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An axiomatization of the human development index

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Abstract In 2010 the UNDP unveiled a new methodology for the calculation of the *Human Development Index* (HDI). In this paper I investigate the normative and practical properties of this change vis a vis the original formulation of the HDI in 1990. The main conceptual innovation of the new index can be summarized as follows: the new HDI penalizes both low and uneven achievements across all dimensions of human development, whereas the old formulation is not sensitive to such uneven development. In practice, however, both methodologies agree considerably in terms of how they rank countries, but when they differ, the new methodology produces results more consistent with what the HDI is intended to measure: *human development and capabilities*, as conceptualized by Sen (Commodities and capabilities. Elsevier, Oxford 1985).

1 Introduction

In 2010, in the context of the 20th anniversary of the publication of the first Human Development Report (HDR), the United Nations Development Program unveiled a new methodology for computing the Human Development Index (HDI). This new methodological change has received a fair amount of attention in the news media and in academic and policy circles.

This attention is warranted, given the significance the HDI has as the flagship indicator of multidimensional development worldwide. As the New York Times wrote on May 10th, 2010: "So far only one measure has succeeded in challenging the hegemony of growth-centric thinking. This is known as the HDI, which turns 20 this year."

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Analysis of media coverage in the 2000's shows a significant increase in the use of the HDI to challenge—and to praise—government performance. In 2009 alone, after the 2009 HDR was launched, the HDR website was visited nearly 3 million times, and almost half a million copies of the report were downloaded.[1]

The HDI is a multidimensional index and, as such, exhibits different combinations of dimensional achievements that lead to the same level of 'human development' as computed by such index. The question then arises as to the kinds of tradeoffs that have been embedded in the new methodology. If the HDI is to be used properly in policy circles worldwide it is important that we all strive to comprehend the rationale behind those implied tradeoffs.

This paper aims at illustrating the rationale behind the methodology chosen by means of identifying all of the normative principles it satisfies. In other words, this paper provides a full axiomatization of the new HDI. Providing such normative characterization is essential for the success of the updated methodology, so that it becomes clear to all what the HDI intends to measure, what it does not intend to measure, what are the principles behind the proposed measurement. This is the only way to ensure that the end users of the HDI will ultimately trust and understand the insights it generates. Because a full axiomatization of the formula used before 2010 was not known, in this paper I also provide such axiomatization. This exercise can help us clarify the conceptual similarities and differences between the old and new formulations.

The structure of the rest of this paper is the following. Section 2 explains what the HDI intends to measure. In going about determining how to perform such measurement Sect. 3 lays out some fundamental principles one may want the HDI to satisfy. Section 4 shows how the new HDI formula follows from those principles, thus providing a normative justification for such measurement. Section 5 performs conceptual and practical comparisons between the new and the old methodologies, emphasizing the ways in which they are alike, and the way in which they differ. Section 6 concludes.

2 Foundations

2.1 What is the HDI? What does it intend to measure?

The HDI is an index that tracks the *capabilities* available to the individuals in a society.

2.2 What are capabilities?

Most simply, a person's capabilities are the list of things that person can do or be in his or her life. More broadly, the term *capabilities* refers to the opportunities a person has to exercise his or her "freedom to attain different kinds of alternative lives between which a person can choose."[2]

[1] UNDP (2010a), p. 14.

[2] Sen (2008), p. 23.

The capabilities approach to the measurement of development thus attempts to keep track of whether the capabilities of the members of a given society are expanding over time.

An extensive literature on capabilities exists, stemming from the seminal work by Sen (1985). I will not survey it here. It is, nevertheless, important to insist, before turning to how this measurement is conducted in practice, that the Human Development Index has always been an attempt to measure capabilities, since the index was launched in 1990 with Sen's close assistance.[3] From the 1990 HDR: The HDI is "an index that captures the three essential components of human life (. . .) longevity and knowledge refer to the formation of human capabilities, and income is a proxy measure for the choices people have in putting their capabilities to use."[4]

2.3 How can one measure these capabilities?

To understand the measurement problem it is important to introduce the basic building block in the capability approach: the idea of a *functioning*. In Sen's own words:

"The primitive notion in the approach is that of functioning—seen as constitutive elements of living. A functioning is an achievement of a person: what she manages to do or be (. . .) The capability of a person is a derived notion. It reflects the various combinations of functionings he or she can achieve."[5]

Thus to measure the capability of a person we need to think in terms of the possible lifestyles that are within reach of that person in light of the opportunities that are open to him or her to function well along a number of basic dimensions of living. "Just as the so-called 'budget-set' in the commodity space represents a person's freedom to buy commodity bundles, the 'capabilities set' in the functioning space reflects the person's freedom to choose from possible livings."[6]

Here it is important to bring forth an elementary fact: Even though the budget set is a multidimensional object, it's size, in a well-functioning market economy with stable (relative) prices, can be indexed by the person's real personal income, which is to say that a single number (real income) can tell us if the budget set is expanding or contracting over time.

Hence we can see an analogy: the HDI intends to be, to the set of capabilities available to the individuals in a country, what the GNP index intends to be to the collective budget set of the individuals in that country: an admittedly crude way of keeping track if those sets of capabilities are expanding or contracting over time.

To do this in practice we need to think of which human functionings we wish to keep track of in a country, and how we want to think about how those functionings transform into capabilities.[7] "Some functionings are very elementary, such as being

[3] For an account of the collaboration between Mahbud ul Haq and Sen that led to the publication of the first HDR in 1990 see UNDP (2010b) and Sen (2003a), pp. vii–xiii.

[4] UNDP (1990), p. 14. This is one of many references that can be given on the matter. See, e.g., the references in the reader compiled by Fukuda-Parr and Kumar (2003).

[5] Sen (2003b), p. 5.

[6] Sen (1995), p. 40.

[7] This is to some extent as in the approach of 'household production functions' developed by Becker (1976) and Lancaster (1966), but the capabilities approach goes well beyond that in the inclusion of functionings

adequately nourished, being in good health, and so on, and these may be strongly valued by all, for obvious reasons. Others may be more complex, but still widely valued, such as achieving self-respect, or being socially integrated."[8]

In choosing what dimensions of human functioning to track for the construction of a capabilities index the HDI must balance being broad in scope with the ability to gather data of sufficient quality for a large number of countries on a frequent (annual) basis. From the outset, the HDI keeps track of people's abilities (i) to live a long and healthy life, (ii) to have access to knowledge and (iii) to have command over the resources that would allow them to participate in community life and to make the choices that would permit them to live a full, meaningful life. From the 1993 HDR:

> *The three dimensions of the HDI relate to one or many capabilities that they are expected to capture. Thus, longevity captures the capability of leading a long and healthy life. Educational attainment captures the capability of acquiring knowledge, communicating and participating in the life of the community. Access to resources needed for a decent standard of living captures the capability of leading a healthy life, guaranteeing physical and social mobility, communicating and participating in the life of the community (including consumption).*[9]

Notice that the first two are elementary, intrinsically valuable, functionings: life, and education, for their own sake. Command over resources, however, has a different status in the capabilities approach, as "commodity ownership is rarely sought for itself, since commodities are means to other ends (...) For example, having food helps us to be nourished, to enjoy eating, to entertain friends, and so forth."[10] One is interested in keeping track of the set of functionings that become available to the individual thanks to commodity ownership, not necessarily in keeping track of commodity ownership for its own sake.

