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On relative egalitarianism

Yves Sprumont

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Abstract We reconsider the problem of aggregating individual preference orderings into a single social ordering when alternatives are lotteries and individual preferences are of the von Neumann–Morgenstern type. *Relative egalitarianism* ranks alternatives by applying the leximin ordering to the distributions of 0–1 normalized utilities they generate. We propose an axiomatic characterization of this aggregation rule.

1 Introduction

The problem of aggregating individual preference orderings into a single social ordering admits interesting solutions in contexts where Arrow (1963) Independence of Irrelevant Alternatives can be suitably weakened.[1] An important example is when social alternatives are lotteries and individual preferences obey the von Neumann–Morgenstern axioms. Dhillon (1998) and Dhillon and Mertens (1999) showed how anonymity and several weak axioms that are implied by Arrow's axiom lead to *relative utilitarianism*, which ranks social alternatives according to the sum of 0–1 normalized utilities they generate for the individuals who are not completely indifferent between all alternatives.[2] Because relative utilitarianism does not require any a priori knowledge of individual utilities, it may be regarded as a truly "operational" variant of Harsanyi (1955) classical utilitarianism.

Yet, relative utilitarianism remains subject to some of the traditional criticisms formulated against classical utilitarianism. If there are just two pure alternatives, *a* and *b*,

[1] See Fleurbaey and Maniquet (2011) for a theory of fair preference aggregation in such contexts.

[2] Further arguments in favor of relative utilitarianism were proposed by Karni (1998) and Segal (2000).

Y. Sprumont (✉)
Département de Sciences Économiques and CIREQ, Université de Montréal,
C.P. 6128, Succursale Centre-ville, Montreal, QC H3C 3J7, Canada
e-mail: yves.sprumont@umontreal.ca

and society consists of just two individuals with opposite von Neumann–Morgenstern preferences over the lotteries between a and b, relative utilitarianism deems all lotteries equally good: flipping a fair coin is not better than choosing, say, b. Just like classical utilitarianism, relative utilitarianism cares only about final outcomes and shows no concern for procedural fairness. Critics such as Diamond (1967) and Sen (1970) argue that a lottery between a and b is superior to b because it gives all individuals a chance to get their preferred alternative.

In response to that criticism, we propose an egalitarian counterpart to relative utilitarianism. *Relative egalitarianism* ranks social alternatives by applying the leximin ordering to the distributions of 0–1 normalized utilities they generate. Like relative utilitarianism, this procedure requires no knowledge of individual utilities but, unlike relative utilitarianism, it embodies a genuine concern for procedural fairness.

We offer an axiomatic characterization of relative egalitarianism which rests mainly on two properties: Preference for Compromise and Independence of Inessential Expansions.

In the two-individual, two-alternative problem discussed above, the reason why society might prefer a lottery between a and b to either of these pure alternatives is that it permits some form of *compromise* between decisions that seem too radical. But in many social choice problems, compromises may exist which are not lotteries. If all individuals have strict preferences between a and b, let us formally define a compromise between these two alternatives to be any alternative strictly preferred to b (but not to a) by the supporters of a and strictly preferred to a (but not to b) by the supporters of b. A compromise could be a lottery between a and b but it need not be. Preference for Compromise says that if all individuals have strict preferences between a and b, then society should deem any compromise between a and b at least as good as the worst of these two alternatives. This is a strengthening of Diamond and Sen's suggestion that randomizing between two alternatives should be regarded at least as good as choosing the worst of the two.

Independence of Inessential Expansions is a weakening of Arrow's independence condition which requires that society's preference over a set of alternatives be unaffected by the addition of new alternatives that no individual finds better than his most preferred alternative or worse than his least preferred one.

We show that in combination with fairly standard conditions—namely, the Pareto Principle, Anonymity, and Separability—Preference for Compromise and Independence of Inessential Expansions characterize relative egalitarianism.

A few comments on the literature are in order at this point. The first modern formulation of relative egalitarianism is generally credited to Kalai and Smorodinsky (1975).[3] It is phrased in the context of classical bargaining theory, where a choice rule selects a utility vector from each feasible set.[4] Key to the axiomatization of the

[3] Kalai and Smorodinsky (1975) considered the two-individual case. Their solution was extended to n individuals by Imai (1983).

[4] Nash's (1950) solution was meant to describe the likely outcome of a bargaining situation between rational agents. But part of the literature generated by Nash's article tends to reinterpret bargaining solutions as fair arbitration rules that might be used by a benevolent third party. We have such a normative interpretation of the Kalai–Smorodinsky solution in mind when comparing it to our relative leximin aggregation rule. We regard the latter as a fair preference aggregation rule that might be used by a benevolent third party.

Kalai–Smorodinsky solution is the so-called Restricted Monotonicity condition: an expansion of the feasible set that does not affect the individuals' minimal and maximal utilities should lead to a (weakly) higher utility vector.

Thanks to the work of Roemer (1986), it is by now well understood that defining choice rules on feasible utility sets rather than on the actual collective choice problems generating these sets is extremely restrictive because the same feasible utility set can arise from very different combinations of preferences and resource constraints. More importantly, the actual content of most axioms exceeds their intended meaning. The Restricted Monotonicity axiom, for instance, is usually motivated by imagining that the expansion of the feasible utility set results from an increase in resources (and no change in preferences) although this expansion could also result from a mere change in preferences. In the latter case, the requirement that the utility vector selected after the change dominates the one before the change is meaningless.

These criticisms prompted a reformulation of the main bargaining solutions as collective choice rules defined on economic environments: see among others Roemer (1988), Rubinstein et al. (1992) and de Clippel (2008). This last paper is closely related to our work since it contains an axiomatization of a reformulation of the Kalai–Smorodinsky solution.[5] The central axioms are a variant of Restricted Monotonicity and an independence axiom asking that the solution be unaffected by changes in preferences outside the feasible set. The main differences with our work are that (i) de Clippel works on a restricted (though interesting) domain where social alternatives are (lotteries over) allocations of private goods whereas we consider an abstract set of pure alternatives, (ii) he is concerned with choice rules rather than preference aggregation rules, and (iii) his axiomatization is restricted to the two-individual case (and seems difficult to generalize) whereas our axiomatization is valid for any number of individuals.

Another paper discussing a relatively egalitarian choice rule is Nehring (2000). Nehring's framework is very different from both de Clippel's and ours. It is an individual decision-theoretic model à la Savage where an act is a mapping from the set of states of the world to some ordered set of consequences. Mixed acts—i.e., lotteries over acts—are available. The decision-maker is assumed to be completely ignorant of the state of the world in the sense that he prefers an act x over an other act y if and *only if* the consequence attached to x is better than that attached to y in every state. A choice rule selects a mixed act from every closed and convex subset of simple acts. Nehring axiomatizes what he calls the Simultaneous Expected-Utility Maximization (or SIMEU) choice rule. This rule associates with each extremal prior belief over states of the world a fictitious decision-maker whose preferences over mixed acts are determined by this belief (and the fixed ordering of consequences); it then selects from every feasible set the undominated mixed act which equalizes the 0–1 utilities of these fictitious decision-makers. Because Nehring's main axioms are meant to express consequences of the complete ignorance assumption, his axiom system is markedly different from de Clippel's and ours. His WAREP axiom, however, is similar to our axiom of Independence of Inessential Expansions.

[5] The paper is also—and primarily—concerned with the Nash solution.

