Soc Choice Welf (2015) 45:97–121
DOI 10.1007/s00355-014-0868-x

Reaching consensus: solidarity and strategic properties in binary social choice

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Received: 12 January 2014 / Accepted: 27 December 2014 / Published online: 6 January 2015

Abstract We study solidarity and strategic properties in binary social choice. We consider both the standard setting with strict preferences and the "full" preference domain which allows for indifference. Two solidarity properties drive our investigation: "Welfare dominance under preference replacement", which says that when the preferences of one agent change, the other agents all weakly gain or all weakly lose; and "population monotonicity", which requires the same conclusion when one agent leaves. We identify the families of rules satisfying these properties on each preference domain. Additionally requiring efficiency characterizes the "consensus" rules in each case. We also relate welfare dominance to other properties. Two results highlight the role of indifference: Welfare dominance implies "anonymity" when preferences are strict, but not otherwise; "group strategy-proofness" implies welfare dominance when indifference is allowed, but not otherwise. Finally, we introduce a "duality" operator which structures the space of rules and extends our results to a model in which rules may select neither alternative. Only in this case are our solidarity properties consistent with "neutrality".

JEL Classification D63 · D71

1 Introduction

In the classic social choice problem, a group of agents must choose among a set of alternatives. We consider the special case where two alternatives are available, a setting that has spawned considerable literature since May's seminal work (May 1952).

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Choosing between two alternatives covers many important situations, such as two-candidate elections, up-down votes on legislation, and yes-no decisions about building a new public facility. Extending previous work, we search for rules with desirable solidarity and incentive properties on two preference domains. In the first case, we assume all agents have strict preferences over the alternatives, and in the second case, we allow agents to be indifferent between the alternatives. This model admits a rich variety of rules, especially when agents may express indifference. Moreover, our solidarity requirements are compatible with other attractive properties. Our analysis can help policy-makers choose among rules.

Our approach is axiomatic and emphasizes solidarity. As a general principle, solidarity says when the environment changes, agents should all gain together or all lose together. The principle is compelling when applied to agents who bear no responsibility for the change. We analyze two such situations: Changes in the preference of other agents and changes in the population of agents. Our first requirement, "welfare dominance under preference replacement", considers a change in the preferences of one agent. It says that when the preferences of one agent change, the welfares of the other agents should move in the same direction. Our second requirement, "population monotonicity", applies when the population changes. It says that when one agent leaves, the welfares of the remaining agents should be in the same direction. In our model, *population monotonicity* implies *welfare dominance* (Lemma 3).

We identify a class of eight rules satisfying *welfare dominance* when all preferences are strict (Theorem 1) and a large class of rules satisfying the property on the "full" preference domain, which allows for indifference (Theorem 2). Allowing for variable populations, we generalize these results to characterize the rules satisfying *population monotonicity* (Theorem 3). In each case, only the "consensus" rules are also *efficient*. A consensus rule selects an alternative when all agents with a strict preference favor that alternative and selects a default otherwise. Highlighting the role of indifference, we relate our solidarity properties to another desirable property, "anonymity", which says that the names of the agents should not matter. On the strict preference domain, each of our solidarity properties implies *anonymity*. Surprisingly, allowing indifference overturns these results: On the full preference domain, neither solidarity property implies *anonymity*.

Turning to incentives, we find that *welfare dominance* is compatible with "strategy-proofness", which says that no agent should be able to gain by misreporting her preferences, as well as "group strategy-proofness", the requirement that no group of agents be able to jointly misrepresent their preferences so that at least one member gains and no member loses. Encouragingly, the consensus rules satisfy both incentive properties (Manjunath 2012). On the strict preference domain, *strategy-proofness* and *group strategy-proofness* coincide, and as we show, neither is logically related to *welfare dominance*. Including indifference changes these results dramatically: The two notions of *strategy-proofness* no longer coincide and, surprisingly, *group strategy-proofness* now implies *welfare dominance* (Proposition 2). *Strategy-proofness* requires that truth-telling be a dominant strategy. A stronger strategic requirement, "secure strategy-proofness" additionally requires that the outcome of truth-telling be the only Nash equilibria of the related implementation game. Unfortunately, this property is satisfied only by undesirable rules: Dictatorship and constant rules on the strict preference domain and only the constant rules on the full preference domain (Theorem 5).

To better understand the structure of the space of rules, we introduce a notion of "duality". This investigation uncovers additional relationships among axioms and provides a new interpretation for "neutrality", the requirement that the labels of the alternatives should not matter. Implicitly, duality measures the "bias" of a rule in favor of one alternative over the other. Since *neutrality* is incompatible with either of our solidarity axioms in our framework, we extend the model to allow rules to sometimes choose neither alternative. Our results extend naturally to this setting, and once again, lead to the consensus rules.

1.1 Related literature

The solidarity principle embodied by *welfare dominance* has important implications in several models.[1] In a binary social choice model similar to ours but with side payments, *welfare dominance* is the key axiom characterizing the "egalitarian" rule (Moulin 1987). In a model with a continuum of alternatives over which agents have single-peaked preferences, the combination of *welfare dominance* and *efficiency* identifies a class of "target" rules (Thomson 1993). When two alternatives are to be selected, only the "left peaks" and "right peaks" rules satisfy both properties (Miyagawa 2001). If the alternatives have a tree structure, no rule for choosing two alternatives satisfies both axioms (Umezawa 2012). In a generalized public decision model, the combination of *welfare dominance* and *efficiency* together with "replication indifference" implies *population monotonicity* as well as the existence of a special alternative that acts as a default selection (Gordon 2007). Applied to rationing a divisible resource and to allocating objects with money, *welfare dominance* is particularly demanding. In these models, it is incompatible with other standard properties except in special cases (Thomson 1997, 1998).

Incentive properties are similarly restrictive. In a social choice setting with three or more alternatives, the results are disappointing. On the strict preference domain, the only *strategy-proof* rules with *full range* are dictatorships (Gibbard 1973; Satterthwaite 1975), and on the full preference domain these properties are satisfied only by the slightly larger class of "sequential dictatorships" (Larsson and Svensson 2006). Restricted to two alternatives, however, rules significantly different from dictatorships satisfy both properties. On the strict preference domain, it is possible to move quite far from dictatorship. Here, *strategy-proofness* and *full range* are satisfied by a class of rules called "voting by committees" (Barberà et al. 1991). Allowing indifference makes available an even more diverse class of rules defined by "extended committees" (Larsson and Svensson 2006). It is even possible to strengthen the incentive requirement to *group strategy-proofness*, which leads back to the consensus rules (Manjunath 2012).

Binary social choice has received considerable attention, and continues to be an active area of research. May's Theorem characterizes simple majority voting by

[1] See Thomson (1999) for a survey.

anonymity, *monotonicity*, and *neutrality* (May 1952). With a slightly weaker version of *monotonicity*, a family of voting rules based on absolute and relative majorities satisfy these properties (Llamazares 2013). Although restricted to two alternatives, the voting rules in this family are allowed to select neither alternative. Moving from binary choice to the model with single-peaked preferences over an interval, each of the "generalized Condorcet-winners" is *group strategy-proof* (Moulin 1984). The target rules fall in this family (Thomson 1993).

In Sect. 2, we introduce the model. We study the implications of *welfare dominance population monotonicity* in Sect. 3 and of incentive properties in Sect. 4. We introduce *duality* in Sect. 5, extend the model to allow rules which choose neither option in Sect. 6, and conclude in Sect. 7.

2 Model

Let $A \equiv \{a, b\}$ be the set of alternatives. Let $N \in \mathcal{N}$ be a set of agents drawn from a set of potential agents $\mathcal{N}$. We assume throughout that $|N| \geq 3$. Each agent $i \in N$ has a preference relation R_i over A. Let R^a represent preference for a over b, R^b represent preference for b over a, and R^{ab} represent indifference between a and b. The **full preference domain** is $\boldsymbol{\mathcal{R}} \equiv \{R^a, R^b, R^{ab}\}$ and the **strict preference domain** is $\boldsymbol{\mathcal{R}^*} \equiv \{R^a, R^b\}$. We will also write $a\ P_i\ b$ if $R_i = R^a$, $a\ R_i\ b$ if $R_i \in \{R^a, R^{ab}\}$, $a\ I_i\ b$ if $R_i = R^{ab}$, and similarly if $R_i = R^b$ or $R_i \in \{R^b, R^{ab}\}$.

An **economy** is a preference profile $R \in \mathcal{R}^N$. For each $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv (R_i)_{i \in N \setminus N'}$ so $R = (R_{N'}, R_{-N'})$. Abusing notation slightly, we write R_{-i} for $R_{-\{i\}}$, R_{-ij} for $R_{-\{i,j\}}$, and so on. A **rule** $\varphi\colon \mathcal{R}^N \to A$ selects one alternative for each economy. For each $\alpha \in \{a, b, ab\}$, let $\boldsymbol{N_\alpha(R)} \equiv \{i \in N : R_i = R^\alpha\}$ be the set of agents with preferences R^α. We distinguish three types of economies. For each $R \in \mathcal{R}^N$, we say that there is **disagreement at R** and call R a **disagreement economy** if $N_a(R) \neq \emptyset$ and $N_b(R) \neq \emptyset$; we say that there is **consensus at R** and call R a **consensus economy** if $N_a(R) \neq \emptyset = N_b(R)$ or $N_a(R) = \emptyset \neq N_b(R)$; and we call R the **indifference economy** if $N_{ab}(R) = N$. Finally, let $\boldsymbol{\Pi^N}$ be the set of permutations of N. For each $R \in \mathcal{R}^N$ and each $\pi \in \Pi^N$, let $\boldsymbol{\pi(R)} \equiv (R_{\pi(i)})_{i \in N}$.

