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Complexities of electing diverse committees

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Abstract Electing a committee introduces constraints beyond excellence, such as ensuring a balance of gender, tenure, talent, and other characteristics. The difficulties are captured by an actual example where every voter desired gender diversity on a committee and voted accordingly, but only men were elected. After developing the properties of certain methods that avoid these problems, other needs in this area are described.

1 Complexity of selecting committees

Considerable progress has been made in understanding what causes standard voting paradoxes and how to select rules that avoid many of these difficulties. But a topic that remains essentially untouched involves the common experience of selecting committees. New types of fascinating complexities (accompanied with different kinds of strategic actions, paradoxical outcomes, likelihood computations, and so forth) reflect the reality that such elections involve a combination of voter priorities.

By this we mean that while each voter's personal ranking of the candidates may suffice for traditional settings, when selecting committees, individual preferences can be accompanied with tacit expectations that an elected committee will satisfy still other constraints. To ensure needed diversity, for instance, everyone may want an academic committee consisting of both tenured and untenured faculty, or of both

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men and women. Other conditions might involve mixtures of talents; e.g., because of personal skills, voters might want to have either Bob or Sue on the committee, but not both because they would offer similar perspectives.

This mixture of priorities affects other choice settings. In electing new members for the US National Academy of Sciences, each of the six NAS classes selects and ranks their quota of candidates. This ranking is placed on the election ballot that is then sent to the general membership. The ranking matters; this is because it serves as the class' recommendation to the general membership, so a higher ranked candidate is more likely to be elected. To use specific numbers, each year class 5 (consisting of the four sections of Anthropology, Psychology, Social and Political Science, and Economic Sciences) places a ranked list of 14 names on this ballot from which nine will be elected to the NAS (in 2012, the numbers to be elected and listed on the ballot were increased). A Class Membership Committee (CMC) has the responsibility of selecting and ranking the class' candidates from lists proposed by each section. Accompanying each CMC member's ranking of the relative merits of the candidates, then, is the need for a balanced representation over the four sections. The ideal situation is to elect at least two members from each section.

While it is easy to create hypothetical examples to illustrate various kinds of difficulties that can arise in these settings, actual paradoxical and strategic behaviors are more informative. An interesting case described by Ratliff (2006) is where everyone voting in a Wheaton College election wanted to have a committee of women and men, but only men were elected. The way in which this occurred and how it was avoided in a subsequent election is described in Ratliff (2006), but the source of the problem is indicated below in Sect. 2.

A related concern is how certain voting rules, when combined with this diversity intent, create strategic incentives for voters who normally may not want to vote strategically. This happens in the class 5 CMC example where each CMC member votes for nine candidates (from around 20 choices) to determine the class' ranked list of 14 for the NAS ballot. While each voter ranks the candidates according to distinction, the voter also wants to ensure that at least two candidates from his or her section will be in the top nine. But the voting rule fails to address both needs, so it introduces incentives to vote strategically.

The feature encouraging strategic behavior is that a CMC ballot cannot distinguish top-ranked from secondary choices. To circumvent this weakness, a CMC member representing Section A may vote in a way to protect at least two Section A candidates (the diversity constraint). The obvious strategy is to vote for the top two or three candidates from area A, three or four other top candidates (who represent excellence and will be elected anyway), and then strategically "protect" Section A candidates by "wasting" the remaining votes by voting for candidates who probably would not win.[1] A danger associated with this (and any strategic action) is a final outcome that need not reflect the voters' intent. An indicator as to whether several CMC voters voted in this strategic manner is if a lower ranked candidate from some section is ranked over a

[1] This is not hypothetical; when one of us (DGS) tallied CMC ballots for two different years, it was clear that some voters had adopted this strategic voting behavior.

clearly preferred one. That such an outcome is not unusual is manifested by the NAS rule allowing members from an affected section to appeal to the CMC to reverse the positioning of these two candidates.

Returning to the choice problem, the number of reasonable conditions that can accompany committee selections probably is without limit, so it is premature to try to examine the full problem. Thus our contributions are targeted toward those special but common settings where the universal intent is to elect a committee that satisfies a specified diversity characteristic such as including both women and men, or (motivated by the CMC example) a balanced representation from each section of a group. A second contribution is to identify unexpected problems and issues that are involved in selecting committees.

As an example, our analysis introduces a novel twist on strategic voting. While strategic voting cannot be avoided, we should avoid voting rules that encourage strategic voting just to achieve a universally accepted diversity constraint. With the analogy that "a padlock is intended to encourage honest people to remain honest," a criterion for selecting a voting rule should be that it does not encourage voters who wish to vote sincerely to vote strategically. With our emphasis on sincere voting, this means that sincere voting should not lead to outcomes that everyone in the group finds to be obviously inferior.

As motivated by the examples above, our analysis assumes that the voters have a preference for the overall composition of the committee that cannot be decomposed into preferences on the individual candidates. This perspective differs from that found in much of the literature on committee elections. A common view is consistent with the idea of proportional representation as nicely expressed by Chamberlin and Courant (1983), where each voter has a representative on the deliberative body to reflect their views and interests. There has also been work exploring barriers to diverse representation in legislatures with single-member districts [e.g. see Fréchette et al. (2008) for a discussion of the impact of a gender parity law in the French Assembly]. We are interested, instead, in situations where the voters' preferences are for the composition of the entire selected group, which is a distinct concern from proportional representation or single-member districts.

Another approach is taken by Barberà et al. (1991, 2005) where it is assumed that the voters' preferences are separable into preferences on individual candidates. However, the Wheaton election described in Ratliff (2006) demonstrates that this is not always possible. Voters ranked the eight possible committees, and for over half of the voters, their top-ranked and bottom-ranked committees had at least one candidate in common. This demonstrates the complexity of these issues in that it may be incorrect to assume that voters' preferences are separable into preferences on individual candidates.

