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# On weighted utilitarianism and an application

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**Abstract** This paper discusses and evaluates some axiomatic approaches to weighted utilitarianism. It then offers a new and direct proof for weighted utilitarianism under full comparability which yields, as a corollary, U. Ebert's result on rank-weighted utilitarianism.

## 1 Introduction

As in most axiomatic studies of social choice, we are interested in the problem of a social planner aggregating individual preferences over a set $X$ into a social ordering. We shall take cardinal data—utilities—to be available to the planner, furthermore requiring fairly stringent interpersonal comparability of data. We refrain from imposing further specifications or structure on $X$.

Utilitarianism is the focus of our attention. Several reasons underlie its appeal as a theory of distributive justice. Given cardinal information satisfying some sort of compatibility criteria, it is a very intuitive functional form to posit; its tractability and normative transparency are also encouraging. Moreover, the structural parallels between utilitarianism and utility maximization of the individual decision maker invite examination. Of course, utilitarianism has a host of well-documented issues. It has, for example, no regard for distributional concerns unless everyone has the same utility function—but this is then "the marginal tail wagging the total dog" (Sen 1979). Utility functions are also limited in their ability to capture notions like altruism. Other critiques malign the inadequacy of marginal utility as an index of moral importance (Sen 1979) or follow the Rawlsian argument that an individual in an original position might pay close(r) attention to the minimum utility that a good renders over a population

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(Rawls 2005). Such thought loosely underlies the leximin rule, which of course has its own pitfalls.

Weighted utilitarianism—which we specifically concern ourselves with here—shares many of the same benefits and concerns. It provides for an introduction of exogenous data by the planner, a parameterizing vector $\lambda \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$ of "welfare weights" attached to particular individuals. These parameters measure, in social comparisons between goods, a global sensitivity to any particular individual's utility differential across the goods. Modulo appropriate assumptions on comparability of interpersonal utility, the welfare weights reflect (social) decision-making power of individuals in society. A social planner's exogenously formed views on individuals' importances could be introduced and manifested this way. This extra parameter of richness is a key vindication of weighted utilitarianism. For example, the encoded view of the planner could feature pivotally in a model that dynamically connects voting (or social choice rules) to output in a discrete temporal setting. One might ask, according to various metrics, whether allocating produce to a society's most productive individuals leads (in some discrete temporal setting) to better outcomes than other assignments of importance. Path dependencies could be investigated, and weighted utilitarianism (or some counterpart in voting) would be the most tractable formulation with which to introduce such assumptions and views. Less speculatively, weighted utilitarianism stars prominently in Harsanyi's celebrated aggregation theorem (Harsanyi 1955). Its parameter of richness emerges as mathematically organic and normatively interesting in the growing literature on Bayesian aggregation, where a social planner aggregates individuals' orderings on Savagean acts or lotteries thereof; here weighting— which arises naturally from Pareto type conditions—allows exogenous and simultaneous assessments of individuals' tastes and beliefs (c.f. Mongin 1995; Gajdos et al. 2007).

Functionally speaking, the still-versatile linearity restriction that weighted utilitarianism brings to generalized utilitarianism is quite powerful: it is the only one consistent with preservation of units. Furthermore, normalizing the (usually strictly positive) weights gives an extremely clear analogy with expected utility theory, which confers axiomatic insights.

The objective of this paper, therefore, is to glean further insight into the axiomatic properties of weighted utilitarianism. Many approaches have been pioneered to characterize weighted utilitarianism under the powerful and interpretationally sparse informational requirement of transition-scale measurability, which rather strangely prohibits interpersonal comparisons of utility; this also suggests a puzzling importance of informational requirements in the difference between leximin (and leximax) rules and utilitarianism. Other approaches warrant excessively strong non-ordinal axioms. We make the case for the stringent informational requirement of full comparability,[1] which allows for interpersonal comparisons of gains and losses as well as levels, despite the more restrictively small equivalence classes it imposes upon our cardinal data, and the corresponding difficulty it creates in proofs. Maskin (1978) characterizes utilitarianism under full comparability; his characterization, coupled with a paper of Hammond

[1] This seems fair because generalized utilitarianism can be rather naturally—and ordinally—obtained without recourse to informational requirements at all. This is essentially the force of Debreu's theorem on additive separability (c.f. Fishburn 1970).

(1976), suggests continuity as underlying differences between utilitarianism and the leximin rules. It is unclear a priori, what weakening of Maskin's hypotheses (which, rather palatably, are entirely ordinal save for the informational requirement) produces weighted utilitarianism and how it does so. Deschamps and Gevers wrote a set of two papers (1977, 1978) culminating in a joint characterization of leximin, leximax, and weighted utilitarian rules. A corollary of their result—as suggested by d'Aspremont and Gevers (2002)—*is* a characterization of weighted utilitarianism, but the proof of their stronger result is incredibly indirect and non-reliant on continuity. We thus proceed from Maskin's derivation of generalized utilitarianism and offer a new and direct proof of weighted utilitarianism, revealing more strongly the centrality of continuity and total comparability in the process.

For those unconvinced or uninterested by this axiomatization exercise in its own right, our proof trivializes a natural recasting, in the social welfare functional (SWFL) setting, of a result of Ebert (1988) characterizing rank-weighted utilitarianism. A quick axiomatic restriction of this class of SWFLs to weight lower-ranked utilities more intensely produces the generalized Gini family of SWFLs, which, at first blush, possess strongly different moral content; indeed, a limit case of this family is the maximin rule. As such, our result has an application in a (perhaps) more morally palatable setting.

The paper proceeds as follows: Sect. 2 defines preliminaries, discusses some axioms and informational requirements, and presents relevant old results; Sect. 3 presents our direct proof of weighted utilitarianism, while Sect. 4 concludes with our application to Ebert's theorem and the generalized Gini family.

