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Probabilistic assignment of indivisible goods with single-peaked preferences

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Abstract We consider the problem of assigning indivisible goods among a group of agents with lotteries when the preference profile is single-peaked. Unfortunately, even on this restricted domain of preferences, *equal treatment of equals*, *stochastic dominance efficiency*, and *stochastic dominance strategy-proofness* are incompatible.

1 Introduction

We consider the problem of allocating indivisible goods, or "objects," among a group of agents when each agent is supposed to receive exactly one of them. Each agent has a complete, transitive, and strict binary relation, or "preference relation," over objects. Examples of such situations are abundant: assigning offices in a university to its faculty, allocating time slots for access to a facility to potential users, placing students in public schools, and assigning organs to patients.

Not surprisingly, the indivisibility of objects causes serious difficulties in achieving fairness. For instance, suppose there are two desirable objects to be allocated to two agents, and that they prefer the same object. It is clear that each of the two possible allocations will violate any reasonable notion of fairness. One may want to use monetary transfers to restore fairness. However, as in the examples mentioned above, monetary transfers are often not possible or desirable. They may even be illegal (think of organs). In this paper, in an attempt to restore fairness, we resort to lotteries over allocations. Thus, we aim to achieve fairness in a probabilistic sense.

The analysis of this problem is initiated by Hylland and Zeckhauser (1979). A "probabilistic allocation," or simply an "allocation," is a matrix with columns indexed by agents and rows indexed by objects, each entry representing the probability with

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which the corresponding agent receives the corresponding object. We refer to what each agent receives, i.e., her column, as her "assignment." For each problem, a rule recommends an allocation.

We require rules to satisfy some efficiency and fairness requirements. An allocation is *stochastic dominance efficient* (henceforth *sd-efficient*) (Bogomolnaia and Moulin 2001) if it is not (first order) stochastically dominated for all agents by any other allocation. (This property is commonly referred to as "ordinal efficiency."[1]) Next is a minimal requirement of fairness. A rule satisfies *equal treatment of equals* if agents with the same preferences receive the same assignment.

Since preferences are generally unknown, we require of a rule that it should give agents the incentive to announce their true preferences. A rule is *stochastic dominance strategy-proof* (henceforth *sd-strategy-proof*) (Bogomolnaia and Moulin 2001) if for each agent and each false announcement that she could make about her preference, her assignment when she tells the truth either stochastically dominates (according to her true preference) her assignment when she lies, or the two assignments are the same.[2]

Two rules, the "random priority rule" (Abdulkadiroğlu and Sönmez 1998) and the "serial rule" (Bogomolnaia and Moulin 2001), are central in the literature.[3] The random priority rule is described as follows: (i) for each order on the set of agents, let each agent choose her best object among the remaining ones when her turn comes; and (ii) assume all orders to be equally likely and take the average of the allocations associated with all orders. For the serial rule, each object is considered as an infinitely divisible good whose supply is 1. Agents "consume" objects progressively and at equal "speeds." Each agent starts with her most preferred object, and when the supply of that object is exhausted, moves on to her next most preferred object among those that are still available, until the supply of that object is exhausted, at which point she turns to her most preferred object among those that are still available, and so on. (The precise description of this rule is in Sect. 2.)

The random priority rule behaves well from the strategic view point (it is *sd-strategy-proof*), but it is not *sd-efficient*. On the other hand, the serial rule behaves well from the view point of efficiency (it is *sd-efficient*) but it is not *sd-strategy-proof*. Both rules satisfy *equal treatment of equals*, however.[4]

There are good reasons why the above two rules do not simultaneously satisfy *equal treatment of equals*, *sd-efficiency*, and *sd-strategy-proofness*. Indeed, for more than three agents, no rule satisfies all three together (Bogomolnaia and Moulin 2001).

Axioms have less force when the domain of problems under consideration is restricted. Crès and Moulin (2001) and Bogomolnaia and Moulin (2002) consider the case when the rankings over the objects are the same for all agents, but the

[1] The other axioms involving comparisons by means of stochastic dominance are also prefixed by the abbreviation "sd" (this terminology is suggested by Thomson 2008).

[2] One can also consider a weak form of *sd-strategy-proofness*: for each agent and each false announcement that she could make, her assignment when she lies does not stochastically dominate (according to her true preference) her assignment when she tells the truth. We discuss this property in Sect. 4.

[3] They are referred to as the "random serial dictatorship" and the "probabilistic serial rule" in Abdulkadiroğlu and Sönmez (1998) and Bogomolnaia and Moulin (2001), respectively.

[4] For the proofs of these statements, see Bogomolnaia and Moulin (2001).

ranking of the "null object" (interpreted as receiving no object) may differ from agent to agent.[5] An example is when the objects are time slots in which to obtain a service, each agent prefers being served earlier but faces a deadline beyond which the service is useless. In that case, the axioms are compatible; however, the serial rule is the only rule to satisfy them (Bogomolnaia and Moulin 2002).