This reality about the complicated nature of voters' preferences suggests the remedy of requiring each voter to rank all possible committees. Unfortunately, such an approach quickly becomes impractical as the number of possibilities becomes extremely large for even moderately sized committees. Selecting a four-person committee from eight candidates, for example, requires ranking 70 different possibilities. In our approach, voters need only specify their top-ranked committee.

Our contributions identify wide classes of easily used rules that achieve the specified objectives in electing committees. The wealth of emerging voting methods puts forth several new questions, such as developing ways to identify strengths and weaknesses of these different possibilities.

The paper is organized as follows. Section 2 states the main results. Section 3 introduces a geometric framework and includes the proofs of Theorems 1, 2 and 4. Section 4 outlines unsolved issues with larger and unconstrained committees. Section 3 contains the proof of Theorem 3.

2 Committees of three

It is trivial to design a ballot to guarantee representation from different parts of an organization. Within a university, for instance, a ballot may require each voter to vote for one person from each division or school. To see what can happen, the following example lists the candidates for a three-person university committee where each voter must select one candidate from each of the three listed categories.

Social sciences	**Natural sciences**	**Humanities**
Ann	Carole	Ellen
Bob	David	Fred

(1)

A majority vote determines who is elected from each section. With an added, universally accepted diversity intent, such as to have a mixed gender outcome, each voter would vote for at least one man and one woman.

But even should each voter's vote reflect this objective, the outcome can violate the intent. To see this with an example, each voter in the three-voter profile

Voter1	**Voter2**	**Voter3**
Ann, David, Fred	Bob, Carole, Fred	Bob, David, Ellen

(2)

adheres to this goal, but the Bob, David, Fred outcome (each receives a 2:1 vote) does not. This particular form of election that splits a voter's preference for the overall composition into preferences on individual candidates illustrates what caused the problems with the above referenced Wheaton College election.

It is reasonable to wonder whether this phenomenon, which violates the universal intent of all voters, is rare enough to be safely ignored. It cannot; expect this unintended behavior to arise around once in every 15–30 elections. So, while not ubiquitous, this difficulty arises frequently enough to cause concern.

Theorem 1 *In electing a three-person committee from among candidates who are slotted in three divisions, suppose each division has two candidates where each represents one of two different categories (e.g., tenured and untenured faculty). Determine the selected candidate from each division by a majority vote; each voter votes in all three elections. To reflect a universal intent of electing a diverse committee, an admissible ballot must include at least one candidate from each category.*

If each profile is considered to be equally likely (Impartial Anonymous Culture, or IAC), the likelihood of electing a homogeneous committee consisting of candidates from only one category, given by $Prob(H)$, ranges from approximately 0.036 with *three voters to over 0.061 with 25 voters, and approaches* $\frac{1}{16} = 0.0625$ *as the number of voters increases to infinity.*

If each voter is equally likely to choose each committee (Impartial Culture, or IC), then $Prob(H)$ *ranges from approximately 0.0556 with three voters to over 0.0837 with 25 voters, and approaches* 0.0877 *as the number of voters increases to infinity.*

The way in which Theorem 1 is proved (Sect. 3) makes it clear that the likelihood of this bothersome behavior increases with the number of slotted divisions and candidates. Precise probability values, of course, are not of practical interest; the important message is that this behavior cannot be ignored. Combining Theorem 1 with the Wheaton College and CMC experiences underscores the importance of analyzing this concern.

2.1 A class of solutions

As the Eqs. 1, 2 example suggests, a way to achieve such a universally accepted objective is to tally ballots so that an added premium is given to each voter's diversity candidate. Staying with this example, if the premium assigned to a ballot's diversity candidate is $\lambda > 1$ points, then, with the Eq. 2 profile, choosing $\lambda < 2$ results in an all male committee; selecting $\lambda > 2$ creates an all female committee. This calculation suggests that it is necessary to assign $\lambda = 2$ points to the "diversity" candidate and one point to each of the other two candidates.

Surprisingly, this (2, 1, 1) choice is a general solution (the reason, as explained in Sect. 3, is that the Eq. 2 profile plays a central role in creating these problems). But beyond the (2, 1, 1) rule, there is a continuum of other possibilities. The wealth of choices permits selecting a rule that can make other subtle distinctions among the candidates. One choice, for example, is (3, 2, 1) where the diversity candidate receives 3 points, and a distinction is made between the two non-diversity choices.

In what follows, our rules are impartial with respect to the two categories. That is, the rule makes no distinction about which category the diversity candidate represents. Thus, if the rule assigns weights (2, 1, 1) to the committee {Ann, David, Fred}, it will assign weights (2, 1, 1) to the committee {Bob, Carole, Ellen}.

Theorem 2 *In electing a three-person committee from among candidates who are slotted in three divisions, suppose each division has two candidates representing two different categories (e.g., tenured and untenured faculty, or men and women, or Nordics and non-Nordics). To reflect a universal intent shared by all voters to elect a committee with representation coming from each of these two categories, an admissible ballot must have at least one candidate from each category (so, two candidates are from one category and the "diversity" candidate is from the second category). The diversity objective always can be achieved by assigning weights if and only if the weight* w_1 *assigned to the diversity candidate equals the sum of the weights* w_2 *and* w_3 *assigned to the other two candidates.*

To illustrate with a tenured, untenured setting, suppose a twelve-voter profile has three preferring (t, u, u), three preferring (u, t, u), five preferring (u, u, t) and one preferring (t, t, u). The majority vote over each section creates the homogeneous untenured committee (u, u, u). By using the (2, 1, 1) method, the tenured representative, "t," is the diversity candidate for the first eleven ballots, and the untenured "u" is the diversity choice for the last ballot. A count shows that the tenured representative receives 7 points in the first two sections, while the untenured choice receives 8 points. In the final section, the tenured beats the untenured candidate by 10:8 to elect the (u, u, t) committee.