This is to be contrasted with the more standard, resource-based approach to social welfare measurement:

> *Whereas the human development approach values capabilities related to, say, health, nutrition and basic education as ends in themselves—and income only as a means to achieve these—human resource development (like 'human capital investment') is based on precisely the opposite valuation. This approach assesses investment in human capital—including health, nutrition and education—entirely in terms of the extra income or output the investment generates, judging it to be worthwhile if the rate of return exceeds the capital cost. By contrast, proponents of the human development approach would argue for the enhancement of people's ability to read and write, or to be well-nourished and healthy,*

footnote 7 continued
that cannot be easily seen as detached objects that the person or the household happen to 'own' or 'produce.' See Sen and Hawthorn (1989), p. 104.

8 Sen (2008), p. 24.

9 UNDP (1993), p. 105.

10 Sen (2008), p. 24.

even if the conventionally measured economic return to investment in literacy, or improved food intake and health care, were zero.[11]

This is not to say that the standard resource-based approach is useless or even wrong. Rather, to point out that it does differ in very important respects from the capabilities-based approach.

It is thus vitally important to understand how the (intrinsically valuable) capacities for being able to live a long life and being educated combine with (instrumentally valuable) income in creating the set of possible livings that we call the capabilities set of the individuals in a society. To this task we turn our attention now.

3 Transforming health, education and income into capabilities

Until 2009 the HDI was calculated as the arithmetic mean of suitably normalized values for life expectancy, educational attainment, and income but ever since the publication of the first HDR in 1990 many researchers have pondered whether this was the best way to calculate the HDI. In spite of the improvement that the HDI implied with respect to the mere comparison of per capita GNP values, the old HDI was subject to a number of well-grounded criticisms. Herrero et al. (2010b) performed a compilation of the most important criticisms and summarize them as follows:[12]

(a) *Incomplete dimensional coverage*. There are some relevant aspects of human development that are missing, such as social integration or sustainability.
(b) *Inadequate measurement of the included dimensions*. Even though this is partly a practical matter (availability of data), it is not clear that the variables used to approximate health, education and material wellbeing are [necessarily] the most sensible ones.
(c) *The lack of concern for distributive issues*. It is only natural to think that the level of human development should compute not only "the size of the cake," but also the way in which it is distributed.
(d) *The meaning of the comparability between achievements in the three variables involved*. This makes it difficult to interpret the HDI, (with it being) an average value (of disparate indicators).
(e) *The justification behind the additive structure of the index*. Aggregating the different components by the arithmetic mean has strong implications on their substitutability and makes the index dependent on (conceptually irrelevant details about) the normalization chosen for the different components.[13]

[11] Anand and Sen (1994a), p. 2.

[12] Herrero et al. (2010b), based on the contributions by Anand and Sen (1994a,b), Hicks (1997); Sagar and Najam (1998), Osberg and Sharpe (2002), Phillipson and Soares (2001), Pinilla and Goerlich (2004), Foster et al. (2005), Becker et al. (2005), Stiglitz et al. (2009) and Herrero et al. (2004).

[13] For example, in the old HDI an extra year of expected life would be deemed to contribute as much to the development of any country as ten extra months of expected schooling (instead of a full year), simply because the years of schooling across countries oscillate over a narrower range than life expectancy does. Normatively, however, it is not warranted that a dimensional achievement is more valuable for development simply because most countries have similar levels of attainment in that dimension.

(f) *A limited understanding of the normative properties of the formula.* This makes it difficult to analyze the suitability of this index vis a vis other alternatives.

In what follows I stay within the framework of considering three core dimensions on which there is agreement about their importance, and about how well measured they are in practice, therefore sidestepping points (a) and (b) above, but address points (d), (e) and (f) in detail.[14]

Apart from the references brought forth above, other attempts to propose conceptually sensible alternatives to the formulation adopted by the UNDP (1990) include Desai (1991), Moreno-Ternero and Roemer (2006), Seth (2009, 2010), Chakravarty (2003), to which I return in Section 7, and, of course, the important contributions of Herrero et al. (2010a,b), on which the work I present below is eminently based.

3.1 Setting

The starting point here is a collection of countries, each characterized by aggregate level of achievements, or *human functionings*, in health, h, education, e, and income, y. Let H, E, Y denote the corresponding spaces to which they belong. In what follows assume that H and E are closed intervals $[h^o, h^*]$ and $[e^o, e^*]$. Y, on the other hand, is best thought of as $[y^o, \infty)$, although in practice one also picks an upper bound for y, called y^* and thus $Y = [y^o, y^*]$.

The interpretation of the lower bounds (possibly zero) on the values of h, e and y is that these should be thought as normative values below which subsistence is not known to be possible. The interpretation of the upper bounds is that these are the highest level any society has been known to achieve in those dimensions.

Let $\Omega = H \times E \times Y$. A *capabilities index* is a continuous single-valued mapping $C{:}\Omega \rightarrow \mathbb{R}$ with the following interpretation: the capabilities set of the society with achievements given by (h, e, y) is at least as large as that of the society with achievements given by (h', e', y') if and only if $C(h,e,y) \geq C(h',e',y')$.

Associated with each capabilities index C there are three *partial capability measurement functions* $C_h{:}H \rightarrow [0,1]$, $C_e{:}\, E \rightarrow [0,1]$ and $C_y{:}Y \rightarrow [0,1]$, and a *capabilities aggregator* $I{:}\,[0,1]^3 \rightarrow \mathbb{R}$ such that, by composition

$$C(h, e, y) \equiv I(C_h(h), C_e(e), C_y(y)).$$

The functions C_h, C_e and C_y are intended to capture *health capabilities*, *education capabilities* and *income capabilitites*, that is, the manner in which 'health functionings,' 'education functionings' and 'income functionings,' each taken in isolation, contribute towards enhancing the capabilities set of the people in a society.