We will also allow the population to vary. To accommodate changes in the population, we extend the definition of an economy to include the set of agents as well as preferences. Let $\boldsymbol{\mathcal{E}} \equiv \{(N, R) : N \in \mathcal{N}, R \in \mathcal{R}^N\}$. A **variable-population economy** is a pair $(N, R) \in \mathcal{E}$ and a **variable-population rule** is a mapping $\varphi\colon \mathcal{E} \to A$. Each variable-population rule is a collection of fixed-population rules, which we call the components of the rule.

2.1 Axioms

In this section, we introduce desirable properties for rules. Let φ be a rule.

According to our first axiom, whenever it is possible to make at least one agent better off without making another agent worse off, we should do so. In terms of our model, if there is an alternative which all agents find at least as desirable as the other alternative and at least one agent prefers, then the rule should select it.[2]

Efficiency for each $R \in \mathcal{R}^N$, if $N_a(R) \neq \emptyset = N_b(R)$, then $\varphi(R) = a$, and if $N_a(R) = \emptyset \neq N_b(R)$, then $\varphi(R) = b$.

While desirable, *efficiency* may be unattainable. Our next axiom imposes a weaker requirement. If all agents prefer the same alternative, then the rule should select it.

Unanimity $\varphi(R^a, \dots, R^a) = a$ and $\varphi(R^b, \dots, R^b) = b$.

Efficiency implies *unanimity*. On the strict preference domain, the axioms are equivalent. While *unanimity* is very mild and only applies to two economies, the following axiom is even weaker. The rule should not select the same alternative in all economies.

Full range there is a pair $\{R, R'\} \subseteq \mathcal{R}^N$ such that $\varphi(R) = a$ and $\varphi(R') = b$.

Our next requirement is that the names of the agents should not matter. When the identities of the agents are shuffled, the rule should select the same alternative.

Anonymity for each $R \in \mathcal{R}^N$ and each $\pi \in \Pi^N$, we have $\varphi(R) = \varphi(\pi(R))$.

Applied to variable-population rules, *anonymity* requires invariance when the set of potential agents is permuted.

Our next two axioms concern potential manipulation. First, no agent should gain by reporting false preferences.

Strategy-proofness for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi(R) \; R_i \; \varphi(R'_i, R_{-i})$.

In addition, we may be concerned with manipulation by groups of agents. No group of agents should be able to misrepresent their preferences so that one member of the group is better off and no member of the group is worse off.

Group strategy-proofness for each $R \in \mathcal{R}^N$, each $S \subseteq N$, and each $R'_S \in \mathcal{R}^S$, if there is $i \in S$ such that $\varphi(R'_S, R_{-S}) \; P_i \; \varphi(R)$, then there is $j \in S$ such that $\varphi(R) \; P_j \; \varphi(R'_S, R_{-S})$.

On the strict preference domain, *strategy-proofness* and *group strategy-proofness* are equivalent.[3] Also, *group strategy-proofness* and *full range* together imply *efficiency*.

Strategy-proofness guarantees implementation in dominant strategies.[4] However, in a given economy, the direct revelation game associated with a rule may still have undesirable Nash equilibria. Our next axiom strengthens *strategy-proofness* by requiring a rule to be doubly implementable in dominant strategies and in Nash equilibria.

[2] *Efficiency* and *unanimity* are also known as the "strong Pareto principle" and "weak Pareto principle" respectively.

[3] A weaker notion of *group strategy-proofness* requires that all misrepresenting agents gain. This condition is equivalent to *strategy-proofness* very generally (Barberà et al. 2010, 2012; Le Breton and Zaporozhets 2009).

[4] That is, it is a dominant strategy for each agent to truthfully report her preferences, and this constitutes a Nash equilibrium.

To define the axiom formally, let $\mathbf{\Gamma}^{\varphi} \equiv (\varphi, \mathcal{R})$ be the game form associated with φ and for each $R \in \mathcal{R}^N$, let $\boldsymbol{NE(\Gamma^{\varphi}, R)}$ be the set of Nash equilibria of the direct revelation game associated with φ at R. That is, for each pair $\{R, R'\} \subseteq \mathcal{R}^N$, $R' \in NE(\Gamma^{\varphi}, R)$ if for each $i \in N$ and each $R_i'' \in \mathcal{R}$, we have $\varphi(R')\ R_i\ \varphi(R_i'', R_{-i}')$.[5]

Secure strategy-proofness φ is *strategy-proof* and for each pair $\{R, R'\} \subseteq \mathcal{R}^N$, if $R' \in NE(\Gamma^{\varphi}, R)$, then $\varphi(R') = \varphi(R)$.

In addition to *strategy-proofness*, *secure strategy-proofness* further requires that, in each economy, all Nash equilibrium outcomes of the direct revelation game coincide.

Our final axioms, and the central axioms of our study, require forms of solidarity. Our first solidarity axiom applies to economies with fixed populations. When the preferences of one agent change, all of the agents whose preferences are fixed should be affected in the same direction: Either no agent is made worse off or no agent is made better off.[6]

Welfare dominance under preference replacement for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R_i' \in \mathcal{R}$, either (i) for each $j \in N\backslash\{i\}$, we have $\varphi(R)\ R_j\ \varphi(R_i', R_{-i})$ or (ii) for each $j \in N\backslash\{i\}$, we have $\varphi(R_i', R_{-i})\ R_j\ \varphi(R)$.

In addition to replacement of one agent's preferences, we may consider the effect simultaneously replacing the preferences of a group of agents. Applied to such cases, the axiom would require solidarity among the agents whose preferences are fixed. However, these notions are equivalent in our model with two alternatives.

Welfare dominance alone imposes no restrictions when the population changes. For *anonymous* rules, *welfare dominance* requires solidarity when one agent replaces another agent. Following this line of reasoning, we may also consider the effect when an agent leaves. This leads to our second solidarity axiom. We require that when one agent leaves, all of the agents who remain should be affected in the same direction: Either no agent is made worse off or no agent is made better off.[7]

Population monotonicity for each $(N, R) \in \mathcal{E}$ and each $i \in N$, either (i) for each $j \in N\backslash\{i\}$, we have $\varphi(N\backslash\{i\}, R_{N\backslash\{i\}})\ R_j\ \varphi(N, R)$ or (ii) for each $j \in N\backslash\{i\}$, we have $\varphi(N, R)\ R_j\ \varphi(N\backslash\{i\}, R_{N\backslash\{i\}})$.

2.2 Rules

We present several classes of rules. Within each class, we identify a specific rule by a collection of parameters which we call the signature of the rule.

The simplest rules choose the same alternative in each economy. There are two such rules, one associated with each alternative.

[5] Saijo et al. (2007) introduce this property as "secure implementability" and allow for indirect mechanisms. However, they show that it is without loss of generality to consider only direct revelation mechanism.

[6] *Welfare dominance* has been studied in a several models under various names such as "agreement" or "replacement principle". See Thomson (1999) for additional details.

[7] *Population monotonicity* was originally introduced in the context of bargaining (Thomson 1983a, b) and has recently been considered in social choice (Bu 2013).

Constant rule associated with alternative $\boldsymbol{d} \in \boldsymbol{A}$, $\bar{\boldsymbol{C}}^{\boldsymbol{d}}$: For each $R \in \mathcal{R}^N$, $\bar{C}^d(R) \equiv d$.

Of course, the constant rules are not *efficient*, nor do they have *full range*. Our next rules are more responsive. Each rule is identified by a pair of parameters, $(d_1, d_2) \in A^2$. The first parameter specifies the choice of the rule in each disagreement economy; the second parameter specifies the choice of the rule when all agents are indifferent. In each consensus economy, the rule follows the consensus.

Consensus rule with defaults $\boldsymbol{(d_1, d_2) \in A^2}$, $\boldsymbol{C^{d_1,d_2}}$ for each $R \in \mathcal{R}^N$,

$$C^{d_1,d_2}(R) \equiv \begin{cases} a & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) = \emptyset \\ b & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_1 & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_2 & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) = \emptyset \end{cases}.$$

In the signature of a consensus rule, we call d_1 the **disagreement default** and d_2 the **indifference default**. While the disagreement default is a parameter of the rule rather than a parameter of the problem, its role is similar to that of a status quo: One privileged alternative is selected unless it receives no support. In all, there are four consensus rules, although those differing only by their indifference default are welfare-equivalent.

The next rules are clearly undesirable. Yet, they are essential to consider for a complete understanding of our axioms. Like a consensus rule, one of these rules makes the same selection in each disagreement economy. In each consensus economy, however, the rule opposes the consensus. In the indifference economy, a second default applies.

Anti-consensus rule with defaults $\boldsymbol{(d_1, d_2) \in A^2}$, $\boldsymbol{\hat{C}^{d_1,d_2}}$ for each $R \in \mathcal{R}^N$,

$$\hat{C}^{d_1,d_2}(R) \equiv \begin{cases} b & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) = \emptyset \\ a & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_1 & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_2 & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) = \emptyset \end{cases}.$$

In addition to following each consensus or opposing each consensus, a rule may follow a consensus in some economies and oppose a consensus in others. A rule in the next class selects the same alternative in each consensus economy, whether the consensus supports or opposes that alternative, and selects the other alternative in each disagreement economy. Again, a second default covers the indifference economy.

Mixed-consensus rule with defaults $\boldsymbol{(d_1, d_2) \in A^2}$, $\boldsymbol{\tilde{C}^{d_1,d_2}}$ for each $R \in \mathcal{R}^N$,

$$\tilde{C}^{d_1,d_2}(R) \equiv \begin{cases} A\backslash\{d_1\} & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) = \emptyset \\ A\backslash\{d_1\} & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_1 & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_2 & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) = \emptyset \end{cases}.$$

On the strict preference domain, the indifference defaults for rules in each of the previous three classes never apply. On this domain, we simplify our notation. Denoting the disagreement default by d, we write $\boldsymbol{C^d}$ for a consensus rule, $\boldsymbol{\hat{C}^d}$ for an anti-consensus rule, and $\boldsymbol{\tilde{C}^d}$ for a mixed-consensus rule.