## 2 Preliminaries

### 2.1 A setting for social choice and utilitarianism

Let $X$ be the set of social alternatives, where $|X| \geq 3$. Let $N = \{1, 2, 3, \ldots, n\}$ for $n \geq 3$ be the set of individuals in the society, and $\mathcal{O}(X)$ be the set of all orderings of $X$. Consider the set

$$\mathcal{U} = \{u : X \times N \to \mathbb{R}, \text{ with } u \text{ bounded}\}.$$

Denote a generic element of the set $\mathcal{U}$ as $u = u(\cdot, \cdot)$; we will write $u(x, i) = u_i(x)$ to be the utility the $i$-th person gets from choice $x$. We will assume that each person has an ordering $\succeq_i \in \mathcal{O}(X)$ representable by such a function $u_i : X \to \mathbb{R}$.[2] Now, we will define a social welfare functional (SWFL):

**Definition 1** A social welfare functional $f$ is a map $f : \mathcal{U} \to \mathcal{O}(X)$.

We denote by $f(u) = R$ the image of $u(\cdot, \cdot)$ in $\mathcal{O}(X)$;[3] the corresponding strict part of $f(u)$ will be denoted by $P$, and the purely symmetric part by $I$.

[2] Here we are of course discarding such pathological examples as lexicographic preferences, but this is standard.

[3] As per the language of d'Aspremont and Gevers (2002), we are implicitly making the assumption of *domain universality* on $f$ here in our definition of a SWFL. A weaker condition to impose on the domain

As per d'Aspremont and Gevers (1977), we will always require two foundational properties of the SWFLs we consider:

**Axiom 1** (*Independence* (*I*)) For all $u, u' \in \mathcal{U}$ and for all $B \subseteq X$, we have

$$u(x,\cdot) = u'(x,\cdot)\text{for all } x \in B \Rightarrow f(u)|_B = f(u')|_B.$$

**Axiom 2** (*Strong Pareto Property* (*SP*)) (1) For all $x, y \in X$, for all $u \in \mathcal{U}$, if for all $i \in N$, we have $u_i(x) = u_i(y)$ then $xIy$. (2) Moreover, if for all $i \in N$ we have $u_i(x) \geq u_i(y)$ and there exists even a single $j$ such that $u_j(x) > u_j(y)$ then $xPy$.

d'Aspremont and Gevers further demonstrate that these two properties, taken together, imply a third one called extended neutrality (XNE), according to which all that matters to $f$'s collective choice are the *utility levels* of individuals. In other words, we have:

**Axiom 3** (*Extended Neutrality* (*XNE*)) For all $x, y \in X$, for all $u \in \mathcal{U}$: If $w, z \in X$ and $\tilde{u} \in \mathcal{U}$ such that for all $i \in N$,

$$u_i(x) = \tilde{u}_i(w)$$
$$u_i(y) = \tilde{u}_i(z)$$

then, we have that $xRy \Leftrightarrow w\tilde{R}z$.

It is natural to think of a binary relation $R^*$ on $\mathbb{R}^n$ where for all $v, w \in \mathbb{R}^n$ we have[4]

$$vR^*w \Leftrightarrow \text{ there exist } x, y \in X, u \in U \text{ such that for all}$$
$$i \in N, u_i(x) = v_i, u_i(y) = w_i, \text{ and } xRy.$$

A lemma of d'Aspremont and Gevers tells us that this induced relation on $\mathbb{R}^n$ is thankfully an order in the presence of XNE (and hence in the presence of I and SP):

**Lemma 1** (d'Aspremont and Gevers 1977) *If $f$ satisfies XNE, then $R^*$ is an ordering on $\mathbb{R}^n$.*

So in the presence of axioms I and SP (as well as domain universality), we have reduced ourselves to thinking about functions $f : \mathbb{R}^n \to \mathbb{R}$ (again, recall $|N| = n$ so we could rewrite statements above with $\mathbb{R}^N$ to explicitly include data of individuals' names), which is evocative of a social welfare function. In these settings,[5] d'Aspremont

Footnote continued 3
$\mathcal{D} \subset \mathcal{U}$ of a SWFL is that of *domain attainability*: For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ there exist $x, y, z \in X$ and there exist $u \in \mathcal{D}$ such that $u(x,\cdot) = \vec{u}$, $u(y,\cdot) = \vec{v}$ and $u(z,\cdot) = \vec{w}$.

[4] In the presence of domain universality, XNE is equivalent to the more satisfying condition of *formal welfarism* that we may take to define $R^*$ in our setting: For all $v, w \in \mathbb{R}^n$, for all $x, y \in X$ for all $u \in U$: If $u_i(x) = v_i$ and $u_i(y) = w_i$ for all coordinates $i$, then we have $vR^*w \Leftrightarrow xRy$.

[5] The specified settings are in fact unnecessarily strong. A theorem of Sen (1977) shows, for example, that in light of domain universality, only Pareto indifference is required to obtain formal welfarism. This will be silently used in our survey of old results but is a relatively unimportant point.

and Gevers (2002) would call $R^*$ a social welfare ordering (SWO) and say that it is the *formally welfare associate* of $f$; the purpose of this clarification is firstly to dispel confusion about the direct presentation of results from d'Aspremont and Gevers (notably that they often eschew axioms present in the SWFL case), and secondly to presage some comments in a later part of the paper.

Having established some "baseline" axioms, let us present several others, with a few interpretational remarks:

**Axiom 4** (*Separability* (*S*)) For all $u, \tilde{u} \in \mathcal{U}$, if there exists $M \subseteq N$ such that for all $i \in M, u_i(\cdot) = \tilde{u}_i(\cdot)$ while for all $j \in N \setminus M$, for all $x, y \in X$, we have $u_j(x) = u_j(y)$ and $\tilde{u}_j(x) = \tilde{u}_j(y)$, then $f(u) = f(\tilde{u})$.

This axiom rather reasonably states that those individuals indifferent between $x, y \in X$ will not influence the social choice between those two items; the level of the utility in the indifference does not matter either. This property is also known as (total) separability. It is key to the additivity of the representation across individuals. The following axiom is self-explanatory:

**Axiom 5** (*Anonymity* (*A*)) Let $\sigma$ be any permutation of $N$. If, for some $u, u' \in \mathcal{U}$ we have $u_i(x) = u'_{\sigma(i)}(x)$ for all $1 \leq i \leq n$ and all $x \in X$, then it must be the case that $f(u) = f(u')$.

This axiom is also rather straightforward:

**Axiom 6** (*Minimal Individual Symmetry* (*MIS*)) For any two $i, j \in N$, there exists $u \in \mathcal{U}$ and $x, y \in X$ such that

$$\text{for all } l \notin \{i, j\}, u_l(x) = u_l(y)$$

and

$$u_i(x) > u_i(y), \quad u_j(x) < u_j(y)$$

with $xIy$.