[14] Point (c), regarding the distribution of achievements of these variables across the population in these countries, has been addressed by the UNDP by the launching, in 2010, of a HDI (the Inequality Adjusted Human Development Index) that fully takes those distributional considerations into account. The methodology and rationale for the computation of such index is explained in detail in Alkire and Foster (2010). In a related contribution, Hartgen and Klasen (2012), compute Human Development Indices at the household level for a sample of 15 countries. These household indices can then be used for constructing country level Human Development Indices that are affected by the level of inequality in human development existing across households.

The following definitions will be useful in what follows. For h in H, e in E and y in Y define:

- $\Delta C_h(h, \Delta_h) := C_h(h + \Delta_h) - C_h(h)$, where Δ_h is *feasible*, that is, it is such that $h + \Delta_h$ is in H.
- $\Delta C_e(e, \Delta_e) := C_e(e + \Delta_e) - C_e(e)$, where Δ_e is *feasible*, that is, it is such that $e + \Delta_e$ is in E.
- $\Delta C_y(y, \Delta_y) := C_y(y + \Delta_y) - C_y(y)$, where $\Delta_y = y \cdot d_y$ and d_y is *feasible*, that is, it is such that $y(1 + d_y)$ is in Y.

These are the measures of *partial capabilities growth*, associated with feasible changes in the 'functionings' h, e or y.

Similarly, define measures of *aggregate capabilities growth*, for $(C_h, C_e, C_y) \in [0, 1]^3$ and feasible ΔC_h, ΔC_e and ΔC_y, as follows:

- $\Delta I_h(C_h, \Delta C_h, C_e, C_y) := I(C_h + \Delta C_h, C_e, C_y) - I(C_h, C_e, C_y)$,
- $\Delta I_e(C_h, C_e, \Delta C_e, C_y) := I(C_h, C_e + \Delta C_e, C_y) - I(C_h, C_e, C_y)$ and
- $\Delta I_y(C_h, C_e, C_y, \Delta C_y) := I(C_h, C_e, C_y + \Delta C_y) - I(C_h, C_e, C_y)$

3.2 The principles behind the measurement

We make the following assumptions about C:

Monotonicity: For each (h, e, y), (h', e', y') in Ω with $(h, e, y) >> (h', e', y')$ we have $C(h,e,y) > C(h', e', y')$.

The first assumption establishes that the capabilities index increases when all of the arguments increase simultaneously.

Independence: Let (h, e, y), (h', e', y') be in Ω with $h, h' > h^o; e, e' > e^o$ *and* $y, y' > y^o$. Then

- $C(h, e, y) \geq C(h, e', y')$ implies $C(h', e, y) \geq C(h', e', y')$,
- $C(h, e, y) \geq C(h', e, y')$ implies $C(h, e', y) \geq C(h', e', y')$ and
- $C(h, e, y) \geq C(h', e', y)$ implies $C(h, e, y') \geq C(h', e', y')$.

The second assumption states that if the capabilities set of society (h, e, y) is at least as large as that of society (h', e', y), then this relation holds for all common values of income, y. Similarly for health and for education, with respect to the corresponding other two variables.

Subsistence: For all h, h' in H, e, e' in E and y, y' in Y

- $C(h, e, y) \geq C(h', e', y^o)$,
- $C(h, e, y) \geq C(h', e^o, y')$ and
- $C(h, e, y) \geq C(h^o, e', y')$.

The third assumption states that there are no trade-offs between any of the dimensions of achievement when the members of society are at their worst levels in any of the dimensions.[15]

Partial capabilities growth: For all h, h' in H, e, e' in E and y, y' in Y, and feasible values for Δh, Δe and d_y

[15] This assumption is called 'Minimum lower boundedness' in Herrero et al. (2010a).

- $\Delta C_h\,(h, \Delta h) = \Delta C_h\,\left(h', \Delta h\right)$,
- $\Delta C_e\,(e, \Delta e) = \Delta C_e\,\left(e', \Delta e\right)$ and
- $\Delta C_y\,\left(y, y \cdot d_y\right) = \Delta C_y\,\left(y', y' \cdot d_y\right)$.

The fourth assumption establishes that different functionings contribute to capabilities enhancement differently. Starting from h, the amount of change in health achievements required to produce an increase in *health capabilities* of a certain size is independent of h (and similarly for education, regarding *education capabilities*, and starting from e). Income changes contribute to capabilities in a different way: starting from y, the amount of income change required to produce an increase in *income capabilities* of a certain size is proportional to y.

Scale: Let (h, e, y) be such that $C_h\,(h) = C_e\,(e) = C_y\,(y) = c$. Then $C\,(h, e, y) = c$.

The fifth assumption states that if the partial capabilities measures all take the same value, the capabilities index takes that value as well.

Aggregation symmetry: Let $\left(C_h, C_e, C_y\right)$ be a vector of partial health, education and income capabilities and let $\pi\,\left(C_h, C_e, C_y\right)$ be a permutation of $\left(C_h, C_e, C_y\right)$. Then $I\,\left(C_h, C_e, C_y\right) = I\,\left(\pi\,\left(C_h, C_e, C_y\right)\right)$.

The sixth assumption states that health, education and income *capabilities* contribute equally to the aggregate.[16]

4 The 2010 HDI

These assumptions are necessary and sufficient to pin down the functional form for the 2010 HDI.

Theorem 1 *The following definitions are equivalent:*

- The 2010 HDI is the capabilities index that satisfies *Monotonicity, Independence, Subsistence, Partial Capabilities Growth, Aggregation Symmetry and Scale*.
- The 2010 HDI is defined as $HDI\,(h, e, y) = I\,\left(C_h\,(h), C_e\,(e), C_y\,(y)\right)$ with

$$I\left(C_h, C_e, C_y\right) = C_h^{1/3} \cdot C_e^{1/3} \cdot C_y^{1/3}$$

and $C_h\,(h) = \frac{h - h^o}{h^* - h^o}$, $C_e\,(e) = \frac{e - e^o}{e^* - e^o}$, $C_y\,(y) = \frac{\log y - \log y^o}{\log y^* - \log y^o}$.

The proof is in the Appendix.

4.1 Remarks

4.1.1 Functionings and capabilities

Theorem 1 pins down both the functional form of I as well as those of the partial capability measures C_h, C_e and C_y. While each of the different functionings contributes in their own unique way towards the enhancement of capabilities, the contribution each

[16] This is not to say that the raw variables *themselves* contribute equally to aggregate capabilities, a point to which I return in Sect. 4.2 below.

has on the overall capabilities set is affected by the contributions of the other variables as well.