2 Framework

Let A be an infinite reference set of *pure* (*social*) *alternatives* and let $\mathcal{A}$ denote the set of nonempty finite subsets of A. For each $X \in \mathcal{A}$, let $\Delta(X)$ be the set of lotteries on X, that is, $\Delta(X) = \left\{a \in [0,\ 1]^X \mid \sum_{x \in X} a(x) = 1\right\}$. If $x \in X$, we abuse notation and also denote by x the lottery in $\Delta(X)$ assigning probability 1 to x. Let $\mathcal{R}(X)$ and $\mathcal{R}_*(X)$ denote respectively the set of all preference orderings and the subset of von Neumann–Morgenstern preference orderings over $\Delta(X)$. Write $\mathcal{R} = \cup_{X \in \mathcal{A}} \mathcal{R}(X)$ and $\mathcal{R}_* = \cup_{X \in \mathcal{A}} \mathcal{R}_*(X)$.

Let $N = \{1, \ldots, n\}$ be a fixed finite set of individuals. A (*social choice*) *problem* is a list $(X,\ R)$ where $X \in \mathcal{A}$ and $R \in (\mathcal{R}_*(X))^N$. We call R a *preference profile*. The set of all problems is denoted by $\mathcal{P}$. An (*aggregation*) *rule* is a mapping $\mathbf{R} : \mathcal{P} \to \mathcal{R}$ such that $\mathbf{R}(X,\ R) \in \mathcal{R}(X)$ for every $(X,\ R) \in \mathcal{P}$.

We make five comments on the above framework.

(1) As in Dhillon and Mertens (1999), the set X is meant to include the pure social alternatives that are both feasible and just. The term "just" is used here in the weak sense of "ethically acceptable" and could be approximated by "lawful". We call X the *set of acceptable pure alternatives* and $\Delta(X)$ the *set of acceptable alternatives*. When there is no risk of confusion, we refer to either set as the "acceptable set". Defining X is a fundamental ethical issue which cannot be addressed without further knowledge of the nature of the alternatives in A.

(2) We interpret $\mathbf{R}(X,\ R)$ as the ordering over $\Delta(X)$ that should guide society's choices when individual preferences are given by the profile R. We refer to it as *society's preference*. Note that society's preference over $\Delta(X)$ is constrained to depend only upon individual preferences *over that set*. This is a serious restriction. But it is a natural one because A is a large unstructured set and individual preferences over lotteries involving arbitrary alternatives in A may therefore be difficult to elucidate. Moreover, since no structure is imposed on A, there is no natural reference point outside X which could help define the aggregation rule.

(3) When the set of acceptable alternatives expands, the social preference over the originally acceptable alternatives is allowed to change: if $X \subseteq X'$ and the preference profile R' over $\Delta(X')$ coincides over $\Delta(X)$ with the profile R, $\mathbf{R}(X',\ R')$ need not coincide with $\mathbf{R}(X,\ R)$ on $\Delta(X)$.[6]

(4) Even though individual preferences are of the von Neumann–Morgenstern type, society's preference is not restricted to be of that type. The set of possible aggregation rules is therefore much larger than in Dhillon and Mertens (1999).

(5) All individual von Neumann–Morgenstern preferences over the acceptable alternatives are admissible. This is in line with Arrow's universal domain assumption and guarantees that the aggregation rules we discuss are not restricted to a particular type of social choice problem. In applications, however, it may be desirable to impose restrictions on preferences. For instance, if X is the set of acceptable allocations in a

[6] In Dhillon and Mertens (1999), the acceptable set (which they denote by A) is kept fixed. No axiom linking social preferences across different acceptable sets is used to characterize relative utilitarianism. Yet, if the acceptable set is allowed to expand, the social preference recommended by relative utilitarianism over the originally acceptable alternatives will typically be affected.

private-good economy, it is natural to require that preferences be selfish, as de Clippel (2008) does. We believe that our result can be reformulated in such restricted contexts but such a reformulation is not straightforward. Our proof does rely on the universal domain assumption.

3 A theorem on relative egalitarianism

What we call *relative egalitarianism*—or, more precisely, the *relative leximin aggregation rule*—is the rule which ranks alternatives by applying the leximin ordering to the distributions of 0–1 normalized utilities they generate. This section proposes a set of axioms leading to that rule.

For any $v \in [0,1]^N$, let v^* denote the vector in $[0,1]^N$ obtained by reordering the coordinates of v in nondecreasing order. The leximin ordering $\succsim^L$ on $[0,1]^N$ is defined by letting $v \succsim^L w$ if and only if either there exists $j \in N$ such that $v_i^* = w_i^*$ for all $i < j$ and $v_j^* > w_j^*$ (in which case we write $v \succ^L w$) or $v_i^* = w_i^*$ for all $i \in N$ (in which case we write $v \sim^L w$).

Given $X \in \mathcal{A}$, we denote by R_0 the complete indifference relation on $\Delta(X)$. If $(X, R) \in \mathcal{P}$ and $i \in N$, we write R_i instead of $R(i)$ and denote by $\overline{A}(X, R_i)$ and $\underline{A}(X, R_i)$ the sets of best and worst alternatives in $\Delta(X)$ according to R_i. We let $u(., X, R_i) : \Delta(X) \to [0,1]$ be the *normalized von Neumann–Morgenstern numerical representation* of R_i: if $R_i \neq R_0$, then $u(a, X, R_i) = \alpha \Leftrightarrow aI_i\left[\alpha\overline{a} + (1-\alpha)\underline{a}\right]$ for any $\overline{a} \in \overline{A}(X, R_i)$ and $\underline{a} \in \underline{A}(X, R_i)$; if $R_i = R_0$, then $u(a, X, R_i) = 1$ for all $a \in \Delta(X)$. The *relative leximin* aggregation rule $\mathbf{R}^L$ is defined as follows: for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, $a\mathbf{R}^L(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim^L (u(b, X, R_1), \ldots, u(b, X, R_n))$.

Our axiomatic characterization of $\mathbf{R}^L$ uses five axioms. The first two are familiar conditions. As usual, P_i and I_i denote the strict preference and indifference relations associated with the individual preference R_i and $\mathbf{P}(X, R)$ and $\mathbf{I}(X, R)$ denote the strict social preference and indifference relations associated with $\mathbf{R}(X, R)$. Let $\Pi(N)$ be the set of permutations on N. If $\sigma \in \Pi(N)$, $\sigma R \in (\mathcal{R}_*(X))^N$ denotes the preference profile such that $(\sigma R)_{\sigma(i)} = R_i$ for all $i \in N$.

Pareto Principle. If $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$, and aR_ib for all $i \in N$, then $a\mathbf{R}(X, R)b$. If, in addition, aP_jb for some $j \in N$, then $a\mathbf{P}(X, R)b$.

Anonymity. For all $(X, R) \in \mathcal{P}$ and $\sigma \in \Pi(N)$, $\mathbf{R}(X, R) = \mathbf{R}(X, \sigma R)$.

Next we state our central axiom. Preference for Compromise expresses the ethical judgement that it is desirable to compromise between two social alternatives over which individual preferences are antagonistic.

Preference for Compromise. Let $(X, R) \in \mathcal{P}$, $a, b, c \in \Delta(X)$, and $\varnothing \subsetneq S \subsetneq N$. If aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N\backslash S$, then $c\mathbf{R}(X, R)a$ or $c\mathbf{R}(X, R)b$.

Alternative c can be regarded as a compromise between a and b because all agents find c intermediate between a and b: all agree that c is strictly better than the worst of a and b and not better than the best of the two (even though they disagree on the ranking of a and b). Since c is a compromise between a and b, society should find c at least as good as the worst of a and b. In particular, if society is indifferent between a and b, then it should find c at least as good as either of these two alternatives.