Returning to the original unrestricted domain, the above result raises the question of how much one has to restrict the domain so that the axioms become compatible. Obviously, the domain just described is very narrow. We then consider a slightly more general domain: when each agent has single-peaked preferences, i.e., there is an order on the object set such that each agent has a single most preferred object and becomes monotonically worse off when moving away from her most preferred object. For instance, returning to the example when the objects are time slots in which to receive a service, one can think of a situation where each agent has a most preferred time to be served and becomes progressively worse off as she is served further and further away from her most preferred time. We show that, unfortunately, even on this restricted domain, *equal treatment of equals*, *sd-efficiency*, and *sd-strategy-proofness* are incompatible.[6]

Our paper belongs to the growing literature on "probabilistic assignment" (e.g., Hylland and Zeckhauser 1979; Bogomolnaia and Moulin 2001; Katta and Sethuraman 2006; Kojima 2009; Che and Kojima 2010; Kojima and Manea 2010; Heo 2011).[7] Several variations of the problem have been considered. Of particular interest is when objects are identical, each agent may receive several copies of the object, and each agent's preference is single-peaked over the number of copies he consumes (Sasaki 1997; Ehlers and Klaus 2003; Kureishi and Mizukami 2007; Hatsumi and Serizawa 2009). The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 presents the result. Section 4 discusses the result.

2 The model

Let $\boldsymbol{N} \equiv \{1, 2, \ldots, n\}$ be a set of **agents**. A typical agent is denoted by $i \in N$. Let $\boldsymbol{O} \equiv \{o_1, o_2, \ldots, o_m\}$ be a finite set of distinct indivisible goods, or **objects**. A typical object is denoted by $k \in O$. Each agent is supposed to receive exactly one object. We assume that $|O| = |N|$.[8] Each agent $i \in N$ has a complete, transitive, and strict binary relation $\boldsymbol{R_i}$ over objects. We refer to R_i as agent i's **preference relation**. Let $\boldsymbol{\mathcal{R}}$ be a domain of preferences. Let $R \equiv (R_i)_{i \in N}$ be a preference profile. Let $\boldsymbol{\mathcal{R}^N}$ be a domain of preference profiles. Since we vary neither N nor O, we simply write a **problem** as a list $R \in \mathcal{R}^N$.

[5] This domain cannot be seen as a restriction of the domain considered by Bogomolnaia and Moulin (2001) because the null object is included in the model.

[6] Our model in Sect. 2 does not include a null object. However, our result holds when there are copies of a null object. This fact is discussed in Sect. 4.

[7] These problems are commonly referred to as "random assignment problems."

[8] The case $|O| \neq |N|$ is discussed in Sect. 4.

We consider the following restriction on preferences. Let Φ^O be the set of strict orders on O. Let $\prec \in \Phi^O$. Without loss of generality, let O be ordered in such a way that

$$o_1 \prec o_2 \cdots \prec o_m.$$

For each $i \in N$, $R_i \in \mathcal{R}$ is **single-peaked** on O (with respect to $\prec$) if and only if one of the following three conditions holds:

(i) there is $t \in \{2, \ldots, m-1\}$ such that

$$o_t \; R_i \; o_{t-1} \; R_i \; o_{t-2} \; \ldots \; R_i \; o_1 \text{ and } o_t \; R_i \; o_{t+1} \; R_i \; o_{t+2} \; \ldots \; R_i \; o_m,$$

(ii) $o_m \; R_i \; o_{m-1} \; R_i \; o_{m-2} \; \ldots \; R_i \; o_1$,
(iii) $o_1 \; R_i \; o_2 \; R_i \; o_3 \; \ldots \; R_i \; o_m$.

A preference profile $R \in \mathcal{R}^N$ is single-peaked on O if and only if for each $i \in N$, R_i is single-peaked on O. Let $\boldsymbol{\mathcal{R}}^N_{sp}$ be a domain of single-peaked preference profiles.

Working with a restricted class of preferences should increase our chance of obtaining positive results, and one could hope that the strong restriction of single-peakedness would help, as it has in a variety of other contexts. Unfortunately, in our case, it does not (Theorem 1).

We represent a "deterministic" allocation as a matrix of 0's and 1's, with columns indexed by agents and rows indexed by objects: a 0 entry in a cell means that the agent indexing the column does not receive the object indexing the row, and a 1 means that she does. Formally, a **deterministic allocation** is a matrix $M \equiv [M_{ik}]_{i \in N, k \in O}$ such that

(i) for each $i \in N$ and each $k \in O$, $M_{ik} \in \{0, 1\}$,
(ii) for each $i \in N$, $\sum_{k \in O} M_{ik} = 1$, and
(iii) for each $k \in O$, $\sum_{i \in N} M_{ik} = 1$.

We consider probabilistic distributions over deterministic allocations. Such an allocation is also represented as a matrix, but this time each entry is a number between 0 and 1, interpreted as the probability with which the agent indexing the column receives the object indexing the row. A **probabilistic allocation** is a matrix $M \equiv [M_{ik}]_{i \in N, k \in O}$ such that

(i) for each $i \in N$ and each $k \in O$, $0 \leq M_{ik} \leq 1$,
(ii) for each $i \in N$, $\sum_{k \in O} M_{ik} = 1$, and
(iii) for each $k \in O$, $\sum_{i \in N} M_{ik} = 1$.