Both positive and bothersome properties of these rules can be extracted by using the simple geometry (Sect. 3) that is developed to prove Theorem 2. Possible resolutions for a potentially troubling property also follow from this geometry.

2.2 Other weights

When tallying ballots, a premium is assigned to each ballot's diversity candidate. While the (2, 1, 1) choice satisfies Theorem 2, so does (3, 2, 1) where three points are assigned to a diversity candidate and a ranking distinction is made between the other two candidates. This continuum of choices characterized by

$$(w_1, w_2, w_3) \quad \text{where} \quad \mathrm{w_1 = w_2 + w_3}, \quad \mathrm{w_j} > 0, \tag{3}$$

either creates a need to further filter the possibilities, or introduces an opportunity to select the weights to achieve other objectives such as allowing voters to record finer differences among candidates. Our (2, 1, 1) choice reflects a natural neutrality where further distinctions are not considered.

If the adopted weights do not satisfy Eq. 3, the outcome could be homogeneous even if each voter casts a diversity ballot (Theorem 2). On the other hand, if the weights provide some advantage for each ballot's diversity choice, they mitigate the Theorem 1 conclusion by making these undesired behaviors less likely to occur.

A group, for example, may find the (2, 1, 1) tallying approach to be overly extreme; perhaps the best they can accept are the weights $(3, 2, 2)\left[\text{equivalently, } \left(\frac{3}{2}, 1, 1\right)\right]$. While this choice allows negative conclusions, it significantly reduces the likelihood that they can occur. In fact, the closer $(\lambda, 1, 1)$ is to (2, 1, 1), the smaller the likelihood of paradoxical outcomes.

Theorem 3 *With the probability measures and assumptions of Theorem 1, the likelihood of electing a homogeneous committee with* $(\lambda, 1, 1)$*,* $1 \leq \lambda \leq 2$ *is bounded by* $g(2-\lambda)Pr(H)$ *where the continuous function* $g(x)$ *depends on the probability measure (IAC or IC), and satisfies* $g(1) = 1$*,* $g(0) = 0$*, and* $g(2-\lambda) \to 0$ *as* $\lambda \to 2$*.*

While we have not computed the form of g, there are strong reasons to believe (described below in Sect. 5) that, at least for IAC, $g(2-\lambda)$ eventually behaves like a multiple of $(2-\lambda)^2$. This would mean that by using (3, 2, 2) (so $\lambda = 1.5$), the probability of incurring problems is approximately one-fourth as likely as reported in Theorem 1; i.e., expect problems to arise only in about one out of 60–120 elections. Problems can occur, but they can be expected to take a long while before happening.

Theorem 2 extends in a natural fashion to $k \geq 3$ divisions where a candidate is to be selected from each division. While we have not addressed the likelihood issue should a method not satisfying Theorem 4 be used, a version of Theorem 3 most surely holds.

Theorem 4 *In electing a k person committee from among candidates who are slotted in k divisions, suppose each division has two candidates representing two different categories. To reflect a universal intent shared by all voters to elect a committee with representatives from each of these two categories, an admissible ballot must have at least one candidate from each category. The diversity objective always can be achieved if and only if individual weights assigned to diversity candidates are greater than or equal to the weights assigned to non-diversity candidates and the sum of weights assigned to the candidates of each category are equal.*

With $k = 5$ and the groups of athletes (A) and non-athletes (N), a way to tally a (A, N, N, A, N) ballot is to assign $(3, 2, 2, 3, 2)$ points. But, once $k \geq 5$, problems can arise if ballots have different numbers of diversity candidates. For instance, if the total number of points on a ballot is fixed (with the above, the point total would be 12), then a (N, A, A, A, A) ballot would be tallied using $(6, 1.5, 1.5, 1.5, 1.5)$. The doubled weight, from 3 to 6, provides an incentive for a voter to strategically vote for only one diversity candidate. A way to eliminate this problem is to require each voter to vote for a fixed number of diversity candidates.

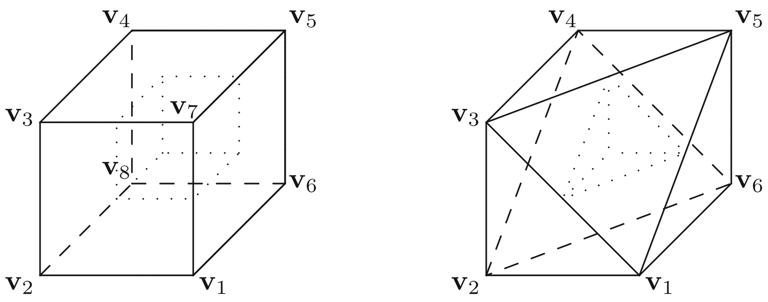
3 Using geometry to find properties

It is interesting how Theorem 2 is related to the "least-squares" approach commonly used in statistics. To explain the connection, because data cannot be expected to precisely satisfy a desired expression, such as lying on a line $y = ax + b$, the selected line (i.e., the choice of "a" and "b") is the one that comes the closest to satisfying the data. To determine this line, the least square's approach orthogonally projects the data into an appropriate subspace; the projection is intended to filter out errors.

With the committee problem, if majority vote outcomes satisfy the desired diversity condition, they may be accepted. But should the tallies fail to satisfy the desired condition, then, as with "least-squares," a way to filter out errors is to orthogonally project the tallies into an appropriate subspace where these conditions are satisfied; the projected outcome is what is used. As developed below, this projection is equivalent to using the $(2, 1, 1)$ rule.

To illustrate with the Theorem 2 conditions, assume that the diversity requirement is a mixed gender requirement. To list the eight possible committees, specify the gender of each candidate from each division; e.g., (m, w, w) means that a man represents the first division, and women represent the other two. This representation can be converted into a voting scheme in the three-dimensional $\mathbb{R}^3$ by replacing the gender with $+1$ for a man and -1 for a woman; e.g., this transforms (m, w, w) to $(1, -1, -1)$. The six admissible ballots have differences in signs, while the two inadmissible ballots are $(1, 1, 1)$ and $(-1, -1, -1)$ representing, respectively, an all man and an all woman committee.

