Soc Choice Welf (2013) 40:871–911
DOI 10.1007/s00355-011-0648-9

ORIGINAL PAPER

A characterization of the uniform rule with several commodities and agents

Shuhei Morimoto · Shigehiro Serizawa · Stephen Ching

Received: 20 April 2011 / Accepted: 22 December 2011 / Published online: 26 January 2012

Abstract We consider a problem of allocating infinitely divisible commodities among a group of agents. More specifically, there are several commodities to be allocated and agents have continuous, strictly convex, and separable preferences. We establish that a rule satisfies *strategy-proofness*, *unanimity*, *weak symmetry*, and *non-bossiness* if and only if it is the uniform rule. This result extends to the class of continuous, strictly convex, and multidimensional single-peaked preferences.

1 Introduction

We consider a problem of allocating several infinitely divisible commodities among a group of agents. We assume that each agent has a continuous, strictly convex, and "separable" preference. A preference is *separable* if the preference over consumption of each commodity is not affected by the consumption levels of the other commodities. To study this allocation problem, we conduct an axiomatic analysis of (allocation) "rules". A *rule* is a function which assigns a feasible allocation for each preference profile.

Preferences are usually private information. Agents may strategically misrepresent their preferences to obtain assignments they prefer. As a result, the assigned alloca-

S. Morimoto (✉) · S. Serizawa
Institute of Social and Economic Research, Osaka University, 6-1, Mihogaoka, Ibaraki 567-0047, Japan
e-mail: morimoto@iser.osaka-u.ac.jp

S. Serizawa
e-mail: serizawa@iser.osaka-u.ac.jp

S. Ching
School of Economics and Finance, The University of Hong Kong, Pokfulam Road, Pokfulam, Hong Kong
e-mail: steve.ching@hku.hk

tions may not be socially desirable with respect to the agents' true preferences. Thus, it is important for a rule to give agents the incentive to represent their preferences truthfully. The property is called *strategy-proofness*.[1] Our goal is to identify the class of *strategy-proof* rules which yield socially desirable allocations.

In the one-commodity case, it is well-known that the rule called the "uniform rule" is *strategy-proof*. For this rule, agents are allowed to choose their consumption subject to a common upper or lower bound, and the common bound is chosen so as to attain feasibility. In this article, we characterize a multiple-commodity version of the uniform rule by *strategy-proofness* and the following axioms.

The first axiom is *unanimity*: if the sum of the peak amounts of each commodity is equal to the supply of the commodity, then each agent's assignment should be equal to his own peak vector. Indeed, there is no conflict on how to allocate the commodities in such a case, and agents unanimously would agree that all of them are allocated their peak amounts respectively. *Unanimity* is weaker than *Pareto efficiency*. This weaker efficiency axiom is used here, because *strategy-proofness* and *Pareto-efficiency* with a mild fairness axiom are incompatible in the multiple-commodity case.[2] This is an important difference between the one- and multiple-commodity models.

The second axiom is *weak symmetry*: two agents with the same preferences should receive assignments between which they are indifferent. If two agents have the same preference but one of them obtains a better assignment, then this agent is deemed to be favored. The axiom of *weak symmetry* excludes such unfair treatments.

The third axiom is *nonbossiness*: the change of an agent's preference does not alter allocations unless it alters his own assignment. Satterthwaite and Sonnenschein (1981) motivated *nonbossiness* as an axiom of simplicity. Since then, *nonbossiness* has been employed by many authors and accepted at face value.[3]

We offer an interpretation of *nonbossiness* in the spirit of "consistency". A typical application of *consistency* assumes that an agent is departed with his assignment. Such a departure leaves the amounts to be allocated to the remaining agents unchanged. *Consistency* requires the same allocation be given to the remaining agents.

The assumption of *nonbossiness* can be interpreted as not a departure of an agent, but a replacement of the agent with a new agent and the new agent is given the same assignment. Such a replacement again leaves the amounts to be allocated to all other agents unchanged. *Nonbossiness* requires the same allocation be given to all other agents. Hence, *nonbossiness* is in the spirit of *consistency* and can be called *replacement consistency*.[4]

We establish that *on the class of continuous, strictly convex, and separable preferences, a rule satisfies strategy-proofness, unanimity, weak symmetry, and nonbossiness*

---

[1] *Strategy-proofness* requires that, in the normal form game induced by the rule, it is a weakly dominant strategy for each agent to reveal his true preference.

[2] See Proposition 1 in Sect. 2.

[3] For example, Barberà and Jackson (1995), Serizawa (1996), Barberà et al. (1997), Fleurbaey and Maniquet (1997), Pápai (2000), Schummer (2000), Miyagawa (2001), Svensson and Larsson (2002), Ehlers and Klaus (2003), Bogomolnaia et al. (2005), Hatfield (2009), etc.

[4] See Thomson (1990, 2004) for comprehensive reviews of the literature on *consistency*.

if and only if it is the uniform rule. This result extends to the class of continuous, strictly convex, and "multidimensional single-peaked" preferences.[5]

Sprumont (1991) gave the first axiomatic characterization of the uniform rule, which was introduced in a general (dis)equilibrium model with fixed prices (Bénassy 1982). In the one-commodity case, he showed that the uniform rule is the only rule satisfying *strategy-proofness*, *Pareto-efficiency*, and *anonymity*.[6] He also showed that *anonymity* can be replaced by *no-envy*.[7] Ching (1994) strengthened both characterizations by weakening *anonymity* or *no-envy* to *weak symmetry*.[8]

It is desirable to analyze the uniform rule in the multiple-commodity case so that it can be applied to Bénassy (1982) general equilibrium model with fixed prices. Consider an economy with money and several (nonmonetary) commodities in which money consumption is not bounded. Agents have preferences which are continuous, strictly convex, separable, and linear with respect to money. If prices are exogenously fixed as in Bénassy (1982), then preferences only on consumption of commodities are induced, and they are continuous, strictly convex, and separable. Our result can be applied to this class of economies.[9]

Also, consider a situation where agents contribute several types of input to joint production process. The total amount of each input is fixed, and the contribution of each type of input is rewarded proportionally. An agent receives a quantity of output equal to the sum of the rewards to his contributions. If agents' preferences are continuous, strictly convex, separable, and linear with respect to money, then preferences only on contribution of inputs are continuous, strictly convex, and separable. Our result can be also applied to the problem of how agents contribute each type of input.

Moulin (1980) characterized the class of *strategy-proof* voting schemes in one-dimensional public alternative model where agents have single-peaked preferences: a rule satisfies *strategy-proofness* and *unanimity* if and only if it is a so-called "generalized median voter scheme". Border and Jordan (1983) established that when the space of public alternatives is multidimensional and each agent has continuous, strictly convex, separable, and star-shaped[10] preferences, a rule is *strategy-proof* and *unanimous* if and only if it can be decomposed into a product of one-dimensional rules, each

[5] A preference is *multidimensional single-peaked* if there is an ideal consumption point $p \equiv (p_1, \ldots, p_m)$, called the *peak*, and for any two distinct consumption bundles $x \equiv (x_1, \ldots, x_m)$ and $y \equiv (y_1, \ldots, y_m)$, whenever x_ℓ is between y_ℓ and p_ℓ for each dimension $\ell = 1, \ldots, m$, x is strictly preferred to y. The domain of continuous, strictly convex, and separable preferences is a subclass of the multidimensional single-peaked domain.

[6] *Anonymity* requires that if two agent's preferences are switched, then their assignments should be switched too.

[7] *No-envy* requires that no agent should prefer anyone else's assignment to his own.

[8] The one-commodity allotment problem has been analyzed from a wide variety of viewpoints. See, for example, Thomson (1994a,b, 1995, 1997), Otten et al. (1996), Barberà et al. (1997), Ching and Serizawa (1998), Massó and Neme (2001, 2007), and Serizawa (2006).

[9] Example 1 in Sect. 2 describes the application of the model to Bénassy's (1982) model in more detail.

[10] A preference is *star-shaped* if there is an ideal consumption point p such that for any bundle x differing from p, and any real number $a \in (0, 1)$, $a \cdot p + (1 - a) \cdot x$ is strictly preferred to x and p is strictly preferred to $a \cdot p + (1 - a) \cdot x$. Note that, in our model, if a preference is continuous and strictly convex, then it is star-shaped.

of which is a generalized median voter scheme. As Border and Jordan (1983) generalized Moulin's (1980) result to a multidimensional model, we generalize the results of Sprumont (1991) and Ching (1994) to a model with multiple private commodities.

Amorós (2002) analyzes the same model as ours. Assuming that there are only two agents, he shows that a rule defined on the class of multidimensional single-peaked preferences satisfies *strategy-proofness*, *same-sideness*,[11] and *no-envy* (alternatively, *symmetry*[12]) if and only if it is the uniform rule.[13] In the two-agent case, since the assignment of one agent determines that of the other agent, *nonbossiness* automatically holds, and one agent's consumption can be treated as if it were a public alternative. Accordingly, the model of Amorós (2002) can be treated as a special case of Border and Jordan's (1983) model, and his result can be derived from theirs,[14] although his proof differs from theirs. On the other hand, when there are more than two agents, Border and Jordan's result cannot be applied.

It is worthwhile to emphasize the difficulty of characterizing *strategy-proof* rules in models with several private goods and agents as in ours. In private good models with more than two agents, the assignment of one agent leaves freedom for other agents' consumptions. When there is only one private good, this freedom is relatively limited. On the other hand, when there are multiple private goods and agents as in our model, such freedom is much wider, and this fact makes characterizing *strategy-proof* rules more difficult. Since *unanimity* is much weaker than *Pareto-efficiency*, the characterization becomes even more difficult.

This paper is organized as follows. Section 2 explains the model and the main results. Section 3 is devoted to the proof of the results in Sect. 2. Section 4 provides concluding remarks.