Every probabilistic allocation can be written as a convex combination of deterministic allocations (this combination may not be unique) (Birkhoff 1946; von Neumann 1953). Let $\boldsymbol{\mathcal{M}}$ be the set of all probabilistic allocations. For each $i \in N$, her **probabilistic assignment** in $M \in \mathcal{M}$ is a vector $M_i \equiv [M_{ik}]_{k \in O}$, i.e., the ith column of M. A **rule** is a function that associates with each problem a matrix in $\mathcal{M}$. A generic rule is denoted φ.

From here, we simplify the expressions "a probabilistic allocation" to "an allocation" and "a probabilistic assignment" to "an assignment."

The following are three examples of rules. Let Φ^N be the set of strict orders on N.
Priority rule associated with $\prec \in \Phi^N$ $Pri^{\prec}$: For each $R \in \mathcal{R}^N$, each agent chooses her best object among the remaining ones according to the order $\prec$.
Random priority rule, RP (Abdulkadiroğlu and Sönmez 1998)**:** For each $R \in \mathcal{R}^N$, $RP(R) \equiv \frac{1}{|N|!} \sum_{\prec \in \Phi^N} Pri^{\prec}(R)$.
Serial rule, Sr (Bogomolnaia and Moulin 2001)**:** Consider each object as an infinitely divisible good, whose supply is 1. Let $R \in \mathcal{R}^N$. Agents consume objects at the same rate, each starting with her most preferred object. The object(s) whose supply is exhausted first is (are) the object(s) that is (are) ranked first by the greatest number of agents. When the supply of an object is exhausted, each agent who ranked this object first turns to her second most preferred object (joining those agents who rank that object first), or to her third most preferred object if the supply of her second most preferred object happens to reach exhaustion at the same time, and so on. Consumption goes on until the supply of some other object is exhausted, and so. The process continues until the supplies of all objects are exhausted. In the end, the fractions of the various objects that each agent has consumed, interpreted as probabilities that she will receive these objects, determine her assignment.[9]

We consider three requirements on rules. Again, let φ be an arbitrary rule.

- If two agents have the same preference, their assignments should be the same.

Equal treatment of equals: For each $R \in \mathcal{R}^N$ and each pair $i, j \in N$, if $R_i = R_j$, then $\varphi_i(R) = \varphi_j(R)$.

The next definition requires specifying how an agent compares two assignments.

An assignment $\boldsymbol{M_i} \equiv [M_{ik}]_{k \in O}$ for $i \in N$ **stochastically dominates** another assignment $\boldsymbol{M'_i} \equiv [M'_{ik}]_{k \in O}$ for $i \in N$ **at $\boldsymbol{R_i}$** (or M_i is at least as sd-desirable as M'_i at R_i), which we write $\boldsymbol{M_i\ R_i^{sd}\ M'_i}$, if for each $k \in O$,

$$\sum_{\{x \in O : x R_i k\}} M_{ix} \geq \sum_{\{x \in O : x R_i k\}} M'_{ix}.$$

If strict inequality holds for some k, then we write $\boldsymbol{M_i\ P_i^{sd}\ M'_i}$.
Note that two different assignments M_i and M'_i may not be comparable in the stochastic dominance sense.

Given the stochastic comparison described above, we define a notion of Pareto domination.

An allocation $\boldsymbol{M} \equiv [M_{ik}]_{i \in N, k \in O}$ **stochastically Pareto dominates** another allocation $\boldsymbol{M'} \equiv [M'_{ik}]_{i \in N, k \in O}$ **at $\boldsymbol{R}$** if

(i) for each $i \in N$, $M_i\ R_i^{sd}\ M'_i$, and
(ii) for some $i \in N$, $M_i\ P_i^{sd}\ M'_i$.

- The rule should select an allocation that is not stochastically Pareto dominated by any other allocation. For each $R \in \mathcal{R}^N$, let $\boldsymbol{Eff^{sd}(R)} \equiv \{M \in \mathcal{M} |$ there is no $M' \in \mathcal{M}$ such that M' stochastically Pareto dominates M at $R\}$.[10]

[9] For a formal statement, see Bogomolnaia and Moulin (2001).
[10] See Bogomolnaia and Moulin (2001) for detailed discussion of this axiom.

Stochastic dominance efficiency, (simply, **sd-efficiency**)**:** For each $R \in \mathcal{R}^N$, $\varphi(R) \in Eff^{sd}(R)$.

- Let $i \in N$ and fix the other agents' preferences. The rule should select an allocation such that, according to agent i's true preference, she finds her assignment when she tells the truth at least as sd-desirable as her assignment when she lies. Note that this requires the two assignments to be comparable in the stochastic dominance sense.[11]

Stochastic dominance strategy-proofness, (simply, **sd-strategy-proofness**)**:** For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R)\ R_i^{sd}\ \varphi_i(R'_i, R_{-i})$.[12]

3 Result

We consider the case when the preference profile is single-peaked. We provide two such examples.

Example 1 Consider the problem of allocating time slots for access to a facility to its potential users. Each user has a most preferred time to use the facility and becomes progressively worse off as she uses it further and further away from her most preferred time. One could think that too early may not be good because she may not be ready for the service, but having to wait too long is costly for her.

