension of $R$ and a modification of the transitive closure of $R$. The motivation for this is that $R$ is so strong that stable sets defined by it may not exist in some circumstances while the transitive closure of $R$ is so weak that a generalized stable set defined by it disagrees with stable set even if $R$ is acyclic.[3] With $R^{\omega}$, we revisit the notion of stable sets and get an extension of stable sets that we call *extended stable sets*. It will be verified that the solution of extended stable sets not only solves every abstract decision problem, but also exactly equals to the solution of stable sets when $R$ is acyclic.

The outline of this paper is as follows. In Sect. 2, we recall some basic notations and definitions concerning abstract decision problems and represent the concepts of stable set and of generalized stable set. In Sect. 3, we first offer an extended dominance relation $R^{w}$, then compare it with $R$ and the transitive closure of $R$. Subsequently, the concept of extended stable sets and the technique of contraction are introduced. Sects. 4 and 5 are respectively devoted to the characteristics of extended stable sets and their relations with stable sets and generalized stable sets. We give a short concluding remark and directions for further research in Sect. 6.

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[1] That is, for any two alternatives $x$ and $y$ contained in this solution, if $x$ dominates $y$ (directly or indirectly), then $y$ also dominates $x$ (directly or indirectly).

[2] That is, an alternative inside a solution should not be dominated by any alternative outside this solution.

[3] i.e., never $x_1 R x_2, x_2 R x_3, \ldots, x_{m-1} R x_m, x_m R x_1$ where $x_1, x_2, \ldots, x_m \in X$.

## 2 Notations, definitions and solutions

Consider an abstract decision problem $(X, R)$. A dominance relation $R$ defined on $X$ is called

asymmetric : $\forall x, y \in X$, if $xRy$, then not $yRx$;
complete : $\forall x, y \in X$, if $x \neq y$, then $xRy$ or $yRx$;
transitive : $\forall x, y, z \in X$, if $xRy$ and $yRz$, then $xRz$;
cyclic : $\exists\, x_1, x_2, \ldots, x_m \in X$ such that $x_1Rx_2, x_2Rx_3, \ldots, x_{m-1}Rx_m, x_mRx_1$;
acyclic : if it is not cyclic.

We exclusively focus on abstract decision problems with a nonempty finite $X$ and asymmetric dominance relation $R$ on $X$. Neither transitivity nor completeness are required. Denote by $\Omega(X)$ the set of these abstract decision problems on $X$.

For any $T \subseteq X$, let $R_T$ denote the sub-relation of $R$ with the restriction on $T$. That is, for any $x, y \in T$, $xR_Ty$ if $xRy$ and $x, y \in T$. For any $(X, R) \in \Omega(X)$, $(X, R)$ can be represented by a digraph where $X$ is vertex set and $R$ is the set of directed edges.

Let $\mu(X, R)$ be the set of alternatives which are maximal, where an alternative is *maximal* if it is not dominated by any other alternative. That is,

$$\mu(X, R) = \{x \in X |\, not\ yRx,\ \ for\ any\ y \in X\}.$$

Any solution that includes $\mu(X, R)$ is called *core-inclusive*.

It is well-known that $\mu(X, R)$ may not exist if $R$ is cyclic. The solution of stable sets was offered in Von Neumann and Morgenstern (1944) which can be treated as an extension of $\mu(X, R)$.

Let $(X, R) \in \Omega(X)$. A set $V \subseteq X$ is called a *stable set* if

(i) $\forall x, y \in V$, not $xRy$;
(ii) $\forall y \in X \backslash V$, there is an $x \in V$ such that $xRy$.

Let $S(X, R)$ denote the set of all stable sets of $(X, R)$.

Condition (i), called *internal stability*, expresses the fact that no alternative $x$ in $V$ is dominated by any other alternative $y$ in $V$. Condition (ii), called *external stability*, says that for any alternative outside $V$, there is an alternative in $V$ that dominates it.

The main shortcoming of the solution of stable sets is that it may fail to exist if the dominance relation $R$ is cyclic. To deal with cyclic cases, Van Deemen (1991) proposed the concept of generalized stable sets. Its point of departure is the transitive closure of $R$.