Two remarks are in order. First, one may wish to require that society's preference for compromise be strict: if $\varnothing \subsetneq S \subsetneq N$, aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$, then $c\mathbf{P}(X, R)a$ or $c\mathbf{P}(X, R)b$. While we strongly believe that this strict version of Preference for Compromise is desirable, we do not impose it. We will see that it follows from the weak version and our other axioms. Second, it is important to realize that the alternative c in the axiom is a compromise between a and b in a possibly very poor, "unbalanced" sense. It could happen that the von Neumann–Morgenstern utility of every agent at c is very close to his von Neumann–Morgenstern utility at the worst of a and b, meaning that a switch from either a or b to the compromise alternative c benefits the supporters of the move much less than it hurts its opponents. In particular, c need not be a lottery between a and b: Preference for Compromise goes beyond the requirement that randomizing between two alternatives be at least as good as choosing the worst of the two.[7] It is stronger than "procedural fairness" à la Diamond and Sen.

We now turn to our weakening of Arrow's independence axiom. As pointed out in Comment 3 in Sect. 2, the definition of an aggregation rule allows the social preference over a given subset of alternatives to vary with the set of acceptable alternatives. Such flexibility is necessary in order to construct "fair" aggregation rules. In particular, it is needed if society values compromise. To see this, suppose again that the acceptable set consists of the lotteries between two pure alternatives, a and b. If society is composed of two individuals with opposite von Neumann–Morgenstern preferences over $\Delta(\{a, b\})$, very basic requirements (such as the usual Anonymity and Neutrality axioms) force social indifference between a and b. But if a third pure alternative, c, becomes acceptable and individual 1 strictly prefers a to b to c while 2 strictly prefers c to b to a, the strict version of Preference for Compromise requires that society should now strictly prefer b to a in $\Delta(\{a, b, c\})$: the reason is that b may now be regarded as a compromise between the two agents while a has become a more extreme alternative.

In this example, adding the pure social alternative c to the set $\{a, b\}$ alters the preference aggregation problem in an essential way because it changes the worst possible outcome for individual 1 and the best outcome for 2. Independence of Inessential Expansions says that society's preferences over $\Delta(X)$ should be unaffected by the addition of new alternatives that leave the best and worst outcomes of all individuals unchanged—what we call an "inessential expansion" of the acceptable set. Our formal requirement is the following.

Independence of Inessential Expansions. Let $(X, R), (X', R') \in \mathcal{P}$ be two problems such that $X \subseteq X'$ and R' coincides with R on $\Delta(X)$. If $\overline{a} R'_i a' R'_i \underline{a}$ for all $i \in N$ and all $\overline{a} \in \overline{A}(X, R_i)$, $a' \in X'$, $\underline{a} \in \underline{A}(X, R_i)$, then $\mathbf{R}(X', R')$ coincides with $\mathbf{R}(X, R)$ on $\Delta(X)$.

While we argued that the social preference should be allowed to vary when the individuals' best or worst alternatives change, it is not clear why it should remain unaffected by "inessential expansions" of the acceptable set. We do not think there are compelling ethical reasons to insist on this requirement. As is the case with other

[7] Epstein and Segal (1992) propose a related condition. In a different context (where individual utilities are assumed to be available), they submit that if society is indifferent between two alternatives that are not Pareto-equivalent, a 50–50 lottery between these alternatives should be strictly better than either of them.

independence axioms (such as those proposed by Arrow 1963; Hansson 1973; Young and Levenglick 1978), the primary justification is practical convenience. An aggregation rule satisfying Independence of Inessential Expansions is relatively simple to use because the social ranking of alternatives is only affected by fairly radical changes in the environment.

The restriction to those expansions which do not change the best and worst outcomes of all individuals, though definitely somewhat ad hoc, is rather natural and seems hard to avoid. It is present in Nehring (2000) WAREP axiom as well as in the various versions of the Restricted Monotonicity axiom used by Kalai and Smorodinsky (1975), Imai (1983) and de Clippel (2008). In fact, Independence of Inessential Expansion is the direct counterpart of Nehring's WAREP axiom—expressed for aggregation rules rather than choice rules. Restricted Monotonicity, on the other hand, not only requires that "inessential" expansions should not distort social decisions but also imposes a solidarity requirement that has no counterpart in our axiom.[8]

Our last axiom is again a familiar one.

Separability. Let $X \in \mathcal{A}$, $a, b \in \Delta(X)$, and $S \subseteq N$. Let $R, R' \in (\mathcal{R}_*(X))^N$ be such that $R_i = R'_i$ for all $i \in S$ and aI_jb and aI'_jb for all $j \in N \setminus S$. Then $a\mathbf{R}(X, R)b$ if and only if $a\mathbf{R}(X, R')b$.

Separability says that the social ranking of two alternatives should be independent of the preferences of all unconcerned individuals, namely those who are indifferent between the two alternatives in any case. This property, adapted from Fleming (1952), has a long tradition in social choice theory: see d'Aspremont (1985) for references.

We are now ready to state our main result.

Theorem *The aggregation rule* $\mathbf{R}$ *satisfies the Pareto Principle, Anonymity, Preference for Compromise, Independence of Inessential Expansions, and Separability if and only if* $\mathbf{R} = \mathbf{R}^L$.

It is important to remember that no utility information is available in our framework. An aggregation rule is a purely ordinal procedure transforming every profile of individual preference orderings into a social ordering. Our axioms thus perform a double task: (1) they select the 0–1 normalized von Neumann–Morgenstern representation of preferences as the adequate measure of individual welfare and (2) they force the use of the leximin criterion to compare welfare vectors.

Very roughly, the Pareto Principle, Anonymity, Preference for Compromise, and Separability jointly impose the leximin criterion. The argument is relatively straightforward because, as we stressed earlier, Preference for Compromise is a very powerful condition that recommends even the poorest compromises between social alternatives.

Independence of Inessential Expansions, on the other hand, is responsible for pinning down the 0–1 normalization. This turns out to be a very delicate task. If we do not impose Independence of Inessential Expansions, (i) the adequate numerical representation of an individual's preference need not be a von Neumann–Morgenstern

[8] The solidarity requirement present in Restricted Monotonicity is what drives the egalitarian character of the Kalai–Smorodinsky solution. This role is played by Preference for Compromise in our characterization of relative egalitarianism.

utility function and (ii) it could vary with the preferences of the other individuals. The examples below illustrate these difficulties.

Example 1 Given $(X, R) \in \mathcal{P}$ and $i \in N$, define $v(., X, R_i) : \Delta(X) \to [0, 1]$ by letting $v(a, X, R_i) = \frac{\mu(\{b \in \Delta(X) | aR_ib\})}{\mu(\Delta(X))}$ for all $a \in \Delta(X)$, where μ is the Lebesgue measure on $\mathbb{R}^{|X|-1}$. That is, an individual's utility from alternative a is measured by the proportion of acceptable alternatives that he does not consider better than a. Note that even though $v(., X, R_i)$ is not linear, it is a numerical representation of the von Neumann–Morgenstern preference R_i. For all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, let $a\mathbf{R}(X, R)b \Leftrightarrow (v(a, X, R_1), \ldots, v(a, X, R_n)) \succsim^L (v(b, X, R_1), \ldots, v(b, X, R_n))$, where $\succsim^L$ is again the leximin ordering on $[0, 1]^N$. This aggregation rule satisfies all our axioms except Independence of Inessential Expansions. Separability is met because the numerical representation of an individual's preference does not depend on the preferences of the others.

The difficulty exemplified by this rule is proper to our framework. In Dhillon and Mertens (1999), the assumption that society's preference obeys the independence axiom prevents the construction of comparable examples. The "utilitarian" ordering $a\mathbf{R}(X, R)b \Leftrightarrow \sum_{i \in N} v(a, X, R_i) \geq \sum_{i \in N} v(b, X, R_i)$ is generally not a von Neumann–Morgenstern preference over $\Delta(X)$.