Remark 1 On the strict preference domain, the constant, consensus, anti-consensus, and mixed-consensus rules constitute a class of eight rules. These rules are distinguished by their choices in three types of economies: The consensus economy for a, the consensus economy for b, and all disagreement economies.

Our next class of rules includes each of the rules previously defined. On the full preference domain, we can now distinguish among consensus economies favoring the same alternative. This leads us to generalize our notion of defaults in consensus economies. A **consensus mapping** is a function $D\colon 2^N \backslash \{\emptyset\} \to A$ which assigns one alternative to each non-empty subset of agents. Let $\mathcal{D}$ be the collection of consensus mappings.

We define a rule by selecting two consensus mappings and two defaults. The signature of the rule is a collection $(D^a, D^b, d_1, d_2) \in \mathcal{D}^2 \times A^2$, which specifies a choice for each consensus economy favoring a, each consensus economy favoring b, all disagreement economies, and the indifference economy.

Generalized mixed-consensus rule with signature $\boldsymbol{(\mathcal{D}^a, \mathcal{D}^b, d_1, d_2)}$, $\boldsymbol{C^{D^a, D^b, d_1, d_2}}$ for each $R \in \mathcal{R}^N$,

$$C^{D^a, D^b, d_1, d_2}(R) \equiv \begin{cases} D^a(N_a(R)) & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) = \emptyset \\ D^b(N_b(R)) & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_1 & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) \neq \emptyset \\ d_2 & \text{if } N_a(R) = \emptyset \text{ and } N_b(R) = \emptyset \end{cases}.$$

Remark 2 The constant, consensus, anti-consensus, and mixed-consensus rules are examples of generalized mixed-consensus rules. On the strict preference domain, each generalized mixed-consensus rule is outcome-equivalent to one of these eight rules.

Our final class of rules give some agents priority over others. For each rule in this class, one special agent is chosen. If she prefers one alternative, then the rule selects it. If she is indifferent, then the rule proceeds to a second special agent and so on. If all agents are indifferent, then the rule selects an indifference default.

Serial dictatorship rule relative to $\boldsymbol{\pi \in \Pi^N}$ *with indifference default* $\boldsymbol{d}$, $\boldsymbol{D^{\pi,d}}$ for each $R \in \mathcal{R}^N$,

$$D^{\pi,d}(R) \equiv \begin{cases} a & \text{if } \pi(1) \in N_a(R) \\ b & \text{if } \pi(1) \in N_b(R) \\ a & \text{if } \pi(1) \in N_{ab}(R) \text{ and } \pi(2) \in N_a(R) \\ b & \text{if } \pi(1) \in N_{ab}(R) \text{ and } \pi(2) \in N_b(R) \\ \vdots & \\ d & \text{if } N_{ab} = N \end{cases}.$$

Each serial dictatorship[8] is *efficient* and *strategy-proof*. On the strict preference domain, the first special agent always prefers one alternative, so only the first two cases are relevant. On this domain, we call the rule a **dictatorship** and denote the rule by $\boldsymbol{D^i}$ where $i \in N$ is the first special agent.

3 Solidarity properties

In this section, we search for rules satisfying our solidarity requirements. We begin by identifying all rules that satisfy *welfare dominance* on the strict preference domain (Theorem 1) and on the full preference domain (Theorem 2). Allowing the population to vary, we extend these results to characterize the rules satisfying *population monotonicity* (Theorem 3). In each case, additionally requiring *efficiency* leads to the consensus rules. Finally, contrasting the strict and full preference domains, we relate our solidarity properties to *anonymity*.

3.1 Welfare dominance

We begin with the fixed-population model. Our first lemma shows that *welfare dominance* requires a uniform choice in all disagreement economies. Although Lemma 1 refers only to *welfare dominance*, it plays a central role in our analysis of *population monotonicity* as well.

Lemma 1 *If a rule satisfies* welfare dominance*, then it selects the same alternative in each disagreement economy.*

Proof Let $\{R, R'\} \subseteq \mathcal{R}^N$ be such that $N_a(R)$, $N_b(R)$, $N_a(R')$, and $N_b(R')$ are non-empty. Let $i \in N_a(R)$ and $j \in N_b(R)$. By *welfare dominance*, $\varphi(R^a, R^b, R'_{-ij}) = \varphi(R)$. There are four cases.
Case 1: $R'_i = R^a$ and $R'_j = R^b$. Then $R' = (R^a, R^b, R'_{-ij})$ and $\varphi(R') = \varphi(R^a, R^b, R'_{-ij}) = \varphi(R)$.
Case 2: $R'_i = R^b$ and $R'_j = R^b$. Then $R' = (R^b, R^b, R'_{-ij})$. Since $N_a(R') \neq \emptyset$, there is $k \in N \backslash \{i, j\}$ such that $R'_k = R^a$. Then by *welfare dominance*, $\varphi(R') = \varphi(R^b, R^b, R'_{-ij}) = \varphi(R^a, R^b, R'_{-ij}) = \varphi(R)$.
Case 3: $R'_i = R^a$ and $R'_j = R^a$. Then $R' = (R^a, R^a, R'_{-ij})$. Since $N_b(R') \neq \emptyset$, there is $k \in N \backslash \{i, j\}$ such that $R'_k = R^b$. Then by *welfare dominance*, $\varphi(R') = \varphi(R^a, R^a, R'_{-ij}) = \varphi(R^a, R^b, R'_{-ij}) = \varphi(R)$.
Case 4: $R'_i = R^b$ and $R'_j = R^a$. Since $|N| \geq 3$, there is $k \in N \backslash \{i, j\}$. By *welfare dominance*, $\varphi(R^a, R_{-k}) = \varphi(R)$ so $\varphi(R^a, R_{-k}) = \varphi(R^a, R^b, R'_{-ij})$. Then by repeated application of *welfare dominance*, $\varphi(R) = \varphi(R^a, R^b, R^a, R'_{-ijk}) = \varphi(R^b, R^b, R^a, R'_{-ijk}) = \varphi(R^b, R^a, R^a, R'_{-ijk}) = \varphi(R^b, R^a, R'_{-ij})$. □

[8] We define serial dictatorship rules according to a fixed order. Instead, a rule may allow the order of later dictators may depend on the preferences of earlier dictators. While this approach often leads to a larger class of rules (e.g., Larsson and Svensson 2006), there is no difference in our model.

Lemma 2 identifies several rules that satisfy *welfare dominance* on both preference domains.

Lemma 2 *The constant rules, consensus rules, and anti-consensus rules satisfy* welfare dominance.

Proof We prove the result on the strict preference domain; the argument extends immediately to the full preference domain because the conclusion of *welfare dominance* is trivially satisfied by all agents who are indifferent between the alternatives. Let $R \in (\mathcal{R}^*)^N$, $i \in N$, and $R'_i \in \mathcal{R}^*$.

Constant rules: Since the constant rules are independent of preferences, they satisfy *welfare dominance*.

Consensus rules: If $C^a(R) = C^a(R'_i, R_{-i})$, then *welfare dominance* is satisfied so suppose instead that $C^a(R) = a$ and $C^a(R'_i, R_{-i}) = b$, relabeling if necessary. By definition of a consensus rule, $N_b(R'_i, R_{-i}) = N$ and $N_a(R'_i, R_{-i}) = \emptyset$. Therefore, for each $j \in N\backslash\{i\}$, we have $C^a(R'_i, R_{-i})\ R_j\ C^a(R)$ as required by *welfare dominance*.

Anti-consensus rules: If $\hat{C}^a(R) = \hat{C}^a(R'_i, R_{-i})$, then *welfare dominance* is satisfied so suppose instead that $\hat{C}^a(R) = a$ and $\hat{C}^a(R'_i, R_{-i}) = b$, relabeling if necessary. By definition of an anti-consensus rule, $N_b(R'_i, R_{-i}) = \emptyset$ and $N_a(R'_i, R_{-i}) = N$. Therefore, for each $j \in N\backslash\{i\}$, we have $\hat{C}^a(R)\ R_j\ \hat{C}^a(R'_i, R_{-i})$, as required by *welfare dominance*.

Mixed-consensus rules: If $\tilde{C}^a(R) = \tilde{C}^a(R'_i, R_{-i})$, then *welfare dominance* is satisfied, so suppose instead that $\tilde{C}^a(R) = a$ and $\tilde{C}^a(R'_i, R_{-i}) = b$, relabeling if necessary. By definition of a mixed-consensus rule, $N_a(R'_i, R_{-i}) = \emptyset$ or $N_a(R'_i, R_{-i}) = N$. In the first case, for each $j \in N\backslash\{i\}$, we have $\tilde{C}^a(R'_i, R_{-i})\ R_j\ \tilde{C}^a(R)$. In the second case, for each $j \in N\backslash\{i\}$, we have $\tilde{C}^a(R)\ R_j\ \tilde{C}^a(R'_i, R_{-i})$. Therefore, $\tilde{C}^a$ satisfies *welfare dominance*. □

Notably, the dictatorship rules do not satisfy *welfare dominance*.

Example 1 **Dictatorship rules violate welfare dominance.** Consider D^1. Let $R \in (\mathcal{R}^*)^N$ be such that $R_2 = R^a$ and $R_i = R^b$ for each $i \in N\backslash\{2\}$. Let $R'_1 = R^a$. Then $D^1(R) = b$ and $D^1(R'_1, R_{-1}) = a$. Since $a\ P_2\ b$ while $b\ P_3\ a$, this violates *welfare dominance*.