Next, we can present a more technical requirement tremendously important to our methods in this paper. First, a definition:

**Definition 2** Say an ordering $\succeq$ on a topological space $X$ is continuous if for all $y \in X$, the sets $\{x \in X : x \succ y\}$ and $\{x \in X : y \succ x\}$ are open.

Recalling Lemma 1, we define:

**Axiom 7** (*Continuity* (*C*)) A SWFL $f$ satisfying I and SP is continuous if the induced ordering $R^*$ on $\mathbb{R}^n$ is continuous with respect to the usual metric topology.

This axiom has a few interpretations. Maskin (1978) explains, for example, that if everyone thinks two alternatives are rather similar to each-other, then their relationships to other alternatives in the social ordering will be similar. d'Aspremont and

Gevers (2002) explain it as an axiom "relieving potential anxiety about slight measurement errors in evaluation profiles."

Finally, we introduce a few axioms regarding the comparability of interpersonal data;[6] precisely because we have cardinal data and not just ordinal data, we want to accord it meaning (beyond, e.g. unrestricted monotone transformations in each coordinate). Essentially, we are introducing $f$-equivalence classes on $\mathcal{U}$ of varying stringencies to "quotient out" irrelevant information. For example, we might have:

**Axiom 8** (*Invariance with respect to individual increasing transformations*) For all $u, u' \in \mathcal{U}$, for all $n$-tuples of increasing functions $\phi_i : \mathbb{R} \to \mathbb{R}$, if for all $i \in N$ and $x \in X$ it is the case that

$$u_i(x) = \phi_i(u_i'(x))$$

then $f(u) = f(u')$.

This axiom clearly forbids interpersonal comparisons of utility and gives us very undiscriminating equivalence classes to work with.[7] Interpersonal comparisons make sense, however, if you require all $\phi_i := \psi$; this condition is known as co-ordinality. The utilitarian social planner needs some way of comparing gains and losses across individuals, and hence the following axiom is often used:

**Axiom 9** (*Transition-Scale Measurability* (*TSM*)) For all $u, u' \in \mathcal{U}$, for all $a \in \mathbb{R}^n$, if for all $i \in N$, and $x \in X$, we have

$$u_i(x) = a_i + u_i'(x)$$

then $f(u) = f(u')$.

While the unit of measurement cannot be changed, interpersonal comparability is destroyed. Although a very bizarre requirement from the interpretational standpoint, TSM does *greatly* improve the ease of many proofs. A much more natural and discerning informational requirement is that of full comparability:

**Axiom 10** (Full Comparability (FC)) For all $u, u' \in \mathcal{U}$, for all $b > 0$ and $a \in \mathbb{R}$, if for all $i \in N$, $x \in X$, we have

$$u_i(x) = bu_i'(x) + a$$

then $f(u) = f(u')$.

Interpersonal comparisons of both levels and gains and losses is possible in this framework. While arguably imposing *too fine* of an informational requirement it seems vastly more reasonable than TSM.

[6] To avoid clunkiness, we could have stated the properties far more intelligibly in terms of the associate SWO on $\mathbb{R}^n$. But given our operation in the SWFL framework, we opt for consistency.

[7] On the other hand, such statements give much more mileage as hypotheses in theorems than their more refined counterparts.

We should stress that while these “informational requirement” properties are intended as stipulations on comparisons the social planner can make, they nonetheless possess tremendous power in facilitating proofs and distinguishing between various SWFLs. As Gaertner (Gaertner 2009) claims, for example, “ the real difference between a utilitarian and a Rawlsian rule stems from the different informational requirements.”

Now we will, without much commentary, define the families of SWFLs relevant to this paper.

**Definition 3** A SWFL is utilitarian if and only if for all $x, y \in X$, and $u \in \mathcal{U}$, we have

$$xRy \Leftrightarrow \sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y).$$

**Definition 4** A SWFL is said to be of the generalized utilitarian (c.f. Blackorby et al.) family if there exist increasing functions $g_i : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in X$, and $u \in \mathcal{U}$, we have

$$xRy \Leftrightarrow \sum_{i=1}^{n} g_i(u_i(x)) \geq \sum_{i=1}^{n} g_i(u_i(y)).$$

We are specially interested, in this paper, with the following families of SWFLS:

**Definition 5** A SWFL is said to be of the weighted utilitarian family if there exist $\lambda \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$ whereby for all $x, y \in X$, for all $u \in \mathcal{U}$ we have

$$xRy \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_i(x) \geq \sum_{i=1}^{n} \lambda_i u_i(y).$$

Perhaps more agreeable than the above is:

**Definition 6** A SWFL is said to be of the rank-weighted utilitarian family if there exist $\lambda \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$ whereby for all $x, y \in X$, for all $u \in \mathcal{U}$ we have

$$xRy \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_{(i)}(x) \geq \sum_{i=1}^{n} \lambda_i u_{(i)}(y),$$

where $u_{(i)}(x)$ denotes the $i$-th order statistic of $(u_1(x), u_2(x), \ldots, u_n(x))$. [8]

Finally, we may impose a restriction upon the coefficients in the rank-weighted family to obtain:

**Definition 7** A SWFL is said to be of the generalized Gini family if it is of the rank-weighted family and satisfies $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$.

[8] For example, $u_{(1)}(x) = \min_i u_i(x)$ and $u_{(n)}(x) = \max_i u_i(x)$.

Notice that Rawlsian maximin, for example, is a special (limiting) case of the rank-weighted utilitarian and Gini families.

## 2.2 Survey of results

We will now take stock of various axiomatizations of weighted utilitarianism in the literature, largely following the survey presented in d'Aspremont and Gevers (2002).

Several results exist for the rather arbitrary informational requirement of TSM. For example, take the following theorem of Blackwell and Girshick, slightly adapted by d'Aspremont and Gevers:

**Theorem 1** (Blackwell and Girshick 1954 and/or d'Aspremont and Gevers 2002) *If a SWFL satisfies I, SP, TSM, and C then it is a member of the weighted utilitarian family with $\lambda_i > 0$ for all $i$. The result holds also if SP is switched with WP,*[9] *TSM is switched to Inv$(a_i + bu_i)$*[10] *and MIS is added.*

*Remark 1* The role of continuity in this result is completely secondary. It is used only to generate indifference hyperplanes; for all $u$, the asymmetric part of $R = f(u)$ is characterized entirely by WP and TSM, and strict positivity of $\lambda$ is guaranteed by MIS.