Example 2 Partition $\mathcal{R}_*$ into two sets $\mathcal{R}_*^1, \mathcal{R}_* \setminus \mathcal{R}_*^1$, each containing preferences differing from complete indifference. For instance, $\mathcal{R}_*^1$ could be the set of preferences with at most two indifference classes of pure alternatives. For each $(X, R) \in \mathcal{P}$ and $i \in N$, define $\beta_i(X, R) = \left|\left\{j \in N \setminus \{i\} \mid R_j \in \mathcal{R}_* \setminus \mathcal{R}_*^1\right\}\right|$ and define $w_i(., X, R) : \Delta(X) \to [0, 1]$ by letting $w_i(a, X, R) = u(a, X, R_i)^{2^{\beta_i(X, R)}}$ for all $a \in \Delta(X)$. Note that $w_i(., X, R)$ is a numerical representation of the von Neumann–Morgenstern preference R_i that changes with the preferences of the agents $j \in N \setminus i$. Let $a\mathbf{R}(X, R)b \Leftrightarrow (w_1(a, X, R), \ldots, w_n(a, X, R)) \succsim^L (w_1(b, X, R), \ldots, w_n(b, X, R))$ for all $a, b \in \Delta(X)$. Again, this aggregation rule satisfies all our axioms but Independence of Inessential Expansions. To see why Separability is satisfied, consider two problems $(X, R), (X, R')$ where $R_i = R'_i$ for all $i \neq 1$, $R_1 \in \mathcal{R}_*^1$, and $R'_1 \in \mathcal{R}_* \setminus \mathcal{R}_*^1$. Then, for all $i \neq 1$,

$$
\begin{aligned}
w_i(., X, R') &= u(., X, R_i)^{2^{\beta_i(X, R')}} \\
&= u(., X, R_i)^{2^{\beta_i(X, R)+1}} \\
&= (u(., X, R_i)^{2^{\beta_i(X, R)}})^2 \\
&= (w_i(., X, R))^2,
\end{aligned}
$$

that is, the numerical representations of the preferences of all agents other than 1 are modified according to a *common* increasing transformation when 1's preference changes from R_1 to R'_1. Using the fact that the leximin ordering $\succsim^L$ is separable (in the usual sense formally defined just before Step 2.2 in the proof of our theorem) it follows easily that for all $a, b \in \Delta(X)$ such that aI_1b and aI'_1b, $a\mathbf{R}(X, R)b \Leftrightarrow a\mathbf{R}(X, R')b$.

The two examples above show that Independence of Inessential Expansions cannot simply be dispensed with. But one might hope that it could be replaced with some weaker and less ad hoc axiom. A natural candidate is the following requirement.

Independence of Redundant Alternatives. Let $(X, R), (X', R') \in \mathcal{P}$ be two problems such that $X \subseteq X'$ and R' coincides with R on $\Delta(X)$. If for all $a' \in X'$ there exists $a \in \Delta(X)$ such that $a' I_i' a$ for all $i \in N$, then $\mathbf{R}(X', R')$ coincides with $\mathbf{R}(X, R)$ on $\Delta(X)$.

This weakening of Independence of Inessential Expansions merely says that adding alternatives that are Pareto equivalent to some originally acceptable alternatives does not change the social ranking over the original acceptable set. It is the translation in our framework of the property bearing the same name in Dhillon and Mertens (1999). The rules in Examples 1 and 2 above violate this requirement. Yet, the following example shows that Independence of Redundant Alternatives cannot replace Independence of Inessential Expansions in the statement of our theorem.

Example 3 For each $(X, R) \in \mathcal{P}$, let $U(X, R)$ denote the normalized utility set generated by (X, R), that is, $U(X, R) = \{(u(a, X, R_1), \ldots, u(a, X, R_n)) \mid a \in \Delta(X)\}$. Let $\alpha(X, R) = (\alpha_1(X, R), \ldots, \alpha_n(X, R))$ be the unique maximizer of $\prod_{i \in N} z_i$ over all $z \in U(X, R)$. For each $i \in N$, define the function $v_i(., X, R) : \Delta(X) \to [0, 1]$ by $v_i(a, X, R) = u(a, X, R_i)^{\alpha_i(X, R)}$ for all $a \in \Delta(X)$. This function is another example of a numerical representation of R_i which changes with the preferences of the agents other than i. The rule $a\mathbf{R}(X, R)b \Leftrightarrow (v_1(a, X, R), \ldots, v_n(a, X, R_n)) \succsim^L (v_1(b, X, R), \ldots, v_n(b, X, R))$ for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$ satisfies the Pareto Principle, Anonymity, Preference for Compromise, and Independence of Redundant Alternatives. When $n = 2$, it also (trivially) meets Separability. Since our theorem does apply to the two-agent case, this example shows that Independence of Redundant Alternatives cannot replace Independence of Inessential Expansions in its statement.

An interesting but seemingly difficult question is whether Independence of Redundant Alternatives can replace Independence of Inessential Expansions in the statement of our theorem if we add the assumption that $n \geq 3$.

All the axioms used in our theorem are independent.

(1) The constant complete indifference rule $a\mathbf{I}(X, R)b$ for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$ satisfies all the axioms except the Pareto Principle. Contrary to the relative leximin rule, the constant indifference rule violates the strict version of Preference for Compromise described earlier. For an example satisfying that version along with Anonymity, Independence of Inessential Expansions, and Separability, consider the opposite of the relative leximin rule defined as follows: for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow (1 - u(a, X, R_1), \ldots, 1 - u(a, X, R_n)) \succsim^L (1 - u(b, X, R_1), \ldots, 1 - u(b, X, R_n))$.

(2) An example of a rule violating only Anonymity is relative serial dictatorship: for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim^{(1,\ldots,n)} (u(b, X, R_1), \ldots, u(b, X, R_n))$, where $\succsim^{(1,\ldots,n)}$ is the lexicographic ordering on $[0, 1]^N$ corresponding to the natural ordering over N.

(3) As already explained, the rules in Examples 1 and 2 satisfy all our axioms but Independence of Inessential Expansions.

(4) Relative utilitarianism $a\mathbf{R}(X, R)b \Leftrightarrow \sum_{i \in N} u(a, X, R_i) \geq \sum_{i \in N} u(b, X, R_i)$ for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$ violates only Preference for Compromise.

(5) The lexicographic combination of relative maximin and relative utilitarianism, $\mathbf{R}^{MU}$, defined by letting $a\mathbf{R}^{MU}(X, R)b$ if and only if (i) $\min_{i \in N} u(a, X, R_i) > \min_{i \in N} u(b, X, R_i)$ or (ii) $\min_{i \in N} u(a, X, R_i) = \min_{i \in N} u(b, X, R_i)$ and $\sum_{i \in N} u(a, X, R_i) \geq \sum_{i \in N} u(b, X, R_i)$, violates only Separability.

We conclude this section with a brief discussion of the role of Separability. Even though the axiom is used rather heavily in the proof provided in the next section, a different (and more cumbersome) argument offered in Sprumont (2009) shows that the other four axioms are strong enough to imply the *relative maximin principle*: if the aggregation rule $\mathbf{R}$ satisfies the Pareto Principle, Anonymity, Preference for Compromise, and Independence of Inessential Expansions, then for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, $\min_{i \in N} u(a, X, R_i) > \min_{i \in N} u(b, X, R_i) \Rightarrow a\mathbf{P}(X, R)b$. Moreover, these four axioms imply Separability when $n = 2$. The role of Separability, therefore, is only to bridge the gap between relative maximin and relative leximin when $n \geq 3$. The proof in Sprumont (2009) also shows that Separability can be replaced in our theorem with the following weaker requirement.[9]

Weak Separability. Let $X \in \mathcal{A}$, $a, b \in \Delta(X)$, $R \in (\mathcal{R}_*(X))^N$, $i \in N$, and let (R_0, R_{-i}) denote the profile obtained from R by replacing R_i with the complete indifference relation R_0. If $a\mathbf{P}(X, (R_0, R_{-i}))b$ and aI_ib, then $a\mathbf{P}(X, R)b$.