2 The model and the results

2.1 Model setting

Let $M \equiv \{1, \ldots, m\}$ be a set of infinitely divisible commodities. For each commodity $\ell \in M$, there is an amount $W_\ell \in \mathbb{R}_{++}$ to be allocated. Let $W \equiv (W_1, \ldots, W_m) \in \mathbb{R}^m_{++}$. Let $N \equiv \{1, \ldots, n\}$ be a set of agents. Assume that $2 \leq n < \infty$. For each $i \in N$, agent i's consumption set is $X \equiv \{x^i \in \mathbb{R}^m_+ \mid \text{for each } \ell \in M,\ 0 \leq x^i_\ell \leq W_\ell\}$, and

[11] *Same-sideness* requires that for each commodity, if the sum of the peak amounts of the commodity is greater (smaller) than, or equal to, the supply of the commodity, then each agent's assignment of the commodity should be smaller (greater) than, or equal to, his own peak amount of the commodity. It is a necessary condition for *Pareto-efficiency*.

[12] *Symmetry* requires that two agents with the same preference should receive the same assignment.

[13] Sasaki (2003) also showed that the uniform rule is the most efficient rule among all *strategy-proof* rules in the two-agent and multiple-commodity model.

[14] Since *same-sideness* implies *unanimity*, Border and Jordan's (1983) result implies that a rule satisfies *strategy-proofness* and *same-sideness* if and only if it can be decomposed into a product of one-dimensional rules, each of which is a generalized median voter scheme. *No-envy* (alternatively, *symmetry*) implies that the one-dimensional rule for each commodity is the uniform rule.

agent i's (consumption) bundle is a vector $x^i \equiv (x^i_\ell)_{\ell\in M} \in X$. For each $\ell \in M$, let $X_\ell \equiv [0, W_\ell]$ and $X_{-\ell} \equiv \prod_{\ell'\neq\ell}[0, W_{\ell'}]$.

Each agent $i \in N$ has a complete and transitive preference relation R^i on X. Let P^i be the strict preference relation associated with R^i, and I^i the indifference relation. Given a preference R^i and a bundle $x \in X$, the upper contour set of R^i at x is the set $UC(R^i, x) \equiv \{y \in X \mid y \; R^i \; x\}$, and the lower contour set of R^i at x is the set $LC(R^i, x) \equiv \{y \in X \mid x \; R^i \; y\}$. The preference R^i is continuous if for each $x \in X$, $UC(R^i, x)$ and $LC(R^i, x)$ are closed. It is strictly convex if for each $x \in X$, each pair $\{y, z\} \subset UC(R^i, x)$, and each $a \in (0, 1)$, $y \neq z$ implies $ay + (1-a)z \; P^i \; x$. We assume that preferences are continuous and strictly convex.[15] Given a preference R^i, let $p(R^i) \equiv \{x \in X \mid \text{for each } y \in X, x \; R^i \; y\}$ be the set of preferred consumptions according to R^i. Since R^i is continuous and strictly convex, $p(R^i)$ is a singleton. We call $p(R^i)$ the peak of R^i and write $(p_\ell(R^i))_{\ell\in M} \equiv p(R^i)$. We also define two additional properties of preferences.

Definition 1 A preference relation R^i on X is **separable** if for each $\ell \in M$, each pair $x^i_\ell, \hat{x}^i_\ell \in X_\ell$, and each pair $x^i_{-\ell}, \hat{x}^i_{-\ell} \in X_{-\ell}$, $(x^i_\ell, x^i_{-\ell}) \; R^i \; (\hat{x}^i_\ell, x^i_{-\ell})$ if and only if $(x^i_\ell, \hat{x}^i_{-\ell}) \; R^i \; (\hat{x}^i_\ell, \hat{x}^i_{-\ell})$.

Definition 2 A preference relation R^i on X is **multidimensional single-peaked** if $p(R^i)$ is a singleton, and for each pair $x^i, \hat{x}^i \in X$ such that $x^i \neq \hat{x}^i$, whenever for each $\ell \in M$, either $p_\ell(R^i) \geq x^i_\ell \geq \hat{x}^i_\ell$ or $p_\ell(R^i) \leq x^i_\ell \leq \hat{x}^i_\ell$, we have $x^i \; P^i \; \hat{x}^i$.

Let $\mathcal{R}$ denote the class of continuous, strictly convex, and separable preference relations on X. Any such relation is multidimensional single-peaked. Thus, the class of continuous, strictly convex, and separable preference relations is a subclass of continuous, strictly convex, and multidimensional single-peaked preference relations.

A **feasible allocation** is a list $x \equiv (x^i)_{i\in N} \in X^n$ such that $\sum_{i\in N} x^i = W$. Note that free disposal is not assumed. Let $\boldsymbol{Z}$ be the set of feasible allocations.

A **preference profile** is a list $R \equiv (R^1, \ldots, R^n) \in \mathcal{R}^n$. An allocation rule, or simply a rule, is a function $f : \mathcal{R}^n \to Z$. Let R^{-i} be a list of preferences for all agents except for agent i, that is, $R^{-i} \equiv (R^j)_{j\in N\setminus\{i\}}$. We often write the profile $(R^1, \ldots, R^{i-1}, \bar{R}^i, R^{i+1}, \ldots, R^n)$ as $(\bar{R}^i, R^{-i})$. Let $f^i(R) \equiv (f^i_1(R), \ldots, f^i_m(R))$ be the bundle assigned to agent i by f when the preference profile is R.

Example 1 describes the application of the model to Bénassy's (1982) general equilibrium model with fixed prices.

Example 1 Consider an economy with money and m commodities in which money consumption is not bounded. Let $\hat{R}^i$ be the agent i's continuous, strictly convex, and separable preference defined on $\mathbb{R}^m_+ \times \mathbb{R}$, where the last coordinate denotes agent i's money consumption t^i. Assume that preferences are linear in money consumption, that is, for each preference $\hat{R}^i$, there is a function U^i on $\mathbb{R}^m_+$ such that for each $(x^i, t^i) \in \mathbb{R}^m_+ \times \mathbb{R}$, and each $(\hat{x}^i, \hat{t}^i) \in \mathbb{R}^m_+ \times \mathbb{R}$,

$$\left(x^i, t^i\right) \hat{R}^i \left(\hat{x}^i, \hat{t}^i\right) \text{ if and only if } U^i\left(x^i\right) + t^i \geq U^i\left(\hat{x}^i\right) + \hat{t}^i.$$

[15] If R^i is continuous and strictly convex, then for each $x \in X$, the set $UC(R^i, x)$ is strictly convex. The converse is not true.

Also, assume that prices $p \in \mathbb{R}^m_{++}$ of m commodities are exogenously fixed as in Bénassy (1982). For each preference $\hat{R}^i$ on $\mathbb{R}^m_+ \times \mathbb{R}$, the preference R^i on X is induced as follows: for each pair $x, \hat{x} \in X$,

$$x \ R^i \ \hat{x} \text{ if and only if } (x, t^i) \ \hat{R}^i \ (\hat{x}, \hat{t}^i),$$

where $t^i = p \cdot (W^i - x)$, $\hat{t}^i = p \cdot (W^i - \hat{x})$, and W^i is agent i's endowment vector. Then, the induced preferences on X are continuous, strictly convex, and separable.

2.2 Axioms for rules

We introduce the axioms for rules. Let f denotes a general rule in this section. The first axiom we introduce is *strategy-proofness*, an incentive property: by misrepresenting his preferences, no agent should obtain an assignment that he prefers.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, $f^i(R) \ R^i \ f^i(\hat{R}^i, R^{-i})$.

The next three axioms are related to efficiency. The first efficiency axiom is *Pareto-efficiency*. An allocation $x \in Z$ is **Pareto-efficient for R** if there is no $y \in Z$ such that, for each $i \in N$, $y^i \ R^i \ x^i$, and for some $j \in N$, $y^j \ P^j \ x^j$. For each $R \in \mathcal{R}^n$, let $P(R)$ be the set of Pareto-efficient allocations for R.

Pareto-efficiency: For each $R \in \mathcal{R}^n$, $f(R) \in P(R)$.

The second efficiency axiom is *same-sideness*: for each commodity, if the sum of the peak amounts of the commodity is greater than, or equal to, the supply of the commodity, then each agent's assignment of the commodity should be smaller than, or equal to, his own peak amount of the commodity, and conversely.

Same-sideness: For each $R \in \mathcal{R}^n$ and each $\ell \in M$,

(i) if $\sum_{i \in N} p_\ell(R^i) \geq W_\ell$, then for each $i \in N$, $f^i_\ell(R) \leq p_\ell(R^i)$, and
(ii) if $\sum_{i \in N} p_\ell(R^i) \leq W_\ell$, then for each $i \in N$, $f^i_\ell(R) \geq p_\ell(R^i)$.

In the one-commodity case, *same-sideness* is equivalent to *Pareto-efficiency*.[16] In the multiple-commodity case, *Pareto-efficiency* implies *same-sideness*, but the converse is not true. Example 2 illustrates this fact.

Example 2 Let $N \equiv \{1, 2\}$ and $M \equiv \{1, 2\}$. Let f be the rule defined as follows.[17] For each $R \in \mathcal{R}^2$, each $i \in \{1, 2\}$, and each $\ell \in \{1, 2\}$,

$$f^i_\ell(R) \equiv \begin{cases} \frac{p_\ell(R^i) \cdot W_\ell}{p_\ell(R^1) + p_\ell(R^2)} & \text{if } p_\ell(R^1) + p_\ell(R^2) > 0 \\ \frac{W_\ell}{2} & \text{otherwise.} \end{cases}$$

Then, f is *same-sided*. Let $R \in \mathcal{R}^2$ be such that for each $\ell \in \{1, 2\}$, $p_\ell(R^1) = p_\ell(R^2) = W_\ell$, and there is a bundle $y^1 \in Z$ such that $y^2 = W - y^1$ and for each

16 See Sprumont (1991).

17 This rule is called **Proportional Rule**.

$i \in \{1, 2\}$, $y^i \ P^i \ (\frac{W_1}{2}, \frac{W_2}{2})$. Then, $f^1(R) = f^2(R) = (\frac{W_1}{2}, \frac{W_2}{2})$. However, $f(R)$ is Pareto-dominated by y, contradicting *Pareto-efficiency*.