Following the geometric analysis of pairwise voting developed in Saari (1995), the eight vectors (representing six admissible and two inadmissible ballots) define the vertices of the Fig. 1a cube. The $\mathbf{v}_j$ vertices represent ballots supporting the following committees:



(a) Cube **(b)** Truncated cube

Fig. 1 Cube and rankings

Vertex	**Committee**	**Vertex**	**Committee**
$\mathbf{v}_1$	(1, 1, −1)or(m, m, w)	$\mathbf{v}_4$	(−1, −1, 1)or(w, w, m)
$\mathbf{v}_2$	(1, −1, −1)or(m, w, w)	$\mathbf{v}_5$	(−1, 1, 1)or(w, m, m)
$\mathbf{v}_3$	(1, −1, 1)or(m, w, m)	$\mathbf{v}_6$	(−1, 1, −1)or(w, m, w)
$\mathbf{v}_7$	(1, 1, 1)or(m, m, m)	$\mathbf{v}_8$	(−1, −1, −1)or(w, w, w)

(4)

If p_j represents the proportion of all voters who prefer committee j, $j = 1, \ldots, 6$, then a profile becomes $\mathbf{p} = (p_1, \ldots, p_6)$ where $\sum_{j=1}^{6} p_j = 1$. The election outcome $\mathbf{V}$ is

$$\mathbf{V} = \sum_{j=1}^{6} p_j \mathbf{v}_j. \tag{5}$$

Because the summation for each slot involves $+1$ and -1 terms, the outcomes describe the difference of standard tallies for each pair of candidates. Namely, a (x, y, z) point in the cube represents normalized differences in tallies for candidates from each section.

Illustrating with Eq. 1,

$$\begin{aligned} x &= (\text{Number of votes for Bob} - \text{Number of votes for Ann})/\text{Number of votes} \\ y &= (\text{Number of votes for David} - \text{Number of votes for Carole})/\text{Number of votes} \\ z &= (\text{Number of votes for Fred} - \text{Number of votes for Ellen})/\text{Number of votes} \end{aligned} \tag{6}$$

Thus, with the Eq. 2 preferences, the three listed committees are $\mathbf{v}_1, \mathbf{v}_3$, and $\mathbf{v}_5$, corresponding to the preferences of Voters 3, 2, and 1, respectively. Then $\mathbf{p} = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)$, and by using Eq. 5, the normalized difference in tallies is

$$\mathbf{V} = \frac{1}{3}(1, 1, -1) + \frac{1}{3}(1, -1, 1) + \frac{1}{3}(-1, 1, 1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Notice how these $\mathbf{V}$ values agree with the previously determined 2:1 tallies. This particular election outcome (with an all male committee) is in the cube's positive orthant, which is defined by the positive x, y, and z axes; it is the small Fig. 1a cube designated by dotted lines.

It follows from Eq. 5 that the set of all possible tallies is the convex hull of the six admissible vertices as given in Fig. 1b [A template to create a physical example of this hull is given in Saari (1995, p. 100). Conversely, any point in this Fig. 1b region with rational coefficients is attainable with a profile. As such, points in the positive and negative orthants of this figure (and only these points) correspond to outcomes that violate the diversity condition, even though all ballots respect this condition. The portion of the positive orthant in this convex hull is the tetrahedron indicated by the dotted lines in Fig. 1b and defined by the vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

3.1 Connection with majority voting and the proof of Theorem 1

The above construction differs from what is in Saari (1995) in that the vertices in Saari (1995) represent rankings of alternatives. This similarity allows each profile over committees to be identified in a one-to-one manner with a profile over three candidates. The representation from Saari (1995) [also see Saari (2008, Chap. 2) where this translation is used to explain cyclic behavior] follows:

Vertex	**Ranking**	**Vertex**	**Ranking**	**Vertex**	**Ranking**
$\mathbf{v}_1$	$A \succ B \succ C$	$\mathbf{v}_2$	$A \succ C \succ B$	$\mathbf{v}_3$	$C \succ A \succ B$
$\mathbf{v}_4$	$C \succ B \succ A$	$\mathbf{v}_5$	$B \succ C \succ A$	$\mathbf{v}_6$	$B \succ A \succ C$

(7)

The remaining vertices, $(1, 1, 1)$ and $(-1, -1, -1)$, represent, respectively, the cyclic rankings $A \succ B, B \succ C, C \succ A$ and $B \succ A, C \succ B, A \succ C$. In this manner, a mixed gender committee is identified with a transitive ranking while a common gender committee corresponds to a cyclic ranking. This connection also explains the centrality of the Eq. 2 profile by identifying it with the Condorcet triplet $A \succ B \succ C$, $B \succ C \succ A$, $C \succ A \succ B$, which now is known to be completely responsible for all possible majority vote paired comparisons difficulties Saari (2008).

This one-to-one linear translation (i.e., a "name-change") immediately proves Theorem 1. Namely, any result about cyclic behavior in the traditional paired comparison voting setting transfers immediately to the same result about electing a homogeneous committee. Thus Theorem 1 follows directly from a result in Gehrlein (2002) that the probability of a pairwise majority winner with three candidates and n voters, assuming IAC and n odd, is given by $\frac{15(n+3)^2}{16(n+2)(n+4)}$; in the limit as n becomes very large, the likelihood of a cycle (or, in our setting, a homogeneous committee) is $\frac{1}{16}$. The results for IC also follow directly from calculations given in Table IV of Gehrlein (2002) for the probability of a pairwise majority winner with three candidates assuming IC.