4 A proof of the theorem

An implication of the Pareto Principle is the following well-known condition.

Pareto Indifference. If $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$, and aI_ib for all $i \in N$, then $a\mathbf{I}(X, R)b$.

We begin with two lemmas that combine Pareto Indifference with Independence of Inessential Expansions. The first lemma shows that these two requirements jointly imply a strong form of neutrality. Let $\Pi(A)$ denote the set of permutations on A. If $(X, R) \in \mathcal{P}$, $\pi \in \Pi(A)$, and $a \in \Delta(X)$, then $a^\pi \in \Delta(\pi(X))$ is the lottery on $\pi(X)$ given by $a^\pi(\pi(x)) = a(x)$ for all $x \in X$ and the preference profile $R^\pi \in (\mathcal{R}_*(\pi(X)))^N$ is defined by $a^\pi R_i^\pi b^\pi \Leftrightarrow aR_ib$ for all $i \in N$ and $a, b \in \Delta(X)$.

Neutrality. For all $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(A)$, $a\mathbf{R}(X, R)b \Leftrightarrow a^\pi\mathbf{R}(\pi(X), R^\pi)b^\pi$.

Denoting by $\Pi(X)$ the set of permutations on $X \in \mathcal{A}$, Neutrality implies that for all $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow a^\pi\mathbf{R}(X, R^\pi)b^\pi$.

Lemma 1 *If the aggregation rule* $\mathbf{R}$ *satisfies Pareto Indifference and Independence of Inessential Expansions*, *then* $\mathbf{R}$ *satisfies Neutrality.*

Proof Let $\mathbf{R}$ satisfy Pareto Indifference and Independence of Inessential Expansions. Let $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(A)$. We prove that $a\mathbf{R}(X, R)b \Rightarrow a^\pi\mathbf{R}(\pi(X), R^\pi)b^\pi$. The converse implication follows immediately since $a =$

[9] Weak Separability is the translation in our framework of the strict part of the condition dubbed "consistency" in Dhillon and Mertens (1999).

$(a^\pi)^{\pi^{-1}}, b = (b^\pi)^{\pi^{-1}}$, $X = \pi^{-1}(\pi(X))$, and $R = (R^\pi)^{\pi^{-1}}$. Let us thus assume that

$$a\mathbf{R}(X, R)b. \tag{4.1}$$

Step 1. *We prove that* $a^\pi \mathbf{R}(\pi(X), R^\pi)b^\pi$ *if* $\pi(X) \cap X = \emptyset$.

Let $\overline{X} = X \cup \pi(X)$. For each $i \in N$, let $\overline{R}_i$ be the von Neumann–Morgenstern preference over $\Delta(\overline{X})$ which coincides with R_i on $\Delta(X)$ and is such that $x\overline{I}_i\pi(x)$ for all $x \in X$. This is well defined because $\pi(X) \cap X = \emptyset$. Observe that $\overline{R}_i$ coincides with R_i^π on $\Delta(\pi(X))$. Moreover, $\overline{a}\overline{R}_i x\overline{R}_i \underline{a}$ for all $\overline{a} \in \overline{A}(X, R_i) \cup \overline{A}(\pi(X), R_i^\pi)$, $x \in \overline{X}$, and $\underline{a} \in \underline{A}(X, R_i) \cup \underline{A}(\pi(X), R_i^\pi)$. Let $\overline{R} = (\overline{R}_1, \ldots, \overline{R}_n)$. Applying Independence of Inessential Expansions to (4.1),

$$a\mathbf{R}(\overline{X}, \overline{R})b. \tag{4.2}$$

Since $a^\pi \overline{I}_i a$ and $b^\pi \overline{I}_i b$ for all $i \in N$, Pareto Indifference implies $a^\pi \mathbf{I}(\overline{X}, \overline{R})a$ and $b^\pi \mathbf{I}(\overline{X}, \overline{R})b$. Hence from (4.2),

$$a^\pi \mathbf{R}(\overline{X}, \overline{R})b^\pi. \tag{4.3}$$

Applying Independence of Inessential Expansions to (4.3) and recalling that $\overline{R}$ coincides with R^π on $\Delta(\pi(X))$, we obtain $a^\pi \mathbf{R}(\pi(X), R^\pi)b^\pi$.

Step 2. *We prove that* $a^\pi \mathbf{R}(\pi(X), R^\pi)b^\pi$.

Choose $\rho \in \Pi(A)$ such that $\rho(X) \cap X = \rho(X) \cap \pi(X) = \emptyset$. By Step 1, (4.1) implies

$$a^\rho \mathbf{R}(\rho(X), R^\rho)b^\rho. \tag{4.4}$$

Next consider the permutation $\pi \circ \rho^{-1} \in \Pi(A)$. Since $(\pi \circ \rho^{-1})(\rho(X)) \cap \rho(X) = \emptyset$, Step 1 and (4.4) imply

$$(a^\rho)^{\pi\circ\rho^{-1}} \mathbf{R}((\pi \circ \rho^{-1})(\rho(X)), R^\rho)^{\pi\circ\rho^{-1}})(b^\rho)^{\pi\circ\rho^{-1}}. \tag{4.5}$$

By definition, $(\pi \circ \rho^{-1})(\rho(X)) = \pi(X)$. Moreover, $(a^\rho)^{\pi\circ\rho^{-1}} = a^\pi$ since $(a^\rho)^{\pi\circ\rho^{-1}}(\pi(x)) = (a^\rho)^{\pi\circ\rho^{-1}}((\pi \circ \rho^{-1})(\rho(x))) = a^\rho(\rho(x)) = a(x)$ for all $x \in X$. Likewise, $(b^\rho)^{\pi\circ\rho^{-1}} = b^\pi$ and $(R^\rho)^{\pi\circ\rho^{-1}} = R^\pi$. Hence (4.5) reduces to $a^\pi \mathbf{R}(\pi(X), R^\pi)b^\pi$. □

It is worth noting that the above proof does not use the full force of Independence of Inessential Expansions. In fact, the axiom can be replaced with Independence of Redundant Alternatives in the statement of Lemma 1.

Our second lemma describes the class of rules satisfying Pareto Indifference and Independence of Inessential Expansions. It seems natural to conjecture that these two properties suffice to force society to compare alternatives by ranking the vectors of individual normalized utilities they generate according to a fixed ordering on $[0, 1]^N$. This conjecture is not quite correct. While the aggregation rule must indeed rely solely on the individual normalized utilities, it turns out that the criterion used to aggregate these utilities may in fact depend upon the preference profile. But this dependence is severely restricted. Given $(X, R) \in \mathcal{P}$, define $N_0(X, R) = \{i \in N \mid R_i = R_0\}$. For each $S \subseteq N$, let $\mathcal{P}_S = \{(X, R) \in \mathcal{P} \mid N_0(X, R) = S\}$: this is the set of problems where the individuals indifferent between all acceptable alternatives are the members of S.

Lemma 2 *The aggregation rule* $\mathbf{R}$ *satisfies Pareto Indifference and Independence of Inessential Expansions if and only if for each* $S \subseteq N$ *there exists an ordering* $\succsim_S$ *on* $[0, 1]^N$ *such that for all* $(X, R) \in \mathcal{P}_S$ *and* $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim_S (u(b, X, R_1), \ldots, u(b, X, R_n))$.