For any $x, y \in X$, say $x$ $\tau$-dominates $y$, denoted by $xR^{\tau}y$, if there is a sequence of $x, x_1, x_2, \ldots, x_m, y \in X$ such that $xRx_1, x_1Rx_2, \ldots, x_{m-1}Rx_m, x_mRy$.

Let $(X, R) \in \Omega(X)$. A set $V \subseteq X$ is called a *generalized stable set* if

(i) $\forall x, y \in V$, not $xR^{\tau}y$;
(ii) $\forall y \in X \backslash V$, there is an $x \in V$ such that $xR^{\tau}y$.

Let $GS(X, R)$ denote the set of all generalized stable sets of $(X, R)$.

Note that $GS(X, R) = S(X, R^{\tau})$ for any $(X, R) \in \Omega(X)$. Condition (i), called *generalized internal stability*, states that no alternative in $V$ can be $\tau$-dominated by another alternative $y$ in $V$. This condition is stronger than internal stability. Condition

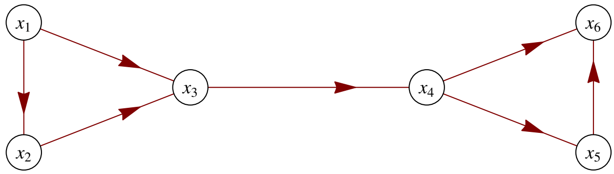



**Fig. 1** Example 1

(ii) expresses that any alternative outside $V$ is $\tau$-dominated by an alternative $x$ in $V$. This condition is weaker than external stability.

One main advantage of the solution of generalized stable sets is that it can deal with every abstract decision problem. However, this solution may not coincide with the stable set of $(X, R)$ if dominance relation $R$ is acyclic.

*Example 1* Consider $(X, R)$ described in Fig. 1. As for this example, we have that $S(X, R) = \{\{x_1, x_4\}\}$ *while* $GS(X, R) = \{\{x_1\}\}$.

Clearly, $\{x_1\}$ is the unique generalized stable set and equals to $\mu(X, R)$. But, it is not a stable set.

## 3 Extended stable sets

Hereby, we propose a new dominance relation that is an extension of $R$ and a modification of the transitive closure of $R$. For any $x, y \in X$, we say $x$ and $y$ are *equipotent*, denoted by $xI^\tau y$, if either $x = y$ or $xR^\tau y$ and $yR^\tau x$. In general, if $xR^\tau y$ is counterbalanced by $yR^\tau x$, then it is difficult to find arguments for choosing between $x$ and $y$. The new dominance relation is defined as follows.

**Definition 1** Let $(X, R) \in \Omega(X)$ and $x, y \in X$. Say $x$ $\omega$-dominates $y$, denoted by $xR^\omega y$, if there are $z, w \in X$ such that $xI^\tau z$, $zRw$ and $wI^\tau y$.

The basic idea behind $R^\omega$ is as follows: we say $xR^\omega y$ if one of the equipotent alternatives of $x$ dominates an equipotent alternative of $y$. Usually, given $(X, R) \in \Omega(X)$ and $x, y \in X$, it is possible that $xR^\omega y$ but not $xRy$. Moreover, we may have $xR^\tau y$ while neither $xR^\omega y$ nor $xRy$. The intuitive differences among $R$, $R^\tau$ and $R^\omega$ can be illustrated by the following example.

*Example 2* Consider $(X, R) \in \Omega(X)$ where $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $R$, $R^\tau$ and $R^\omega$ are respectively described in Figs. 2, 3 and 4.

In Example 2, we have $x_1R^\omega x_5$, but not $x_1Rx_5$. Moreover, we have $x_1R^\tau x_6$ while neither $x_1Rx_6$ nor $x_1R^\omega x_6$. The relations between $R$, $R^\tau$ and $R^\omega$ are summarized in the following Theorem.[4]

**Theorem 1** *For any* $(X, R) \in \Omega(X)$*, we have* $R \subseteq R^\omega \subseteq R^\tau$.

[4] For any $(X, R^1), (X, R^2) \in \Omega(X)$, say $R^1 \subseteq R^2$ if $xR^1y$ implies $xR^2y$ for any $x, y \in X$.