In fact, on the strict preference domain, the converse of Lemma 2 is true. On this domain, *welfare dominance* characterizes a class of eight rules: The constant, consensus, anti-consensus, and mixed-consensus rules. Among these, only the consensus rules are *unanimous*.

Theorem 1 *On the strict preference domain:*

(a) A rule satisfies welfare dominance *if and only if it is a constant, consensus, anti-consensus, or mixed-consensus rule.*
(b) A rule satisfies welfare dominance *and* unanimity *if and only if it is a consensus rule.*

Proof (a) By Lemma 2, each of the named rules satisfies *welfare dominance*. To prove the converse, let φ be a rule satisfying *welfare dominance*. Let $R \equiv (R^a, \ldots, R^a)$, $R' \equiv (R^b, \ldots, R^b)$, and $R'' \equiv (R^a, R^b, \ldots, R^b)$. Let $d_1 \equiv \varphi(R)$, $d_2 \equiv \varphi(R')$, and $d_3 \equiv \varphi(R'')$. Now let $R''' \in (R^*)^N$. If $R''' \neq R$ and $R''' \neq R'$, then $N_a(R''') \neq \emptyset$ and $N_b(R''') \neq \emptyset$. By Lemma 1, $\varphi(R''') = \varphi(R'') = d_3$. Therefore, φ is completely determined by d_1, d_2, and d_3. There are four cases:
Case 1: $d_1 = d_2 = d_3$. Then φ is a constant rule.
Case 2: $d_1 = a$ and $d_2 = b$. Then φ is a consensus rule.
Case 3: $d_1 = b$ and $d_2 = a$. Then φ is an anti-consensus rule.
Case 4: $d_1 = d_2 \neq d_3$. Then φ is a mixed-consensus rule.

(b) The consensus rules are *unanimous* while the constant, anti-consensus, and mixed-consensus rules are not. The conclusion follows by (a). □

We now extend the preference domain to allow for indifference and again search for rules satisfying *welfare dominance*. While each of the previously identified rules continues to satisfy *welfare dominance* on the full preference domain, the entire class of such rules is considerably larger. On this domain, *welfare dominance* characterizes the generalized mixed-consensus rules. As on the strict preference domain, only the consensus rules are also *efficient*.

Theorem 2 *On the full preference domain:*

(a) A rule satisfies welfare dominance *if and only if it is a generalized mixed-consensus rule.*
(b) A rule satisfies welfare dominance *and* efficiency *if and only if it is a consensus rule.*

Proof (a) Let φ satisfy *welfare dominance*. Let $R \in \mathcal{R}^N$ be such that $R_1 = R^a$, $R_2 = R^b$, and $R_i = R^{ab}$ for each $i \in N \backslash \{1, 2\}$. Let $d \equiv \varphi(R)$. By Lemma 1, φ selects d in each disagreement economy. Therefore, φ is a generalized mixed-consensus rule.

Let φ be a generalized mixed-consensus rule. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. If both R and (R'_i, R_{-i}) are disagreement economies, then $\varphi(R) = \varphi(R'_i, R_{-i})$. Suppose this is not the case. Without loss of generality, suppose $N_a(R) = \emptyset$, relabeling if necessary. There are two cases: Either for each $j \in N\backslash\{i\}$, we have $\varphi(R)\ R_j\ \varphi(R'_i, R_{-i})$; or for each $j \in N\backslash\{i\}$, we have $\varphi(R'_i, R_{-i})\ R_j\ \varphi(R)$. In either case, *welfare dominance* is satisfied.

(b) We have argued previously that the consensus rules satisfy *welfare dominance* (Lemma 1). By construction, the consensus rules are *efficient*. We now prove the converse.

Let φ satisfy *welfare dominance* and *efficiency*. Then φ is a generalized mixed-consensus rule defined by a default family $(\mathcal{D}^a, \mathcal{D}^b, d_1, d_2)$. Let $S \in 2^N \backslash \{\emptyset\}$. Let $R \in \mathcal{R}^N$ be such that $N_a(R) = S$ and $N_b(R) = \emptyset$. Then by *efficiency*, $\varphi(R) = a$ and so $\mathcal{D}^a(S) = a$. Next, let $R' \in \mathcal{R}^N$ be such that $N_a(R') = \emptyset$ and $N_b(R) = S$. Then, again by *efficiency*, $\varphi(R') = b$ and so $\mathcal{D}^b(S) = a$. Since this is true for each $S \in 2^N \backslash \{\emptyset\}$, we have $\varphi = C^{d_1, d_2}$ and φ is a consensus rule. □

Theorem 2(b) is false if *efficiency* is replaced by *unanimity*; several generalized mixed-consensus rules follow a unanimous choice but oppose the consensus in other economies. In fact, the class of generalized mixed-consensus rules is large, and the number of rules in the class grows exponentially with the size of the population.[9]

3.2 Population monotonicity

We now allow the population to vary and search for *population monotonic* rules. In contrast with other models,[10] *population monotonicity* is stronger than *welfare dominance* (Lemma 3). Theorem 3, which characterizes the class of *population monotonic* rules, extends our results from the fixed-population model.

Lemma 3 Population monotonicity *implies* welfare dominance.

Proof We prove the contra-positive. Let φ be a rule that violates *welfare dominance*. Then there is $(N, R) \in \mathcal{E}$ with $|N| \geq 3$, $\{i, j, k\} \subseteq N$, and $R'_i \in \mathcal{R}$ such that $\varphi(N, R'_i, R_{-i})\ P_j\ \varphi(N, R)$ and $\varphi(N, R)\ P_k\ \varphi(N, R'_i, R_{-i})$. Let $R' \equiv (R'_i, R_{-i})$. Let $N' \equiv N\backslash\{i\}$. Then $R_{N'} = R'_{N'}$ so $\varphi(N', R'_{N'}) = \varphi(N', R_{N'})$. There are two cases.
Case 1: $\varphi(N', R_{N'}) = \varphi(N, R)$. Then $\varphi(N', R_{N'}) \neq \varphi(N, R')$ and so $\varphi(N, R')\ P_j$ $\varphi(N', R_{N'})$ while $\varphi(N', R_{N'})\ P_k\ \varphi(N, R')$.
Case 2: $\varphi(N', R_{N'}) \neq \varphi(N, R)$. Then $\varphi(N', R_{N'}) = \varphi(N, R')$ and so $\varphi(N', R_{N'})\ P_j$ $\varphi(N, R)$ while $\varphi(N, R)\ P_k\ \varphi(N', R_{N'})$.

In either case, *population monotonicity* is violated and so φ is not *population monotonic*. □

The generalized mixed-consensus rules can be extended to *population monotonic* rules. To do so, we select one such rule for each population, with the restriction that each rule have the same disagreement default. In fact, this technique characterizes the class of *population monotonic* rules. Consistent with our previous results, additionally requiring *efficiency* leaves us with rules whose components are consensus rules.

Theorem 3 *On the full preference domain:*

(a) A rule satisfies population monotonicity *if and only if its components are generalized mixed-consensus rules with a common disagreement default.*
(b) A rule satisfies population monotonicity *and* efficiency *if and only if its components are consensus rules with a common disagreement default.*

Proof (a) First, we show that each such rule is *population monotonic*. Let $d \in A$. For each $N \in \mathcal{N}$, let φ^N be a generalized mixed-consensus rule defined on $\mathcal{R}^N$ with disagreement default d. For each $(N, R) \in \mathcal{E}$, let $\varphi(N, R) \equiv \varphi^N(R)$. To show that the extended rule φ is *population monotonic*, let $N \in \mathcal{N}$, $i \in N$, and $R \in \mathcal{R}^N$. Let $N' \equiv N\backslash\{i\}$ and $R' \equiv R_{N'}$ and suppose that $\varphi(N, R) \neq \varphi(N', R_{N'})$. By definition, $\varphi(N, R) = \varphi^N(R)$ and $\varphi(N', R') = \varphi^{N'}(R')$. Moreover, since φ^N and $\varphi^{N'}$ have the same disagreement default, there is no disagreement in the reduced economy. Without loss of generality, suppose $N_a(R') = \emptyset$. There are two cases according to whether

[9] More precisely, for each $N \in \mathcal{N}$, there are $2^{|N|+1}$ generalized mixed-consensus rules.

[10] Gordon (2007) shows that under *efficiency* and "replication indifference", *welfare dominance* implies *population monotonicity* very generally.

$\varphi(N, R) = a$ or $\varphi(N, R) = b$. However, for each $j \in N'$, $b\ R_j\ a$ so *population monotonicity* is satisfied in either case.

Next, we show that the components of each *population monotonic* are as claimed. Let $\hat{\varphi}$ be a *population monotonic rule*. By Lemma 3, $\hat{\varphi}$ satisfies *welfare dominance*. In particular, for each $N \in \mathcal{N}$ there is $\hat{\varphi}^N$ such that for each $R \in \mathcal{R}^N$, $\hat{\varphi}(N, R) = \hat{\varphi}^N(R)$. For each $N \in \mathcal{N}$, let $\hat{d}_N$ be the disagreement default associated with $\hat{\varphi}^N$. To see that the disagreement defaults are the same, let $\{N', N''\} \subseteq \mathcal{N}$ and let $N \equiv N' \cup N''$. Since there are two alternatives, either $\hat{d}_N = \hat{d}_{N'}$ or $\hat{d}_N = \hat{d}_{N''}$. Without loss of generality, suppose $\hat{d}_N = \hat{d}_{N'}$. Let $R \in \mathcal{R}^N$ be such that there is a pair $\{i, j\} \subseteq N''$ with $i \in N_a(R)$ and $j \in N_b(R)$. Beginning from the economy (N, R), we remove each agent $k \in N \backslash N''$ in turn. Since $\{i, j\} \subseteq N''$, there is disagreement in each economy in the sequence. Applying *population monotonicity* at each step, we conclude that $\hat{d}_N = \hat{\varphi}(N, R) = \hat{\varphi}(N', R') = \hat{d}_{N''}$. Since $\hat{d}_N = \hat{d}_{N'}$, we have $\hat{d}_{N''} = \hat{d}_{N'}$.