The third efficiency axiom is *unanimity*: if the sum of the peak amounts of each commodity is equal to the supply of the commodity, then each agent's assignment should be equal to his own peak vector.

Unanimity: For each $R \in \mathcal{R}^n$, if for each $\ell \in M$, $\sum_{i \in N} p_\ell(R^i) = W_\ell$, then for each $i \in N$, $f^i(R) = p(R^i)$.

Obviously, *same-sideness* implies *unanimity*. It is the weakest of our three axioms related to efficiency.

Our next four axioms have to do with fairness. The first fairness axiom is *no-envy*: no agent should prefer anyone else's assignment to his own. The second fairness axiom is *anonymity*: if two agents' preferences are switched, then their assignments should be switched too.

No-envy (Foley 1967): For each $R \in \mathcal{R}^n$ and each $i, j \in N$, $f^i(R) \ R^i \ f^j(R)$.

Anonymity: For each $R \in \mathcal{R}^n$, each $i, j \in N$, and each $\hat{R}^i, \hat{R}^j \in \mathcal{R}$, if $\hat{R}^i = R^j$ and $\hat{R}^j = R^i$, then $f^i(\hat{R}^i, \hat{R}^j, R^{-i,j}) = f^j(R)$.

The third fairness axiom is *symmetry*: two agents with the same preferences should receive equal assignments.

Symmetry: For each $R \in \mathcal{R}^n$ and each $i, j \in N$, if $R^i = R^j$, then $f^i(R) = f^j(R)$.

Note that *anonymity* implies *symmetry*. The fourth fairness axiom is *weak symmetry*: two agents with the same preferences should receive assignments between which they are indifferent.

Weak symmetry: For each $R \in \mathcal{R}^n$ and each $i, j \in N$, if $R^i = R^j$, then $f^i(R) \ I^i \ f^j(R)$.

Either *no-envy* or *symmetry* implies *weak symmetry*. *Weak symmetry* is the weakest of our four axioms concerning fairness. In the one-commodity case, for any rule satisfying *Pareto-efficiency*, *weak symmetry* is equivalent to *symmetry*.[18] However, in the multiple-commodity case, *same-sideness* and *weak symmetry* do not imply *symmetry*.

Next is an axiom introduced by Satterthwaite and Sonnenschein (1981), known as *nonbossiness*: when an agent's preferences change, if his assignment remains the same, then the chosen allocation should remain the same.

Nonbossiness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, if $f^i(\hat{R}^i, R^{-i}) = f^i(R)$, then $f(\hat{R}^i, R^{-i}) = f(R)$.

A usual interpretation of R^i to $\hat{R}^i$ in *nonbossiness* is a preference change of agent i. Alternatively, we can interpret it as agent R^i being replaced by agent $\hat{R}^i$. The hypothesis $f^i(\hat{R}^i, R^{-i}) = f^i(R)$ can then be regarded as the new agent $\hat{R}^i$ receiving agent R^i's original assignment. This leaves the amounts to be allocated to all other agents $j \neq i$ unchanged ($W - f^i(\hat{R}^i, R^{-i}) = W - f^i(R^i, R^{-i})$). Now we can apply the idea of *consistency*, which is to give the same allocation to all other agents, i.e.

18 See Ching (1994).

$f^{-i}(\hat{R}^i, R^{-i}) = f^{-i}(R)$. This conclusion is the same as the one obtained from *nonbossiness*, so *nonbossiness* is in the spirit of *consistency* and can be specifically called *replacement consistency*.

By *feasibility*, the following holds.

Remark 1 If there are only two agents, then any rule is *nonbossy*.

2.3 Uniform rule

Now, we introduce the uniform rule, a rule that is central to our paper. Under this rule, for each commodity, agents are allowed to choose their consumption subject to a common upper or lower bound, and the common bound is chosen so as to attain feasibility.

Uniform rule, U: For each $R \in \mathcal{R}^n$, each $\ell \in M$, and each $i \in N$,

$$U^i_\ell(R) = \begin{cases} \min\{p_\ell(R^i), \lambda_\ell(R)\} & \text{if } \sum_{j\in N} p_\ell(R^j) \geq W_\ell \\ \max\{p_\ell(R^i), \lambda_\ell(R)\} & \text{if } \sum_{j\in N} p_\ell(R^j) \leq W_\ell, \end{cases}$$

where $\lambda_\ell(R)$ solves $\sum_{j\in N} U^j_\ell(R) = W_\ell$.

Example 3 illustrates the definition.

Example 3 Let $N \equiv \{1, 2, 3, 4\}$, $M \equiv \{1, 2\}$, and $(W_1, W_2) \equiv (10, 20)$. Let $R \in \mathcal{R}^4$ be such that $p(R^1) = (3, 5)$, $p(R^2) = p(R^3) = (2, 2)$, and $p(R^4) = (5, 6)$. Then, $\sum_{i\in N} p_1(R^i) > W_1$ and $\sum_{i\in N} p_2(R^i) < W_2$. We calculate $\lambda_1(R) = 3$ and $\lambda_2(R) = 4.5$. Then, $U^1(R) = (3, 5)$, $U^2(R) = U^3(R) = (2, 4.5)$, and $U^4(R) = (3, 6)$.

2.4 Main results

In this section, we state our main results. First, as we mentioned in Introduction, we explain that *strategy-proofness* and *Pareto-efficiency* with a mild fairness axiom are incompatible in the multiple-commodity case.

It is known that in a pure exchange economy, when agents have selfish, monotonic, continuous, strictly convex, homothetic, and smooth preferences, there is no *strategy-proof*, *Pareto-efficient* and *weakly symmetric* rule (Serizawa 2002). Since the class of selfish, monotonic, continuous, strictly convex, homothetic, and smooth preferences is a subclass of continuous, strictly convex, and separable preferences, this impossibility result can be applied to our model to obtain the same conclusion.[19] Thus, we have the following impossibility result.

Proposition 1 *There is no strategy-proof, Pareto-efficient, and weakly symmetric rule on the domain of continuous, strictly convex, and separable preferences.*

[19] See Anno and Sasaki (2009) for detailed discussion.

Because of this impossibility result, if *strategy-proofness* and *weak symmetry* are to be kept, *Pareto-efficiency* needs to be weakened. We weaken it to *unanimity* and obtain the following characterization of the uniform rule.

Theorem 1 *A rule defined on the domain of continuous, strictly convex, and separable preferences satisfies strategy-proofness, unanimity, weak symmetry, and nonbossiness if and only if it is the uniform rule.*

The proof of Theorem 1 is in Sect. 3.

Remark 2 The *only if* part of Theorem 1 fails if we drop any of the first three axioms. The proportional rule[20] satisfies *unanimity*, *(weak) symmetry*, and *nonbossiness*, but not *strategy-proofness*. The queuing rules[21] satisfy *strategy-proofness*, *unanimity*, and *nonbossiness*, but not *(weak) symmetry*. The equal distribution rule[22] satisfies *strategy-proofness*, *(weak) symmetry*, and *nonbossiness*, but not *unanimity*. However, it is an open question whether *nonbossiness* is dispensable or not.

We briefly explain the difficulties to show the indispensability of *nonbossiness*. Example 4 illustrates a *strategy-proof*, *unanimous*, and *bossy* rule.

---

20 **Proportional rule, Pro**: For each $R \in \mathcal{R}^n$, each $\ell \in M$, and each $i \in N$,

$$Pro^i_\ell(R) = \begin{cases} \frac{p_\ell(R^i)\cdot W_\ell}{\sum_{j\in N} p_\ell(R^j)} & \text{if } \sum_{j\in N} p_\ell(R^j) > 0 \\ \frac{W_\ell}{n} & \text{otherwise.} \end{cases}$$

21 **Queuing rule, Q^π, associated with the permutation π on N**: For each $R \in \mathcal{R}^n$ and each $\ell \in M$,

$$\begin{aligned} Q^{\pi(1)}_\ell(R) &= p_\ell(R^{\pi(1)}) \\ Q^{\pi(2)}_\ell(R) &= \min\left\{p_\ell(R^{\pi(2)}),\ W_\ell - Q^{\pi(1)}_\ell(R)\right\} \\ Q^{\pi(3)}_\ell(R) &= \min\left\{p_\ell(R^{\pi(3)}),\ W_\ell - Q^{\pi(1)}_\ell(R) - Q^{\pi(2)}_\ell(R)\right\} \\ &\vdots \\ Q^{\pi(n)}_\ell(R) &= W_\ell - \sum_{j=1}^{n-1} Q^{\pi(j)}_\ell(R). \end{aligned}$$

22 **Equal distribution rule, E**: For each $R \in \mathcal{R}^n$, each $\ell \in M$, and each $i \in N$,

$$E^i_\ell(R) = \frac{W_\ell}{n}.$$

Example 4 Let $N \equiv \{1,2,3\}$ and $M \equiv \{1,2\}$. Let g be a rule such that for each $R \in \mathcal{R}^3$ and each $\ell \in M$,

$$g^1_\ell(R) = p_\ell(R^1)$$
$$g^2_\ell(R) = \begin{cases} \min\left\{p_\ell(R^2),\ W_\ell - g^1_\ell(R)\right\} & \text{if } W\ P^1\ (0,0), \\ W_\ell - g^1_\ell(R) - g^3_\ell(R) & \text{if } (0,0)\ R^1\ W, \end{cases}$$
$$g^3_\ell(R) = \begin{cases} W_\ell - g^1_\ell(R) - g^2_\ell(R) & \text{if } W\ P^1\ (0,0), \\ \min\left\{p_\ell(R^3),\ W_\ell - g^1_\ell(R)\right\} & \text{if } (0,0)\ R^1\ W. \end{cases}$$

Note that, since agent 1 is a dictator under the rule g, this rule is not *weakly symmetric*.[23] The important feature of this rule is that the assignments of agent 2 and 3 depend on the agent 1's preferences, that is, the assignment of an agent is determined by the preferences of someone else. By constructing this kind of rules, we can find many *bossy* rules that are *strategy-proof* and *unanimous*. Thus, it might be possible to construct a *bossy* rule satisfying the first three properties in Theorem 1. At the same time, however, since we need to take into account the requirement of *weak symmetry* in addition to *strategy-proofness* and *unanimity*, it is too complicated to construct such a rule.