3.2 Geometry and the proofs of Theorem 2 and 4

The Fig. 1a, b cubes have a special "transitivity plane" Saari (1995, 2008) defined by

$$x + y + z = 0. \tag{8}$$

This plane removes from profiles all traces of what can cause cyclic election effects. In terms of committee selections, profiles in this plane are stripped of all Eq. 2 type components that contribute to an undesired homogeneous outcome. Thus projecting an outcome, or a profile, into this plane removes all traces of what creates the committee problem.

The meaning of Eq. 8 can be illustrated with the integer profile $\mathbf{p} = (1, 0, 3, 0, 2, 6)$ (one voter prefers the committee (u, u, t), three prefer (u, t, u), two prefer (t, u, u), and six prefer (t, u, t)). Using the Eq. 6 formulation, where each section's male candidate is replaced with u, the normalized tally differences are

$$x = \frac{4-8}{12} = -\frac{4}{12}, \quad y = \frac{9-3}{12} = \frac{6}{12}, \quad z = \frac{5-7}{12} = -\frac{2}{12}, \tag{9}$$

which elect the (t, u, t) committee and satisfy Eq. 8. This Eq. 8 expression, then, means that the sum of tally-differences between tenured and untenured candidates over the three sections is equal to zero. But Eq. 8 requires some terms to be positive and others negative, so it ensures a diversity outcome (the differences in tallies over the three sections are, respectively, $4-8=-4$, $9-3=6$, and $5-7=-2$, but the differences in (2, 1, 1) tallies are, respectively, -6, 9, and -3. The reason for the $\frac{3}{2}$ multiple is explained below).

The importance of this plane can be further illustrated with the $(\lambda, 1, 1)$ rule, where the ballots become

$(1, 1, -\lambda)$ for $\mathbf{v}_1$, $(\lambda, -1, -1)$ for $\mathbf{v}_2$, $(1, -\lambda, 1)$ for $\mathbf{v}_3$, $(-1, -1, \lambda)$ for $\mathbf{v}_4$, $(-\lambda, 1, 1)$ for $\mathbf{v}_5$, and $(-1, \lambda, -1)$ for $\mathbf{v}_6$.

As it is easy to show for $\lambda \neq 2$, the convex hull of these points (i.e., the space of admissible election outcomes) meets both the positive and negative orthants (to show this for $1 \leq \lambda < 2$, the outcome for profile $\mathbf{p}_1 = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)$ is in the positive orthant while the outcome for $\mathbf{p}_2 = \left(0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)$ is in the negative orthant. If $\lambda > 2$, then the outcome for each of these two profiles now resides in the opposite orthant. This means that for each $\lambda \neq 2$, the associated convex hull meets one of these orthants to ensure undesired outcomes). Thus, with any $(\lambda, 1, 1)$ rule where $\lambda \neq 2$, there exist election outcomes that violate the diversity condition.

The same argument and assertion hold for (w_1, w_2, w_3) ballots that do not satisfy Eq. 3. But if the Eq. 3 equality is satisfied, then all ballots must lie on the transitivity plane (Eq. 8), so their convex hull (the set of all possible election outcomes) misses the positive and the negative orthants. As all possible election outcomes satisfy the diversity condition, this completes the proof of Theorem 2.

A similar geometric argument easily extends to prove Theorem 4 for k person committees with candidates slotted in k divisions where each division has two candidates representing two different categories. Call the candidates from one category $u_1, u_2, \ldots, u_k$ and the candidates from the second category $t_1, t_2, \ldots, t_k$ where we have u_i versus t_i in the ith slot. Similar to Eq. 9, we can create a point in k dimensions where the ith coordinate corresponds to the normalized tally differences between u_i and t_i. If the sign of the ith component is positive, this indicates that u_i is elected to the committee and if the sign is negative then t_i is elected.

So, an outcome with all positive coordinates corresponds to the homogeneous committee $\{u_1, u_2, \dots, u_k\}$ whereas an outcome with all negative coordinates corresponds to the homogeneous committee $\{t_1, t_2, \dots, t_k\}$. Any outcome with both positive and negative coordinates corresponds to a diverse committee. Thus, for any rule, the admissible ballots determine vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{2^k-2}$ that contain both positive and negative coordinates; the convex hull of these points determines the space of admissible election outcomes.

First, assume that the sums of the weights assigned to the candidates of each category are equal. This means that the sum of the coordinates for each vertex $\mathbf{v}_i$ is equal to zero, which implies that each vertex lies in the hyperplane in $\mathbb{R}^k$ through the origin that is orthogonal to the vector $(1, 1, 1, \dots, 1) \in \mathbb{R}^k$. Thus, the $\{\mathbf{v}_i\}$ convex hull also lies in this hyperplane, implying that every admissible outcome corresponds to a vector that is orthogonal to $(1, 1, 1, \dots, 1)$. Therefore, every admissible outcome must contain both positive and negative components, and, because any convex combination of vectors orthogonal to $(1, 1, 1, \dots, 1)$ also is orthogonal to this vector, the diversity objective will always be achieved.

Now suppose the sums of the weights assigned to the candidates in the two categories are not equal for some particular admissible committee. This defines a vertex $\mathbf{v}_1 = (x_1, x_2, \dots, x_k)$ where $x_1 + x_2 + \cdots + x_k \neq 0$. Because we have assumed that our rules are impartial with respect to the two categories, we know that the rule also allows the following vertices:

$$\begin{aligned}
\mathbf{v}_2 &= (x_2, x_3, \dots, x_{k-1}, x_k, x_1)\\
\mathbf{v}_3 &= (x_3, x_4, \dots, x_k, x_1, x_2)\\
&\vdots\\
\mathbf{v}_k &= (x_k, x_1, \dots, x_{k-2}, x_{k-1})
\end{aligned}$$

The point determined by the ballots $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is

$$\left(\frac{x_1 + x_2 + \cdots + x_k}{k}, \frac{x_1 + x_2 + \cdots + x_k}{k}, \dots, \frac{x_1 + x_2 + \cdots + x_k}{k}\right)$$

We know each coordinate is non-zero, and therefore all must have the same sign. Thus, the outcome is a homogeneous committee, and the diversity criterion is not met. This completes the proof of Theorem 4.