(b) By Theorem 2(b), the components of each such rule are consensus rules. By (a), these components have a common disagreement default. □

Combining the results of Theorems 1 and 3, we are able to identify the subclasses of rules that satisfy *population monotonicity* or the combination of *population monotonicity* and *unanimity* on the strict preference domain. Once again, the key requirement of *population monotonicity* is that the components have a common disagreement default.

Theorem 4 *On the strict preference domain:*

(a) A rule satisfies population monotonicity *if and only if its components are constant, consensus, mixed-consensus, or anti-consensus rules with a common disagreement default.*

(b) A rule satisfies population monotonicity *and* unanimity *if and only if its components are consensus rules with a common disagreement default.*

As Theorems 3 and 4 show, *population monotonicity* adds to *welfare dominance* a uniformity across populations. When an agent who prefers one alternative becomes indifferent, she no longer has an interest in the outcome. Intuitively, we may think of her as "leaving" the economy. *Welfare dominance* applies to this hypothetical population change. When we allow actual population changes, *population monotonicity* applies. Theorems 3 and 4 show that when considered from either perspective, solidarity restricts rules in the same way.

3.3 Solidarity and anonymity

Depending on the preference domain, our solidarity properties may or may not imply *anonymity*. These results highlight the role of indifference.

Since the consensus rules are *anonymous*, Theorem 2 shows that *welfare dominance* and *efficiency* together imply *anonymity*. *Efficiency* is required for this conclusion on the full preference domain but not on the strict preference domain.

Proposition 1 *On the strict preference domain,* welfare dominance *implies* anonymity.

Proof Let φ satisfy *welfare dominance* and let $\pi \in \Pi^N$. Let $R \in (\mathcal{R}^*)^N$ and let $R' \equiv \pi(R)$. By construction, $|N_a(R)| = |N_a(R')|$ and $|N_b(R)| = |N_b(R')|$. Suppose by way of contradiction that $\varphi(R) \neq \varphi(R')$. Then $N_a(R) \neq N_a(R')$ and $N_b(R) \neq N_b(R')$. In particular, since $N_a(R) \cup N_b(R) = N_a(R') \cup N_b(R') = N$, we have $N_a(R) \neq \emptyset$ and $N_b(R) \neq \emptyset$. Let $i \in N_a(R)$, $j \in N_b(R)$, and $k \in N\backslash\{i, j\}$. By *welfare dominance*, $\varphi(R^a, R_{-k}) = \varphi(R)$. Without loss of generality, then, we may assume $R_k = R^a$ so $|N_a(R)| \geq 2$.

Let $S \subseteq N$ be the set of agents whose preferences change: $S \equiv \{h \in N : R_h \neq R'_h\}$. Then $|S \cap N_a(R)| = |S \cap N_b(R')|$ and $|S|$ is even. List the agents of S as $(s_1, s_2, \ldots, s_{2m})$ so that adjacent agents have opposite preferences: For each $k = 1, \ldots, m$, $s_{2k-1} \in N_a(R)$ and $s_{2k} \in N_b(R)$. Let $R^{(0)} \equiv R$, $R^{(1)} \equiv (R'_{s_1}, R_{-s_1})$, $R^{(2)} \equiv (R'_{s_1}, R'_{s_2}, R_{-s_1 s_2})$, and so on to $R^{(k)} \equiv (R'_S, R_{-S}) = R'$. For each $l = 1, \ldots, k$, $N_a(R^{(l)}) \neq \emptyset$ and $N_b(R^{(l)}) \neq \emptyset$, so *welfare dominance* implies $\varphi(R^{(l)}) = \varphi(R^{(l-1)})$. But then $\varphi(R) = \varphi(R^{(0)}) = \varphi(R^{(k)}) = \varphi(R')$, which contradicts our initial assumption. □

A key step in the proof is deducing that both $N_a(R)$ and $N_b(R)$ are non-empty. This implication does not hold on the full preference domain because in that case there is a third set $N_{ab}(R)$ to consider. Example 2 illustrates a rule that satisfies *welfare dominance* (and is *unanimous*) yet is not *anonymous*.

Example 2 **A rule satisfying welfare dominance and unanimity but not anonymity.** For each $R \in \mathcal{R}^N$, define φ by

$$\varphi(R) \equiv \begin{cases} a & \text{if } 1 \in N_a(R) \text{ and } N_b(R) = \emptyset \\ b & \text{otherwise} \end{cases}.$$

For each $R \in \mathcal{R}^N$, if $1 \in N_a(R)$, then $\varphi(R) = a$ and if $1 \in N_b(R)$, then $\varphi(R) = b$ so φ satisfies *unanimity*. To see that φ satisfies *welfare dominance*, let $R \in R^N$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $\varphi(R) = a$ and $\varphi(R'_i, R_{-i}) = b$. By definition, $1 \in N_a(R)$ and $N_b(R) = \emptyset$. Then $N_b(R'_i, R_{-i})\backslash\{i\} = \emptyset$ and for each $j \in N\backslash\{i\}$ we have $\varphi(R)\ R_j\ \varphi(R'_i, R_{-i})$. Of course, φ distinguishes agent 1 and so is not *anonymous*.

The rule in Example 2 is not *efficient* and so shows that *efficiency* is not implied by the combination of *welfare dominance* and *unanimity*. The next example shows that *efficiency* is not implied even when *anonymity* is included.

Example 3 **A rule satisfying welfare dominance, anonymity, and unanimity but not efficiency.** For each $R \in \mathcal{R}^N$, define φ by

$$\varphi(R) \equiv \begin{cases} a & \text{if } N_a(R) = N \\ b & \text{otherwise} \end{cases}.$$

The rule is a generalized mixed-consensus rule and so satisfies *welfare dominance*. It is *anonymous* and *unanimous* essentially by definition. To see that φ is not *efficient*, let $R \in \mathcal{R}^N$ be such that that $N_{ab}(R) = \{1\}$ and $N_a(R) = N\backslash\{1\}$. Then $\varphi(R) = b$ in violation of *efficiency*.

Since *population monotonicity* implies *welfare dominance* (Lemma 3), Proposition 1 shows that on the strict preference domain, *population monotonicity* implies *anonymity* as well. On the full preference domain, however, these properties are independent.

Example 4 **A population monotonic and efficient rule that is not anonymous.** For each $N \in \mathcal{N}$, let $C_N^{d_1,d_2}$ be the consensus rule with defaults (d_1, d_2) defined on $\mathcal{R}^N$. For each $(N, R) \in \mathcal{E}$, define φ by

$$\varphi(N, R) \equiv \begin{cases} C_N^{a,a}(R) & \text{if } 1 \in N \\ C_N^{a,b}(R) & \text{otherwise} \end{cases}.$$

Since agent 1 plays a special role, φ is not *anonymous*. However, φ is an extended generalized mixed-consensus rule and so satisfies *population monotonicity* by Theorem 3.

Example 4 shows that, in contrast with the fixed-population model, *anonymity* is not even implied under *efficiency*. Of course, the possible violations of *anonymity* have no welfare significance, as they are limited to changes in the indifference default.

4 Strategic properties

In this section, we turn to incentive properties and relate these properties to our solidarity properties. The literature provides ready characterizations of *strategy-proof* rules with *full range* on both the strict and full preference domains: They are described as "voting by committees" (Barberà et al. 1991) and "voting by extended committees" (Larsson and Svensson 2006). Moreover, as examples of serial dictatorship and anti-consensus rules show, *strategy-proofness* is logically independent from our solidarity properties on either the strict or full preference domain. Thus, we turn immediately to stronger strategic requirements.

On the full preference domain, we uncover a surprising connection between normative and strategic properties: *Group strategy-proofness* implies *welfare dominance* (Proposition 2). Strengthening our strategic requirement to *secure strategy-proofness*, we show that only undesirable rules remain admissible. On the strict preference domain, only the dictatorship and constant rules are *securely strategy-proof* and on the full preference domain, only the constant rules satisfy the property (Theorem 5).

4.1 Group strategy-proofness

On the strict preference domain, *strategy-proofness* and *group strategy-proofness* are equivalent (Le Breton and Zaporozhets 2009). Thus, the rules satisfying *group strategy-proofness*) and *full range* are also "voting by committees" rules (Barberà et al. 1991). On the full preference domain, however, these relationships change. For example, when indifferences are allowed, the serial dictatorship rules continue to be *strategy-proof* but are no longer *group strategy-proof*. Intuitively, *group strategy-proofness* becomes more demanding because an indifferent agent may be able to influence the outcome by misrepresenting her preferences, thereby benefiting other without cost to herself.

On the strict preference domain, neither incentive property is logically related to *welfare dominance* or *anonymity*. Allowing for indifference, however, yields a surprise: On the full preference domain, *welfare dominance* is a necessary condition for *group strategy-proofness*. Proposition 2 summarizes.

Proposition 2 *(a) On the strict preference domain,* welfare dominance *neither implies nor is implied by* group strategy-proofness.
(b) On the full preference domain, group strategy-proofness *implies* welfare dominance.

Proof (a) The dictatorship rules are *group strategy-proof* but violate *welfare dominance*. The anti-consensus rules satisfy *welfare dominance* but are not *group strategy-proof*.

(b) We prove the contra-positive. Suppose φ violates *welfare dominance*. Then there are $R \in \mathcal{R}^N$, $\{i, j, k\} \subseteq N$, and $R'_i \in \mathcal{R}$ such that $\varphi(R'_i, R_{-i})\ P_j\ \varphi(R)$ and $\varphi(R)\ P_k\ \varphi(R'_i, R_{-i})$. If $R_i = R^{ab}$, then i and j can manipulate at R. Similarly, if $R'_i = R^{ab}$, then i and k can manipulate at (R'_i, R_{-i}). In either case, φ is not *group strategy-proof*.