Our result extends to the domain of continuous, strictly convex, and multidimensional single-peaked preferences. The following is a corollary of Theorem 1.

Corollary 1 *A rule defined on the domain of continuous, strictly convex, and multidimensional single-peaked preferences satisfies strategy-proofness, unanimity, weak symmetry, and nonbossiness if and only if it is the uniform rule.*

Proof of Corollary 1 Let $\mathcal{R}_M$ be the class of continuous, strictly convex, and multidimensional single-peaked preferences. Let f be a rule on $\mathcal{R}^n_M$ satisfying the four axioms. Let $R \in \mathcal{R}^n_M$. Let $\hat{R} \in \mathcal{R}^n$ be such that for each $i \in N$, (i) $p(\hat{R}^i) = p(R^i)$, and (ii) $UC(\hat{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\hat{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$.[24] Then, by *strategy-proofness*, $f^1(\hat{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\hat{R}^1, R^{-1}) = f(R)$. Repeating this argument for $i = 2, \ldots, n$, we have $f(\hat{R}) = f(R)$. By Theorem 1, $f(\hat{R}) = U(\hat{R})$. Since the uniform rule is *peak-only*, $U(R) = U(\hat{R})$. Hence, $f(R) = U(R)$. □

In the two-agent case, by Remark 1, any rule is *nonbossy*. As we mentioned above, *unanimity* is weaker than *same-sideness*, and *weak symmetry* is weaker than either *no-envy* or *symmetry*. Thus, we obtain Amorós' (2002) result as a corollary of Theorem 1.

23 Note that the rule g is *Pareto-efficient*. Thus, Example 4 also illustrates a *strategy-proof* and *Pareto-efficient* rule other than the queuing rules. By constructing this kind of *bossy* rules, we can find many *strategy-proof* and *Pareto-efficient* rules other than the queuing rules.

24 The condition (ii) means that $\hat{R}^i$ is a strict Maskin monotonic transformation of R^i at $f^i(R)$ (Maskin 1999). See Fact A in Appendix A for the existence of such transformations.

Corollary 2 (Amorós 2002) *Assume that there are two agents. A rule defined on the domain of multidimensional single-peaked preferences satisfies strategy-proofness, same-sideness, and no-envy (alternatively, symmetry) if and only if it is the uniform rule.*

Remark 3 Although the uniform rule is *Pareto-efficient* in the one-commodity case, it is not in the multiple-commodity case as illustrated in Example 5 below.

Remark 4 In many models with private goods, *group strategy-proofness* follows from *strategy-proofness* and *nonbossiness*[25] (Barberà and Jackson (1995) for pure exchange economies; Serizawa (1996) for public goods economies). Furthermore, the uniform rule is *group strategy-proof* in the one-commodity case. Under the domain condition called "indirect sequential inclusion", *strategy-proofness* coincides with *group strategy-proofness* in the two-agent case (Barberà et al. 2010).[26]

However, as Example 5 below illustrates, the uniform rule is not *group strategy-proof* in the multiple-commodity case. Thus, we cannot replace *nonbossiness* with *group strategy-proofness* in Theorem 1. Example 5 also illustrates that *group strategy-proofness* does not follow from *strategy-proofness* and *nonbossiness* in the multiple-commodity case. In the one-commodity case, the domain of single-peaked preferences satisfies *indirect sequential inclusion*. On the other hand, in the multiple-commodity case, the preference domain studied in this paper does not satisfy this domain condition.[27] Thus, we cannot apply their results to our model. This is an interesting difference between the one- and multiple-commodity model.

Example 5 Let $N \equiv \{1, 2\}$, $M \equiv \{1, 2\}$, and $R \in \mathcal{R}^2$ be such that $p(R^1) \equiv (0, 0) \equiv p(R^2)$, $\left(\frac{2W_1}{3}, \frac{W_2}{3}\right)$ P^1 $\left(\frac{W_1}{2}, \frac{W_2}{2}\right)$ and $\left(\frac{W_1}{3}, \frac{2W_2}{3}\right)$ P^2 $\left(\frac{W_1}{2}, \frac{W_2}{2}\right)$. Then, $U^1(R) = \left(\frac{W_1}{2}, \frac{W_2}{2}\right)$ and $U^2(R) = \left(\frac{W_1}{2}, \frac{W_2}{2}\right)$. Let $\hat{R} \in \mathcal{R}^2$ be such that $p(\hat{R}^1) \equiv \left(\frac{2W_1}{3}, 0\right)$ and $p(\hat{R}^2) \equiv \left(0, \frac{2W_2}{3}\right)$. Then, $U^1(\hat{R}) = \left(\frac{2W_1}{3}, \frac{W_2}{3}\right)$ and $U^2(\hat{R}) = \left(\frac{W_1}{3}, \frac{2W_2}{3}\right)$. Thus, for each $i \in N$, $U^i(\hat{R})$ P^i $U^i(R)$. That is, the uniform rule is not *group strategy-proof*. Also, since $U(\hat{R})$ Pareto-dominates $U(R)$, the uniform rule is not *Pareto-efficient*.

3 Proof of theorem

In this section, we prove Theorem 1. It is easy to check that the uniform rule is *strategy-proof*, *unanimous*, and *weakly symmetric*. Furthermore, we can easily verify that the single-commodity uniform rule is *nonbossy*. Since the multiple-commodity uniform rule applies the single-commodity uniform rule commodity by commodity, it is also *nonbossy*. Thus, the *if* part of Theorem 1 holds.

[25] **Group strategy-proofness**: For each $R \in \mathcal{R}^n$, there do not exist $\hat{N} \subset N$ and $\hat{R}^{\hat{N}} \in \mathcal{R}^{|\hat{N}|}$ such that for each $i \in \hat{N}$, $f^i(\hat{R}^{\hat{N}}, R^{-\hat{N}})$ P^i $f^i(R)$, where $|A|$ denotes the cardinality of set A.

[26] More precisely, they study the public alternative model with more than two agents. In our model, two-agent case can be regarded as a public good model as mentioned in Introduction. Thus, we can apply their arguments to our model for the two-agent case. The similar results are also founded in Le Breton and Zaporozhets (2009).

[27] See Appendix B for an example illustrating the violation of *indirect sequential inclusion* of our preference domain.

We turn to the *only if* part. That is, we pick a rule f satisfying the four axioms, and prove that f is the uniform rule. We present an informal sketch of the proof in Sect. 3.1, and discuss the key points and difficulties of the proof in Sect. 3.2. As in the proof of Corollary 1, we repeatedly use strict Maskin monotonic transformations in the proof of Theorem 2. See also Fact A in Appendix A for the existence of such transformations.

3.1 Sketch of proof of only if part

The proof consists of five Steps.

Step I: We show that *strategy-proofness*, *unanimity* and *nonbossiness* imply *same-sideness* in Lemma 1.

Next, let $\ell \in M$, and let $x_{-\ell}$ be a feasible allocation except for commodity ℓ. We concentrate on the restricted domain of preference profiles such that, except for commodity ℓ, each agent i's peak amount is equal to the fixed assignment $x^i_{-\ell}$. Given $\ell \in M$, $x_{-\ell} \in Z_{-\ell} \equiv \left\{(x^1_{-\ell}, \ldots, x^n_{-\ell}) \in (X_{-\ell})^n \mid \sum_{i \in N} x^i_{-\ell} = W_{-\ell}\right\}$, and $i \in N$, let $\bar{\mathcal{R}}^i(x_{-\ell}) \equiv \left\{R^i \in \mathcal{R} \mid p_{-\ell}(R^i) = x^i_{-\ell}\right\}$ and $\bar{\mathcal{R}}^N(x_{-\ell}) \equiv \prod_{i \in N} \bar{\mathcal{R}}^i(x_{-\ell})$. Then, $\bar{\mathcal{R}}^N(x_{-\ell})$ denotes the restricted domain. For simplicity, given $\ell \in M$, $R \in \mathcal{R}^n$, and $i \in N$, we write $p_{-\ell}(R^i) \equiv (p_{\ell'}(R^i))_{\ell' \neq \ell}$.

Note that each preference profile R in $\bar{\mathcal{R}}^N(x_{-\ell})$ is unanimous, i.e., $\sum_{i \in N} p_{-\ell}(R^i) = \sum_{i \in N} x^i_{-\ell} = W_{-\ell}$. Thus, it follows from *same-sideness* (Lemma 1) that on $\bar{\mathcal{R}}^N(x_{-\ell})$, the allocation chosen by the rule always coincides with the allocation x except for commodity ℓ, that is, for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, $f_{-\ell}(R) = p_{-\ell}(R) = x_{-\ell}$. Since the allocation except for commodity ℓ is fixed on $\bar{\mathcal{R}}^N(x_{-\ell})$ in this way, we can regard the restriction of f to $\bar{\mathcal{R}}^N(x_{-\ell})$ as if it were a single-commodity rule for commodity ℓ.

Step II: We show that the single-commodity rule for commodity ℓ on the restricted domain $\bar{\mathcal{R}}^N(x_{-\ell})$ derived in Step I (the restriction of f to $\bar{\mathcal{R}}^N(x_{-\ell})$) satisfies several standard properties.

Lemma 2 establishes two properties, "*own peak-onlyness*" and "*peak-onlyness*". *Own peak-onlyness* says that if an agent's preferences change but his peak amounts remain the same, then his assignment should remain the same. *Peak-onlyness* says that if all agents' preferences change but the peak profiles remain the same, then the chosen allocation should remain the same. *Peak-onlyness* implies *own peak-onlyness*. These properties are defined as follows.

Own peak-onlyness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, if $p(\hat{R}^i) = p(R^i)$, then $f^i(R) = f^i(\hat{R}^i, R^{-i})$.