The theorem's proof has at its heart the supporting hyperplane theorem: $\{x \in \mathbb{R}^n : xR^*0\}$ is shown to be concave (using TSM and WP) and closed, whence TSM essentially gives the result.[11] A continuity argument dispenses of the symmetric concerns, and MIS (or SP) guarantees positivity of all $\lambda_i$.

Introducing anonymity into the mix eliminates the need for MIS (which is a strictly weaker condition), continuity, and strong Pareto (if one $\lambda_i = 0$ then all would be):

**Theorem 2** (d'Aspremont and Gevers 1977) *If a SWFL satisfies I, WP, TSM, and A, then it is utilitarianism.*

The proof of this result is extremely simple, but it again relies heavily on the very arbitrary assumption of TSM. Let us turn now to alternative informational assumptions.

The first (pseudo)-candidate is an approach of Blackorby et al. (2002), which is deeply unsatisfying, especially given our penchant for more ordinally phrased axioms. In the SWO framework, his axiom says the following:

**Axiom 11** (*Incremental Equity* (*IE*)) Let $e_j$ be the $j$-th standard basis vector in $\mathbb{R}^n$. Then, for all $u \in \mathbb{R}^n$, $\delta \in \mathbb{R}$, and for all pairs $i, j \in N$, we have $(u + \delta e_j)I^*(u + \delta e_k)$.

In other words, the welfare function doesn't care about the recipient of a utility increase and clearly cares only about the sum. That WP and IE together characterize a

[9] The Weak Pareto axiom, WP, states that for all $x, y \in X$, for all $u \in \mathcal{U}$, if $u_i(x) > u_i(y)$ for all $i \in N$, then $xPy$. It manifestly weakens SP.

[10] The notation is very suggestive; $f$ is invariant under such transformations of $u \in \mathcal{U}$—i.e. affine transformations with the same scale factor and varying additive ones. We will employ this useful shorthand henceforth.

[11] The latter part of the proof can be carried out without TSM so long as $f$ is continuous on diag$(\mathbb{R}^n)$.

utilitarian SWO is totally unsurprising, as IE essentially contains the whole content of utilitarianism in the first place! Furthermore, it is even directly clear that IE and WP imply TSM and that A and TSM together imply IE. Finally, in wrapping together IE, A, and TSM so tightly together, this approach leaves little room for innovation towards weighted utilitarianism without some extremely contrived axiom. We are specifically interested in weighted utilitarianism [12] and its offshoots, so this approach will not do.

Mixing invariance requirements provides alternative approaches, c.f. Gevers (1979) and Ebert (1988), of which the following is an example:

**Theorem 3** (d'Aspremont and Gevers 2002) *A SWFL $f$ satisfying I, SP, Inv($\{a_i + bu_i\}$ and $\phi(u_i)$), A, and C is utilitarian.*

Neither is it clear a priori whether dropping anonymity produces weighted utilitarianism, nor is it particularly interesting, in light of the peculiar informational requirement. Let us turn our attention to approaches that actually operate under full comparability; two, specifically, are of note.

The first is one of Deschamps and Gevers (1977, 1978). The pair of papers provide a joint characterization of (weighted) utilitarian, leximin, and leximax rules and achieve a deep understanding of the role of separability for SWOs in the process—all under the assumption of full compatibility.[13] A slight adaptation of the main result is:

**Theorem 4** (Deschamps and Gevers 1977, 1978) *For $n > 2$, a SWO $R^*$ that satisfies SP, MIS, FC, and S is either leximin, leximax, or weak weighted utilitarianism (with all positive weights).*

Crucially, their proofs—which jointly tally some twenty pages in length—do *not* employ continuity—which easily disposes of the leximin and leximax cases; the trade-off is a lot of "preparatory spadework" and a very lengthy proof. A corollary of their strong result is one (essentially) we prove in this paper.

**Corollary 1** (d'Aspremont and Gevers 2002) *For $n > 2$ the SWO $R^*$ satisfies SP, MIS, FC, S and C if and only if it is weighted utilitarianism (with all weights positive).*

Of course, this affords little direct insight, makes minimal use of C, and uses MIS heavily—which for us is unnecessary.

The approach of Maskin (1978) in deriving utilitarianism (although his result is subsumed as a corollary by that of Deschamps and Gevers) is more direct and pivotally involves continuity. Furthermore, coupled with Hammond's result, it "gives a new slant to the contrast between utilitarianism and the lexicographic maximin principle"—a much more satisfying one hinging on continuity rather than informational requirements (recall the remark of Gaertner quoted in Sect. 2.1). Maskin's "trick" is to employ

[12] A goal specifically motivated by our justifications in the introduction.

[13] The papers of Hammond (1976) and Maskin (1978) suggest that at one might have guessed at least SP, MIS, FC, A a priori. The fact that Deschamps and Gevers were able to dilute A into MIS and require *no further conditions* is rather remarkable and a testament to the strength of their result. In fact, MIS is even stronger than what they require in the paper, which is simply an undominated set of individuals of size at least 3.

the decision-theoretic machinery of Debreu's theorem on additive separability (c.f. Fishburn 1970). His proof is enlightening because each step's employment of axioms is transparent—and various steps require few axioms individually. For completeness, we state his result:

**Theorem 5** (Maskin 1978) *A SWFL $f$ satisfies I, SP, FC, S, A, and C if and only if it is utilitarianism.*

Corollary 1 clearly implies that weakening A into MIS is sufficient to deduce weighted utilitarian rule. But this is not obvious from Maskin's proof (after his derivation of generalized utilitarianism), and how to proceed is also unclear. In the following section, we offer a *direct* proof of Corollary 1 which affords more insight into the power of FC and continuity. Our line of argumentation—*unlike those of Maskin or Deschamps and Gevers*—makes trivial an argument for the case of rank-weighted utilitarianism as well.[14]

We close this section with an axiomatic remark: part of the appeal of the approaches of Maskin and Deschamps-Gevers is that their axiomatizations—with the exceptions of full comparability and continuity[15]—can be provided in purely *ordinal* terms. In other words, almost all of these axioms for *utilitarianism* can be conceived of in an Arrovian framework with $f : \mathcal{O}(X)^n \to \mathcal{O}(X)$[16] While nifty, it also suggests something unnatural about axioms like IE which explicitly invoke the vector space structure of $\mathbb{R}^n$ under $R^*$, and as such are less primitive; in contrast, the axiomatizations of Maskin and Deschamps-Gevers squeeze every bit of juice out of an arid informational setting before imposing a fine—but normatively reasonable—set of informational requirements. It also raises the interesting questions of whether given SWFLs can be axiomatized in a strictly ordinal fashion—perhaps handy in situations with no close Euclidean analogues or to appease those who dislike utility functions. In some sense, this brings us full circle to the much-loathed Axiom 8 and fairly insensitive equivalence classes of $\mathcal{U}$ that it creates. Of course, the weary reader might at this point throw his hands up and tailor Leo X's famous proclamation: "Since God has given us" cardinal information, "let us enjoy it."