Example 2 Consider the problem of assigning parking spaces (e.g., parking at a university) that are arranged linearly. Each agent prefers her space to be closer to the building she accesses most frequently (e.g., main library, coffee shop, and building where she works).

For three agents, *equal treatment of equals*, *sd-efficiency*, and *sd-strategy-proofness* are compatible (Bogomolnaia and Moulin 2001).[13] However, for four or more agents, they are not.

Theorem 1 *Domain: single-peaked preferences. Let $n \geq 4$. No rule satisfies the following three requirements:* equal treatment of equals*,* sd-efficiency*, and* sd-strategy-proofness*.*

Proof We divide the proof into two cases.

Case 1: $\boldsymbol{n = 4}$. Let $N = \{1, 2, 3, 4\}$ and $O = \{a, b, c, d\}$. Suppose, by way of contradiction, that there is a rule φ that satisfies the three requirements. We consider the following 9 profiles in $\mathcal{R}^N_{sp}$. Notice that the profiles are all single-peaked on O.[14] Figure 1 illustrates the proof.

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[11] Gibbard (1977) considers strategy-proofness in the context of probabilistic choice of a public good when each agent has von-Neumann Morgenstern preferences.

[12] $R_{-i} \equiv R_{N\setminus\{i\}}$, i.e., the restriction of R to $N\setminus\{i\}$.

[13] The random priority rule satisfies them.

[14] Recall that according to our convention, O is ordered by $a \prec b \prec c \prec d$.

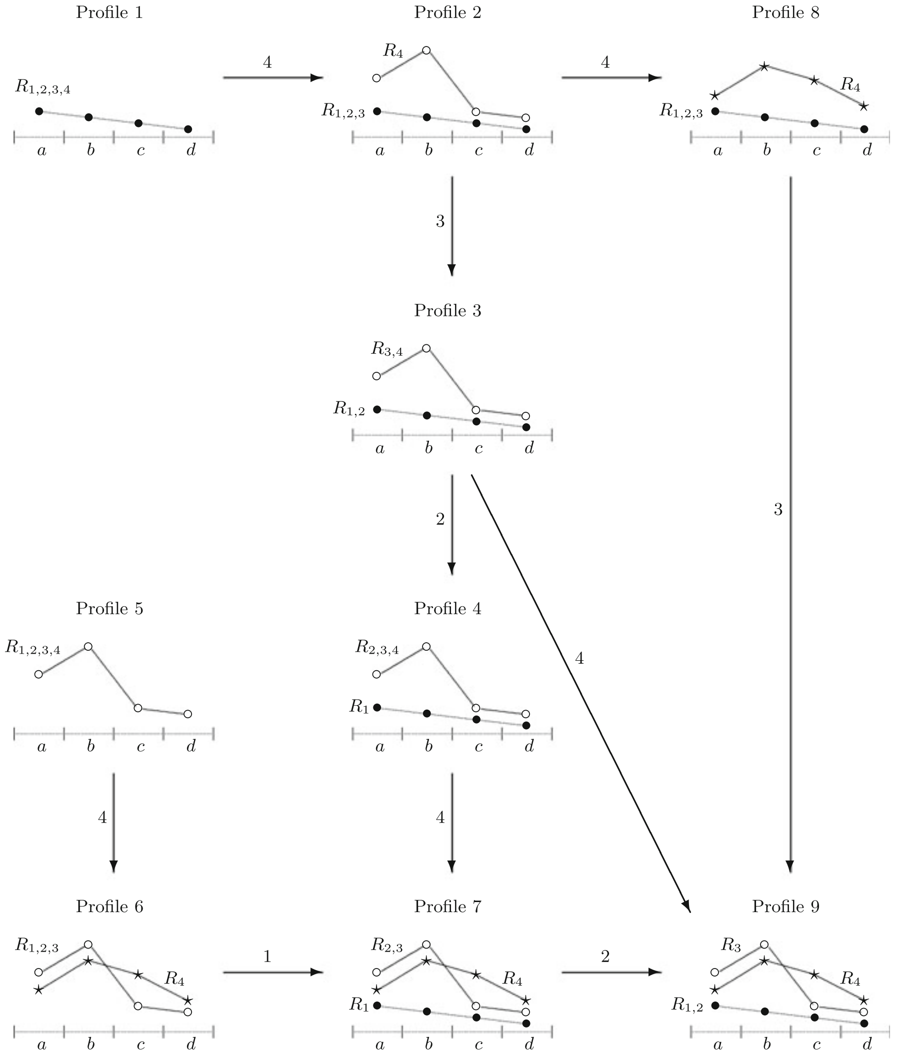

Fig. 1 Illustrating the proof of Theorem 1, Case 1. Each arrow represents how *sd-strategy-proofness* is applied in the proof. A number next to an arrow represents the identity of the agent whose preference is changed

Profile 1: For each $i \in N$, $a \;\; R_i \;\; b \;\; R_i \;\; c \;\; R_i \;\; d$. Then, by *equal treatment of equals*, for each $i \in N$ and each $k \in O$, $\varphi_{ik}(R) = \frac{1}{4}$.