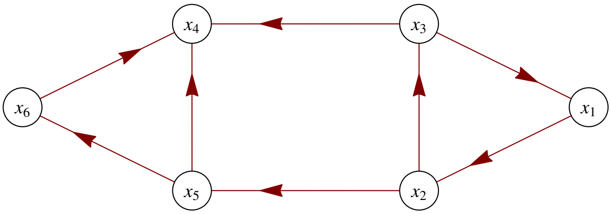
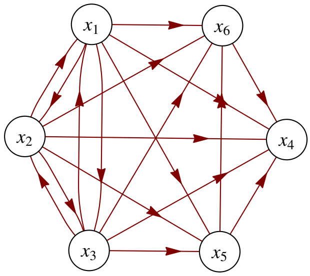



**Fig. 2** $(X, R)$

**Fig. 3** $(X, R^{\tau})$



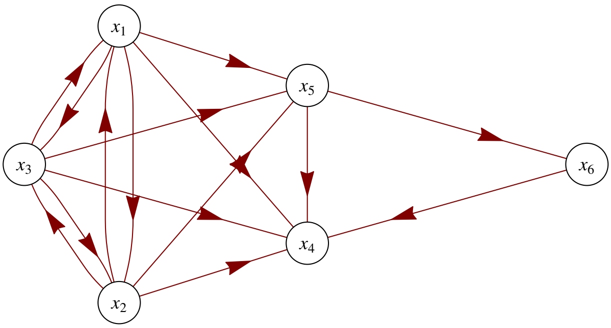




**Fig. 4** $(X, R^{\omega})$

*Proof* Let $x, y \in X$. If $xRy$, then $xR^{\tau}y$. Take $z = x$, $w = y$, then $xR^{w}y$. If $xR^{\omega}y$, there are $z, t \in X$ with $xI^{\tau}z$, $zRt$ and $tI^{\tau}y$. We get $xR^{\tau}y$ by the transitivity of $R^{\tau}$. □

The above theorem shows that $R^{\omega}$ is a weaker dominance relation than $R$ but a stronger dominance relation than the transitive closure of $R$.

Next, we revisit the notion of stable set by using $R^{\omega}$ and obtain an alternative extension of stable set.

**Definition 2** Let $(X, R) \in \Omega(X)$. A set $W \subseteq X$ is called an extended stable set if

(i) $\forall x, y \in W$, not $xR^{\omega}y$;
(ii) $\forall y \in X \backslash W$, there is an $x \in W$ such that $xR^{\omega}y$.

Let $ES(X, R)$ denote the set of all extended stable sets.

It is noted that $ES(X, R) = S(X, R^{\omega})$ for any $(X, R) \in \Omega(X)$. Condition (i), called *extended internal stability*, shows that no alternative $x$ in an extended stable set $W$ is $\omega$-dominated by any other alternative $y$ in $W$. Condition (ii), called *extended external stability*, expresses that for any $y$ outside an extended stable set $W$, there is an alternative $x$ inside $W$ such that $x$ $\omega$-dominates $y$.

Given $(X, R) \in \Omega(X)$ and $x, y \in X$. On the one hand, since not $xR^{\omega}y$ implies not $xRy$ and not $xR^{\tau}y$ implies not $xR^{\omega}y$ (not vice versa), extended internal stability is stronger than internal stability while weaker than generalized internal stability. On the other hand, as $xR^{\omega}y$ means $xR^{\tau}y$ and $xRy$ means $xR^{\omega}y$ (not vice versa), extended external stability is a less demanding constraint than external stability and a more demanding constraint than generalized external stability.

*Example 3* As for Example 1, we have that

$$ES(X, R) = S(X, R) = \{\{x_1, x_4\}\} \textit{ while } GS(X, R) = \{\{x_1\}\}.$$

As for Example 2, there is no stable set. We have that

$$GS(X, R) = \{\{x_1\}, \{x_2\}, \{x_3\}\}$$
$$\textit{while } ES(X, R) = \{\{x_1, x_6\}, \{x_2, x_6\}, \{x_3, x_6\}\}.$$

Finally, we present the technique of contracting a relation which is adopted from digraph theory. Here we use it as a tool for describing the characteristics of extended stable sets.