Suppose instead that $R_i, R'_i \in \mathcal{R}^*$. Let $R''_i = R^{ab}$. Since there are two alternatives, either $\varphi(R''_i, R_{-i}) = \varphi(R)$ or $\varphi(R''_i, R_{-i}) = \varphi(R'_i, R_{-i})$. In the first case, i and j can manipulate at (R''_i, R_{-i}) while in the second case i and k can manipulate at (R''_i, R_{-i}). Thus, φ is not *group strategy-proof*. □

On the full preference domain, the consensus rules are the only *group strategy-proof* rules with *full range* (Manjunath 2012). Based on this characterization and Theorem 2, we have an immediate relationship among axioms:

Proposition 3 *On the full preference domain, the combination of* group strategy-proofness *and* full range *is equivalent to the combination of* welfare dominance *and* efficiency.

4.2 Secure strategy-proofness

We now turn to *secure strategy-proofness*. Since *strategy-proofness* is already a demanding property, we expect *secure strategy-proofness* to be very difficult to satisfy, as our analysis will show. To proceed, we introduce a helpful technical property. Consider a true preference profile R and an alternative preference profile R'. Suppose that each agent is indifferent between the outcome when reporting truthfully and when individually deviating to R'_i. Then, we require the outcomes assigned to R and R' be the same.

Rectangle property for each $N \in \mathcal{N}$ and each pair $\{R, R'\} \subseteq \mathcal{R}^N$, if for each $i \in N$, $\varphi(R'_i, R_{-i})\ I_i\ \varphi(R)$, then $\varphi(R') = \varphi(R)$.

A rule is *securely strategy-proof* if and only if it is *strategy-proof* and satisfies the *rectangle property* (Saijo et al. 2007). Intuitively, the *rectangle property* is the “difference” between *strategy-proofness* and *secure strategy-proofness*.

Since *group strategy-proofness* and *strategy proofness* are equivalent on the strict preference domain, *secure strategy-proofness* implies *group strategy-proofness* by definition. On the full preference domain, this implication continues to hold, but now requires proof.

Proposition 4 *On the full preference domain,* secure strategy-proofness *implies both* group strategy-proofness *and* welfare dominance.

Proof By Proposition 2, *group strategy-proofness* implies *welfare dominance*, so it suffices to show that *secure strategy-proofness* implies *group strategy-proofness*.

Let φ be a *strategy-proof* rule that is not *group strategy-proof*. We will show that φ violates the *rectangle property* and so is not *securely strategy-proof*. Since φ is not *group strategy-proof*, there are $N' \subseteq N$ and $\{R, R'\} \subseteq \mathcal{R}^N$ with $R_{N\setminus N'} = R'_{N\setminus N'}$ such that for each $j \in N'$ and for some $i \in N'$, we have $\varphi(R')\ R_j\ \varphi(R)$ and $\varphi(R')\ P_i\ \varphi(R)$. In particular, $\varphi(R) \neq \varphi(R')$.

We claim that for each $k \in N$, $\varphi(R'_k, R_{-k})\ I_k\ \varphi(R)$. Since there are two alternatives, if $\varphi(R'_k, R_{-k}) \neq \varphi(R)$, then $\varphi(R'_k, R_{-k}) = \varphi(R')$. Also, $R'_k \neq R_k$, so $k \in N'$ and $\varphi(R'_k, R_{-k}) = \varphi(R')\ R_k\ \varphi(R)$. By *strategy-proofness*, $\varphi(R)\ R_k\ \varphi(R'_k, R_{-k}) = \varphi(R')$. Together, $\varphi(R)\ I_k\ \varphi(R')$, as claimed.

For each $k \in N$, $\varphi(R'_k, R_{-k})\ I_k\ \varphi(R)$, so the hypothesis of the *rectangle property* is satisfied. However, $\varphi(R) \neq \varphi(R')$ so the conclusion is violated. □

On the strict preference domain, dictatorship rules are *securely strategy-proof*. However, as Example 5 shows, the consensus rules are not.

Example 5 **The consensus rules are not securely strategy-proof.** Consider C^a. Let $R \in (\mathcal{R}^*)^N$ be such that $N_a(R) = \{1, 2\}$ and $N_b(R) = N\setminus\{1, 2\}$ and let $R'_1 = R'_2 = R^b$. Then $C^a(R) = C^a(R'_1, R_{-1}) = C^a(R'_2, R_{-2}) = a$ while $C^a(R'_1, R'_2, R_{-12}) = b$. This violates the *rectangle property* and so C^a is not *securely strategy-proof*.

The examples of dictatorship and consensus rules show that on the strict preference domain there is no logical relation between *secure strategy-proofness* and *welfare dominance*. On the full preference domain, *secure strategy-proofness* is nearly impossible to satisfy. On this domain, only the consensus rules and constant rules satisfy *group strategy-proofness* (Manjunath 2012). Since the consensus rules violate *secure strategy-proofness* even on the strict preference domain (Example 5), we are left with only the constant rules. Theorem 5 summarizes.

Theorem 5 *(a) On the strict preference domain, a rule is* securely strategy-proof *if and only if it is a dictatorship rule or a constant rule.*
(b) On the full preference domain, a rule is securely strategy-proof *if and only if it is a constant rule.*

5 Duality

In this section, we study the the structure of the space of rules. With two alternatives, a natural symmetry arises with respect to economies, rules, and the alternatives themselves. In fact, as we will explain, the appropriate notion of "self duality" has a straightforward interpretation: The alternatives should be treated symmetrically. More generally, duality helps us to understand the extent to which a rule favors one alternative over the other. Further developing our results, we define a "duality operator" which pairs each rule with a rule that exhibits the opposite bias. To investigate these ideas formally, we need additional notation.

Let φ be a rule. There is a unique rule that selects in each economy the alternative that φ does not select. Formally, the **opposite of $\boldsymbol{\varphi}$, $\boldsymbol{\varphi^{-1}}$**, is such that for each $R \in \mathcal{R}^N$, $\varphi^{-1}(R) \equiv A \backslash \varphi(R)$. Since $|A| = 2$, φ^{-1} is well-defined. We have seen several pairs of opposites. For example, the two constant rules are opposites, and each consensus rule is the opposite of an anti-consensus rule. The opposite of a serial dictatorship rule is a serial "anti-dictatorship" rule: The rule selects the less preferred alternative of the first special agent who is not indifferent.

We next define a symmetry relationship between economies. To each economy we associate a second economy in which each agent's preference is reversed: For each $R \in \mathcal{R}^N$, $\boldsymbol{R^d}$ is the **dual economy of $\boldsymbol{R}$**, if $N_a(R^d) = N_b(R)$ and $N_a(R^d) = N_b(R)$. Next we identify the rule that selects in each economy the alternative that φ does not select in the dual economy, which we call its dual. Formally,

Dual of $\boldsymbol{\varphi}$, $\boldsymbol{\varphi^d}$ for each $R \in \mathcal{R}^N$, $\varphi^d(R) \equiv (\varphi^{-1}(R^d))$.

We say the two properties are dual if whenever a rule satisfies the property, its dual satisfies the other property. A rule φ is **self-dual** if $\varphi^d = \varphi$. *Self-duality* requires that a rule reverse its selection when faced with a dual economy. In particular, *self-duality* implies that the labels of the objects do not matter. That is, *self-duality* is equivalent to the property commonly called **neutrality**. Since the constant rules are not *self-dual*, either property immediately implies *full range*.

Unfortunately, on the full preference domain, no rule is *self-dual*. The reason is that we require rules to select an alternative in each economy, including the indifference economy. Necessarily, the dual of a rule will make the opposite selection in this economy. Our first proposition summarizes the preceding discussion.

Proposition 5 *On the full preference domain, no rule is* self-dual.

On the full preference domain, the dictatorship rules violate *self-duality* only because they must make a default choice in the indifference economy, a technicality that does not arise on the strict preference domain.[11]

Lemma 4 *On the strict preference domain, the dictatorship rules are* self-dual.

Proof Consider D^1. Let $R \in \mathcal{R}^*$ and suppose $D^1(R) = a$. Then $1 \in N_a(R)$ and so $1 \in N_b(R^d)$. Therefore, $D^1(R^d) = b$. □

On the strict preference domain, the dictatorship rules are *strategy-proof* and *efficient*, so *self-duality* is now compatible with these properties. In fact, *efficiency* is implied by the other axioms. This is because *self-duality* implies *full range* and the combination of *strategy-proofness* and *full range* implies *efficiency*.

[11] Majority rule is another rule that violates *self-duality* only rarely. On the strict preference domain with an odd number of agents, majority rule is *self-dual*.

Corollary 1 *On the strict preference domain,* strategy-proofness *and* self-duality *imply* efficiency.

More generally, we may ask which *self-dual* rules are *strategy-proof*. Since these properties together imply *efficiency*, the class of rules satisfying both properties is a subset of the "voting by committees" rules (Barberà et al. 1991). To satisfy *self-duality*, we require that the winning coalitions be the same for both alternatives: $S \subseteq N$ is a winning coalition for a if and only if S is a winning coalition for b.

To better understand the properties of opposites and duals, it is helpful to think in terms of operators, or mappings from the space of rules into itself. Let φ be a rule. The **opposition** operator associates φ with φ^{-1} and the **duality** operator associates φ with φ^d. These operators commute.

Lemma 5 *For each rule* φ*,* $(\varphi^{-1})^d = (\varphi^d)^{-1}$.