Profile 2: For each $i \in \{1, 2, 3\}$, $a \;\; R_i \;\; b \;\; R_i \;\; c \;\; R_i \;\; d$, and $b \;\; R_4 \;\; a \;\; R_4 \;\; c \;\; R_4 \;\; d$. First, we claim that $\varphi_{4c}(R) = \varphi_{4d}(R) = \frac{1}{4}$. Let R_4' be such that $a \;\; R_4' \;\; b \;\; R_4' \;\; c \;\; R_4' \;\; d$. Then, by *sd-strategy-proofness*,

$$\varphi_4(R) \; R_4^{sd} \; \varphi_4(R_{\{1,2,3\}}, R_4') \text{ and } \varphi_4(R_{\{1,2,3\}}, R_4') \; R_4'^{sd} \; \varphi_4(R).$$

Thus, $\varphi_{4c}(R) = \varphi_{4c}(R_{\{1,2,3\}}, R'_4)$ and $\varphi_{4d}(R) = \varphi_{4d}(R_{\{1,2,3\}}, R'_4)$. Invoking our conclusion for Profile 1, $\varphi_{4c}(R) = \varphi_{4d}(R) = \frac{1}{4}$.

Next, we claim that $\varphi_{4a}(R) = 0$. Suppose by way of contradiction that $\varphi_{4a}(R) > 0$. Since $\varphi_{4b}(R) < \frac{1}{2}$, there is $i \in \{1, 2, 3\}$ such that $\varphi_{ib}(R) > 0$. Let $\delta \equiv \min\{\varphi_{4a}(R), \varphi_{ib}(R)\}$. Let $M \in \mathcal{M}$ be such that $M_{4a} = \varphi_{4a}(R) - \delta$, $M_{4b} = \varphi_{4b}(R) + \delta$, $M_{ia} = \varphi_{ia}(R) + \delta$, $M_{ib} = \varphi_{ib}(R) - \delta$, and the other entries are the same as the entries at $\varphi(R)$. Then, it is easy to see that M stochastically Pareto dominates $\varphi(R)$ at R, in violation of *sd-efficiency*. Thus, $\varphi_{4a}(R) = 0$.

Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Profile 3: For each $i \in \{1, 2\}$, $a \ R_i \ b \ R_i \ c \ R_i \ d$, and for each $j \in \{3, 4\}$, $b \ R_j \ a \ R_j \ c \ R_j \ d$. Then, invoking our conclusion for Profile 2 and *sd-strategy-proofness*, $\varphi_{3c}(R) = \varphi_{3d}(R) = \frac{1}{4}$.

We claim that $\varphi_{3a}(R) = 0$. Suppose by way of contradiction that $\varphi_{3a}(R) > 0$. Then, $\varphi_{3b}(R) < \frac{1}{2}$. By *equal treatment of equals*, $\varphi_{4b}(R) < \frac{1}{2}$. Then, there is $i \in \{1, 2\}$ such that $\varphi_{ib}(R) > 0$. As the argument in the previous profile, it is easy to construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R)$ at R, in violation of *sd-efficiency*. Thus, $\varphi_{3a}(R) = 0$.

Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Profile 4: $a \ R_1 \ b \ R_1 \ c \ R_1 \ d$ and for each $i \in \{2, 3, 4\}$, $b \ R_i \ a \ R_i \ c \ R_i \ d$. Then, invoking our conclusion for Profile 3 and *sd-strategy-proofness*, $\varphi_{2c}(R) = \varphi_{2d}(R) = \frac{1}{4}$.

We claim that $\varphi_{1b}(R) = 0$. Suppose by way of contradiction that $\varphi_{1b}(R) > 0$. Then, $\varphi_{1a}(R) < 1$. Then there is $i \in \{2, 3, 4\}$ such that $\varphi_{ia}(R) > 0$. Then, we construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R)$ at R, in violation of *sd-efficiency*. Thus, $\varphi_{1b}(R) = 0$.

Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Profile 5: For each $i \in N$, $b \; R_i \; a \; R_i \; c \; R_i \; d$. Then, by *equal treatment of equals*, for each $i \in N$ and each $k \in O$, $\varphi_{ik}(R) = \frac{1}{4}$.

Profile 6: For each $i \in \{1, 2, 3\}$, $b \; R_i \; a \; R_i \; c \; R_i \; d$, and $b \; R_4 \; c \; R_4 \; a \; R_4 \; d$. Then, invoking our conclusion for Profile 5 and *sd-strategy-proofness*, $\varphi_{4b}(R) = \varphi_{4d}(R) = \frac{1}{4}$. By *sd-efficiency*, $\varphi_{4a}(R) = 0$. Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Profile 7: $a \; R_1 \; b \; R_1 \; c \; R_1 \; d$, for each $i \in \{2, 3\}$, $b \; R_i \; a \; R_i \; c \; R_i \; d$, and $b \; R_4 \; c \; R_4 \; a \; R_4 \; d$. Then, invoking our conclusion for Profile 4 and *sd-strategy-proofness*, $\varphi_{4b}(R) = \frac{1}{3}$ and $\varphi_{4d}(R) = \frac{1}{4}$. By *sd-efficiency*, $\varphi_{4a}(R) = 0$ and $\varphi_{1b}(R) = 0$. Invoking our conclusion for Profile 6 and *sd-strategy-proofness*, $\varphi_{1c}(R) = \frac{1}{6}$ and $\varphi_{1d}(R) = \frac{1}{4}$. Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{7}{12} & \frac{5}{24} & \frac{5}{24} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{24} & \frac{5}{24} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Profile 8: For each $i \in \{1, 2, 3\}$, $a \; R_i \; b \; R_i \; c \; R_i \; d$, and $b \; R_4 \; c \; R_4 \; a \; R_4 \; d$. Then, invoking our conclusion for Profile 2 and *sd-strategy-proofness*, $\varphi_{4b}(R) = \frac{1}{2}$ and $\varphi_{4d}(R) = \frac{1}{4}$. By *sd-efficiency*, $\varphi_{4a}(R) = 0$. Then, by *equal treatment of equals*,