3.3 Properties and alternatives of the (2, 1, 1) rule

To connect (2, 1, 1) with least-squares, the orthogonal projection of a point $\mathbf{v}$ in $\mathbb{R}^3$ to the Eq. 8 transitivity plane is equivalent to finding the unique $\mathbf{q}$ satisfying Eq. 3 and the unique α value so that

$$\mathbf{v} = \alpha(1, 1, 1) + \mathbf{q}. \tag{10}$$

To illustrate, because $\mathbf{v}_1 = (1, 1, -1) = \frac{1}{3}(1, 1, 1) + (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3})$, the orthogonal projection of $\mathbf{v}_1$ is $\mathbf{q} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right)$. Normalizing $\mathbf{q}$ to eliminate fractions yields $(1, 1, -2)$, or the $(2, 1, 1)$ rule (the $\frac{3}{2}\mathbf{q}$ normalization explains the $\frac{3}{2}$ multiple that arose with the Eq. 9 example). As a similar computation holds for all $\mathbf{v}_j$, it follows that the (2, 1, 1) rule is equivalent to using majority votes over pairs, and then, to ensure that the desired diversity requirement is satisfied, to use the orthogonal projection of this outcome. In this manner, (2, 1, 1) is similar to the familiar least-squares rule.

3.3.1 A first difference

The (2, 1, 1) vote projects majority vote tallies, so there will be differences between the majority vote and (2, 1, 1) outcomes. Of importance, all possible differences are found by using Eq. 10, which defines a line passing through $\mathbf{q}$ [which represents the (2, 1, 1) election outcome on the transitivity plane] that is perpendicular to the plane [so, in direction (1, 1, 1)]. All majority vote paired comparisons that project to $\mathbf{q}$ are on this line. So, to discover all possible ways the outcomes between the rules can differ, move the line in a physical model and note when it passes through different regions (different orthants defining difference committees).

An equivalent, simpler way to find all possible differences is to take any profile and modify it by adding multiples of what corresponds to Eq. 2. That is, add to the original profile either multiples of the $\{(u, t, t), (t, u, t), (t, t, u)\}$ type or of the $\{(t, u, u), (u, t, u), (u, u, t)\}$ type. Whatever choice is added, and no matter how large the multiple, the new profile will have the same (2, 1, 1) outcome as the original profile (using the above vector notation, add multiples of either $\mathbf{p}_1 = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)$ or $\mathbf{p}_2 = \left(0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)$ to the original profile). The reason outcomes remain the same is that the (2, 1, 1) rule is a projection that treats these added components as a complete tie.

To illustrate, profile $\mathbf{P}_1$ with {six voters preferring (t, u, u), six preferring (u, t, u), six preferring (u, u, t)} and one preferring (u, t, t) has the same (2, 1, 1) outcome as profile $\mathbf{P}_2$ with {six voters preferring (u, t, t), six preferring (t, u, t), six preferring (t, t, u)} and one preferring (u, t, t). These two profiles differ only by what is in the brackets, which are multiples of either $\mathbf{p}_1 = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)$ or $\mathbf{p}_2 = \left(0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)$. The common (2, 1, 1) outcome for both profiles is (u, t, t) with the respective tallies 14:12, 12:13, 12:13 (as required by Eq. 8, the sum of the differences in $u - t$ tallies over the three sections equals zero).

A feature illustrated by $\mathbf{P}_1$ is that a vast majority of the voters (18 out of 19) appear to want a committee with two untenured faculty, but the elected committee consists of two tenured and one untenured faculty. On the other hand, this (u, t, t) outcome is fully consistent with $\mathbf{P}_2$ where all ballots list a committee with two tenured faculty. What creates this phenomenon is that the votes of the first 18 voters in both profiles (in brackets) cancel resulting in a tie; the tie is broken by the last voter.

To further explain this behavior, Theorem 2 ensures diversity, but it does not ensure that other, new kinds of conditions also will be satisfied. If the goal is to ensure a diverse committee, the feature illustrated by $\mathbf{P}_1$ and $\mathbf{P}_2$ is not troubling. If the goal is to elect a diverse committee that satisfies still another constraint, such as where the committee's composition (e.g., the number of men vs. women, or the number of untenured vs. tenured) reflects, in some manner, the majority opinion, then something different is required.

A natural resolution comes from the Fig. 1 geometry. In either Fig. 1a, b, the region of undesired homogeneous outcomes (given by the dotted lines) has three faces; they are portions of the $x - y$ plane, the $y - z$ plane, and the $x - z$ plane, where each serves as a boundary for a region with a diverse committee. An alternative approach, then, is to project a profile's (x, y, z) point to the nearest face [this approach resembles Dodgson's method; see Dodgson and Abeles (2001) and the series of papers by Ratliff (2001, 2002, 2003)]. From a computational perspective, this is easy to accomplish; with a homogeneous majority vote outcome, merely reverse the outcome for the section with the smallest tally-difference.

To illustrate with $\mathbf{P}_2$, the majority vote tallies (in a $u - t$ form) over the three sections are, respectively, $7 - 12 = -5$, $6 - 13 = -7$, $6 - 13 = -7$ electing a majority vote (t, t, t) committee. To achieve diversity, notice that the closest contest (the smallest tally-difference) is in the first section (with -5), reversing this section's conclusion leads to the committee (u, t, t), which agrees with the $(2, 1, 1,)$ outcome (from a practical perspective, one must expect reversing the outcome of this election would create more voter resistance than announcing a $(2, 1, 1)$ conclusion).