Viewing the HDI as a blend of partial measures of capabilities has a couple of advantages worth mentioning. First, it reiterates that the functions C_h, C_e and C_y are not meant simply to be thought as statistical normalizations of the raw data. Rather, these functions embody specific normative points of view about how the functionings associated with health, education and income achievements transform into capabilities. Second, it helps clarify how the study of inequality ought to be conducted within the capabilities approach to welfare economics: since we are interested in the values the different variables have to the extent that they contribute to capabilities enhancement, it is then inequality in capabilities, rather than inequality in the variables themselves, that which is primarily important to measure and study.[17]

4.1.2 The multiplicative structure

The multiplicative structure implied by the axioms identified is not hard to motivate. An increase in a person's functional ability to live a long life (as made possible, say, by an increase in the life expectancy of that person) clearly enhances the set of possible livings that the individual may adopt, but not by much if that individual has had very limited access to educational opportunities or material resources. Similarly, an increase in a person's educational functioning, or *capacity to know* (as made possible, say, by an increase in the level of educational attainment of that person) makes more lifestyle options open to the person the longer that person expects to live. That is, the effect of any enhancements to the capabilities set of this person in a specific dimension is affected by what is happening to the other dimensions as well.

A general multiplicative formulation for the HDI (albeit not with the specific functional form obtained in Theorem 1 above) had been suggested in the literature, most prominently in the work by Herrero et al. (2010a) and previously had been informally suggested in Desai (1991). A multiplicative formulation and normalizations such as those suggested by Theorem 1 above had been previously advocated in Sagar and Najam (1998), albeit without a theoretical justification for its adoption.

4.1.3 Tradeoffs

A multidimensional index such as the HDI exhibits different combinations of achievements in the different functionings that lead to the same level of 'human development' as computed by such index. These are the so-called tradeoffs between core dimensions embedded in a particular methodology. Whether the tradeoffs implied by the 2010 HDI are sensible deserves serious scrutiny, especially in light of the observations made by Ravallion's (2012) paper "Troubling Tradeoffs in the Human Development Index" about these tradeoffs becoming (more) implausible with the change in methodology. While there is an important element of truth in these assertions it is equally important to point out that, *in practice*, the logarithmic transformation done to income as part of

[17] This is, for example, the point of view implicit in the *Gender Inequality Index* launched by the UNDP in 2010. See Gaye et al. (2013).

the computation of the index contributes considerably more to explain the variations of these tradeoffs across rich and poor countries than the multiplicative structure of the 2010 HDI *per se*. I study these and other matters related to the tradeoffs embedded in the 2010 HDI in a companion paper (Zambrano 2013a), and offer a resolution to the problem posed by Ravallion regarding the general plausibility of the implied tradeoffs in multidimensional development indices in a follow-up paper (Zambrano 2013b).

4.1.4 Units and goalposts

One of the advantages the new HDI formulation has over the old one is that it is insensitive to the choice of units of the different dimensional variables.[18] This has implications for normalization, as what maximum theoretical values one specifies for each of the dimensions will have no effect on the tradeoffs or the implied ranking. Normalization of origins of those variables is, on the other hand, still important, and care must be taken in that this choice is made conscientiously, not arbitrarily, as this is a normative problem (not just a statistical one). To guide our choice of origins it is then best if we remind ourselves the interpretation we're conveying to such origins, and what role those origins plays in the determination of the new HDI.

The values h^o, e^o and y^o are intended to capture subsistence levels for these functionings, that is, levels below which (social) human sustenance is not really feasible. Progress in each of the dimensions is then measured against those subsistence levels. The HDRO has adopted minimum values for life expectancy, educational attainment and income given by 20 years, zero, and 163 PPP $ per year per capita, respectively. Those choices must be reasoned and in regard to these choices, the 2010 HDR states the following:

> *The life expectancy minimum is based on long-run historical evidence from* Maddison (2010) *and* Riley (2005). *Societies can subsist without formal education, justifying the education minimum. A basic level of income is necessary to ensure survival:* $163 *is the lowest value attained by any country in recorded history (in Zimbabwe in 2008) and corresponds to less than 45 cents a day, just over a third of the World's Bank* $1.25 *a day poverty line.*[19]

Regarding the specific values for the thresholds I believe that different individuals could have different points of view as to what those values ought to be. One should always remain open to exploring more fitting choices for those numbers, as well as to studying the different implications those choices may have.[20]

[18] This was not so in the old formulation, as explained at the beginning of Sect. 3 above.

[19] UNDP (2010a), p. 216.

[20] In this respect, it is important to notice that the magnitudes of the tradeoffs under study do not appear to be very sensitive in practice to small changes in the values of the subsistence levels for life expectancy and income.

4.1.5 Education

I have treated the education functioning in this paper as it being measured by a single variable but in practice the UNDP calculates the geometric mean of two separate education indicators (mean years of schooling and expected years of schooling). While the question remains open as to whether this or other form of averaging procedure is more appropriate, it makes very little difference in practice which method is used.

The treatment that income ultimately receives in the HDI deserves its own "Remarks" section.

4.2 The interpretation of log income

This section is devoted to further understanding the differential treatment that income receives in the HDI. To begin with, let's observe that the HDI is asymmetrical by design. Different assumptions were made about how health and education functionings affect the HDI than about how income does so. On the other hand the identified I is symmetrical by design. This is intended to capture the somewhat elementary notion that, once we've found a way to make the core components of the HDI comparable to one another via normalization, the normalized values affect human development equally.

The logarithmic transformation of income arises *as a consequence* of the assumptions identified. It is not assumed at the outset. This is unlike in previous HDI design efforts, where logs were singled out without further explanation as the way to capture the notion that "income transforms into capabilities at a decreasing rate," an idea that has been part of the human development paradigm from the start. The choice of logs based on this criterion makes sense, although there are many other transformations of income with the same property, each of which has the potential to generate a different ranking of countries.

In this paper I have used an assumption (*Partial Capabilities Growth*) that intends to capture how changes in the health, education and income functionings affect the growth in capabilities differently. I gave some intuition for what this assumption does when I introduced it earlier in the paper, and the reader may want to revisit it again before moving on. Below I provide some additional motivation for this assumption.

The main idea behind this assumption is that, because the functionings of being healthy and educated are *ends in themselves*, health and education capabilities grow with a change in health and education achievements in direct proportion to the *absolute* change in these achievements. Income, however, contributes to capabilities only *instrumentally*, that is, indirectly, and income capabilities therefore grow with a change in income in direct proportion to the *relative* change in income. Income levels, *per se*, do not contribute to such income capabilities growth.