Peak-onlyness: For each $R \in \mathcal{R}^n$ and each $\hat{R} \in \mathcal{R}^n$, if for each $i \in N$, $p(\hat{R}^i) = p(R^i)$, then $f(R) = f(\hat{R})$.

Lemma 3 establishes two properties, "*own uncompromisingness*" and "*group uncompromisingness*". *Own uncompromisingness* says that for each commodity ℓ and each agent, if his peak amount of ℓ is smaller (greater) than his assignment of ℓ and his new peak amount of ℓ is also smaller (greater) than, or equal to, his initial assignment of ℓ, then his assignment of ℓ should not change. *Group uncompromisingness* says that for each commodity ℓ and each group of agents, if for each agent in the group, the same assumption holds, then the chosen allocation of ℓ should not change. These properties are defined as follows.

Own uncompromisingness: For each $\ell \in M$, each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$,

if $p_\ell\left(R^i\right) < f_\ell^i(R)$ and $p_\ell\left(\hat{R}^i\right) \leq f_\ell^i(R)$, then, $f_\ell^i\left(\hat{R}^i, R^{-i}\right) = f_\ell^i(R)$,
if $p_\ell\left(R^i\right) > f_\ell^i(R)$ and $p_\ell\left(\hat{R}^i\right) \geq f_\ell^i(R)$, then, $f_\ell^i\left(\hat{R}^i, R^{-i}\right) = f_\ell^i(R)$.

Group uncompromisingness: For each $\ell \in M$, each $R \in \mathcal{R}^n$, each $\hat{N} \subseteq N$, and each $\hat{R}^{\hat{N}} \in \mathcal{R}^{|\hat{N}|}$,

if for each $i \in \hat{N}$, $p_\ell\left(R^i\right) < f_\ell^i(R)$ and $p_\ell\left(\hat{R}^i\right) \leq f_\ell^i(R)$, then, $f_\ell\left(\hat{R}^{\hat{N}}, R^{-\hat{N}}\right) = f_\ell(R)$,
if for each $i \in \hat{N}$, $p_\ell\left(R^i\right) > f_\ell^i(R)$ and $p_\ell\left(\hat{R}^i\right) \geq f_\ell^i(R)$, then, $f_\ell(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f_\ell(R)$.

Step III: We focus on the cases where some agents are almost indifferent to all commodities except for one commodity, say ℓ. We refer to the preferences of such agents as "d-indifferent to all commodities except for commodity ℓ," where d is a small positive number. They are defined as follows.

Definition 3 Given $\ell \in M$, and $d \in (0, \frac{W_\ell}{2n})$, a preference relation R^i on X with peak $p(R^i)$ is d-indifferent to all commodities except for commodity $\boldsymbol{\ell}$ if for each $y_\ell \in X_\ell$ and each $y_{-\ell} \in X_{-\ell}$, we have

$$\text{if } p_\ell\left(R^i\right) \leq y_\ell \leq W_\ell - d, \text{ then } \left(y_\ell, y_{-\ell}\right) P^i \left(y_\ell + d, p_{-\ell}(R^i)\right), \tag{a}$$
$$\text{if } d \leq y_\ell \leq p_\ell\left(R^i\right), \text{ then } \left(y_\ell, y_{-\ell}\right) P^i \left(y_\ell - d, p_{-\ell}\left(R^i\right)\right). \tag{b}$$

For sufficiently small $d > 0$, the indifference surfaces of such a preference at each consumption bundle are located within the distance d from the consumption bundle in the coordinate of commodity ℓ. That is, the indifference surfaces of the preference at each consumption bundle are almost parallel to the hyperplane with normal $e_\ell \equiv \left(0, \ldots, 0, 1, 0, \ldots, 0\right)$, where 1 appears in the ℓth coordinate. Figure 1 illustrates a preference that is d-indifferent to all commodities except for commodity ℓ.

Given $\ell \in M$, $x_\ell^i \in X_\ell$, $x_{-\ell}^i \in X_{-\ell}$, and $d \in \left(0, \frac{W_\ell}{2n}\right)$, let $\mathcal{R}^V\left(x_\ell^i, x_{-\ell}^i, d\right) \subset \mathcal{R}$ be the set of all preferences with peak $p\left(R^i\right) = \left(x_\ell^i, x_{-\ell}^i\right)$ that are d -indifferent to all commodities except for commodity ℓ.

Lemma 4 establishes that *for a strategy-proof and same-sided rule*, *if an agent is almost indifferent to all commodities except for commodity* ℓ, *and if his preference is changed, but he is still almost indifferent to all commodities except for commodity* ℓ, *then his assignment of commodity* ℓ *changes little* (Fig. 2). Lemma 5 establishes that *for a weakly symmetric and same-sided rule*, *if two agents have the same preference*

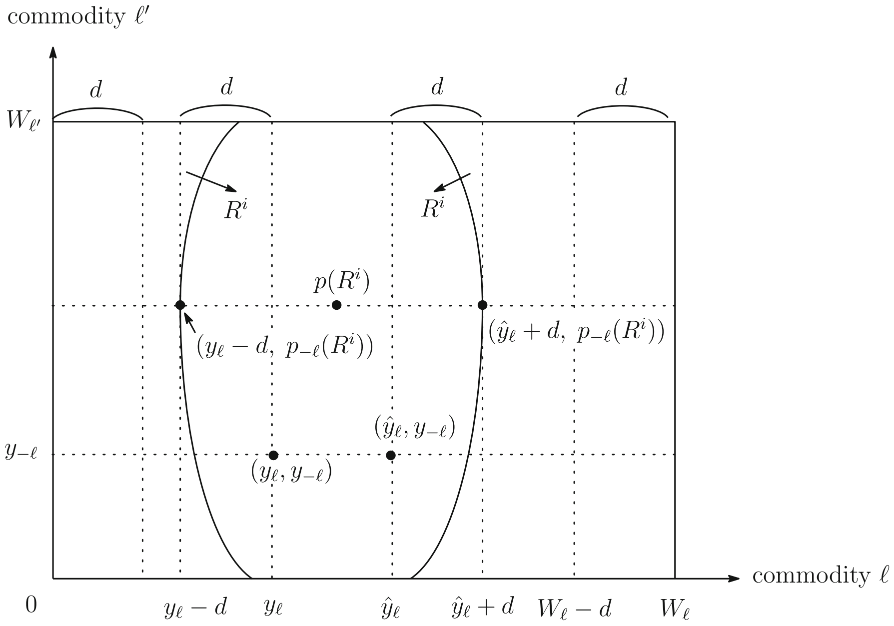

Fig. 1 Illustration of a preference R^i that is d-indifferent to all commodities except for commodity ℓ

and they are almost indifferent to all commodities except for commodity ℓ, *then their assignments of commodity* ℓ *differ little* (Fig. 3). Lemma 6 is the statement implied by Lemmas 4 and 5 for the case where several agents have the same preference and they are almost indifferent to all commodities except for one.

Step IV: We establish that the single-commodity rule for commodity ℓ derived in Step I coincides with the single-commodity uniform rule on the restricted domain $\bar{\mathcal{R}}^N(x_{-\ell})$ (Lemma 7), that is, for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, $f_\ell(R) = U_\ell(R)$. Lemmas 2 to 6 presented in Steps II and III are used to prove Lemma 7.

To present the basic idea of the proof of Lemma 7, we assume that there are only three agents and two commodities, say ℓ and ℓ'. Fix $R \in \bar{\mathcal{R}}^N(x_{-\ell})$. We also assume $\left(p_\ell(R^1), p_\ell(R^2), p_\ell(R^3)\right) = (0, 0, 0)$. Invoking *uncompromisingness* (Lemma 3), we extend the result obtained under this special assumption to more general peak profiles for commodity ℓ. Under these assumptions, the uniform rule prescribes equal division for commodity ℓ, that is, $U_\ell(R) = \left(\frac{W_\ell}{3}, \frac{W_\ell}{3}, \frac{W_\ell}{3}\right)$. Thus, we show that $f_\ell(R) = \left(\frac{W_\ell}{3}, \frac{W_\ell}{3}, \frac{W_\ell}{3}\right)$.

By contradiction, suppose $f_\ell(R) \neq \left(\frac{W_\ell}{3}, \frac{W_\ell}{3}, \frac{W_\ell}{3}\right)$. Then, by *feasibility*, there is an agent, say agent 1, whose assignment of commodity ℓ is greater than equal division.

For each agent i, let $\bar{R}^i$ be a preference that is almost indifferent to all commodities except for commodity ℓ, but whose peak is the same as $p\left(R^i\right)$. Then, since the peak profile remains the same, the new preference profile $\bar{R} \equiv \left(\bar{R}^1, \bar{R}^2, \bar{R}^3\right)$ still belongs to the restricted domain $\bar{\mathcal{R}}^N\left(x_{-\ell}\right)$. Thus, by *peak-onlyness* (Lemma 2), $f\left(\bar{R}\right) = f(R)$.