## 3 The proof

The first step of our proof—and of Maskin's—is the application of Debreu's theorem to the order $R^*$ on $\mathbb{R}^n$ to deduce a generalized utilitarian rule. To the best of our knowledge, Maskin was the first to recognize the use of Debreu's theorem—a decision theoretic tool—in this context. We record the proof because we will return to it later. Note that as generalized utilitarianism is merely the sum of monotone increasing transformations of coordinate functions on utility space, it requires no separate informational assumptions and has an ordinal axiomatization.

14 The theorem itself was already known to U. Ebert, who gave a different, less transparent proof.

15 The latter problem can be avoided if we endow $X$—not terribly unreasonably—with a topology.

16 As we shall see, generalized utilitarianism "fits the bill" of an ordinal characterization if $X$ is a topological space; this is a consequence of Debreu's theorem.

**Theorem 6** (Generalized Utilitarianism (Maskin 1978)) *A SWFL $f$ satisfying I, SP, C, and S is of the generalized utilitarian family.*

*Proof* By formal welfarism, we need only to construct a representation of $R^*$ on $\mathbb{R}^n$. Write $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$; observe that $\mathbb{R}$ is connected and separable. Moreover, SP implies that no component $i \in N$ is null, whence $n \geq 3$ tells us that there are greater than or equal to three non null components. Now, $R^*$ is by supposition a continuous weak order satisfying separability (S); therefore, there exist functions $g_i : \mathbb{R} \to \mathbb{R}$ unique up to common positive affine transformation (where multiplicative constant is the same, but additive constants may be different) such that for all $x, y \in X$,

$$xR^*y \Leftrightarrow \sum_{i=1}^n g_i(u_i(x)) \geq \sum_{i=1}^n g_i(u_i(y)).$$

Condition SP implies that the $g_i$ are strictly monotone increasing. Thus $f$ is generalized utilitarian. □

At this step, Maskin applies A to equate all of the $g_i$ and leans heavily on SP, C, and FC to deduce utilitarianism. As mentioned earlier, Corollary 1 implies that swapping A (used very essentially in Maskin's proof) with MIS is enough to show that the $g_i$ act by scaling as welfare weights $\alpha_i \geq 0$; *why* and *how* this works is hardly clear.

In fact, this dilution of A into MIS is altogether distracting. We dispense with the condition entirely, showing that it is redundant in the presence of I, SP, C, and S:

**Proposition 1** *If a SWFL $f$ satisfies I, SP, S, and C, it must satisfy MIS.*

*Proof* By Theorem 6, $f$ is a generalized utilitarian rule, i.e. $xRy \Leftrightarrow \sum_{i=1}^n g_i(u_i(x)) \geq \sum_{i=1}^n g_i(u_i(y))$ for monotone increasing continuous functions $g_i : \mathbb{R} \to \mathbb{R}$. Write $g : \mathbb{R}^n \to \mathbb{R}$ to be the map $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n g_i(x_i)$ for convenience. We want to show that for all $i, j$ there exist vectors $u, v \in \mathbb{R}^n$ (of utilities) such that $u_i > v_i, u_j < v_j$ and $u_l = v_l$ for all $l \notin \{i, j\}$ whereby $g(u) = g(v)$. Observe that if $u_l = v_l$ for all $l \neq i, j$ then those terms vanish in our setting anyway, so our task is reduced to finding $u, v$ for $i, j$, such that $u_i > v_i, u_j < v_j$ with

$$g_i(u_i) + g_j(u_j) = g_i(v_i) + g_j(v_j).$$

We have thus restricted to a two-dimensional problem. By the uniqueness conditions of Debreu's theorem, we may without loss of generality take $g_i(0) = 0$ for all $1 \leq i \leq n$. Now, each $g_k$, on account of monotonicity and continuity, must surject onto some interval $(-\epsilon_k, \epsilon_k)$; since we are only dealing with $g_i$ and $g_j$, we may take the symmetric interval about 0 to be the same in both cases. To see this, we have $-1 < 0 < 1$ and so $g_i(-1) < 0 < g_i(1)$; continuity ensures by the intermediate value theorem that all values in between $g_i(-1)$ and $g_i(1)$ are attained uniquely by points in $(-1, 1)$ and we can take $\min(|g_i(-1)|, |g_i(1)|)$ to (pedantically) symmetrize the interval upon which $g_i$ surjects (and then we do the same with $g_j$, and intersect to find the interval onto which both functions surject). Hence, using the intermediate value theorem again, we may pick some

$$u_i \text{ such that } g_i(u_i) = \frac{\epsilon}{2}, \qquad u_j \text{ such that } g_j(u_j) = \frac{-\epsilon}{2}$$

$$v_i \text{ such that } g_i(v_i) = \frac{-\epsilon}{2}, \qquad v_j \text{ such that } g_j(v_j) = \frac{\epsilon}{2}$$

whence it is clear that $g_i(u_i) + g_j(u_j) = g_i(v_i) + g_j(v_j) = 0$ with the desired inequalities on the coordinates. Repeating this argument for all pairs $i, j$ yields the result. We thus have minimal individual symmetry. □

In other words, anonymity needn't be weakened; it can be chucked out entirely. Given this refocusing of attentions, we now state and prove our main result:

**Theorem 7** *If a SWFL $f$ satisfies I, SP, S, C, and FC, then it is the principle of weighted utilitarianism.*

*Proof* Without FC, Theorem 6 gives us generalized utilitarianism, i.e a function $g : \mathbb{R}^n \to \mathbb{R}$ where $v \mapsto \sum_{i=1}^{n} g_i(v_i)$ which represents the induced ordering $R^*$ on $\mathbb{R}^n$. Our proof decomposes into a few main steps: (i) apply Debreu's theorem (done), (ii) prove a linear reflection property of the $g_i$, (iii) prove a statement about the local behavior of the $g_i$ and (iv) deduce the linearity of the $g_i$.