For any $(X, R) \in \Omega(X)$ and $T \subseteq X$, let $R^{\tau}_T$ denote the sub-relation[5] of $R^{\tau}$ with the restriction on $T$. We say $T$ is *strongly connected* if $xR^{\tau}_T y$ and $yR^{\tau}_T x$ for any $x, y \in T$. $T$ is called a *strong component* of $(X, R)$ if it is strongly connected and no proper superset of $T$ is strongly connected.

Let $X^* = \{X^*_1, X^*_2, \ldots, X^*_s\}$ be the set of all strong components of $(X, R)$. The *contraction* of $(X, R)$ is denoted by $(X^*, R^{con})$, where $X^*_i R^{con} X^*_j$ for any $X^*_i, X^*_j \in X^*$ if there are $x \in X^*_i$ and $y \in X^*_j$ with $xRy$.

Intuitively, the contraction of $(X, R)$ is the collapsing of each strong component into a single point enriched with a relation between those points. Thus, $(X^*, R^{con})$ is an abstract decision problem where alternatives are strong components and dominance relation is $R^{con}$. Let $\mu(X^*, R^{con})$ denote the set of maximal strong components and $S(X^*, R^{con})$ denote the set of stable sets of $(X^*, R^{con})$.

*Example 4* As for Example 2, the contraction of $(X, R)$ is represented by Fig. 5, where $X^*_1 = \{x_1, x_2, x_3\}$, $X^*_2 = \{x_4\}$, $X^*_3 = \{x_5\}$ and $X^*_4 = \{x_6\}$.

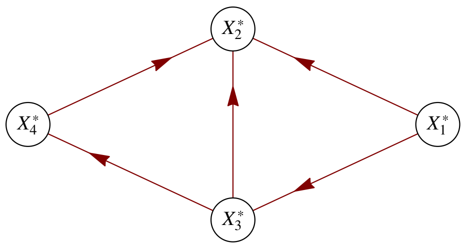
The following lemma will be used in the sequel to show the characterization of extended stable sets. It follows directly from Definition 1 and the notion of contraction.

**Lemma 1** *Let $(X, R) \in \Omega(X)$ and $(X^*, R^{con})$ be the contraction of it. Then*

(i) *for any $X^*_i \in X^*$, $x, y \in X^*_i$ if and only if $xI^{\tau}y$;*

[5] For any $x, y \in T$, $xR^{\tau}_T y$, if and only if, there exist $x, x_1, x_2, \ldots, x_m, y \in T$ such that $xRx_1, x_1Rx_2, \ldots, x_mRy$.

**Fig. 5** $(X^*, R^{con})$




(ii) *for any* $X_i^*, X_j^* \in X^*$, $X_i^* R^{con} X_j^*$ *if and only if* $x_i R^{\omega} x_j$ *for all* $x_i \in X_i^*$ *and* $x_j \in X_j^*$

In the following two sections, we will focus on a characterization of extended stable sets and their relations with stable sets and generalized stable sets.

## 4 A characterization of extended stable sets

The following proposition expresses that any extended stable set is core-inclusive.

**Proposition 1** *Let* $(X, R) \in \Omega(X)$. $\mu(X, R) \subseteq \bigcap_{W \in ES(X,R)} W$.

*Proof* Let $x \in \mu(X, R)$. Suppose there is a $W \in ES(X, R)$ such that $x \notin W$. Then, there exists a $y \in W$ such that $y R^{\omega} x$ which implies $y R^{\tau} x$. Hence, there must be some $z \in X$ with $zRx$, which contradicts $x \in \mu(X, R)$. □

It is easy to verify that if $\mu(X, R)$ is a stable set, it is an extended stable set and a generalized stable set, but not vice versa. Furthermore, if $\mu(X, R)$ is an extended stable set, then it is a generalized stable set while not necessarily a stable set.

The following lemmas will be used to show the characterization of extended stable sets and the existence of extended stable sets.

**Lemma 2** *Let* $(X, R) \in \Omega(X)$. $R^{con}$ *is acyclic.*

The proof of this lemma is given in Behzad et al. (1979, p 327–328).