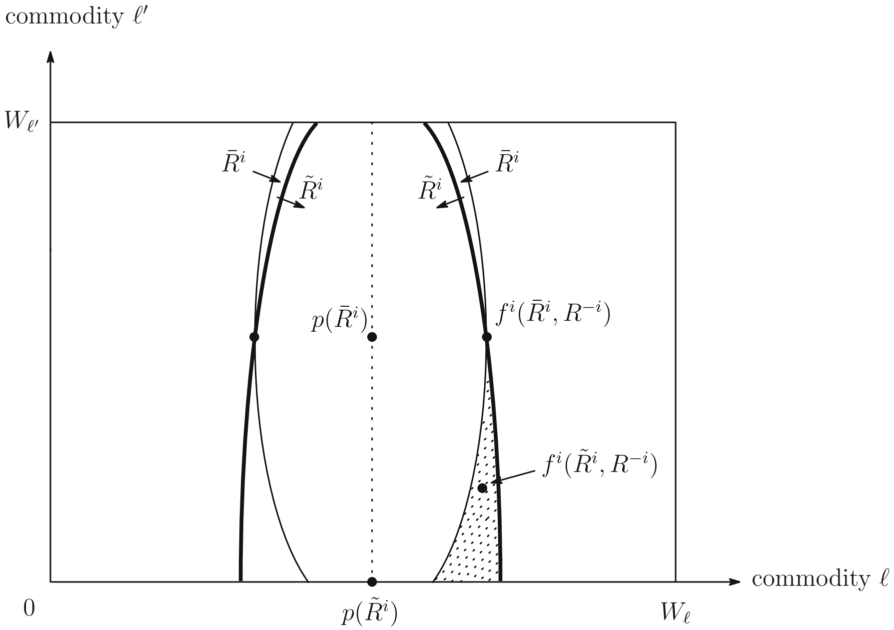

Fig. 2 Illustration of agent i's assignments in Lemma 4 for the case of excess supply. By *same-sideness*, $p_\ell(\bar{R}^i) \leq f^i_\ell(\bar{R}^i, R^{-i})$ and $p_\ell(\tilde{R}^i) \leq f^i_\ell(\tilde{R}^i, R^{-i})$. *Strategy-proofness* implies that $f^i(\bar{R}^i, R^{-i})\bar{R}^i f^i(\tilde{R}^i, R^{-i})$ and $f^i(\tilde{R}^i, R^{-i})\tilde{R}^i f^i(\bar{R}^i, R^{-i})$

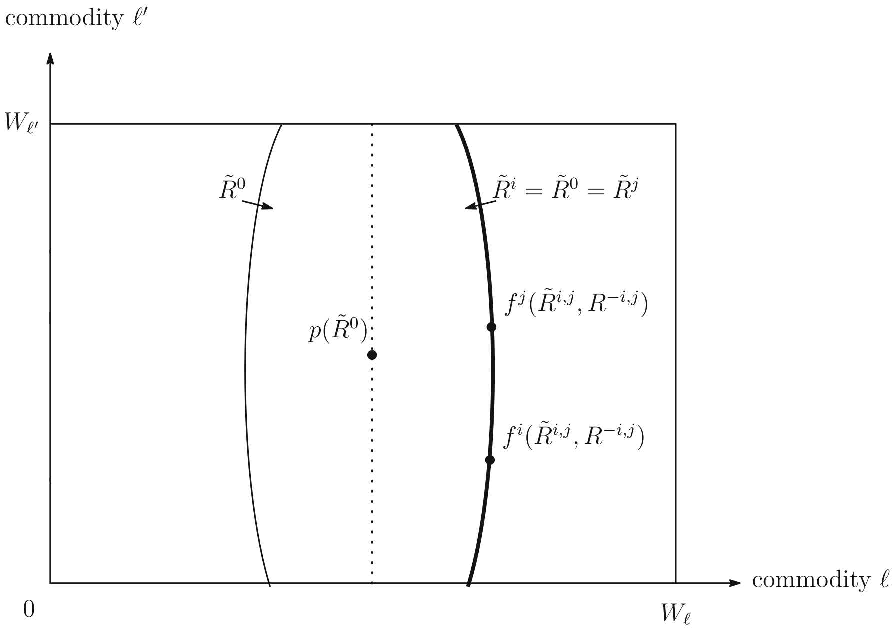

Fig. 3 Illustration of assignments in Lemma 5 for the case of excess supply. By *same-sideness*, $p_\ell(\tilde{R}^0) \leq f^i_\ell(\tilde{R}^{i,j}, R^{-i,j})$ and $p_\ell(\tilde{R}^0) \leq f^j_\ell(\tilde{R}^{i,j}, R^{-i,j})$. *Symmetry* implies that $f^i(\tilde{R}^{i,j}, R^{-i,j})\tilde{I}^0 f^j(\tilde{R}^{i,j}, R^{-i,j})$

Further, let $\tilde{R}^0$ be a preference that is also almost indifferent to all commodities except for commodity ℓ. We replace the preference of each agent with this new preference, inductively.

First, we replace agent 1's preference by $\tilde{R}^1 \equiv \tilde{R}^0$. Then, since $\tilde{R}^1$ and $\bar{R}^1$ both exhibit almost indifference to all commodities except for commodity ℓ, Lemma 4 implies that agent 1's assignment of commodity ℓ changes little, and so is still greater than equal division. Thus, by *feasibility*, there is another agent, say agent 2, whose assignment of commodity ℓ is less than equal division.

Next, we replace agent 2's preference by $\tilde{R}^2 \equiv \tilde{R}^0$. By a similar argument, Lemma 4 implies that agent 2's assignment of commodity ℓ is still less than equal division. Since agents 1 and 2 have the same preference $\tilde{R}^0$, and are almost indifferent to all commodities except for commodity ℓ, Lemma 5 implies that their assignments of commodity ℓ differ little, and so are both less than equal division. Thus, by *feasibility*, the assignment of agent 3 of commodity ℓ is greater than equal division.

Finally, we replace agent 3's preference by $\tilde{R}^3 \equiv \tilde{R}^0$. By a similar argument, Lemma 4 implies that agent 3's assignment of commodity ℓ is still greater than equal division. Since agents 1, 2 and 3 all have the same preference $\tilde{R}^0$, and are also almost indifferent to all commodities except for commodity ℓ, Lemma 5 implies that their assignments of commodity ℓ differ little, and so are all greater than equal division. This is a contradiction to *feasibility*. Therefore, $f_\ell(R) = \left(\frac{W_\ell}{3}, \frac{W_\ell}{3}, \frac{W_\ell}{3}\right)$.

Step V: We complete the proof of Theorem 1 by extending the result of Step IV to the entire domain.

First, let R be a preference profile from the entire domain and let $\ell \in M$. For each agent i, let $\bar{R}^i$ be a strict Maskin monotonic transformation of R^i at $f^i(R)$ such that $p_\ell(\bar{R}^i) = p_\ell(R^i)$ and $p_{-\ell}(\bar{R}^i) = f_{-\ell}(R)$ (Fig. 4).

Next, we replace agent 1's preference with his new preference $\bar{R}^1$. Then, by *strategy-proofness*, his new assignment remains the same. Further, by *nonbossiness*, all the other agents' assignments also remain the same. Repeating this argument for the remaining agents, the chosen allocation under the new preference profile is the same as for the initial allocation. Since the new preference profile belongs to the restricted domain $\bar{\mathcal{R}}^N(f_{-\ell}(R))$ in Step I, it follows from Step IV that the chosen allocation coincides with the allocation under the uniform rule. Therefore, the initial allocation for commodity ℓ also coincides with the allocation under the uniform rule.

3.2 Technical discussion

We refer the literature that established similar results, and discuss the key points and difficulties of our proofs.

3.2.1 Decomposability

The basic structure of our proof is similar to that of Border and Jordan (1983). Indeed, we show that any rule satisfying our axioms can be decomposed into a product of one-dimensional rules, each of which is the single-commodity uniform rule. In the literature on *strategy-proofness*, following Border and Jordan (1983), authors such as

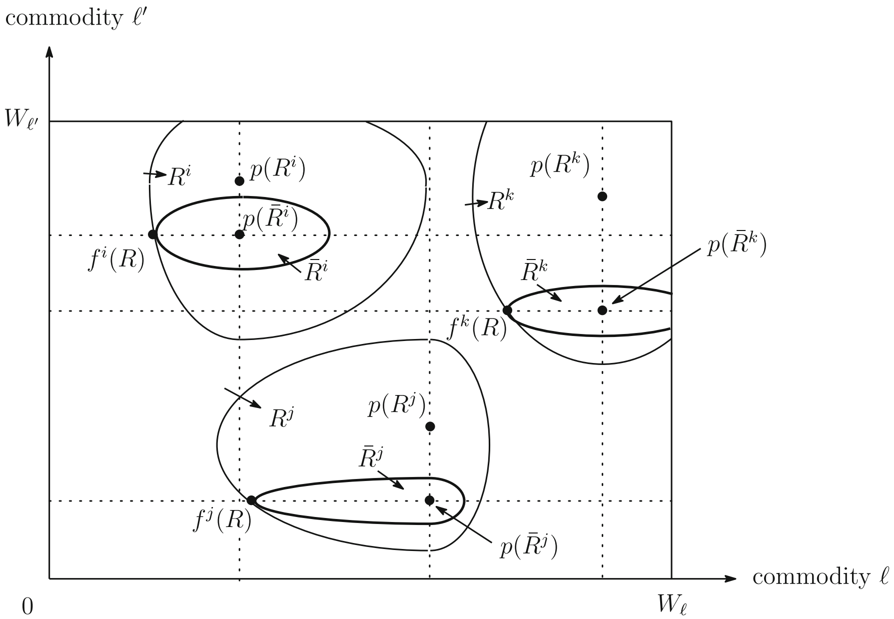

Fig. 4 Illustration of a strict Maskin monotonic transformation $\bar{R}$ of R at $f(R)$

Barberà et al. (1993, 1998), and Le Breton and Sen (1999), etc. also proved decomposability results for *strategy-proof* social choice functions for public alternative models. However, their decomposability results cannot be applied to our private-commodity model.

The multidimensional single-peaked conditions of the two models differ in the following way. Multidimensional single-peakedness of preferences plays an important role in establishing their decomposability results for public alternative models. To apply the decomposability results for public alternative models to our model, we need to assume that each agent has multidimensional single-peaked preferences defined over the entire feasible set $Z \equiv \{(x^1, \ldots, x^n) \in X^n \mid \sum_{i \in N} x^i = W\}$, not only over agent's consumption set X. However, in our private-commodity model, we assume that each agent has multidimensional single-peakedness for preferences defined only over X.[28] In our private-commodity model, agents are only interested in their own assignments, that is, each agent is indifferent to the coordinates representing the assignments of other agents.