Denote by $T_{c,\gamma}$ the positive affine transformation $T_{c,\gamma} : \mathbb{R} \to \mathbb{R}$ which sends $x \mapsto cx + \gamma$ for $c > 0$ and any $\gamma \in \mathbb{R}$. We have the following result: □

**Lemma 2** (Linear Reflection) *For all $i$, there exists some $u_i \in (0, 1)$ such that for all positive affine transformations $T_{c,\gamma}$, we have*

$$g_i(T_{c,\gamma}(1)) = \frac{1}{2}\left(g_i(T_{c,\gamma}(u_i)) + g_i(T_{c,\gamma}(2 - u_i))\right).$$

Choosing $T_{c,\gamma}$ to be the identity transformation, this theorem tells us that the value of $g_i$ at 1 is the average of its values at $2 - u_i$ and that at $u$; this is some sort of restriction on growth. Full comparability is the hero here; it allows us to "smear" this result all over the real line and to appropriately shrink or dilate the size of the interval (symmetric about $g_i$) over which $g_i$ can be seen as an average of the endpoints of. Let us prove the result:

*Proof* We will show the result for $i = 1$; obviously the same proof works for all choice of $i$. Our approach is similar to a part of Maskin's proof. We know that $g_1 + g_2 : \mathbb{R} \to \mathbb{R}$ is a continuous, monotone increasing function with $g_1(0) + g_2(0) = 0$ (by a normalization above). Moreover, we have the inequalities $0 < g_1(1) < g_1(1) + g_2(1)$. Thus, the intermediate value theorem tells us that there exists a unique $u \in (0, 1)$ such that

$$g_1(u) + g_2(u) = g_1(1).$$

Written otherwise, we have that

$$(u, u, 0, \ldots, 0) I^* (1, 0, \ldots, 0).$$

Thus, for any $a > b$ and $c > 0$ we have that

$$((a-b)u+b, (a-b)u+b, b, \ldots, b) I^*(a, b, b, \ldots, b)$$

by full comparability. Picking $a = c$, $b = 0$ we see that

$$g_1(cu) + g_2(cu) = g_1(c).$$

Picking $a = c$ and $b = cu$ we have that

$$g_1(c(1-u)u + cu) + g_2(c(1-u)u + cu) = g_1(c) + g_2(cu) \Rightarrow g_1(c(2-u)u) \\ + g_2(c(2-u)u) = g_1(c) + g_2(cu).$$

Picking $a = (2-u)c$ and $b = 0$ we have that

$$g_1((2-u)cu) + g_2((2-u)cu) = g_1((2-u)c).$$

Now, for an arbitrary $\gamma \in \mathbb{R}$, full comparability allows us to translate the offset equations, telling us that

$$g_1(cu+\gamma) + g_2(cu+\gamma) = g_1(c+\gamma \tag{1}$$

$$g_1(c(2-u)u+\gamma) + g_2(c(2-u)u+\gamma) = g_1(c+\gamma) + g_2(cu+\gamma) \tag{2}$$

$$g_1((2-u)cu+\gamma) + g_2((2-u)cu+\gamma) = g_1((2-u)c+\gamma). \tag{3}$$

The left hand sides of Eqs. (2) and (3) are equal, so we see that

$$g_1(c+\gamma) + g_2(cu+\gamma) = g_1((2-u)c+\gamma),$$

and hence substituting from Eq. (1) tells us that

$$g_1(c+\gamma) + g_1(c+\gamma) - g_1(cu+\gamma) = g_1((2-u)c+\gamma),$$

which simplifies to tell us that

$$g_1((2-u)c+\gamma) + g_1(cu+\gamma) = 2g_1(c+\gamma).$$

This finishes the claim. □

We now have an immediately useful corollary qualitatively alluded to earlier:

**Corollary 2** *For all* $1 \le i \le n$*, the following is true: For all* $\epsilon > 0$*, for all* $x \in \mathbb{R}$*, there exists* $t \in (0, \epsilon)$ *such that*

$$g_i(x) = \frac{g_i(x-t) + g_i(x+t)}{2}. \tag{4}$$

*Proof* Recall that for all $i$ we have some $u_i$ such that

$$g_i(T_{c,\gamma}(1)) = \frac{1}{2}\left(g_i(T_{c,\gamma}(u_i)) + g_i(T_{c,\gamma}(2-u_i))\right).$$

Our problem is simply that of choosing the correct positive affine transformation. Picking

$$c < \frac{\epsilon}{1-u_i}, \quad \gamma = x - c$$

we immediately have

$$\frac{1}{2}\left(g_i(x + c(u_i - 1)) + g_i(x + (1-u_i)c)\right) = g_i(x),$$

and we can observe that by design that the symmetric interval $(x + c(u_i - 1), x + (1-u_i)c)$ about $x$ sits within $(x-\epsilon, x+\epsilon)$. □

Armed with such a characterization as Eq. (4), we strongly suspect that any such $g_i$ must be linear; indeed, it is evocative of the mean value property of harmonic functions. We proceed by an argument reminiscent the proof of the maximum (or minimum) principle of harmonic analysis.

**Lemma 3** *Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function. If, for all $\epsilon > 0$, for all $x \in \mathbb{R}$, there exists $t \in (0, \epsilon)$ such that*

$$f(x) = \frac{f(x-t) + f(x+t)}{2},$$

*then $f$ is affine.*

*Proof* We shall prove that $f$ is affine on any closed interval; this implies the result. Without loss of generality (the exact same argument holds) we may take the interval to have one endpoint at 0, and to contain nonnegative numbers (the argument can easily be flipped); take the interval $[0, H]$ for some $H \in \mathbb{R}^+$. Consider the function

$$\tau(x) = f(x) - \frac{f(H)}{H}x;$$

we shall show that $\tau(x) \equiv 0$ on the interval, which is manifestly equivalent to the claim. First observe that $\tau(0) = \tau(H) = 0$, as we may take $f(0) = 0$ (recall our use of Debreu's theorem) and thus equate our claim to the linearity of $f$. We claim that the maximum of $\tau$ must occur at either 0 or $H$, i.e. that $\tau \leq 0$ on $[0, H]$. To see this, let $M = \max_{x\in[0,H]} \tau(x)$ where $M > 0$. We know that $\tau$ is continuous and hence $M$ cannot occur in some small neighborhoods of zero or $H$, so we could, e.g. write