The logarithmic transformation of income obtained in Theorem 1 above, $C_y = \frac{\log y - \log y^o}{\log y^* - \log y^o}$, may seem mysterious at first sight but since $\frac{\log y - \log y^o}{\log y^* - \log y^o} = \log_{\frac{y^*}{y^o}} \frac{y}{y^o}$ it is easy to provide a natural economic interpretation for C_y: It is computing how many "orders of magnitude" away is income, y, from its subsistence level, y^o, where the

orders of magnitude are computed in economically relevant units: base y^*/y^o, that is, in terms of the size of the income gap to be bridged by a country that is currently at its subsistence level of income.

A final remark is in order. It turns out that, conceptually, the logarithmic transformation interacts well with the multiplicative structure the new HDI in the following sense: it makes the HDI to be very conservative[21] in allowing income to be transformed into capabilities at high income levels (thus giving maximum chance in those cases for educational attainment and life expectancy to matter in the overall determination of the index) and very aggressive[22] in allowing capabilities to shrink as income losses take place at very low income levels (thus giving maximum chance for critically low income levels to matter greatly in the overall determination of the index as well).

In sum, as one looks at the issue of how incomes transform into capabilities from many different angles, it appears that the logarithmic transformation makes intuitive and practical, as well as theoretical, sense.[23]

5 The old and the new HDI, compared

In passing from an additive to a multiplicative formulation for the HDI one has to ponder about the theoretical and practical implications of this change. We have seen that the 2010 HDI can be characterized in terms of *Monotonicity, Independence, Subsistence, Partial Capabilities Growth* and *Scale*. The old, additive HDI, in turn, satisfies *Monotonicity, Independence, Capabilities Growth* and *Scale* but it does not satisfy *Subsistence*. This implies that the citizens in a country could presumably have a rich set of lifestyles available to them (a big capabilities set) even as their health, education or income functionings remain at critically low (subsistence) levels. This did not seem tenable to the designers of the 2010 HDI, and it was one of the reasons behind the move away from an additive formulation in their redesign.

What other principles does the old HDI satisfy? Consider the following principle.

Capabilities growth independence: For all $\left(C_h, C_e, C_y\right), \left(C'_h, C'_e, C'_y\right) \in [0,1]^3$ and feasible values for ΔC_h, ΔC_e and ΔC_y

- $\Delta I_h\left(C_h, \Delta C_h, C_e, C_y\right) = \Delta I_h\left(C'_h, \Delta C_h, C'_e, C'_y\right)$,
- $\Delta I_e\left(C_h, C_e, \Delta C_e, C_y\right) = \Delta I_e\left(C'_h, C'_e, \Delta C_e, C'_y\right)$ and
- $\Delta I_y\left(C_h, C_e, C_y, \Delta C_y\right) = \Delta I_y\left(C'_h, C'_e, C'_y, \Delta C_y\right)$

This assumption establishes that, starting from $\left(C_h, C_e, C_y\right)$, the amount of change in health capabilities required to produce an increase in aggregate capabilities of a certain size is *independent* of $\left(C_h, C_e, C_y\right)$. Similarly for education, and for income capabilities.

[21] Conservative in the sense that the HDI grows more slowly than any polynomial function of income as income grows.

[22] Aggressive in the sense that the HDI shrinks more rapidly than any polynomial function of income as income declines to the normatively determined subsistence level.

[23] Which is not to say there aren't other sensible options, a point to which I return in Sect. 6.

The exact relation between the old and the new HDI turns to be the following: take the axioms that characterize the 2010 HDI, remove *Subsistence* and replace *Independence* with *Capabilities Growth Independence*.[24] One obtains the old, additive HDI as a result. This is a consequence of Theorem 2 below:

Theorem 2 *The following definitions are equivalent:*

- The old (additive) HDI is the capabilities index that satisfies *Monotonicity, Capabilities Growth Independence, Partial Capabilities Growth, Aggregation Symmetry and Scale*.
- The old (additive) HDI is defined as $HDI_a(h, e, y) = I_a\left(C_h(h), C_e(e), C_y(y)\right)$ with

$$I_a\left(C_h, C_e, C_y\right) = \frac{C_h + C_e + C_y}{3}$$

and $C_h(h) = \frac{h-h^o}{h^*-h^o}$, $C_e(e) = \frac{e-e^o}{e^*-e^o}$, $C_y(y) = \frac{\log y - \log y^o}{\log y^* - \log y^o}$.

The proof is in the Appendix.

5.1 The differences, in practice

These conceptual differences notwithstanding, there is considerable convergence between the country rankings produced by the old and the new HDI's, as Fig. 1 below reveals.

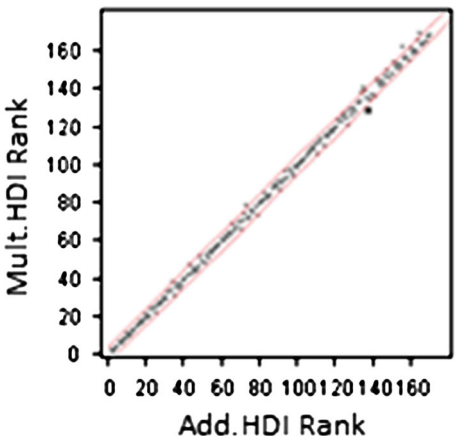

Fig. 1 Rank scatterplots. Source: calculations by the author, based on official UNDP statistics

[24] A much stronger form of independence.

Table 1 The rankings of selected countries according to assorted criteria (2010)

	Multiplicative HDI	Additive HDI	Life exp. rank	Education rank	Income rank
Belize	78	72	41	74	94
Cameroon	131	137	152	119	128
Comoros	140	134	116	138	152
Liberia	162	155	133	125	167

Source: UNDP

This dramatic degree of convergence[25] deserves to be noted for two reasons: First, it was far from obvious that the new formulation would interact with the data in this way. Second, the 2010 HDR makes scant mention of it.[26]

Despite these empirical similarities, the cases for which these two rankings differ substantially illustrate that the new formulation produces more intuitive results. Table 1 presents countries that differ by more than five positions in the two different rankings: *there are only four*.

Consider the case of Liberia, which is near the bottom of the distribution of world income, with a per capita income of 320 PPP dollars per year. With such dramatically low income levels the multiplicative formulation will keep Liberia at the bottom of the 2010 HDI rankings even though it fares comparatively better in the health and education dimensions. The additive HDI is less sensitive to the effects of such very low incomes and thus ranks Liberia seven places higher than the multiplicative version does.

Belize and Comoros also rank better according to their additive HDI vs. their multiplicative HDI in spite of the large disparities that exist between their life expectancy and income ranks. These disparities are 'averaged out' in the additive formulation, whereas the multiplicative formulation penalizes this unevenness and gives both countries a lower rank as a result. Cameroon, on the other hand, ranks *better* according to the multiplicative HDI than in the additive formulation. It is also the country in this sample with the least variance of achievements across dimensions. In short, the multiplicative HDI penalizes both low AND uneven achievements across all dimensions, a feature that is lacking in the additive formulation.