**Lemma 3** *Let* $(X, R) \in \Omega(X)$. *Then* $S(X^*, R^{con})$ *contains a unique set.*

The proof of this lemma is given in Von Neumann and Morgenstern (1944, p 598–599).

The following theorem characterizes the solution of extended stable sets. This result is typically obtained by using the contraction of $(X, R)$.

**Theorem 2** *Let* $(X, R) \in \Omega(X)$ *and* $S(X^*, R^{con}) = \{\{X_1^*, X_2^*, \ldots, X_k^*\}\}$. $W \in ES(X, R)$ *if and only if* $W = \{x_1, x_2, \ldots, x_k\}$, *where* $x_i \in X_i^*$, $1 \leq i \leq k$.

*Proof* Let $V^* = \{X_1^*, X_2^*, \ldots, X_k^*\}$ and $W = \{x_1, x_2, \ldots, x_k\}$, where $x_i \in X_i^*$, $1 \leq i \leq k$. For any $x_i, x_j \in W$, there are distinct $X_i^*, X_j^* \in V^*$ with $x_i \in X_i^*$ and $x_j \in X_j^*$. Since not $X_i^* R^{con} X_j^*$, we get that not $x_i R^{\omega} x_j$ by Lemma 1-(ii). For any $y_t \in X \backslash W$, there exist an $X_t^*$ such that $y_t \in X_t^*$. If $X_t^* \in V^*$, there is a $z_t \in X_t^*$ such that $x_t I^\tau z_t$ and $z_t R y_t$ implying $x_t R^\omega y_t$. Otherwise $X_t^* \notin V^*$. Then there exists an $X_s^* \in V^*$ such that $X_s^* R^{con} X_t^*$, which means $x_s \in X_s^*$ and $y_t \in X_t^*$ with $x_s R^\omega y_t$ by Lemma 1-(ii).

Conversely, let $W \in ES(X, R)$. Denote $W^* = \{X_i^* \in X^* | x \in W,\ x \in X_i^*\}$. Obviously, $W \subseteq \bigcup_{X_i^* \in W^*} X_i^*$. For any $X_i^*, X_j^* \in W^*$, there exist two distinct $x_i, x_j$ with $x_i \in X_i^*, x_j \in X_j^*$ and not $x_i R^\omega x_j$. If $X_i^* R^{con} X_j^*$, then $x_i R^\omega x_j$ by Lemma 1-(ii). Contradiction. Thus, not $X_i^* R^{con} X_j^*$. For any $X_j^* \in X^* \backslash W^*$ and $y_j \in X_j^*$, we have that $y_j \notin W$. Then, there is an $x_i \in W$ and $x_i \in X_i^* \in W^*$ such that $x_i R^\omega y_j$, which indicates $x_i' R^\omega y_j$ for all $x_i' \in X_i^*$. Since $X_i^* \neq X_j^*$, $X_i^* R^{con} X_j^*$ directly follows from Lemma 1-(ii). Thus, $W^* \in S(X^*, R^{con})$. We have to prove that $|W \cap X_i^*| = 1$ for any $X_i^* \in W^*$. If there is an $X_m^* \in W^*$ such that $|W \cap X_m^*| > 1$, then there are two distinct $x, y \in W$ with $x, y \in X_m^* \in W^*$. Thus $x R^\tau y$ and $y R^\tau x$, which implies $x R^\omega y$. Contradiction. □

The next theorem expresses that the solution of extended stable sets is applicable to every abstract decision problem.

**Theorem 3** *For any $(X, R) \in \Omega(X)$, there exists a nonempty extended stable set.*

*Proof* Lemma 2 guarantees that $R^{con}$ is acyclic and Lemma 3 guarantees the existence of $S(X^*, R^{con})$ for any $(X, R) \in \Omega$. By Theorem 2, it can be concluded that nonempty extended stable set always exists. □

By Theorem 2, we may get that extended stable set may not be unique. The following corollary shows the cardinal number of extended stable sets and one feature about the size of extended stable sets.