Proof Let φ be a rule and let $R \in \mathcal{R}^N$. By definition, $\varphi^d(R) = \varphi^{-1}(R^d)$. Computing twice,

$$(\varphi^{-1})^d(R) = (\varphi^{-1})^{-1}(R^d) = A\backslash\varphi^{-1}(R^d) = A\backslash\big(A\backslash\varphi(R^d)\big) = \varphi(R^d) \quad \text{and}$$
$$(\varphi^d)^{-1}(R) = A\backslash\varphi^d(R) = A\backslash\varphi^{-1}(R^d) = A\backslash\big(A\backslash\varphi(R^d)\big) = \varphi(R^d).$$

Comparing results, $(\varphi^{-1})^d(R) = (\varphi^d)^{-1}(R)$. □

Several properties are preserved under opposition.

Proposition 6 *Opposition preserves* full range*,* anonymity*,* welfare dominance*, and* population monotonicity.

Proof Full range: Let φ be a rule satisfying *full range*. Then there is a pair $\{R, R'\} \subseteq \mathcal{R}^N$ such that $\varphi(R) = a$ and $\varphi(R') = b$. Then $\varphi^{-1}(R) = b$ and $\varphi^{-1}(R') = a$.

Anonymity: Let φ be a rule satisfying *anonymity*. Let $R \in \mathcal{R}^N$ and $\pi \in \Pi^N$. By *anonymity*, $\varphi(R) = \varphi(\pi(R))$ and $\varphi(R^d) = \varphi(\pi(R^d))$. Since $\pi(R^d) = (\pi(R))^d$, we have $\varphi^{-1}(R) = \varphi^{-1}(\pi(R))$.

Welfare dominance: Let φ be a rule satisfying *welfare dominance*. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}^N$. By *welfare dominance*, either for each $j \in N\backslash\{i\}$, $\varphi(R)\ R_j\ \varphi(R'_i, R_{-i})$ or for each $j \in N\backslash\{i\}$, $\varphi(R'_i, R_{-i})\ R_j\ \varphi(R)$. Without loss of generality, suppose the first case. Then for each $j \in N\backslash\{i\}$, $\varphi^{-1}(R'_i, R_{-i})\ R_j\ \varphi^{-1}(R)$.

Population monotonicity: Let φ be a rule satisfying *population monotonicity*. Let $(N, R) \in \mathcal{E}$ and $i \in N$. By *population monotonicity*, either for each $j \in N\backslash\{i\}$, $\varphi(N, R)\ R_j\ \varphi(N\backslash\{i\}, R_{N\backslash\{i\}})$ or for each $j \in N\backslash\{i\}$, $\varphi(N\backslash\{i\}, R_{N\backslash\{i\}})\ R_j\ \varphi(N, R)$. In the first case, for each $j \in N\backslash\{i\}$, $\varphi^{-1}(N\backslash\{i\}, R_{N\backslash\{i\}})\ R_j\ \varphi^{-1}(R)$ and in the second case, for each $j \in N\backslash\{i\}$, $\varphi^{-1}(R)\ R_j\ \varphi^{-1}(N\backslash\{i\}, R_{N\backslash\{i\}})$. □

On the other hand, many properties are not preserved. For example, if a rule is *unanimous*, then its opposite is not *unanimous*. Similarly, if a rule is *efficient*, then its opposite is not *efficient*. A rule and its opposite are both *strategy-proofness* if and only if they are constant.

We now turn to the duality operator. Several leading families are closed under *duality*.

Proposition 7 *Each of the following families is closed under duality: (i) generalized mixed-consensus rules, (ii) constant rules, (iii) consensus rules, and (iv) serial dictatorship rules.*

Proof Generalized mixed-consensus rules: Let $(D_1, D_2, d_1, d_2) \in \mathcal{D}^2 \times A^2$ and $\varphi \equiv C^{D_1,D_2,d_1,d_2}$. Without loss of generality, suppose $d_1 = a$. Let $R \in \mathcal{R}^N$ be a disagreement economy. Then R^d is also a disagreement economy and $\varphi(R) = \varphi(R^d) = a$. Therefore, $\varphi^d(R) = \varphi^{-1}(R^d) = b$. Since φ^d selects b in each disagreement economy, it is a generalized mixed-consensus rule. In fact, it is the generalized mixed-consensus rule with signature $(D_2, D_1, A\backslash\{d_1\}, A\backslash\{d_2\})$.

Constant rules: The two constant rules are dual rules, so closure follows immediately.

Consensus rules: Consider C^a. By the previous result, $(C^a)^d$ is a generalized mixed-consensus rule with disagreement default b. Let $R \in \mathcal{R}^N$ be a consensus economy favoring a. Then R^d is a consensus economy favoring b. Therefore, $C^a(R) = (C^a)^d(R) = a$ and $C^a(R^d) = (C^a)^d(R^d) = b$. Therefore, $(C^a)^d = C^b$.

Serial dictatorship rules: The dual of a serial dictatorship rule differs from the original only in the default it selects when all agents are indifferent. Therefore, the dual of a serially dictatorial rule is also a dictatorial rule with the same ordering of agents. □

The proposition implies that *welfare dominance* is preserved under duality. In fact, duality preserves all of the standard properties.

Proposition 8 *Duality preserves* full range*,* anonymity*,* welfare dominance*,* population monotonicity*,* unanimity*,* efficiency*, and* strategy-proofness.

Proof Let φ be a rule. By commutativity of the operators (Lemma 5), for each $R \in \mathcal{R}^N$, we have $\varphi^d(R) = \varphi^{-1}(R)$ and $\varphi^{-1}(R) = \varphi^d(R^d)$. Since *full range*, *anonymity*, *welfare dominance*, and *population monotonicity* are preserved under opposition (Proposition 6), they are also preserved under duality. We verify the remaining properties.

Unanimity: Suppose φ satisfies *unanimity*. Then $\varphi(R^a, \ldots, R^a) = a$ and $\varphi(R^b, \ldots, R^b) = b$. By definition, $\varphi^d(R^a, \ldots, R^a) = \varphi^{-1}(R^b, \ldots, R^b) = a$ and $\varphi^d(R^b, \ldots, R^b) = \varphi^{-1}(R^a, \ldots, R^a) = b$.

Efficiency: Suppose φ satisfies *efficiency*. Without loss of generality, let $R \in \mathcal{R}^N$ be an economy with consensus for a. Then $N_b(R) = N_a(R^d) = \emptyset$. By *efficiency*, $\varphi(R) = a$ and $\varphi(R^d) = b$. Then by definition, $\varphi^d(R) = \varphi^{-1}(R^d) = a$.

Strategy-proofness: Suppose φ^d is not *strategy-proof*. Then there are $R \in \mathcal{R}^N$, $i \in N$, and $R_i' \in \mathcal{R}$ such that $\varphi^d(R_i', R_{-i})\ P_i\ \varphi^d(R)$. Without loss of generality, suppose $\varphi^d(R) = a$ so $\varphi^d(R_i', R_{-i}) = b$ and $R_i = R^b$. By definition, $\varphi^{-1}(R^d) = \varphi^d(R) = a$ so $\varphi(R^d) = b$. Similarly, $\varphi^{-1}(R_i'^d, R_{-i}^d) = \varphi^d(R_i', R_{-i}) = b$ so $\varphi(R_i'^d, R_{-i}^d) = a$. Since $R_i = R^b$, $R_i^d = R^a$. Therefore, $\varphi(R_i'^d, R_{-i}^d) = a\ P_i^d\ b = \varphi(R^d)$ and so φ is not *strategy-proof*. □

6 Extended model allowing no choice

Our model requires a rule to select one alternative in each economy. In some situations, it is possible to postpone a decision and such "resoluteness" is unnecessary. In this section, we modify our model to accommodate rules which select neither alternative in some economies. Formally, we extend the set of alternatives to $\bar{A} \equiv \{\{a\}, \emptyset, \{b\}\} \subseteq \mathcal{P}(A)$. Our rules are now correspondences mapping economies into $\bar{A}$. To keep notation simple, however, we will continue to write a and b instead of $\{a\}$ and $\{b\}$. Reusing notation, we extend the preference relations in our domain $\mathcal{R} = (R^a, R^b, R^{ab})$ so that $a\ P^a\ \emptyset\ P^a\ b$, $b\ P^b\ \emptyset\ P^b\ a$, and $a\ I^{ab}\ \emptyset\ I^{ab}\ b$.[12] Each preference relation has the same interpretation as in the original model. Augmented in this way, our model is a special case of social choice with three alternatives. To apply our notion of duality, we modify our definition of opposite so that for each $R \in \mathcal{R}^N$,

$$\varphi^{-1}(R) \equiv \begin{cases} a & \text{if } \varphi(R) = b \\ b & \text{if } \varphi(R) = a \\ \emptyset & \text{if } \varphi(R) = \emptyset \end{cases}.$$

In this extended model, the signature of a generalized consensus rule has the same form but may now include $\emptyset$ for some defaults. Our main result is that *welfare dominance* is now compatible with *self-duality*. *Self-duality* requires that the disagreement default be the emptyset. On the strict preference domain, adding *self-duality* to *welfare dominance* and *unanimity* identifies a unique rule.

Theorem 6 *On the strict preference domain in the extended model:*

(a) A rule satisfies welfare dominance *and* self-duality *if and only if it is a consensus rule, an anti-consensus rule, or a constant rule with the emptyset as its disagreement default.*
(b) A rule satisfies welfare dominance, self-duality, *and* unanimity *if and only if it is the consensus rule with the emptyset as its disagreement default.*

Proof (a) Trivially, $\bar{C}^{\emptyset}$ satisfies the two properties. We verify the properties for a consensus rule; the argument for an anti-consensus rule is parallel. Since the consensus rules satisfy *welfare dominance* in the original model, $C^{\emptyset}$ satisfies *welfare dominance* in the extended model. Let $R \in \mathcal{R}^N$. If R is a disagreement economy, then so is R^d. In this case, $C^{\emptyset}(R) = \emptyset = C^{\emptyset}(R^d)$. Suppose instead that R is not a disagreement economy. Then, since preferences are strict, it is a consensus economy unanimously favoring one alternative, say a. In this case, $C^{\emptyset}(R) = a$ and $C^{\emptyset}(R^d) = b$ so $(C^{\emptyset})^{-1}(R^d) = a$. Therefore, $C^{\emptyset}$ is *self-dual*.