6 Conclusions

In this paper I have documented the normative and practical reasons behind the introduction in 2010 of a new methodology, on the part of the United Nations Development Program, for the computation of the United Nation's HDI. I reviewed the shortcomings of the old methodology, established principles one would want a capabilities index to

[25] The Spearman and Kendall rank correlation coefficients between the multiplicative and the additive HDI rankings equal 0.9991 and 0.9784, respectively.

[26] The only mention of this convergence in the entire report is the following sentence: "The geometric mean has only a moderate impact on HDI ranks," which shows up in page 217 of the 2010 HDR, in the statistical annex.

satisfy and identified the unique formula consistent with those principles: the formula that was ultimately adopted for the new HDI by the UNDP.

Some people may find one or more of the principles that characterize the 2010 HDI objectionable, thus opening the door for different formulations. I judiciously motivated these assumptions behind the 2010 HDI, and believe these are sensible principles, but certainly other options are possible.[27] One alternative is to stick with the additive formulation, despite its conceptual shortcomings. In this paper I take a look at the ranking implied by the additive formulation, and compare it to the 2010 HDI rankings. In a companion paper (Zambrano 2013a) I also take a look at a family of alternative indices, proposed by Chakravarty (2003), and explore their implied rankings alongside the 2010 HDI rankings. In both cases I investigate the ways in which these rankings are all alike, and the ways in which they differ as well. Each reader can form an opinion about which appears more reasonable.

From the judicious examination of all these alternatives I believe that there is scope to develop, in the future, a formula for the HDI that may be even better than the one adopted, and I am nevertheless confident in that the version of the multiplicative HDI chosen by the UNDP in 2010 is an improvement over the old, additive, HDI, which served its purpose well over its 20 years of active duty. It was also better than the many alternatives that, to my knowledge, made it to the drawing board.

If one were to look towards developing the 'next generation HDI,' in what direction would one look? One could keep *Monotonicity*, *Independence*, *Subsistence, Aggregation Symmetry* and *Scale*, and look for assumptions about how income transforms into capabilities that would be more general than *Partial Capabilities Growth*, but that would contain it as a special case. This approach, further developed in Zambrano (2013b), leads to a parametrized family of multiplicative indices that treat income asymmetrically, as in Sect. 2, from which one would pick one based on reasoned public debate about the plausibility of the different tradeoffs between the core dimensions implied by the different members of that family. These debates can be spirited, and they should, lest we forget these are normative, not technical, choices we're making, which carry the potential of causing great good, or harm, in the world.

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Appendix

Proof of Theorem 1 ($\leftarrow$) Let $HDI(h, e, y) = I\left(C_h(h), C_e(e), C_y(y)\right)$ with $I\left(C_h, C_e, C_y\right) = C_h^{1/3} \cdot C_e^{1/3} \cdot C_y^{1/3}$ and $C_h(h) = \frac{h-h^o}{h^*-h^o}$, $C_e(e) = \frac{e-e^o}{e^*-e^o}$, $C_y(y) = \frac{\log y - \log y^o}{\log y^* - \log y^o}$.

[27] Regardless, these were the principles of design implicit in the formulation adopted in 2010 and it is important for us all to understand what those principles say, if the HDI is to be used properly in policy circles worldwide.

Functions $C_h(h)$, $C_e(e)$ and $C_y(y)$ are all strictly increasing, and so is $I\left(C_h, C_e, C_y\right)$ in all its arguments. Therefore, *Monotonicity* holds. Now let *(h,e,y), (h',e',y')* be in Ω with $h, h' > h^o; e, e' > e^o$ *and* $y, y' > y^o$ and assume $C(h, e, y) \geq C(h, e', y')$. If we multiply each side by $\left[C_h\left(h'\right)/C_h(h)\right]^{\frac{1}{3}}$ we get $C(h', e, y) \geq C(h', e', y')$. Now assume $C(h, e, y) \geq C\left(h', e, y'\right)$. If we multiply each side by $\left[C_e\left(e'\right)/C_e(e)\right]^{\frac{1}{3}}$ we get $C\left(h, e', y\right) \geq C\left(h', e', y'\right)$. Now assume $C(h, e, y) \geq C(h', e', y)$. If we multiply each side by $\left[C_y\left(y'\right)/C_y(y)\right]^{\frac{1}{3}}$ we get $C(h, e, y') \geq C(h', e', y')$. Therefore *Independence* holds. Since $C_h(h^o) = C_e(e^o) = C_y(y^o) = 0$ and $I\left(C_h, C_e, C_y\right)$ is multiplicative, clearly *Subsistence* holds. *Scale* holds since $I(c, c, c) = c$. Now fix (h, e, y) and consider feasible values for $\Delta h, \Delta e$ and d_y. Then $\Delta C_h(h, \Delta h) = \frac{\Delta h}{h^* - h^o}$, $\Delta C_e(e, \Delta e) = \frac{\Delta e}{e^* - e^o}$ and $\Delta C_y\left(y, y \cdot d_y\right) = \frac{\log(1+d_y)}{\log y^* - \log y^o}$. Therefore *Partial Capabilities Growth* holds. *Aggregation Symmetry* is straightforward to verify.

($\rightarrow$) Since C satisfies *Monotonicity*, *Subsistence* and *Independence* it is a consequence of Theorem 1 in Herrero et al. (2010a) that

$$C(h, e, y) = f(h) \cdot g(e) \cdot m(y),$$

where $f(h) : H \rightarrow [0, 1]$, $g(e) : E \rightarrow [0, 1]$ and $m(y) : Y \rightarrow [0, 1]$ are increasing functions such that $f(h^o) = g(e^o) = m(y^o) = 0$ and $f(h^*) = g(e^*) = m(y^*) = 1$. By *Scale* and *Aggregation Symmetry*, $I\left(C_h, C_e, C_y\right) = C_h^{1/3} \cdot C_e^{1/3} \cdot C_y^{1/3}$ with $C_h(h) = f(h)^3$, $C_e(e) = g(h)^3$, $C_y(y) = m(y)^3$. *Partial Capabilities Growth* implies that (i) $C_h(h)$ is of the form $C_h(h) = \alpha_h + \beta_h h$ for some values for α_h and β_h, and $C_h(h^o) = 0$, $C_h(h^*) = 1$ implies that $C_h(h) = \frac{h - h^o}{h^* - h^o}$. (ii) $C_e(e)$ is of the form $C_e(e) = \alpha_e + \beta_e e$ and $C_e(e^o) = 0$, $C_e(e^*) = 1$ implies that $C_e(e) = \frac{e - e^o}{e^* - e^o}$. (iii) $C_y(y) = \frac{\log y - \log y^o}{\log y^* - \log y^o}$. To see this last step let $s(y) = C_y(y)$ and notice that *Partial Capabilities Growth* implies that

$$s\left(y\left(1 + d_y\right)\right) = s(y) + s\left(\left(1 + d_y\right) y^o\right).$$