Proof **Step 1**. *We prove the "if" statement.*

Fix a collection of orderings $\succsim_S$ on $[0, 1]^N$ and suppose that for all $S \subseteq N$, all $(X, R) \in \mathcal{P}_S$ and all $a, b \in X$, $a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim_S (u(b, X, R_1), \ldots, u(b, X, R_n))$. It is obvious that $\mathbf{R}$ satisfies Pareto Indifference. To check Independence of Inessential Expansions, fix (X, R), (X', R') satisfying the premises of the axiom. Then for each $i \in N$ and $c \in \Delta(X)$, $u(c, X, R_i) = u(c, X', R'_i)$. Moreover, since R'_i is the complete indifference relation on $\Delta(X')$ if and only if R_i is the complete indifference relation on $\Delta(X)$, we have $N_0(X, R) = N_0(X', R')$. Therefore, for all $a, b \in \Delta(X)$,

$$\begin{aligned} a\mathbf{R}(X, R)b &\Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \\ &\qquad \succsim_{N_0(X,R)} (u(b, X, R_1), \ldots, u(b, X, R_n)) \\ &\Leftrightarrow (u(a, X', R'_1), \ldots, u(a, X', R'_n)) \\ &\qquad \succsim_{N_0(X',R')} (u(b, X', R'_1), \ldots, u(b, X', R'_n)) \\ &\Leftrightarrow a\mathbf{R}(X', R')b. \end{aligned}$$

Step 2. *We prove the "only if" statement.*

Let $\mathbf{R}$ satisfy Pareto Indifference and Independence of Inessential Expansions. By Lemma 1, $\mathbf{R}$ also satisfies Neutrality. For each $S \subseteq N$, define the binary relations $\succ_S, \sim_S, \succsim_S$ on $[0, 1]^N$ as follows:

(i) $v \succ_S w$ if and only if there exist $(X, R) \in \mathcal{P}_S$ and $a, b \in \Delta(X)$ such that $u(a, X, R_i) = v_i$ and $u(b, X, R_i) = w_i$ for all $i \in N$ and $a\mathbf{P}(X, R)b$,

(ii) $v \sim_S w$ if and only if there exist $(X, R) \in \mathcal{P}_S$ and $a, b \in \Delta(X)$ such that $u(a, X, R_i) = v_i$ and $u(b, X, R_i) = w_i$ for all $i \in N$ and $a\mathbf{I}(X, R)b$,

(iii) $v \succsim_S w$ if and only if $v \succ_S w$ or $v \sim_S w$.

The relations $\succ_S, \sim_S, \succsim_S$ are equivalently defined by replacing $\Delta(X)$ with X in statements (i) and (ii). To see why, fix $v, w \in [0, 1]^N$ and suppose there exist $(X, R) \in \mathcal{P}_S$ and $a, b \in \Delta(X)$ such that $u(a, X, R_i) = v_i$ and $u(b, X, R_i) = w_i$ for all $i \in N$ and $a\mathbf{P}(X, R)b$ (respectively, $a\mathbf{I}(X, R)b$). Choose distinct pure alternatives $a', b' \in A \setminus X$, let $X' = X \cup \{a', b'\}$ and, for each $i \in N$, let R'_i be the von

Neumann–Morgenstern preference on $\Delta(X')$ which coincides with R_i on $\Delta(X)$ and is such that $a' I_i' a$ and $b' I_i' b$. Then $(X', R') \in \mathcal{P}_S$, $a', b' \in X'$, $u(a', X', R_i') = v_i$ and $u(b', X', R_i') = w_i$ for all $i \in N$ and, using Independence of Inessential Expansions and Pareto Indifference, $a'\mathbf{I}(X', R')a\mathbf{P}(X', R')b\mathbf{I}(X', R')b'$ (respectively, $a'\mathbf{I}(X', R')a\mathbf{I}(X', R')b\mathbf{I}(X', R')b'$).

Step 2.1. *We claim that each* $\succsim_S$ *is consistent in* Suzumura (1976) *sense*: if there exist $v^1, \ldots, v^m \in [0, 1]^N$ such that $v^1 \succsim_S \ldots \succsim_S v^m \succsim_S v^1$, then $v^1 \sim_S \ldots \sim_S v^m \sim_S v^1$. Suppose, on the contrary, that, say, $v^1 \succsim_S \ldots \succsim_S v^m \succ_S v^1$. Then there exist $(X^1, R^1), \ldots, (X^m, R^m) \in \mathcal{P}_S$ and $a^1, b^1 \in X^1, \ldots, a^m, b^m \in X^m$ such that

$$a^k\mathbf{R}(X^k, R^k)b^k \quad \text{for } k = 1, \ldots, m-1 \ \text{ and } a^m\mathbf{P}(X^m, R^m)b^m, \tag{4.6}$$

and

$$u(a^k, X^k, R_i^k) = v_i^k \text{ and } u(b^k, X^k, R_i^k) = v_i^{k+1(\text{mod}\, m)} \quad \text{for all } i \in N \text{ and } k = 1, \ldots, m. \tag{4.7}$$

By Neutrality, we may assume that $X^1, \ldots, X^m$ are pairwise disjoint. Let $X = \cup_{k=1}^m X^k$. For each $i \in N$, let $u_i : \Delta(X) \to [0, 1]$ be the von Neumann–Morgenstern utility function such that

$$u_i(x) = u(x, X^{k(x)}, R_i^{k(x)}) \quad \text{for all } x \in X, \tag{4.8}$$

where $k(x)$ is the unique k such that $x \in X^k$. Let R_i be the preference on $\Delta(X)$ represented by u_i and let $R = (R_1, \ldots, R_n)$.
Note that R coincides with R^k on $\Delta(X^k)$ for each k. Moreover, since $N_0(X^1, R^1) = \ldots = N_0(X^m, R^m)$ $(= S)$, (4.8) implies that $\overline{a}R_i x R_i \underline{a}$ for all $i \in N$, all $\overline{a} \in \cup_{k=1}^m \overline{A}(X^k, R_i^k)$, all $x \in X$, and all $\underline{a} \in \cup_{k=1}^m \underline{A}(X^k, R_i^k)$. Applying Independence of Inessential Expansions to (4.6),

$$a^k\mathbf{R}(X, R)b^k \quad \text{for } k = 1, \ldots, m-1 \text{ and } \ a^m\mathbf{P}(X, R)b^m. \tag{4.9}$$

On the other hand, (4.7) and (4.8) imply that $b^k I_i a^{k+1(\text{mod}\, m)}$ for all $i \in N$ and $k = 1, \ldots, m$. By Pareto Indifference,

$$b^k\mathbf{I}(X, R)a^{k+1(\text{mod}\, m)} \quad \text{for } k = 1, \ldots, m,$$

which together with (4.9) contradicts the transitivity of $\mathbf{R}(X, R)$.
Step 2.2. If $S \neq \varnothing$, the relation $\succsim_S$ need not be complete. Since $\succsim_S$ is consistent, however, it has an ordering extension (by Suzumura 1976): denote it $\succsim_S'$. By the very definition of $\succsim_S$, we have that for all $(X, R) \in \mathcal{P}_S$ and $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim_S' (u(b, X, R_1), \ldots, u(b, X, R_n))$. □

We are now ready to proceed to the proof of our theorem.

Proof of the theorem **Step 1**. *We prove the "if" statement.*

The Pareto Principle and Anonymity are obviously met. Independence of Inessential Expansions holds because $u(x, X, R_i) = u(x, X', R_i')$ for all $x \in X$ and $i \in N$ whenever (X, R), (X', R') satisfy the premises of the axiom.