The outcome radically changes with $\mathbf{P}_1$ where the $u - t$ outcomes over sections are, respectively, $13 - 6 = 7$, $12 - 7 = 5$, $12 - 7 = 5$ leading to a majority vote (u, u, u) outcome. The adjustment to create a diverse committee would require reversing the outcome for either section two or section three leading to either the (u, t, u) or the (u, u, t) committee. Either choice differs from the $(2, 1, 1)$ conclusion of (u, t, t) by having two untenured faculty members.

3.3.2 A second difference

The first difference between the majority vote and $(2, 1, 1)$ outcomes reflects properties of the described line passing through $\mathbf{q}$ [the $(2, 1, 1)$ outcome on the transitivity plane]. As illustrated, profiles that lie on one side of the line might cause concern about the $(2, 1, 1)$ outcome.

The only remaining setting is where the base of the line (the $\mathbf{q}$ point) is in a region with a diverse outcome, and the line passes through a different non-homogeneous region defining a different outcome. That this phenomenon must occur follows from the geometry of the transitivity plane, which is tilted with respect to the coordinate system (it is midway between and parallel to the two triangular faces in Fig. 1b). This tilt forces settings where both the majority vote and $(2, 1, 1)$ outcomes elect diverse committees, but they differ. But, while the line can hit another region, it does so only for a small region. This is because the line moves into a region with a homogeneous outcome.

To illustrate with an example, start with where five prefer $\mathbf{v}_1$ and three prefer $\mathbf{v}_6$; that is, five prefer (u, u, t) and three prefer (t, u, t,). Add nine voters where three each prefer $\mathbf{v}_2$, $\mathbf{v}_4$, and $\mathbf{v}_6$. That is, {three prefer four (u, t, t), three prefer (t, t, u), and three prefer (t, u, t)}. The majority vote outcome is (t, u, t), but the (2, 1, 1) outcome is (u, u, t).

The explanation for this difference is essentially the same as the earlier comments. With the original profile, where five prefer (u, u, t) and three prefer (t, u, t,), both the majority vote and (2, 1, 1) methods agree on the (u, u, t) conclusion. The term in the brackets causes the difference; it is of the Eq. 2 type introducing a sub-population setting where the majority vote outcome violates the diversity intent of these nine voters. The majority vote cannot handle and dismiss these terms, but the (2, 1, 1) method can and does so.

4 Unsolved issues: larger and unconstrained committees

While the above constitutes our contributions to the committee problem, for completeness and to provide insight about how to at least mitigate the CMC problem (Sect. 1), we close by outlining other committee selection problems. To review, in the above, a first level of diversity was ensured by separating candidates into sections. Next, ways were found to ensure that a second diversity level, such as mixed tenure or gender condition, would be satisfied. Even a third level of diversity can be handled, as captured by the above discussion about how many members of a committee should be of one type or the other. But it is easy to prove that by imposing too many extra conditions, it becomes impossible to solve this problem. The analysis is similar to solving an algebraic problem with more equations than variables; in general, it cannot be done. In a non-technical manner, the issue is similar to asking students to line up alphabetically (with their last names) and according to height.

The above analysis is possible because each section has two candidates; one from each category. Problems caused by the majority vote reflect an unintended "divide-and-conquer" phenomenon as captured by the Eq. 2 example; while these voters basically agreed on which men candidates should be elected, they divided their voting strength over the women candidates. A similar problem arises if there are more than two candidates in each section.

To see this, change Eq. 1 so that the candidates from each division are three women and one man. Let the women be, respectively, Ann_1, Ann_2, Ann_3; Carole_1, Carole_2, Carole_3; Ellen_1, Ellen_2, Ellen_3. Next modify Eq. 2 to have nine voters: the first three vote in the indicated manner but for the woman with subscript "1," the second three vote in the indicated manner but for the woman with subscript "2," and the last three vote for the woman with subscript "3." With both the majority vote and the (2, 1, 1) method, the outcome is the homogeneous committee of {Bob, David, Fred}. It is not difficult to show that in these kinds of settings, no (λ, 1, 1) will guarantee a diverse committee. Something different, such as using positional voting schemes involving added weights assigned to diversity candidates, probably is required.

The above analysis provides insight about how to handle the concerns motivated by the CMC example where it is reasonable to believe that most voters wish to be nonstrategic and fair. A way to reduce the strong push to be strategic is to allow each voter to designate special consideration for two candidates. With these assumptions, for success and to avoid the "divide-and-conquer" reality, a section should put forth no more than three candidates. To avoid the temptation for strategic voting, the rule should assign weights to the two diversity candidates (i.e., from the voter's section) that agree with the sum of weights given to the other candidates. With the CMC, the ballot would allow each voter to designate two candidates to receive 3.5 points each, while each of the remaining seven on the ballot would receive a point each. In this way and with these assumptions, it is easy to show that the diversity condition will be satisfied.

While such a proposal probably assigns too heavy of a weight for the designated candidate to be accepted, it also follows from an analysis similar to the proof of Theorem 3 that assigning a larger weight for a voter's two top-ranked candidates would reduce the incentives to vote strategically and the likelihood of an unbalanced outcome. In this spirit, a (unsuccessful) proposal put forth to class 5 Saari (2006) to accomplish this objective stated:

> Each voter votes for the number of candidates allotted for the formal ballot (currently nine), and designates which two are top-ranked. The ballots are tallied with two points assigned to each top-ranked candidate and one to each of the remaining candidates. The candidates are ranked according to the number of votes received.

As a closing comment, problems associated with the common experience of electing a committee are far more complex than previously realized. This is an area deserving and requiring much more analysis.