**Corollary 1** *Let $(X, R) \in \Omega(X)$ and $S(X^*, R^{con}) = \{\{X_1^*, X_2^*, \ldots, X_k^*\}\}$.*

(i) $|ES(X, R)| = |X_1^*| \times |X_2^*| \times \cdots \times |X_k^*|$;
(ii) $|W_1| = |W_2|$ *for any* $W_1, W_2 \in ES(X, R)$.

## 5 The relations between extended stable sets, stable sets and generalized stable sets

In the next theorem, we present sufficient conditions for extended stable sets to agree with stable sets and generalized stable sets.

**Theorem 4** *Let $(X, R) \in \Omega(X)$. Then*

(i) *if $R$ is acyclic, then $ES(X, R) = S(X, R)$;*
(ii) *if $R$ is complete, then $ES(X, R) = GS(X, R)$;*
(iii) *if $R$ is transitive, then $ES(X, R) = S(X, R) = GS(X, R)$.*

*Proof* (i) Take $(X, R) \in \Omega$. If $R$ is acyclic, $\{x\} \in X^*$ for any $x \in X$. Then $(X, R) = (X^*, R^{con}) = (X, R^\omega)$. Thus $ES(X, R) = S(X, R)$.

(ii) By Theorem 1, we have $R^\omega \subseteq R^\tau$. Conversely, take any $x, y \in X$ with $x R^\tau y$. Since $R$ is complete, if $x R y$, then $x R^\omega y$. Otherwise, $y R x$. In this case, $x I^\tau y$, which implies $x R^\omega y$. Thus, $R^\tau \subseteq R^\omega$. Therefore, $(X, R^\omega) = (X, R^\tau)$ and $ES(X, R) = GS(X, R)$.

(iii) If $R$ is transitive, $R$ is acyclic. Then, $(X, R^{\omega}) = (X, R) = (X, R^{\tau})$. Hence, $ES(X, R) = S(X, R) = GS(X, R)$. □

It is noted that there may be differences between these three solutions if the underlying dominance relation is cyclic or incomplete.

The following lemma shows the characterization of generalized stable sets. It will be useful in elaborating the relation between extended stable sets and generalized stable sets.

**Lemma 4** *Let* $(X, R) \in \Omega(X)$ *and* $\mu(X^*, R^{con}) = \{X_1^*, X_2^*, \ldots, X_m^*\}$. *Then* $V = \{x_1, x_2, \ldots, x_m\} \in GS(X, R)$ *if and only if* $x_i \in X_i^*$ *where* $1 \leq i \leq m$.

The proof of this lemma is given in Van Deemen (1991, p 258–259).

The following proposition shows that the notion of extended stable sets may be considered as a combination of stable sets and of generalized stable sets.

**Proposition 2** *Let* $(X, R) \in \Omega(X)$ *and* $S(X^*, R^{con}) = \{\{X_1^*, X_2^*, \ldots, X_k^*\}\}$. $W \in ES(X, R)$ *if and only if* $W = \bigcup_{1 \leq i \leq k} V_i$, *where* $V_i \in GS(X_i^*, R_{X_i^*})$.

*Proof* For any $X_i^* \in X^*$, we have that either $X_i^*$ is singleton or $R_{X_i^*}$ is a cycle involving every alternative of $X_i^*$. It is sufficient to prove that for any $X_i^* \in X^*$, $V_i \in GS(X_i^*, R_{X_i^*})$ if and only if $|V_i| = 1$. This follows from Lemma 4. □

The next proposition presents a sufficient and necessary condition for generalized stable sets to coincide with extended stable sets.

**Proposition 3** *Let* $(X, R) \in \Omega(X)$. $GS(X, R) = ES(X, R)$ *if and only if* $\mu(X^*, R^{con}) \in S(X^*, R^{con})$.

*Proof* The proof follows directly from Theorem 2 and Lemma 4. □

The following result accounts for the inclusiveness relation between stable set, extended stable set and generalized stable set.

**Theorem 5** *Let* $(X, R) \in \Omega(X)$ *and* $S(X^*, R^{con}) = \{\{X_1^*, X_2^*, \ldots, X_k^*\}\}$.

(i) *For any* $V \in S(X, R)$, *there is a* $W \in ES(X, R)$ *with* $W \subseteq V$ *if* $V \cap X_i^* \neq \emptyset$ *for any* $1 \leq i \leq k$;
(ii) *For any* $V \in GS(X, R)$, *there is a* $W \in ES(X, R)$ *with* $V \subseteq W$.