3.2.2 *Uniqueness of single-commodity uniform rule*

Step IV is the most important and difficult part in our proof. It says that for each $\ell \in M$, and each allocation $x_{-\ell}$ except for commodity ℓ, the single-commodity allocation rule

[28] Note that only in the case of two agents, the multidimensional single-peakedness of preferences only over X implies the one of preferences over Z. Thus, the results of public alternative models can be applied to only this case.

derived for commodity ℓ in Step I coincides with the single-commodity uniform rule on the restricted domain $\bar{\mathcal{R}}^N(x_{-\ell})$ (Lemma 7). In the one-commodity case, Sprumont (1991) and Ching (1994) already established the similar uniqueness results of the uniform rule. We borrow some of their proof techniques in proving Lemmas 2 and 3. However, we need to develop our own in other parts for several reasons.

One reason is that, as Example 6 below illustrates, *weak symmetry* of the single-commodity rule derived in Step I does not follow from *weak symmetry* of the multiple-commodity rule. Otherwise, we could directly apply their results to obtain Lemma 7.

The *weak symmetry* of the single-commodity rule derived in Step I is defined as follows. Hereafter, we call it *1D weak symmetry*. Let $\ell \in M$ and $x_{-\ell} \in Z_{-\ell}$. Let f be a rule on the entire domain $\mathcal{R}^n$ such that for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, $f_{-\ell}(R) = x_{-\ell}$. Then, the ℓ-th coordinate f_ℓ of the multi-commodity rule f is the derived single-commodity rule. Next, for each $i \in N$ and each $R^i \in \bar{\mathcal{R}}^i(x_{-\ell})$, let $\bar{R}(R^i)$ be a preference relation on $[0, W_\ell]$ such that for each pair $x^i_\ell, y^i_\ell \in [0, W_\ell]$, $x^i_\ell \; \bar{R}(R^i) \; y^i_\ell$ if and only if $(x^i_\ell, x^i_{-\ell}) \; R^i \; (y^i_\ell, x^i_{-\ell})$. Then, f_ℓ is *1D weakly symmetric* if for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ and each $i, j \in N$, if $\bar{R}(R^i) = \bar{R}(R^j)$, then $f^i_\ell(R) \; \bar{I}(R^i) \; f^j_\ell(R)$.[29]

To apply the results of Sprumont (1991) and Ching (1994), we need to assume *1D weak symmetry* defined as above. However, it does not follow from *weak symmetry* of the multiple-commodity rule, as illustrated in Example 6 below.

Example 6 Let $N \equiv \{1, 2\}$ and $M \equiv \{\ell, \ell'\}$. Let f be the rule on $\mathcal{R}^2$ defined as follows: for each $R \in \mathcal{R}^2$, if $R^1 = R^2$, $f^1(R) \equiv f^2(R) \equiv \left(\frac{W_\ell}{2}, \frac{W_{\ell'}}{2}\right)$; otherwise $f^1(R) \equiv p(R^1)$, and $f^2(R) \equiv (W_1, W_2) - p(R^1)$. Note that f satisfies *weak symmetry* of the multiple-commodity rule. Let $x_{-\ell} \equiv (x^1_{-\ell}, x^2_{-\ell}) \equiv \left(\frac{2 \cdot W_{\ell'}}{3}, \frac{W_{\ell'}}{3}\right)$. Let $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ be such that $\bar{R}(R^1) = \bar{R}(R^2)$, and $p_\ell(R^1) \equiv \frac{W_\ell}{3} \equiv p_\ell(R^2)$. Then, since $p_{\ell'}(R^1) = \frac{2 \cdot W_{\ell'}}{3} \neq \frac{W_{\ell'}}{3} = p_{\ell'}(R^2)$, we have $R^1 \neq R^2$. Thus, by the definition of f, $f^1_\ell(R) = p_\ell(R^1)$ and $f^2_\ell(R) = W_\ell - p_\ell(R^1)$. Therefore, $f^1_\ell(R) \; \bar{P}(R^2) \; f^2_\ell(R)$. Accordingly, the derived single-commodity rule f_ℓ violates *1D weak symmetry*.

This fact forces us to employ preference profiles outside $\bar{\mathcal{R}}^N(x_{-\ell})$ even in characterizing the single-commodity rules defined on $\bar{\mathcal{R}}^N(x_{-\ell})$. Notice that while employing such preference profiles, we cannot apply *peak-onlyness* and *uncompromisingness* introduced in Step II.

Another reason is that we cannot depend on *symmetry* to obtain Lemma 7. As we discussed in Sect. 2, in the one-commodity case, *same-sideness* and *weak symmetry* imply *symmetry*. *Symmetry* is key to Ching's (1994) proof of his uniqueness result. However, in the multiple-commodity case, *same-sideness* and *weak symmetry* do not imply *symmetry*.

The above facts make characterizing the uniform rule in our model more difficult. We need to develop new proof techniques to overcome those difficulties. In our proof, *weak symmetry* of the multiple-commodity rule plays a role in the form of Lemma 5, which says that, even if a preference profile is outside $\bar{\mathcal{R}}^N(x_{-\ell})$, whenever two agents

[29] Let $\bar{P}(R^i)$ denote the strict relation associated with $\bar{R}(R^i)$, and $\bar{I}(R^i)$ the indifference relation.

have the same preference and they are almost indifferent to all commodities except for commodity ℓ, their assignments of commodity ℓ are almost equal. That is, owing to Lemma 5, we can obtain an implication approximating "*symmetry* with respect to commodity ℓ". Furthermore, Lemma 4 limits the responsiveness of the agent's assignment to a change of his preference. As we explained in Sect. 3.1, we repeatedly use these observations in our proof.

3.2.3 Nonbossiness

In models with private goods, *nonbossiness* is often added to *strategy-proofness* for tractability. We also assume *nonbossiness* in our characterization. It is worthwhile to discuss this requirement.

In many models with private goods, *group strategy-proofness* follows from *strategy-proofness* and *nonbossiness*. However, as we discussed in Remark 4, *group strategy-proofness* does not follow from *strategy-proofness* and *nonbossiness* in our model.

Under the domain condition called "strict monotonic closedness",[30] *strategy-proofness*, *nonbossiness*, and *weak symmetry* imply *no-envy* (Fleurbaey and Maniquet 1997). However, our domain does not satisfy *strict monotonic closedness*.[31] Thus, we cannot apply this result. Interestingly, in contrast to the multiple-commodity model, the domain of single-peaked preferences of the one-commodity model satisfies *strict monotonic closedness*. This is one of several critical differences between the one- and multiple-commodity models.

3.3 Formal proof of only if part

Step I: In Lemma 1 below, we show *same-sidedness*.

Lemma 1 *If a rule is strategy-proof, unanimous, and nonbossy, then it is same-sided.*

Proof Let f be a rule satisfying the hypotheses. Let $R \in \mathcal{R}^n$ and $\ell \in M$. Assume that $\sum_{h \in N} p_\ell(R^h) \leq W_\ell$. (The same argument applies to the opposite case.) By contradiction, suppose that there is $i \in N$ such that $f_\ell^i(R) < p_\ell(R^i)$. Since $\sum_{h \in N} p_\ell(R^h) \leq$

[30] **Strict monotonic closedness**: For each $R^i, \hat{R}^i \in \mathcal{R}$, and each $a, b \in X$ with $a\ P^i\ b$, there is $\bar{R}^i \in \mathcal{R}$ such that, for each $c \in X \setminus \{a, b\}$, (i) $a\ \hat{R}^i\ c \Rightarrow a\ \bar{P}^i\ c$, (ii) $b\ R^i\ c \Rightarrow b\ \bar{P}^i\ c$, and (iii) $a\ \bar{P}^i\ b$ or $b\ \bar{P}^i\ a$.

[31] **Example illustrating the violation of strict monotonic closedness.** Let $M \equiv \{1, 2\}$. Let $R^i, \hat{R}^i \in \mathcal{R}$ and $a, b \in X$ be such that $p(R^i) \equiv (W_1, 0)$, $p(\hat{R}^i) \equiv (0, W_2)$, $a \equiv (0, 0)$, $b \equiv (W_1, W_2)$, $\left(0, \frac{W_2}{2}\right)\ I^i\ b$, and $\left(\frac{W_1}{2}, W_2\right)\ \hat{I}^i\ a$. Then, $a\ P^i\ b$. By contradiction, suppose that there is $\bar{R}^i \in \mathcal{R}$ that satisfies the conditions (i), (ii), and (iii) of *strict monotonic closedness*. Then $p(\bar{R}^i) \in SUC(\hat{R}^i, a) \cap SUC(R^i, b) \equiv B$, where $SUC(R^i, a)$ is the strict upper contour set of R^i at a.

To see this, suppose on the contrary that $p(\bar{R}^i) \notin B$. Then (A) $p(\bar{R}^i) \notin SUC(\hat{R}^i, a)$ or (B) $p(\bar{R}^i) \notin SUC(R^i, b)$. Consider the case (A). Case (B) is similar. Then $a\ \hat{R}^i\ p(\bar{R}^i)$. Thus, by (i), $a\ \bar{P}^i\ p(\bar{R}^i)$. However, since $p(\bar{R}^i)$ is the peak of $\bar{R}^i$, this is a contradiction. Thus, $p(\bar{R}^i) \in SUC(\hat{R}^i, a) \cap SUC(R^i, b) \equiv B$.

Since $b \notin B$, $p(\bar{R}^i)\ \bar{P}^i\ b$. Then, there is $z^i \in X$ such that $z^i\ \bar{I}^i\ b$, $z_1^i \equiv \frac{2W_1}{3}$, and $p_2(\bar{R}^i) \leq z_2^i < W_2$, which is a contradiction.

Hence, $\mathcal{R}$ violates *strict monotonic closedness*.

W_ℓ, there is $j \in N \setminus \{i\}$ such that $f^j_\ell(R) > p_\ell(R^j)$. Without loss of generality, let $i = 1$ and $j = 2$. For each $k \in N \setminus \{1, 2\}$, let $\hat{R}^k \in \mathcal{R}$ be such that $p(\hat{R}^k) = f^k(R)$. Then, by *strategy-proofness*, $f^3(\hat{R}^3, R^{-3}) = f^3(R)$. By *nonbossiness*, $f(\hat{R}^3, R^{-3}) = f(R)$. Repeating this argument for $k = 4, \ldots, n$, we obtain $f(R^{1,2}, \hat{R}^{-1,2}) = f(R)$.