5 Proof of Theorem 3

To prove Theorem 3, the probability of a homogeneous outcome must be related to the relative size of the portion of the truncated cube in the positive and negative orthants. Indeed, the Theorem 1 $Prob(H)$ value would be (in the limit) the relative volume of these two regions to that of the truncated cube if each rational point in the cube represented a unique profile. Instead, each point is represented by a two-dimensional manifold. The other profiles in this surface (leading to the same differences in paired election outcomes) differ from each other in terms of combinations of pairs of the $A \succ B \succ C$, $C \succ B \succ A$ type (or, with committees, (m, w, w), (w, m, m)) called "reversal configurations" Saari (2008, Chap. 4). It is not difficult to show that the number of profiles in this surface increases as the paired election outcome point moves away from either the $x + y + z = 1$ or $x + y + z = -1$ face (on each face, each point represents a unique normalized profile). Therefore, a measure of changes in the likelihood of a homogeneous committee is given by the shrinkage of relative size of these regions relative to the full region as $\lambda \to 2$. Notice, this assertion hold for any reasonable choice of a probability measure, which means that it includes IAC and IC. This ratio is computed next.

When λ represents the premium for the diversity candidate, the Fig. 1a cube becomes a parallelepiped (a three dimensional figure formed by six parallelograms). With this transformation, the vertices change: Originally one coordinate had a sign that differed from the other two; replace the "1" with λ; e.g., $\mathbf{v}_1 = (1, 1, -\lambda)$. The two remaining vertices are $\mathbf{v}_7 = (2 - \lambda, 2 - \lambda, 2 - \lambda)$ and $\mathbf{v}_8 = -\mathbf{v}_7$.

The volume of this parallelepiped is the value of the determinant with rows defined by $\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v}_3 - \mathbf{v}_2$, and $\mathbf{v}_8 - \mathbf{v}_2$, or $8 - \{2(\lambda - 1)^3 + 6(\lambda - 1)^2\}$. This expression has the required values of eight for $\lambda = 1$ (a cube of side length 2) and zero for $\lambda = 2$. The volume of each truncated tetrahedron (the removed part where none of the points can result from admissible preferences) is $\frac{1}{3}(1+\lambda)^2(2-\lambda)$. As there are two such tetrahedrons, the volume of the truncated parallelepiped is

$$Vol(\lambda) = 8 - \left\{2(\lambda-1)^3 + 6(\lambda-1)^2 + \frac{2(1+\lambda)^2(2-\lambda)}{3}\right\} \tag{11}$$

The vertices of one tetrahedron in the truncated parallelepiped that permit homogeneous committees are $(2-\lambda, 0, 0)$, $(0, 2-\lambda, 0)$ and $(0, 0, 2-\lambda)$. As this region has volume $\frac{(2-\lambda)^3}{6}$, the total volume of such points is $\frac{(2-\lambda)^3}{3}$. Thus, the ratio of the volume of these two regions to that of the truncated parallelepiped is

$$R(\lambda) = \frac{(2-\lambda)^3}{24 - [6(\lambda-1)^3 + 18(\lambda-1)^2 + 2(1+\lambda)^2(2-\lambda)]}. \tag{12}$$

This $R(\lambda)$ value provides a crude estimate of the likelihood of a homogeneous committee (or, in transitive preferences, a cycle). The statements about $g(\lambda)$ in Theorem 3 now follow.

Notice how the $R(1) = \frac{1}{16}$ value is the upper limit for the $Prob(H)$ value. In comparison, $R(\frac{3}{2}) = \frac{1}{100}$ is about one-sixth the $\lambda = 1$ value. Rewriting $R(\lambda) = F(\lambda)(2-\lambda)^2$, both the numerator and denominator of $F(\lambda)$ approach 0 as $\lambda \to 2$. So, to find the limit (e.g., use l'Hopital's rule), it follows that $F(\lambda) \to \frac{1}{36}$, which provides a bound on $g(\lambda)$ in Theorem 3. More precisely,

$$R(\lambda) \sim \frac{(2-\lambda)^2}{36} \to 0 \quad \text{as} \quad \lambda \to 2. \tag{13}$$

The geometry proves that as $\lambda \to 2$, all possible profiles that could cause difficulties with the diversity objective are being squeezed out of the domain. From this it follows that with IAC, IC, or other reasonable measures, the likelihood of difficulties approaches zero as $\lambda \to 2$. The relative size of the problematic profiles, as reflected by Eq. 12, provides a bound on a measure of this behavior.

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References

Barberà S, Sonnenschein H, Zhou L (1991) Voting by committees. Econometrica 59:595–609

Barberà S, Massó J, Neme A (2005) Voting by committees under constraints. J Econ Theory 122:185–205

Chamberlin JR, Courant PN (1983) Representative deliberations and representative decisions: proportional representation and the Borda rule. Am Polit Sci Rev 77:718–733

Dodgson CL, Abeles FF (2011) The pamphlets of Lewis Carroll. In: Carroll L, Abeles F (eds) The political pamphlets and letters of Charles Lutwidge Dodgson and related pieces: a mathematical approach. Lewis Carroll Society of North America, New York

Fréchette G, Maniquet F, Morelli M (2008) Incumbents interests and gender quotas. Am J Polit Sci 52:891–909

Gehrlein WV (2002) Condorcet's paradox and the likelihood of its occurrence: different perspectives on balanced preferences. Theory Decis 52:171–199

Ratliff TC (2006) Selecting committees. Public Choice 126:343–355

Ratliff TC (2001) A comparison of Dodgson's method and Kemeny's rule. Soc Choice Welf 18:79–89

Ratliff TC (2002) A comparison of Dodgson's method and the Borda count. Econ Theory 20:357–372

Ratliff TC (2003) Some startling inconsistencies when electing committees. Soc Choice Welf 2003:433–454

Saari DG (1995) Basic geometry of voting. Springer, New York

Saari DG (2006) Proposal for a change in class V voting rules. Econ Theory 28:265–281

Saari DG (2008) Disposing dictators, demystifying voting paradoxes. Cambridge University Press, Cambridge