*Proof* (i) Let $V \in S(X, R)$. Take $x_i \in V \cap X_i^*$ where $1 \leq i \leq k$. Then we get that $W = \{x_1, x_2, \ldots, x_m\} \in ES(X, R)$ and $W \subseteq V$ by Theorem 2.

(ii) Let $V \in GS(X, R)$ and $\mu(X^*, R^{con}) = \{X_1^*, X_2^*, \ldots, X_m^*\}$. Assume $x_i = X_i^* \cap V$ where $i = 1, 2, \ldots, m$. Then, we have that $V = \{x_1, x_2, \ldots, x_m\}$ by Lemma 4. Since for the unique $V^* \in S(X^*, R^{con})$, $\mu(X^*, R^{con}) \subseteq V^*$. Let $V^* \backslash \mu(X^*, R^{con}) = \{X_{m+1}^*, X_{m+2}^*, \ldots, X_k^*\}$. Take $x_j \in X_j^*$ where $j = m+1, m+2, \ldots, k$ and $W = \{x_1, x_2, \ldots, x_k\}$. Then by Theorem 2, we conclude that $W \in ES(X, R)$ and $V \subseteq W$.

□

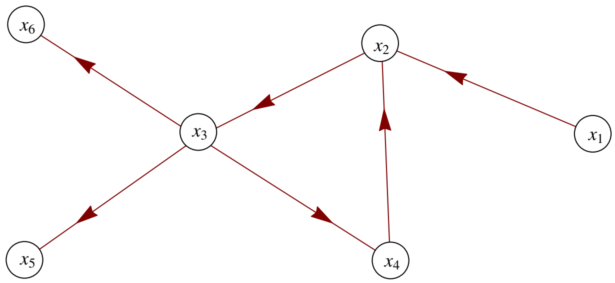



**Fig. 6** $(X, R)$

Let $(X, R) \in \Omega(X)$ with nonempty stable sets. Take $V \in S(X, R)$, there may not exist $W \in ES(X, R)$ with $W \subseteq V$. Moreover, it is possible that for any $V \in S(X, R)$ and $W \in ES(X, R)$, $|V| \leq |W|$. Consider the following example.

*Example 5* Consider $(X, R)$ described in Fig. 6. We have that

$$S(X, R) = \{\{x_1, x_3\}\} \quad and \quad ES(X, R) = \{\{x_1, x_5, x_6\}\}.$$

Obviously,

$$|\{x_1, x_5, x_6\}| > |\{x_1, x_3\}|.$$

As a comparison, it is easy to verify that the size of any generalized stable set is smaller than or equal to the size of any extended stable set.

## 6 Conclusion

In this paper, we introduced an extended dominance relation $R^{\omega}$ on finite nonempty $X$ which is weaker than dominance relation $R$, but stricter than the transitive closure of $R$. Then, we formulated a solution called extended stable set which appears to be a stable set solution defined with respect to $R^{\omega}$. The solution of stable sets defined by Von Neumann and Morgenstern (1944) with respect to $R$ does not permits its existence in all abstract decision problems. By contrast, nonempty extended stable sets always exist for every abstract decision problem. In this aspect, it improves the theory of stable sets. More important, the solution of generalized stable sets defined on the transitive closure of $R$ may fail to coincide with the sable set solution if dominance relation $R$ is acyclic. In this case, the solution of extended stable sets exactly agrees with stable set. This implies that the solution of extended stable sets lies closer to the solution of stable sets than the solution of generalized stable sets.

Our next research step will be to compare extended stable sets with the solutions in which the criteria of stabilities are modified. We will also explore the characterization of extended stable sets in the case that $X$ is infinite. Another research topic in our program is to investigate the relations between extended stable sets, Banks set (see Banks 1985; Penn 2006) and the variations of uncovered set in Duggan (2013). We may explore under which condition an extended stable set is included in the Banks set or the solution of uncovered set.

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