Conversely, let φ be a rule satisfying *welfare dominance* and *self-duality*. By *welfare dominance*, there is $d_0 \in \bar{A}$ such that for each economy $R \in \mathcal{R}^N$ with disagreement, $\varphi(R) = d_0$. Then $\varphi(R^d) = d_0$. By *self-duality*, $\varphi^{-1}(R^d) = d_0 = \varphi(R^d)$ and so $d_0 = \emptyset$. Now consider $R \equiv (R^a, \ldots, R^a)$. By *self-duality*, $(R^b, \ldots, R^b) = \varphi(R^d) =$

[12] This assumption means that preferences are "single-peaked" with respect to the order $(a, \emptyset, b)$.

$\varphi^{-1}(R)$. If $\varphi(R) = a$, then $\varphi = C^{\emptyset}$; if $\varphi(R) = b$, then $\varphi = \hat{C}^{\emptyset}$; and if $\varphi(R) = \emptyset$, then $\varphi = \bar{C}^{\emptyset}$.

(b) In the original model, *welfare dominance* and *unanimity* characterize the consensus rules (Theorem 1(b)). By (a), the default in the extended model must be the emptyset. □

On the full preference domain, the results are similar. Since generalized mixed-consensus rules now have more parameters, *self-duality* imposes correspondingly more requirements. The combination of *welfare dominance*, *self-duality*, and *unanimity* no longer pinpoints a single rule, but only one rule is also *efficient*.

Theorem 7 *On the full preference domain in the extended model:*

(a) A rule satisfies welfare dominance *and* self-duality *if and only if it is a generalized mixed-consensus rule with a symmetric signature and the emptyset as its disagreement default.*

(b) A rule satisfies welfare dominance*,* self-duality*, and* efficiency *if and only if it is the consensus rule with the emptyset as its disagreement default.*

Proof (a) Let $D \in \mathcal{D}$. For each $S \subseteq N$ define D^{-1} by

$$D^{-1}(S) \equiv \begin{cases} a & \text{if } D(S) = b \\ b & \text{if } D(S) = a \\ \emptyset & \text{otherwise} \end{cases}.$$

Consider $C^{D,D^{-1},\emptyset,\emptyset}$. Since each generalized mixed-consensus rule satisfies *welfare dominance* in the original model, $C^{D,D^{-1},\emptyset,\emptyset}$ satisfies *welfare dominance* in the extended model. Let $R \in \mathcal{R}^N$. If R is a disagreement economy or the indifference economy, then $C^{D,D^{-1},\emptyset,\emptyset}(R) = \emptyset$ and $C^{D,D^{-1},\emptyset,\emptyset}(R^d) = \emptyset$. Suppose instead that R is a consensus economy, say in favor of a, and let $S \equiv$

Table 1 Properties satisfied by various rules on the full preference domain

	Serial dictatorship	Constant	Consensus	Anti-consensus	Gen. mixed-consensus
Full-range	+	−	+	+	Some
Unanimity	+	−	+	−	Some
Efficiency	+	−	+	−	Only consensus
Anonymity	−	+	+	−	Some
Welfare-dom.	−	+	+	+	+
Pop. mon.	−	+	+	+	By extension
Stp.	+	+	+	−	Some
Group stp.	−	+	+	−	Some
Secure stp.	−	+	−	−	Only constant

To extend a generalized mixed-consensus to a *population monotonic* rule, we require the disagreement default be uniform across populations

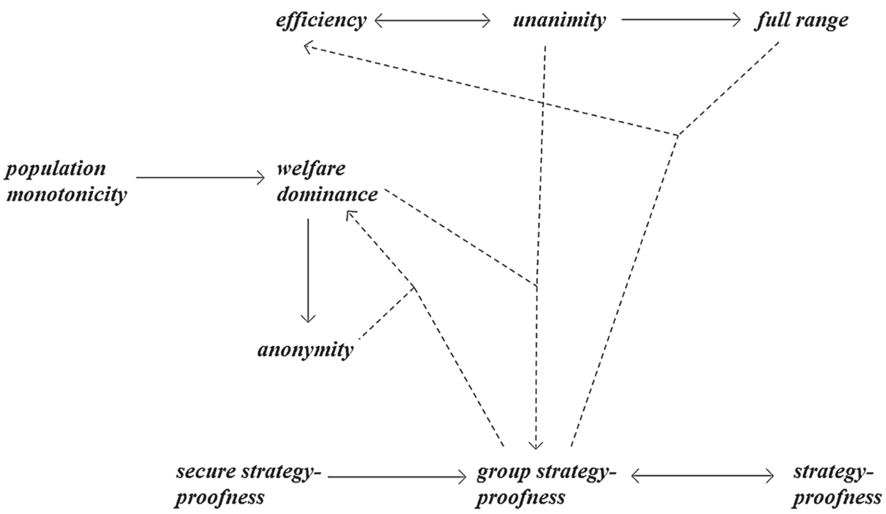

Fig. 1 Relationships among axioms on the strict preference domain

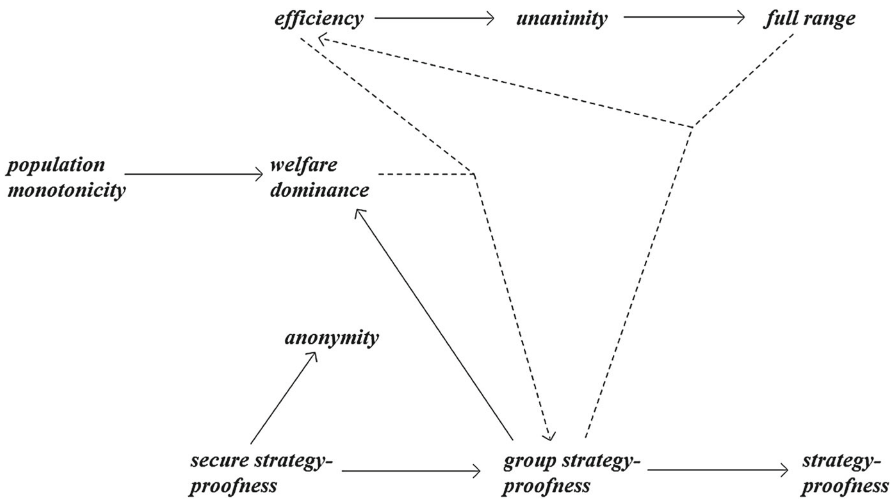

Fig. 2 Relationships among axioms on the full preference domain

$N_a(R)$. Then $C^{D,D^{-1},\emptyset,\emptyset}(R) = D(S)$ and $C^{D,D^{-1},\emptyset,\emptyset}(R^d) = D^{-1}(S)$. Therefore, $C^{D,D^{-1},\emptyset,\emptyset}(R) = (C^{D,D^{-1},\emptyset,\emptyset})^{-1}(R^d)$.

Conversely, let φ be a rule satisfying *welfare dominance* and *self-duality*. For each $S \subseteq N$, define $D(S) \equiv \varphi(R^a_S, R^{ab}_{N\setminus S})$. By *welfare dominance*, there is $d_0 \in \bar{A}$ such that for each disagreement economy $R \in \mathcal{R}^N$, $\varphi(R) = d_0$. Let $R \in \mathcal{R}^N$. If R is a disagreement economy, then $\varphi(R) = d_0$ and $\varphi(R^d) = d_0$ as well. By *self-duality*, $\varphi^{-1}(R^d) = d_0 = \varphi(R^d)$ so $d_0 = \emptyset$. If R is the indifference economy, then $R^d = R$ and *self-duality* implies $\varphi(R) = \varphi(R^d) = \emptyset$. Finally, suppose R is a consensus economy favoring a and let $S \equiv N_a(R)$. Then $\varphi(R) = D(S)$. Now $N_a(R^d) = \emptyset$ and

$N_b(R^d) = S$, so by *self-duality*, $\varphi(R^d) = \varphi^{-1}(R)$. Therefore, $\varphi(R^d) = D^{-1}(S)$. Altogether, $\varphi = C^{D,D^{-1},\emptyset,\emptyset}$.

(b) In the original model, *welfare dominance* and *efficiency* characterize the consensus rules in the original model (Theorem 2(b)). By (a), the default in the extended model must be the emptyset. □

7 Conclusion

We studied the implications of the solidarity and incentive properties in binary social choice. Encouragingly, we uncovered a large class of rules that satisfy our solidarity requirements. Moreover, when combined with *efficiency*, solidarity and strategic considerations lead to the same recommendation: The consensus rules. While these rules are *strategy-proof* and even *group strategy-proof*, they are not *securely strategy-proof*. Unfortunately, only dictatorship and constant rules satisfy this stronger property.

Aside from characterizing rules, comparing preference domains revealed important sensitivities to indifferences. Some relationships that hold on the strict preference domain, such as between *welfare dominance* and *anonymity*, do not carry over to the full preference domain. At the same time, moving to the full preference domain also creates new relationships, as with *group strategy-proofness* and *welfare dominance*, and preserves others, as with *population monotonicity* and *welfare dominance*. Comparing the role of indifference across models remains an open question.

Table 1 summarizes the properties satisfied by the various rules. Figures 1 and 2 summarize the relationships among axioms on the strict and full preference domains.

Acknowledgments I am grateful to William Thomson for invaluable advice and support. I also thank Vikram Manjunath, two anonymous referees, and an associate editor for helpful comments.

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