To check that $\mathbf{R}^L$ satisfies Preference for Compromise, let $(X, R) \in \mathcal{P}$, $a, b, c \in \Delta(X)$, $\varnothing \subsetneqq S \subsetneqq N$ be such that $a\mathbf{R}^L(X, R)b$, aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$. We claim that

$$\min_{i\in N} u(c, X, R_i) > \min_{i\in N} u(b, X, R_i), \tag{4.10}$$

which in turn implies $c\mathbf{P}^L(X, R)b$ (that is, $\mathbf{R}^L$ satisfies the strict version of Preference for Compromise mentioned earlier). Suppose (4.10) does not hold. Let $j \in N$ be such that $u(c, X, R_j) = \min_{i\in N} u(c, X, R_i) \leq \min_{i\in N} u(b, X, R_i)$. Since $a\mathbf{R}^L(X, R)b$, we get $u(c, X, R_j) \leq \min_{i\in N} u(b, X, R_i) \leq \min_{i\in N} u(a, X, R_i)$. In particular, $u(c, X, R_j) \leq u(b, X, R_j)$ and $u(c, X, R_j) \leq u(a, X, R_j)$, that is, bR_jc and aR_jc. Hence $j \notin S$ and $j \notin N \setminus S$, a contradiction.

We omit the standard argument showing that $\mathbf{R}^L$ satisfies Separability.

Step 2. *We prove the "only if" statement.*

Let $\mathbf{R}$ satisfy the Pareto Principle, Anonymity, Independence of Inessential Expansions, Preference for Compromise, and Separability. By Lemma 2, there exists a collection of orderings $\succsim_S$ on $[0, 1]^N$ such that for all $S \subseteq N$, all $(X, R) \in \mathcal{P}_S$ and all $a, b \in \Delta(X)$,

$$a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim_S (u(b, X, R_1), \ldots, u(b, X, R_n)). \tag{4.11}$$

Step 2.1. *For all* $(X, R) \in \mathcal{P}$ *and all* $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), \ldots, u(a, X, R_n)) \succsim_\emptyset (u(b, X, R_1), \ldots, u(b, X, R_n))$.

Suppose not. Then there exist $S \subseteq N$, $(X, R) \in \mathcal{P}_S$, and $a, b \in \Delta(X)$ such that one of the following statements holds:

$$a\mathbf{P}(X, R)b \text{ and } (u(b, X, R_1), \ldots, u(b, X, R_n)) \succsim_\emptyset (u(a, X, R_1), \ldots, u(a, X, R_n)), \tag{4.12}$$

$$a\mathbf{R}(X, R)b \text{ and } (u(b, X, R_1), \ldots, u(b, X, R_n)) \succ_\emptyset (u(a, X, R_1), \ldots, u(a, X, R_n)). \tag{4.13}$$

Assume (4.12). Without loss of generality, suppose also that $|X| \geq 3$. (If $|X| < 3$, simply choose $\overline{X} \in \mathcal{A}$ such that $X \subseteq \overline{X}$ and $|\overline{X}| \geq 3$. For each $i \in N$ let $\overline{R}_i$ be a von Neumann–Morgenstern preference on $\Delta(\overline{X})$ coinciding with R_i on $\Delta(X)$ and such that $\overline{a}\overline{R}_i x \overline{R}_i \underline{a}$ for all $\overline{a} \in \overline{A}(X, R_i)$, $x \in \overline{X}$, $\underline{a} \in \underline{A}(X, R_i)$. Then $a\mathbf{P}(\overline{X}, \overline{R})b$ by Independence of Inessential Expansions and $(u(b, \overline{X}, \overline{R}_1), \ldots, u(b, \overline{X}, \overline{R}_n)) = (u(b, X, R_1), \ldots, u(b, X, R_n)) \succsim_\emptyset$

$(u(a, X, R_1), \ldots, u(a, X, R_n)) = (u(a, \overline{X}, \overline{R}_1), \ldots, u(a, \overline{X}, \overline{R}_n))$, so that the argument below would apply with $(\overline{X}, \overline{R})$ instead of (X, R).)

Let $R' = (R'_1, \ldots, R'_n)$ be a profile of preferences over $\Delta(X)$ such that

$$R'_i \neq R_0 \text{ and } aI'_i bR'_i x \text{ for all } x \in X \text{ and all } i \in S,$$
$$R'_i = R_i \text{ for all } i \in N \setminus S.$$

Such a profile exists because $\left|\overline{X}\right| \geq 3$. By construction, $(X, R') \in \mathcal{P}_\emptyset$ and

$$u(a, X, R'_i) = u(a, X, R_i) = u(b, X, R'_i) = u(b, X, R_i) = 1 \quad \text{for all } i \in S,$$
$$u(a, X, R'_i) = u(a, X, R_i) \text{ and } u(b, X, R'_i) = u(b, X, R_i) \quad \text{for all } i \in N \setminus S,$$

so that (4.12) implies $(u(b, X, R'_1), \ldots, u(b, X, R'_n)) \succsim_\emptyset (u(a, X, R'_1), \ldots, u(a, X, R'_n))$.

But by Separability, $a\mathbf{P}(X, R)b$ implies $a\mathbf{P}(X, R')b$, contradicting (4.11) for $S = \emptyset$. Essentially the same argument applies if we assume (4.13) instead of (4.12). This completes Step 2.1.

The rest of the proof consists in showing that $\succsim_\emptyset$ is the leximin ordering on $[0, 1]^N$. Some further terminology is needed at this point. Using the notation $\geq, >, \gg$ for vector inequalities, we say that $\succsim_\emptyset$ is *strictly monotonic* if, for all $v, w \in [0, 1]^N$, $v > w$ implies $v \succ_\emptyset w$. If $v \in [0, 1]^N$ and $\sigma \in \Pi(N)$, let σv be the vector obtained by permuting the coordinates of v according to the permutation σ, i.e., $(\sigma w)_{\sigma(i)} = w_i$ for all $i \in N$. We call $\succsim_\emptyset$ *symmetric* if, for all $v, w \in [0, 1]^N$ and $\sigma \in \Pi(N)$, $v \succsim_\emptyset w$ if and only if $\sigma v \succsim_\emptyset \sigma w$. We say that $\succsim_\emptyset$ is *compromising* if, for all $v, w, z \in [0, 1]^N$ such that $v_i \neq w_i$ for all $i \in N$,

$$v \wedge w \ll z \leq v \vee w \Rightarrow z \succsim_\emptyset v \text{ or } z \succsim_\emptyset w. \tag{4.14}$$

Finally, we call $\succsim_\emptyset$ *separable* if, for all $S \subseteq N$ and $v, v', w, w' \in [0, 1]^N$ such that $v_i = v'_i$ and $w_i = w'_i$ for all $i \in S$ and $v_j = w_j$ and $v'_j = w'_j$ for all $j \in N \setminus S$, we have $v \succsim_\emptyset w$ if and only if $v' \succsim_\emptyset w'$.

Step 2.2. *The ordering $\succsim_\emptyset$ is strictly monotonic, symmetric, compromising, and separable.*

These properties follow directly from the Pareto Principle, Anonymity, Preference for Compromise, and Separability, respectively. Let us check, for instance, that the ordering $\succsim_\emptyset$ is strictly monotonic. Fix $v, w \in [0, 1]^N$ such that $v > w$. Choose $a, b \in A$ and $(X, R) \in \mathcal{P}_\emptyset$ such that $u(a, X, R_i) = v_i$ and $u(b, X, R_i) = w_i$ for all $i \in N$. Since $(u(a, X, R_1), \ldots, u(a, X, R_n)) = v > w = (u(b, X, R_1), \ldots, u(b, X, R_n))$, the Pareto Principle implies $a\mathbf{P}(X, R)b$. It follows from Step 2.1 that $v \succ_\emptyset w$, as desired. We omit the similar proof that Anonymity, Preference for Compromise, and Separability imply, respectively, that $\succsim_\emptyset$ is symmetric, compromising, and separable.