$M = \max_{x \in (\delta, H-\delta)} \tau(x)$ for some $\delta > 0$. Define $s = \sup_{0<t<H}\{t : \tau(t) = M\}$. We know that for all $\epsilon > 0$ there exists $t \in (0, \epsilon)$ such that

$$f(s) = \frac{f(s-t) + f(s+t)}{2}.$$

Hence, we see that

$$\frac{f(s+t) - \frac{(s+t)f(H)}{H} + f(s-t) - \frac{(s-t)f(H)}{H}}{2} = f(s) - \frac{sf(H)}{H},$$

or in other words, that we also have

$$\tau(s) = \frac{\tau(s-t) + \tau(s+t)}{2};$$

here we may pick $\epsilon$ as small as we like, so long as, e.g. $\epsilon < H - \delta - s$. By definition of $M$, $\tau(s-t) \leq M$, and *by definition of* $s$ it must be the case that $\tau(s+t) < M$. Therefore, we have

$$\tau(s) < \frac{M + M}{2} = M$$

which is indeed a contradiction. Thus, we see that $\tau \leq 0$ on $[0, H]$. But applying this same argument to $-\tau$, we see that $-\tau \leq 0$ on $[0, H]$, i.e. that $\tau \geq 0$. Thus, $\tau \equiv 0$ on $[0, H]$ which means that

$$f(x) = x\frac{f(H)}{H} \text{ on } [0, H]$$

meaning that $f$ is linear, as desired. □

We thus see that for all $1 \leq i \leq n$ that $g_i$ is a linear function, i.e. $g_i(x) = \lambda_i x$ (to beat the horse dead, you might say that we can repeat on a countably dense set of $H_i$ and deduce that $f(H_i)/H_i$ remains fixed, which ascertains the linearity of $g_i$) ; because $g_i$ are strictly increasing (by SP), it must be the case that $\lambda_i > 0$. Thus, we see that $g$ is a linear function $\mathbb{R}^n \to \mathbb{R}$ and that weighted utilitarianism holds, i.e. for all $x, y \in X$ and for all $u \in \mathcal{U}$, we have that $xRy$ if and only if

$$\sum_{i=1}^{n} \lambda_i u_i(x) \geq \sum_{i=1}^{n} \lambda_i u_i(y),$$

with all $\lambda_i > 0$.

We conclude the section with a few remarks on our result and its proof:

(1) We have provided (to the author's knowledge) the only direct proof of weighted utilitarianism under full comparability. As argued earlier, work hitherto done in other informational settings is less reasonable in its assumptions.

(2) We have shown that diluting A to MIS is unnecessary; we can discard A altogether. In this (admittedly limited) sense, our result is also the most parsimoniously axiomatized one.[17]
(3) Getting to "generalized utilitarianism" essentially exhausts the explicit need for I, SP, and S. Certainly, Corollary 2 and Lemma 3 make no direct invocations of these properties; while we have not explicitly completed the exercise, non-constancy and continuity of $g_i$ is likely sufficient to replicate the results of Lemma 2 (along with FC, rather than an explicit use of monotonicity). This "division of labor" between the axioms more transparently reveals their use.
(4) This proof makes immediate an extension to rank-weighted utilitarianism.

## 4 An application: rank-weighted utilitarianism

The proof we have given of weighted utilitarianism suggests an approach to tackling *rank-weighted* utilitarianism, subfamilies of which could be more agreeable from a moral standpoint.[18]

### 4.1 The result

First, we must introduce a definition and a rather strange axiom, due to Ebert, and described in an SWO context by d'Aspremont-Gevers. Let

$$\mathcal{G}_n = \{w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n : w_1 \leq w_2 \leq \ldots \leq w_n\}$$

be the subset of well-ordered vectors in $\mathbb{R}^n$. Then, we make the following definition:

**Definition 8** For any $u \in \mathcal{U}$, let

$$\mathcal{G}_u = \{x \in X : (u_1(x), \ldots, u_n(x)) \in \mathcal{G}_n\}.$$

In order to axiomatize SWFLs that act on the order statistics of individual evaluation levels, rather than the individual levels themselves, it makes sense that we might restrict properties (or axioms) to families of sets looking like $\mathcal{G}_n$. The following is such an axiom that restricts separability:

**Axiom 12** (*SWFL $\mathcal{G}_n$–separability*) For all $u, v \in \mathcal{U}$, if there exists $M \subset N$ such that for all $i \in M$,

$$u_i|_{\mathcal{G}_u \cap \mathcal{G}_v} = v_i|_{\mathcal{G}_u \cap \mathcal{G}_v}$$

and for all $j \in N \setminus M$,

$$u_j|_{\mathcal{G}_u \cap \mathcal{G}_v} \text{ is constant and } v_j|_{\mathcal{G}_u \cap \mathcal{G}_v} \text{ is constant}$$

then we have that

[17] More importantly, perhaps, it is most "focused."

[18] Particularly in the case of the generalized Gini family.

$$f(u)|_{\mathcal{G}_u \cap \mathcal{G}_v} = f(v)|_{\mathcal{G}_u \cap \mathcal{G}_v}.$$

In an SWFL context this is a mouthful, so we will state the SWO version, due to Ebert and rephrased by d'Aspremont and Gevers:

**Axiom 13** *(SWO $\mathcal{G}_n$–separability)* For all $u, v, u', v' \in \mathcal{G}_n$, if there exists $M \subset N$ such that for all $i \in M$, $u_i = v_i$ and $u'_i = v'_i$ whereas for all $j \in N \setminus M$, $u_j = u'_j$ and $v_j = v'_j$, then $uR^*v \Leftrightarrow u'R^*v'$.