Now let $a = d_y \cdot y$. Then $s\left(\left(1 + \frac{a}{y}\right) y\right) = s(y) + s\left(\left(1 + \frac{a}{y}\right) y^o\right)$. The steps below are based on Lady (2005):

Step 1: Show $s'(y) = \frac{K}{y}$ for some constant K. To see that this is so notice that

$$\begin{aligned} s'(y) &= \lim_{a \to 0} \frac{s(y + a) - s(y)}{a} = \lim_{a \to 0} \frac{s\left(y\left(1 + \frac{a}{y}\right)\right) - s(y)}{a} \\ &= \lim_{a \to 0} \frac{s(y) + s\left(\left(1 + \frac{a}{y}\right) y^o\right) - s(y)}{a} \\ &= \lim_{a \to 0} \frac{s\left(\left(1 + \frac{a}{y}\right) y^o\right) - s(y^o)}{a} = \lim_{a \to 0} \frac{s\left(y^o + \frac{a}{y} y^o\right) - s(y^o)}{a} \end{aligned}$$

$$= \lim_{a\to 0} \frac{s\left(y^o + \frac{y^o}{y}a\right) - s\left(y^o\right)}{\frac{y}{y^o}\frac{y^o}{y}a}$$

$$= \frac{y^o}{\mathrm{y}} \lim_{a\to 0} \frac{s\left(y^o + \frac{y^o}{y}a\right) - s\left(y^o\right)}{\frac{y^o}{y}a} = \frac{y^o}{\mathrm{y}} \lim_{\frac{y^o}{y}a\to 0} \frac{s\left(y^o + \frac{y^o}{y}a\right) - s\left(y^o\right)}{\frac{y^o}{y}a}$$

$$= \frac{y^o}{\mathrm{y}} s'\left(y^o\right)$$

with the desired constant K given by $y^o s'\left(y^o\right)$. Notice that $K \neq 0$ because otherwise $s'(y) = 0$ and s would be a constant function, which cannot be since $s\left(y^o\right) = 0$ and $s\left(y^*\right) = 1$. Indeed, the same argument shows that $K > 0$ and that therefore s is a strictly increasing function.

Step 2: Show that $s\left(y^r \cdot y^o\right) = rs\left(y \cdot y^o\right)$. To see this notice that, by the Chain Rule,

$\frac{d}{dy} s\left(y^r \cdot y^o\right) = \frac{K}{y^r \cdot y^o} y^o r y^{r-1} = r\frac{K}{y} = rs'(y)$, and $s'\left(y \cdot y^o\right) = \frac{K}{y \cdot y^o} y^o = s'(y)$.

The implication is that, since $rs\left(y \cdot y^o\right)$ and $s\left(y^r \cdot y^o\right)$ have the same derivative, they differ by a constant, that is, $rs\left(y \cdot y^o\right) = s\left(y^r \cdot y^o\right) + Q$, for some number Q. Letting $y = 1$ shows $Q = 0$, since $s\left(y^o\right) = 0$.

Step 3: Show that $\ell = s(y)$ if and only if $\left(\frac{y^*}{y^o}\right)^\ell = \frac{y}{y^o}$, that is, ℓ is the logarithm of $\frac{y}{y^o}$ with respect to the base $\frac{y^*}{y^o}$.

To see this notice that, by the conclusion from Step 2 above, $\left(\frac{y^*}{y^o}\right)^\ell = \frac{y}{y^o}$ implies $s(y) = s\left(\left(\frac{y^*}{y^o}\right)^\ell y^o\right) = \ell \cdot s\left(\frac{y^*}{y^o} \cdot y^o\right) = \ell \cdot s\left(y^*\right) = \ell$.

On the other hand, $\ell = s(y)$ implies, as shown above, that $s(y) = s\left(\left(\frac{y^*}{y^o}\right)^\ell y^o\right)$ and since s is strictly increasing it follows that $y = \left(\frac{y^*}{y^o}\right)^\ell y^o$.

I have thus shown that $C_y(y) = s(y) = \log_{\frac{y^*}{y^o}} \frac{y}{y^o} = \frac{\log y - \log y^o}{\log y^* - \log y^o}$ and the proof is complete.

To separate the properties consider the following indices:

(1) The HDI_a. Satisfies *Monotonicity*, *Independence*, *Scale*, *Aggregation Symmetry* and *Partial Capabilities Growth* but not *Subsistence*.

(2) $C(h, e, y) = 0$. Satisfies *Subsistence*, *Independence*, *Scale, Aggregation Symmetry* and *Partial Capabilities Growth* but not *Monotonicity*.

(3) $C(h, e, y) = \frac{h-h^o}{h^*-h^o} \cdot \frac{e-e^o}{e^*-e^o} \cdot \frac{\log y - \log y^o}{\log y^* - \log y^o}$. Satisfies *Subsistence*, *Independence*, *Monotonicity*, *Partial Capabilities Growth* and *Aggregation Symmetry* but not *Scale*.

(4) $C(h, e, y) = \left(\frac{\log h - \log h^o}{\log h^* - \log h^o}\right)^{1/3} \cdot \left(\frac{\log e - \log e^o}{\log e^* - \log e^o}\right)^{1/3} \cdot \left(\frac{y - y^o}{y^* - y^o}\right)^{1/3}$. Satisfies *Subsistence*, *Independence*, *Monotonicity*, *Aggregation Symmetry*, *Scale*, but not *Partial Capabilities Growth*.

(5) $C(h,e,y) = min\left\{\frac{h-h^o}{h^*-h^o}, \frac{e-e^o}{e^*-e^o}, \frac{\log y-\log y^o}{\log y^*-\log y^o}\right\}$ Satisfies *Subsistence*, *Partial Capabilities Growth*, *Monotonicity*, *Aggregation Symmetry*, *Scale*, but not *Independence.*

(6) $C(h,e,y) = \left(\frac{h-h^o}{h^*-h^o}\right)^a \cdot \left(\frac{e-e^o}{e^*-e^o}\right)^b \cdot \left(\frac{\log y-\log y^o}{\log y^*-\log y^o}\right)^c$ with $a+b+c=1$ and $a \neq b \neq c$. Satisfies *Subsistence*, *Independence*, *Monotonicity*, *Partial Capabilities Growth, Scale* but not *Aggregation Symmetry*.