$$\varphi(R) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Profile 9: For each $i \in \{1, 2\}$, $a \; R_i \; b \; R_i \; c \; R_i \; d$, $b \; R_3 \; a \; R_3 \; c \; R_3 \; d$, and $b \; R_4 \; c \; R_4 \; a \; R_4 \; d$. Then, invoking our conclusion for Profile 3 and *sd-strategy-proofness*, $\varphi_{4b}(R) = \frac{1}{2}$ and $\varphi_{4d}(R) = \frac{1}{4}$. By *sd-efficiency*, $\varphi_{4a}(R) = 0$. Thus, $\varphi_{4c}(R) = \frac{1}{4}$. Invoking our conclusion for Profile 7 and *sd-strategy-proofness*, $\varphi_{2c}(R) = \frac{5}{24}$. Then, by *equal treatment of equals*, $\varphi_{1c}(R) = \frac{5}{24}$. Invoking our conclusion for Profile 8 and *sd-strategy-proofness*, $\varphi_{3c}(R) = \frac{1}{4}$. However,

$$\varphi_{1c}(R) + \varphi_{2c}(R) + \varphi_{3c}(R) + \varphi_{4c}(R) = \frac{5}{24} + \frac{5}{24} + \frac{1}{4} + \frac{1}{4} < 1,$$

a contradiction with $\varphi(R) \in \mathcal{M}$.[15]

Case 2: $\boldsymbol{n > 4}$. Let $h \in \mathbb{N}$ and $n = 4 + h$. Let $N = \{1, 2, 3, 4, 5, \ldots, 4 + h\}$ and $O = \{a, b, c, d, o_1, \ldots, o_h\}$. Suppose, by way of contradiction, that there is a rule φ that satisfies the three requirements. For each Profile $l \in \{1, \ldots, 9\}$ in Case 1, we construct a preference Profile $R^l \in \mathcal{R}^N_{sp}$ as the following three conditions are satisfied:

(i) for each $i \in \{1, \ldots, 4\}$, the preference ordering on $\{a, b, c, d\}$ is the same as in Profile $l \in \{1, \ldots, 9\}$ and those objects are her best four objects,
(ii) for each $j \in \{5, \ldots, 4 + h\}$, her best object is o_{j-4}, and
(iii) all profiles are single-peaked on O.[16]

For instance, let for each $i \in \{1, \ldots, 4\}$, the preference ordering on $\{a, b, c, d\}$ be the same as in Profile $l \in \{1, \ldots, 9\}$ and for each $k \in \{a, b, c, d\}$,

$$k \; R^l_i \; o_1 \; R^l_i \; o_2 \; R^l_i \; \cdots \; R^l_i \; o_h.$$

Let for each $j \in \{5, \ldots, 4 + h\}$,

$$o_{j-4} \; R^l_j \; o_{j-5} \; R^l_j \; \cdots \; R^l_j \; o_1 \; R^l_j \; d \; R^l_j \; c \; R^l_j \; b \; R^l_j \; a \; R^l_j \; o_{j-3} \; R^l_j \; o_{j-2} \cdots \; R^l_j \; o_h.$$

Thus, for each $l \in \{1, \ldots, 9\}$, we have the following preference profile.

R^l_1	R^l_2	R^l_3	R^l_4	R^l_5	R^l_6	$\cdots$	R^l_j	$\cdots$	R^l_{3+h}	R^l_{4+h}
Profile l in Case 1				o_1	o_2		o_{j-4}		o_{h-1}	o_h
$l \in \{1, \ldots, 9\}$				d	o_1		o_{j-5}		o_{h-2}	o_{h-1}
				c	d		$\vdots$		$\vdots$	$\vdots$
				b	c		o_1		$\vdots$	$\vdots$
o_1	o_1	o_1	o_1	a	b		d		$\vdots$	$\vdots$
o_2	o_2	o_2	o_2	o_2	a		c		$\vdots$	$\vdots$
o_3	o_3	o_3	o_3	o_3	o_3		b		o_1	$\vdots$
o_4	o_4	o_4	o_4	o_4	o_4		a		d	o_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		o_{j-3}		c	d
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		o_{j-2}		b	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		a	b
o_h	o_h	o_h	o_h	o_h	o_h		o_h		o_h	a

Notice that the profiles we consider are all single-peaked on O. It is straightforward to see that by *sd-efficiency*, for each Profile $l \in \{1, \ldots, 9\}$ and each $j \in \{5, \ldots, 4 + h\}$, $\varphi_{j o_{j-4}}(R^l) = 1$. Then, a similar argument as in Case 1 leads to a contradiction. ◻

[15] The assignments we obtain at Profiles 1–9 coincide with those made by the serial rule except for Profiles 7 and 9.