This restriction of separability only to objects in $\mathcal{G}_n$ is rather bizarre and unsatisfying. Its normative content or prescription is some strange mangling of that of usual S; it doesn't make much moral sense without the context of actual S. In the presence of an *anonymity* axiom, however, at least this axiom's technical content takes shape. For any specified element of $X$, the induced vector of utilities $(u_1(x), u_2(x), \ldots, u_n(x))$ can be permuted (without affecting social rank) to belong to $\mathcal{G}_n$. As separability holds here, we can apply Debreu's theorem[19] to deduce a generalized, rank-dependent utilitarianism where

$$xRy \Leftrightarrow \sum_{i=1}^{n} g_i(u_{(i)}(x)) \geq \sum_{i=1} g_i(u_{(i)}(y))$$

for monotone increasing functions $g_i : \mathbb{R} \to \mathbb{R}$. The problem of linearization now is identical to the one we solved earlier—we thus get rank-weighted utilitarianism as a corollary; the conditions we used above to generate weighted utilitarianism will then naturally grant us rank-weighted utilitarianism (note, in particular, that the affine transformations encoded in full separability preserve membership in $\mathcal{G}_n$—or alternatively in SWFL lingo, $\mathcal{G}_u = \mathcal{G}_{a+bu}$ for $a \in \text{diag}(\mathbb{R}^n)$ and $b > 0$). In essence, the funny axiom of (either SWO or SWFL) $\mathcal{G}_n$—separability yields rank-dependence of the SWFL. Now we provide a more formal elocution of our argument:

**Corollary 3** (Ebert 1988) *If $n \geq 3$ and a SWFL $f$ satisfies I, SP, A, $\mathcal{G}_n$—separability, C, and FC, then $f$ is the principle of rank-weighted utilitarianism.*

*Proof* By formal welfarism we need only to construct a representation of $R^*$ on $\mathbb{R}^n$. Anonymity allows us to permute any vector $v \in \mathbb{R}^n$ such that it belongs to $\mathcal{G}_n$. We may then write, given the representability of $R^*$ by some function $F$, that $F(v) = F^{\text{ord}}(v_{\text{ord}})$ where $v_{\text{ord}}$ is the permuted version of $v$ lying in $\mathcal{G}_n$ and $F^{\text{ord}} : \mathcal{G}_n \to \mathbb{R}$ which is continuous and monotone increasing in each coordinate. Define the following subset of $\mathcal{G}_n$:

$$\mathcal{S}_n = \{w \in \mathcal{G}_n : w_1 < w_2 < \ldots < w_n\}.$$

For any $w \in \mathcal{S}_n$, observe that any suitably small neighborhood of it inside $\mathcal{S}_n$ sits within some product $X = \prod_{i=1}^{n} X_i$ of open intervals (where $X \subset \mathcal{S}_n$ as well);

[19] This step of the argument is also due to Ebert.

applying Debreu's theorem on any such space $X$ (where separability certainly holds) gives us (locally) the rank-dependent generalized utilitarian representation

$$F^{\text{ord}}(v_{\text{ord}}) = \sum_{i=1}^{n} g_i(v_{(i)}),$$

unique up to common positive affine transformation, where $g_i$ are monotone increasing and continuous. Separability and connectedness of $\mathcal{G}_n$, along with (pivotally) continuity of $R^*$ on $\mathbb{R}^n$ allow us to extend the above representation to the entirety of $\mathcal{G}_n$.

Equipped with this rank-dependent generalized utilitarian representation, the arguments of Lemma 2, Corollary 2, and Lemma 3 yield the result. □

*Remark 2* Although Ebert's original proof (and ours) invokes the $n \geq 3$ condition, Corollary 3 can be strengthened to ignore this requirement, which exists strictly for application of Debreu's theorem. The case $n = 2$ can be proven directly without any resort to separability, c.f Theorem 4.7 of d'Aspremont and Gevers (2002). As such, we prove a result strictly weaker than what is already known.

On its own, rank-weighted utilitarianism is uninteresting and even arbitrary, but subject to the constraint $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n-1$, it produces, as alluded to earlier, a more ethically interesting and agreeable system of SWFLs. Here, higher welfare weights are accorded to those "worse off" in utility terms; this parallels, e.g. the motivation behind Sen's Relative Equity axiom (Sen 1976). The family also encompasses a broad range of views: Rawlsian maximin, for example, is achieved in the limit. The axiomatic condition guaranteeing this condition on the coefficients is known to be (c.f. Ebert 1988) the property of progressive transfers:

**Axiom 14** (*SWO Progressive Transfers* (*PT*)) For all $u \in \mathbb{R}^n$: for all $1 \leq i < j \leq n$ and $\epsilon > 0$ such that $u_{(i)} + \epsilon < u_{(i+1)}$ and $u_{(j)} - \epsilon > u_{(j-1)}$, we have

$$(u_{(1)}, u_{(2)}, \ldots, u_{(i-1)}, u_{(i)} + \epsilon, u_{(i+1)}, \ldots, u_{(j-1)}, u_{(j)} - \epsilon, u_{(j+1)}, \ldots, u_n) P^* u.$$

While transparent and natural in Ebert's setup (where it is easy to conceptualize the income vectors $u, v$), this axiom seems "forced" in the same way that IE is—it seems unreasonable and oddly specific for the social planner to incidentally keep $\sum_{i=1}^{n} u_i(x)$ in his mind the whole time, possibly in the circumstance of a non-utilitarian SWFL in the first place! The condition also seems to impose unduly linearizing constraints on $f$'s representation; we suspect a weaker condition could be substituted. Given our strong preference for ordinal axioms that aren't informational constraints (which we can somewhat naturally think of as statements about the units of cardinal data), we would hope that PT is equivalent—in the presence of our other axioms—to some ordinal condition other than Schur concavity, which seems qualitatively arbitrary in its own right.

A final remark is in order: as alluded to earlier, Ebert's theorem is not exactly as stated above. He formulates his problem in the context of income distribution, imposing conditions on a function $W : (\mathbb{R}^n)^+ \to \mathbb{R}$ that represents a social planner's preference on various $n$-tuples of income distributions for society. Particularly in this limited capacity, we object to some of his axioms—whether fundamental[20] or not; the Pareto property seems questionable (whether $(\epsilon + N, \epsilon, \ldots, \epsilon) P (\epsilon, \ldots, \epsilon)$ may be debated, for example), while each of scale and additive invariance (on pairwise comparisons) seems unreasonable if we take the view point that past a certain level (or merely at high levels) of societal affluence, equality of income distribution should matter more than average income. We point out that d'Aspremont and Gevers's choice of embedding of his theorem into the SWO (or SWFL, as we have) context skirts these criticisms by moving into utility space. As such, although the preliminaries are denser, we hope that the reader sees some independent worth in this applied digression on rank-weighted utilitarianism vis-a-vis Ebert's paper.

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[20] He bundles $W$'s continuity, Pareto property, symmetry (as a function $(\mathbb{R}^n)^+ \to \mathbb{R}$ which we see as anonymity), and "independence with respect to ordered vectors" ($\mathcal{G}_n$-separability) together as a baseline property $W$ that all welfare orderings in the paper possess.