□

Proof of Theorem 2 ($\leftarrow$) Let $HDI_a(h,e,y) = I_a\left(C_h(h), C_e(e), C_y(y)\right)$ with $I_a\left(C_h, C_e, C_y\right) = \frac{C_h+C_e+C_y}{3}$ and $C_h(h) = \frac{h-h^o}{h^*-h^o}, C_e(e) = \frac{e-e^o}{e^*-e^o}, C_y(y) = \frac{\log y-\log y^o}{\log y^*-\log y^o}$.

Functions $C_h(h), C_e(e)$ and $C_y(y)$ are all strictly increasing, and so is $I_a\left(C_h, C_e, C_y\right)$ in all its arguments. Therefore, *Monotonicity* holds. Now let *(h,e,y), (h',e',y')* be in Ω with $h, h' > h^o$; $e, e' > e^o$and $y, y' > y^o$ and assume $C(h,e,y) \geq C(h,e',y')$. If we add $\frac{1}{3}\left(C_h\left(h'\right) - C_h(h)\right)$ to each side we get $C(h',e,y) \geq C(h',e',y')$. *Scale* holds since $I_a(c,c,c) = c$. Now fix $(h,e,y)$and consider feasible values for $\Delta h, \Delta e$ and d_y. Then $\Delta C_h(h,\Delta h,e,y) = \frac{1}{3}\frac{\Delta h}{h^*-h^o}$, $\Delta C_e(h,e,\Delta e,y) = \frac{1}{3}\frac{\Delta e}{e^*-e^o}$ and $\Delta C_y\left(h,e,y,y\cdot d_y\right) = \frac{1}{3}\frac{\log(1+d_y)}{\log y^*-\log y^o}$. Thus *Partial Capabilities Growth* holds. Now fix (C_h, C_e, C_y) and consider feasible values for ΔC_h ΔC_e and ΔC_y. Then $\Delta I_h\left(C_h, \Delta C_h, C_e, C_y\right) = \frac{1}{3}\Delta C_h$, $\Delta I_e\left(C_h, C_e, \Delta C_e, C_y\right) = \frac{1}{3}\Delta C_e$ and $\Delta I_y\left(C_h, C_e, C_y, \Delta C_y\right) = \frac{1}{3}\Delta C_y$. Therefore, *Capabilities Growth Independence* holds. *Aggregation Symmetry* is straightforward to verify.

($\rightarrow$) Since C satisfies *Monotonicity* and *Capabilities Growth Independence*, for any pair $\left(C_e, C_y\right) \in [0,1]^2$ there exists $\beta_h \in \mathbb{R}_{++}$ and $\alpha_h, \gamma_h \in \mathbb{R}$ such that: $I\left(C_h, C_e, C_y\right) = \alpha_h + \beta_h \cdot C_h + \gamma_h \cdot I\left(1, C_e, C_y\right)$. Now, again by *Monotonicity* and *Capabilities Growth Independence*, for any $C_y \in [0,1]$ and given $C_h = 1$ there exists $\beta_e \in \mathbb{R}_{++}$ and $\alpha_e, \gamma_e \in \mathbb{R}$ such that $I\left(1, C_e, C_y\right) = \alpha_e + \beta_e \cdot C_e + \gamma_e \cdot I\left(1,1,C_y\right)$. *Monotonicity* and *Capabilities Growth Independence*, again, imply that given $C_h = C_h = 1$ there exists $\beta_y \in \mathbb{R}_{++}$ and $\alpha_y, \gamma_y \in \mathbb{R}$ such that $I\left(1,1,C_y\right) = \alpha_y + \beta_y \cdot C_y + \gamma_y \cdot I(1,1,1)$. Nesting these implications we get

$$I\left(C_h, C_e, C_y\right) = \left(\alpha_h + \gamma_h \cdot \alpha_e + \gamma_h \cdot \gamma_e \cdot \left(\alpha_h + \gamma_y\right)\right) + \left(\beta_h \cdot C_h + \gamma_h \cdot \beta_e \cdot C_e + \gamma_h \cdot \gamma_e \cdot \beta_y \cdot C_y\right).$$

Letting $C_h = C_e = C_y = 0$ we learn that $\left(\alpha_h + \gamma_h \cdot \alpha_e + \gamma_h \cdot \gamma_e \cdot \left(\alpha_h + \gamma_y\right)\right) = 0$. By *Aggregation Symmetry*, $\beta_h = \gamma_h \cdot \beta_e = \gamma_h \cdot \gamma_e \cdot \beta_y$and *Scale* implies that $\beta_h + \gamma_h \cdot \beta_e + \gamma_h \cdot \gamma_e \cdot \beta_y = 1$. The consequence is that $I\left(C_h, C_e, C_y\right) = \frac{C_h+C_e+C_y}{3}$. From this point on, the proof that the partial capabilities indices $C_h(h), C_e(e)$ and $C_y(y)$ are of the desired form follows the same steps as in the proof of Theorem 1 above. I omit the details here.

To separate the properties consider the following indices:

(1) The new HDI. Satisfies *Monotonicity*, *Aggregation Symmetry*, *Scale* and *Partial Capabilities Growth* but not *Capabilities Growth Independence*.

(2) $C(h, e, y) = 0$. Satisfies *Capabilities Growth Independence*, *Aggregation Symmetry*, *Scale* and *Partial Capabilities Growth* but not *Monotonicity*.

(3) $C(h, e, y) = \frac{h-h^o}{h^*-h^o} + \frac{e-e^o}{e^*-e^o} + \frac{\log y - \log y^o}{\log y^* - \log y^o}$. Satisfies *Capabilities Growth Independence*, *Monotonicity*, *Partial Capabilities Growth* and *Aggregation Symmetry* but not *Scale*.

(4) $C(h, e, y) = \frac{1}{3}\frac{\log h - \log h^o}{\log h^* - \log h^o} + \frac{1}{3}\frac{\log e - \log e^o}{\log e^* - \log e^o} + \frac{1}{3}\frac{y-y^o}{y^*-y^o}$. Satisfies *Capabilities Growth Independence*, *Monotonicity*, *Scale* and *Aggregation Symmetry* but not *Partial Capabilities Growth*.

(5) $C(h, e, y) = a\frac{h-h^o}{h^*-h^o} + b\frac{e-e^o}{e^*-e^o} + c\frac{\log y - \log y^o}{\log y^* - \log y^o}$ with $a+b+c = 1$ and $a \neq b \neq c$. Satisfies *Capabilities Growth Independence*, *Monotonicity*, *Partial Capabilities Growth* and *Scale* but not *Aggregation Symmetry*.

□

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