[16] Again, recall that according to our convention, O is ordered by $a \prec b \prec c \prec d \prec o_1 \prec o_2 \prec \cdots \prec o_h$.

Note that the above result strengthens the results by Bogomolnaia and Moulin (2001) [p. 310 Theorem 2] since the impossibility holds on a smaller domain of preferences than the one they consider.

Theorem 1 is tight. The serial rule satisfies all the properties but *sd-strategy-proofness*. The random priority rule satisfies all the properties but *sd-efficiency*. For each $\prec \in \Phi^N$, the priority rule associated with $\prec$ satisfies all the properties but *equal treatment of equals*.

4 Discussion

We discuss three variations of our probabilistic assignment problems with single-peaked preferences.

(1) More objects than agents

Suppose that $|O| > |N|$. An allocation here is a $|O| \times |N|$ matrix such that each entry is non-negative, the entries in each row sum to at most one, and the entries in each column sum to one. Then, *equal treatment of equals*, *sd-efficiency*, and *sd-strategy-proofness* should be restated in the obvious way. In this case, a similar result to Theorem 1 holds, i.e., for four or more agents, the three axioms are incompatible even when preferences are single-peaked. The proof is as follows. For four agents, add an object "e" that each agent ranks below "d." For more than four agents, each agent ranks "e" between "d" and "o_1." Notice that preferences are single-peaked with respect to the order $a \prec b \prec c \prec d \prec e$ for the four-agent case and $a \prec b \prec c \prec d \prec e \prec o_1 \prec o_2 \prec \cdots \prec o_h$ for more than four agents. The reader can check that a rule assigns zero probability to "e" for each profile, and that a similar argument as in the proof of Theorem 1 leads to a contradiction.

(2) More agents than objects

Suppose that $|O| < |N|$. Suppose also that there exist infinitely many copies of the "null object" (interpreted as receiving no object), denoted $\varnothing$. Here, some agent has to receive a null object. Each agent has a complete, transitive, and strict binary relation over $O \cup \varnothing$. An allocation here is a $(|O|+1) \times |N|$ matrix $M \equiv [M_{ik}]_{i \in N,\, k \in O \cup \varnothing}$ such that each entry is non-negative, the entries in each row except the row corresponding to the null object sum to at most one, and the entries in each column sum to one. Then, the three axioms should be restated in the obvious way. Here again, a similar result to Theorem 1 holds. The proof consists of replacing "d" with a null object.

(3) Indifference over objects

Suppose that each agent has a complete and transitive (not necessarily strict) binary relation over O. Let $\mathcal{R}^N$ be a domain of such preference profiles. For each agent $i \in N$, the strict preference relation associated with $R_i \in \mathcal{R}^N$ is denoted by P_i and the corresponding indifference relation by I_i. The axioms we studied should be restated in the obvious way. It is well known that introducing indifference changes the nature of the probabilistic assignment problems significantly, and even more, it may change the positive results to negative (Katta and Sethuraman 2006). Now consider the following two requirements on rules.

- Each agent should find her assignment at least as sd-desirable as anybody else's assignment. For each $R \in \mathcal{R}^N$, let $\boldsymbol{F^{sd}(R)} \equiv \{M \in \mathcal{M} | \text{for each pair } i, j \in N,\ M_i\ R_i^{sd}\ M_j\}$.

Stochastic dominance envy-freeness, (simply, **sd-envy-freeness**)**:** For each $R \in \mathcal{R}^N$ $\varphi(R) \in F^{sd}(R)$.
Notice that the above axiom is stronger than *equal treatment of equals*.

- Let $i \in N$ and fix the other agents' preferences. The rule should select an allocation such that, according to agent i's true preference, she never finds her assignment when she lies, to be sd-better than her assignment when she tells the truth.

Weak stochastic dominance strategy-proofness, (simply, **weak sd-strategy-proofness**)**:** For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R_i' \in \mathcal{R}$, it is not the case that $\varphi_i(R_i', R_{-i})\ \ P_i^{sd}\ \ \varphi_i(R)$.
Obviously, the above axiom is weaker than *sd-strategy-proofness*.

When preferences are strict, *sd-envy-freeness*, *sd-efficiency* and *weak sd-strategy-proofness* are compatible (Bogomolnaia and Moulin 2001).[17] However, if indifference is allowed, these three axioms become incompatible (Katta and Sethuraman 2006). A similar result holds even if we restrict the domain to "weakly" single-peaked preferences (for each agent, there is a single most preferred object; no two objects on the left of her peak can be indifferent; the same for the right of her peak; but objects on the left and the right of her peak can be indifferent): for four or more agents, *sd-envy-freeness*, *sd-efficiency*, and *weak sd-strategy-proofness* are incompatible. The proof is as follows. For four agents, let $N = \{1, 2, 3, 4\}$ and $O = \{a, b, c, d\}$. Consider the following two profiles.