Step 2.3. *Let $T \geq 2$, let $z^1, \ldots, z^T \in [0, 1]^N$ be coordinate-by-coordinate distinct* (i.e., $z^t_i \neq z^s_i$ *for all $i \in N$ and $t \neq s$*) *and let $z \in [0, 1]^N$. If $z \gg \wedge_{t=1}^T z^t$, then there exists $t \in \{1, \ldots, T\}$ such that $z \succsim_\emptyset z^t$.*

Consider first the case where $T = 2$. Fix $z^1, z^2, z \in [0, 1]^N$ such that $z_i^1 \neq z_i^2$ for all $i \in N$ and $z \gg z^1 \wedge z^2$. Since $z_i^1 \neq z_i^2$ for all $i \in N$, we have $z^1 \wedge z^2 \ll z^1 \vee z^2$. Let $z' = z \wedge (z^1 \vee z^2)$. By construction, $z^1 \wedge z^2 \ll z' \leq z^1 \vee z^2$. Since $\succsim_\emptyset$ is compromising (in the sense of (4.14)), $z' \succsim_\emptyset z^1$ or $z' \succsim_\emptyset z^2$. But since $\succsim_\emptyset$ is strictly monotonic, $z \succsim_\emptyset z'$. Hence $z \succsim_\emptyset z^1$ or $z \succsim_\emptyset z^2$, as desired.

To complete the proof, proceed by induction on T. Fix $T > 2$ and make the induction hypothesis that, for all $T' \leq T - 1$, all $z^1, \ldots, z^{T'} \in [0, 1]^N$ coordinate-by-coordinate distinct, and all $z \in [0, 1]^N$ such that $z \gg \wedge_{t=1}^{T'} z^t$, there exists $t \in \{1, \ldots, T'\}$ such that $z \succsim_\emptyset z^t$. Fix now T coordinate-by-coordinate distinct vectors $z^1, \ldots, z^T \in [0, 1]^N$ and $z \in [0, 1]^N$ such that $z \gg \wedge_{t=1}^{T} z^t$. Choose $\varepsilon > 0$ such that $z \gg \wedge_{t=1}^{T} z^t + \varepsilon e$, where e is the unit vector in $\mathbb{R}^N$. Since

$$\begin{aligned}\wedge_{t=1}^{T} z^t + \varepsilon e &= \left(\wedge_{t=1}^{T} z^t + \frac{\varepsilon e}{2}\right) + \frac{\varepsilon e}{2} \\ &= \left(\left(\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}\right) \wedge \left(z^T + \frac{\varepsilon e}{2}\right)\right) + \frac{\varepsilon e}{2},\end{aligned}$$

we have $z \gg \left(\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}\right) \wedge \left(z^T + \frac{\varepsilon e}{2}\right)$.

Suppose momentarily that the two vectors $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}$ and $z^T + \frac{\varepsilon e}{2}$ are coordinate-by-coordinate distinct. Using the induction hypothesis (with $T' = 2$), we obtain that

$$z \succsim_\emptyset \wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} \text{ or } z \succsim_\emptyset z^T + \frac{\varepsilon e}{2}. \tag{4.15}$$

Distinguish two cases.

(i) $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} \succsim_\emptyset z^T + \frac{\varepsilon e}{2}$. Then (4.15) implies that $z \succsim_\emptyset z^T + \frac{\varepsilon e}{2}$, hence $z \succsim_\emptyset z^T$, and we are done.

(ii) $z^T + \frac{\varepsilon e}{2} \succ_\emptyset \wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}$. Then (4.15) implies that $z \succsim_\emptyset \wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}$. But since $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} \gg \wedge_{t=1}^{T-1} z^t$ (and $z^1, \ldots, z^{T-1}$ are coordinate-by-coordinate distinct), the induction hypothesis (with $T' = T - 1$) implies that $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} \succsim_\emptyset z^t$ for some $t \in \{1, \ldots, T-1\}$. So $z \succsim_\emptyset z^t$ for some $t \in \{1, \ldots, T-1\}$, and we are done again.

If the two vectors $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}$ and $z^T + \frac{\varepsilon e}{2}$ are not coordinate-by-coordinate distinct, simply pick a small number $\delta > 0$ such that $z \gg \left(\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} + \delta e\right) \wedge \left(z^T + \frac{\varepsilon e}{2}\right)$ and $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} + \delta e$ is coordinate-by-coordinate distinct from $z^T + \frac{\varepsilon e}{2}$, and replace $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2}$ with $\wedge_{t=1}^{T-1} z^t + \frac{\varepsilon e}{2} + \delta e$ in the argument of the previous paragraph.

Step 2.4. *For all* $v, w \in [0, 1]^N$, $\min_{i \in N} v_i > \min_{i \in N} w_i \Rightarrow v \succ_\emptyset w$.

Let $v, w \in [0, 1]^N$ and suppose $\min_{i \in N} v_i > \min_{i \in N} w_i$. Since $\succsim_\emptyset$ is symmetric, we may assume without loss of generality that $v_1 \leq \ldots \leq v_n$ and $w_1 \leq \ldots \leq w_n$. Therefore $v_1 > w_1$. Observe that $\wedge_{\sigma \in \Pi(N)} \sigma w = (w_1, \ldots, w_1)$, so that $v \gg \wedge_{\sigma \in \Pi(N)} \sigma w$. For each σ, choose a small number $\varepsilon_\sigma > 0$ such that (i) $v \gg \wedge_{\sigma \in \Pi(N)} (\sigma w + \varepsilon_\sigma e)$

and (ii) the $n!$ vectors $\sigma w + \varepsilon_\sigma e$, $\sigma \in \Pi(N)$, are coordinate-by-coordinate distinct. By Step 2.3, $v \succsim_\emptyset \sigma w + \varepsilon_\sigma e$ for some $\sigma \in \Pi(N)$. But since $\succsim_\emptyset$ is strictly monotonic and symmetric, $\sigma w + \varepsilon_\sigma e \succ_\emptyset \sigma w \sim_\emptyset w$ for every $\sigma \in \Pi(N)$. Therefore $v \succ_\emptyset w$.

Step 2.5. *The ordering* $\succsim_\emptyset$ *is the leximin ordering on* $[0, 1]^N$.

Let $v, w \in [0, 1]^N$ and assume again, without loss of generality since $\succsim_\emptyset$ is symmetric, that $v_1 \leq \ldots \leq v_n$ and $w_1 \leq \ldots \leq w_n$. Suppose there exists $i \in N$ such that $v_j = w_j$ for all $j \in N$ such that $j < i$ and $v_i > w_i$. Construct $v', w' \in [0, 1]^N$ by letting

$$v'_j = \begin{cases} 1 & \text{for all } j \in N \text{ such that } j < i, \\ v_j & \text{for all } j \in N \text{ such that } j \geq i, \end{cases}$$

and

$$w'_j = \begin{cases} 1 & \text{for all } j \in N \text{ such that } j < i, \\ w_j & \text{for all } j \in N \text{ such that } j \geq i. \end{cases}$$

Since $\succsim_\emptyset$ is separable, $v \succsim_\emptyset w \Leftrightarrow v' \succsim_\emptyset w'$. Since $\min_{j \in N} v'_j = v_i > w_i = \min_{j \in N} w'_j$, Step 2.4 implies $v' \succ_\emptyset w'$. Therefore $v \succ_\emptyset w$. □

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