Profile 1: $b\ \ P_1\ \ a\ \ I_1\ \ c\ \ P_1\ \ d$, for each $i \in \{2, 3\}$, $b\ \ P_i\ \ c\ \ P_i\ \ a\ \ P_i\ \ d$, and $b\ \ P_4\ \ c\ \ P_4\ \ d\ \ P_4\ \ a$.

Profile 2: $b\ \ P_1'\ \ c\ \ P_1'\ \ a\ \ P_1'\ \ d$, for each $i \in \{2, 3\}$, $b\ \ P_i\ \ c\ \ P_i\ \ a\ \ P_i\ \ d$, and $b\ \ P_4\ \ c\ \ P_4\ \ d\ \ P_4\ \ a$.

Both profiles are weakly single-peaked on O.[18] Suppose, by way of contradiction, that there is a rule φ that satisfies the three requirements. For Profile 1, by *sd-envy-freeness*, for each $i \in N$, $\varphi_{ib}(R) = \frac{1}{4}$, $\varphi_{2c}(R) = \varphi_{3c}(R) = \varphi_{4c}(R)$, and $\varphi_{1a}(R) + \varphi_{1c}(R) = \varphi_{2a}(R) + \varphi_{2c}(R) = \varphi_{3a}(R) + \varphi_{3c}(R)$. By *sd-efficiency*, $\varphi_{4a}(R) = 0$ and $\varphi_{1c}(R) = 0$.[19] Then, ${}^t\varphi_1(R) = \left(\frac{5}{9}, \frac{1}{4}, 0, \frac{7}{36}\right)$.[20] For Profile 2, by *sd-envy-freeness*, for each $i \in N$, $\varphi_{ib}(R_1', R_{-1}) = \varphi_{ic}(R_1', R_{-1}) = \frac{1}{4}$ and $\varphi_{1a}(R_1', R_{-1}) = \varphi_{2a}(R_1', R_{-1}) = \varphi_{3a}(R_1', R_{-1})$. By *sd-efficiency*, $\varphi_{4a}(R_1', R_{-1}) = 0$. Then, ${}^t\varphi_1(R_1', R_{-1}) = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right)$. However, $\varphi_1(R_1', R_{-1})\ \ P_1^{sd}\ \ \varphi_1(R)$, a contradiction to φ being *weakly sd-strategy-proof*. For more than four agents, a similar construction as in the proof of Theorem 1 Case 2 leads to a contradiction.

[17] The serial rule satisfies them.

[18] Following our convention, O is ordered by $a \prec b \prec c \prec d$.

[19] If $\varphi_{4a}(R) > 0$, then $\varphi_{4d}(R) < 1$. Then there is $i \in \{1, 2, 3\}$ such that $\varphi_{id}(R) > 0$. Notice that for each $i \in \{1, 2, 3\}$, $a\ \ P_i\ \ d$. Then, it is easy to construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R)$ at R, in violation of *sd-efficiency*. Similar argument holds when $\varphi_{1c}(R) > 0$.

[20] For each vector x, ${}^tx$ represents the transpose of x.

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References

Abdulkadiroğlu A, Sönmez T (1998) Random serial dictatorship and the core from random endowments in house allocation problems. Econometrica 66:689–701

Birkhoff G (1946) Three observations on linear algebra. Univ. Nac. Tucumán. Revista A.5:147–151 (in Spanish)

Bogomolnaia A, Moulin H (2001) A new solution to the random assignment problem. J Econ Theory 100:295–328

Bogomolnaia A, Moulin H (2002) A simple random assignment problem with a unique solution. Econ Theory 19:623–635

Che Y-K, Kojima F (2010) Asymptotic equivalence of probabilistic serial and random priority mechanisms. Econometrica 78: 1625–1672

Crès H, Moulin H (2001) Scheduling with opting out: improving upon random priority. Oper Res 49: 565–577

Ehlers L, Klaus B (2003) Probabilistic assignments of identical indivisible objects and uniform probabilistic rules. Rev Econ Des 8:249–268

Gibbard A (1977) Manipulation of schemes that mix voting with chance. Econometrica 45:665–681

Hatsumi K, Serizawa S (2009) Coalitionally strategy-proof rules in allotment economies with homogeneous indivisible goods. Soc Choice Welf 33:423–447

Heo E-J (2011) Probabilistic assignment problem with multiple demands: a generalization of the serial rule and a characterization, Mimeo

Hylland A, Zeckhauser R (1979) The efficient allocation of individuals to positions. J Political Econ 87: 293–314

Katta A-K, Sethuraman J (2006) A solution to the random assignment problem on the full preference domain. J Econ Theory 131:231–250

Kojima F (2009) Random assignment of multiple indivisible objects. Math Soc Sci 57:134–142

Kojima F, Manea M (2010) Incentives in the probabilistic serial mechanism. J Econ Theory 145:106–123

Kureishi W, Mizukami H (2007) Equal probability for the best and the assignment of identical indivisible objects. Econ Bull 4:1–10

Sasaki H (1997) Randomized uniform allocation mechanism and single-peaked preferences of indivisible good, Working Paper, Waseda University

Thomson W (2008) Strategy-proof allocation rules, Mimeo

von Neumann J (1953) A certain zero-sum two-person game equivalent to the optimal assignment problem. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games, vol 2. Princeton University press, Princeton, pp 5–12