There are two cases.

Case 1 $p_\ell(R^1) - f^1_\ell(R) \geq f^2_\ell(R) - p_\ell(R^2)$.

Let $(\hat{R}^1, \hat{R}^2) \in \mathcal{R}^2$ be such that (i) $p_\ell(\hat{R}^1) = f^1_\ell(R) + f^2_\ell(R) - p_\ell(R^2)$ and $p_\ell(\hat{R}^2) = p_\ell(R^2)$, (ii) for each $i \in \{1, 2\}$, $p_{-\ell}(\hat{R}^i) = f^i_{-\ell}(R)$, and (iii) for each $i \in \{1, 2\}$, $UC(\hat{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\hat{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$, that is, $\hat{R}^i$ is a strict Maskin monotonic transformation of R^i at $f^i(R)$. Then, by *strategy-proofness*, $f^1(R^2, \hat{R}^{-2}) = f^1(R^{1,2}, \hat{R}^{-1,2})$. By *nonbossiness*, $f(R^2, \hat{R}^{-2}) = f(R^{1,2}, \hat{R}^{-1,2})$. Since $f(R^{1,2}, \hat{R}^{-1,2}) = f(R)$, we have $f(R^2, \hat{R}^{-2}) = f(R)$. Similarly, by *strategy-proofness* and *nonbossiness*, $f(\hat{R}) = f(R)$. However, by feasibility, for each $\ell' \in M$, $\sum_{k \in N} p_{\ell'}(\hat{R}^k) = \sum_{k \in N} f^k_{\ell'}(R) = W_{\ell'}$. This contradicts *unanimity*.

Case 2 $p_\ell(R^1) - f^1_\ell(R) < f^2_\ell(R) - p_\ell(R^2)$.

Similarly to Case 1, we derive a contradiction to *unanimity* by using preferences $(\hat{R}^1, \hat{R}^2) \in \mathcal{R}^2$ such that (i) $p_\ell(\hat{R}^1) = p_\ell(R^1)$ and $p_\ell(\hat{R}^2) = f^2_\ell(R) - p_\ell(R^1) + f^1_\ell(R)$, (ii) for each $i \in \{1, 2\}$, $p_{-\ell}(\hat{R}^i) = f^i_{-\ell}(R)$, and (iii) for each $i \in \{1, 2\}$, $UC(\hat{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\hat{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$. □

Step II: First, we show *peak-onlyness* on the restricted domain $\bar{\mathcal{R}}^N(x_{-\ell})$.

Lemma 2 *Let f be a strategy-proof, unanimous, and nonbossy rule. Then, for each $\ell \in M$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, and each $\hat{R} \in \bar{\mathcal{R}}^N(x_{-\ell})$ such that for each $i \in N$, $p(\hat{R}^i) = p(R^i)$, we have $f(\hat{R}) = f(R)$.*

Proof Let $\ell \in M$, $x_{-\ell} \in Z_{-\ell}$, $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, and $\hat{R} \in \bar{\mathcal{R}}^N(x_{-\ell})$. Assume that for each $i \in N$, $p(\hat{R}^i) = p(R^i)$. First, we show $f^1(R) = f^1(\hat{R}^1, R^{-1})$. By *same-sideness* (Lemma 1) and $p(R^1) = p(\hat{R}^1)$, we have $f^1_{-\ell}(R) = x^1_{-\ell} = f^1_{-\ell}(\hat{R}^1, R^{-1})$. Assume that $\sum_{i \in N} p_\ell(R^i) \leq W_\ell$. (The same argument applies to the opposite case.) We first show $f^1_\ell(R) \geq f^1_\ell(\hat{R}^1, R^{-1})$. By contradiction, suppose that $f^1_\ell(R) < f^1_\ell(\hat{R}^1, R^{-1})$. Then, by *same-sideness* (Lemma 1), $p_\ell(\hat{R}^1) = p_\ell(R^1) \leq f^1_\ell(R) < f^1_\ell(\hat{R}^1, R^{-1})$. Thus $f^1(R)\ \hat{P}^1\ f^1(\hat{R}^1, R^{-1})$, contradicting *strategy-proofness*. Thus, $f^1_\ell(R) \geq f^1_\ell(\hat{R}^1, R^{-1})$. Similarly, we can show $f^1_\ell(R) \leq f^1_\ell(\hat{R}^1, R^{-1})$. Hence, $f^1(R) = f^1(\hat{R}^1, R^{-1})$. By *nonbossiness*, $f(R) = f(\hat{R}^1, R^{-1})$. Repeating this argument for $k = 2, \ldots, n$, we get $f(\hat{R}) = f(R)$. □

Next, we show *uncompromisingness* on the restricted domain $\bar{\mathcal{R}}^N(x_{-\ell})$.

Lemma 3 *Let f be a strategy-proof, unanimous, and nonbossy rule. Then, for each $\ell \in M$, and each $x_{-\ell} \in Z_{-\ell}$, we have*

(i) *Own uncompromisingness:*
for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, each $i \in N$, and each $\hat{R}^i \in \bar{\mathcal{R}}^i(x_{-\ell})$,
if $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, then, $f^i(\hat{R}^i, R^{-i}) = f^i(R)$,
if $p_\ell(R^i) > f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \geq f^i_\ell(R)$, then, $f^i(\hat{R}^i, R^{-i}) = f^i(R)$,

(ii) *Group uncompromisingness:*
for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, each $\hat{N} \subseteq N$, and each $\hat{R}^{\hat{N}} \in \prod_{i\in\hat{N}} \bar{\mathcal{R}}^i(x_{-\ell})$,
if for each $i \in \hat{N}$, $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, then, $f(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f(R)$,
if for each $i \in \hat{N}$, $p_\ell(R^i) > f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \geq f^i_\ell(R)$, then, $f(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f(R)$.

Proof Let $\ell \in M$, $x_{-\ell} \in Z_{-\ell}$, and $R \in \bar{\mathcal{R}}^N(x_{-\ell})$.

Proof of (i) Let $i \in N$ and $\hat{R}^i \in \bar{\mathcal{R}}^i(x_{-\ell})$. Assume that $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$. (The same argument applies to the opposite case.) We show $f^i(\hat{R}^i, R^{-i}) = f^i(R)$. By *same-sideness* (Lemma 1), $f^i_{-\ell}(\hat{R}^i, R^{-i}) = f^i_{-\ell}(R)$. Also, by *same-sideness* (Lemma 1) and $p_\ell(R^i) < f^i_\ell(R)$, for each $j \in N \setminus \{i\}$, we have $p_\ell(R^j) \leq f^j_\ell(R)$. Since $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, by feasibility, $p_\ell(\hat{R}^i) + \sum_{j\neq i} p_\ell(R^j) \leq \sum_{j\in N} f^j_\ell(R) = W_\ell$. Thus, by *same-sideness* (Lemma 1), $f^i_\ell(\hat{R}^i, R^{-i}) \geq p_\ell(\hat{R}^i)$. By contradiction, suppose that $f^i_\ell(\hat{R}^i, R^{-i}) \neq f^i_\ell(R)$. There are two cases.
Case 1: $f^i_\ell(\hat{R}^i, R^{-i}) > f^i_\ell(R)$.

Then, $p_\ell(\hat{R}^i) \leq f^i_\ell(R) < f^i_\ell(\hat{R}^i, R^{-i})$. Thus, $f^i(R)\ \hat{P}^i\ f^i(\hat{R}^i, R^{-i})$, contradicting *strategy-proofness*.
Case 2: $f^i_\ell(\hat{R}^i, R^{-i}) < f^i_\ell(R)$.

Let $\bar{R}^i \in \mathcal{R}$ be such that $p(\bar{R}^i) = p(R^i)$ and $f^i(\hat{R}^i, R^{-i})\ \bar{P}^i\ f^i(R)$. Then, by *peak-onlyness* (Lemma 2), $f^i(\bar{R}^i, R^{-i}) = f^i(R)$. Thus, $f^i(\hat{R}^i, R^{-i})\ \bar{P}^i\ f^i(\bar{R}^i, R^{-i})$, contradicting *strategy-proofness*. □

Proof of (ii) Let $\hat{N} \subseteq N$ and $\hat{R}^{\hat{N}} \in \prod_{i\in\hat{N}} \bar{\mathcal{R}}^i(x_{-\ell})$. Assume that for each $i \in \hat{N}$, $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$. (The same argument applies to the opposite case.) Without loss of generality, let $\hat{N} \equiv \{1, \ldots, \hat{n}\}$.

By *own uncompromisingness* (Lemma 3-i), $f^1(\hat{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\hat{R}^1, R^{-1}) = f(R)$. Repeating this argument for $k = 2, \ldots, \hat{n}$, we have $f(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f(R)$. □

Step III: Lemma 4 says that for a *strategy-proof* and *same-sided* rule, and for each commodity, when an agent's preference changes, if both his initial and his new preferences are d-indifferent to all commodities except for that commodity, then his assignments of the commodity differ by at most d.

Lemma 4 *Let f be a strategy-proof and same-sided rule. Let $\ell \in M$, $d \in \left(0, \frac{W_\ell}{2n}\right)$, $i \in N$, $x_\ell^i \in X_\ell$, $\bar{x}_{-\ell}^i, \tilde{x}_{-\ell}^i \in X_{-\ell}$, $\bar{R}^i \in \mathcal{R}^V\left(x_\ell^i, \bar{x}_{-\ell}^i, d\right)$, $\tilde{R}^i \in \mathcal{R}^V\left(x_\ell^i, \tilde{x}_{-\ell}^i, d\right)$, and $R^{-i} \in \mathcal{R}^{n-1}$. Then,*

$$f_\ell^i\left(\bar{R}^i, R^{-i}\right) - d < f_\ell^i\left(\tilde{R}^i, R^{-i}\right) < f_\ell^i\left(\bar{R}^i, R^{-i}\right) + d. \tag{1}$$

Proof Assume that $p_\ell\left(\bar{R}^i\right) + \sum_{j \in N \setminus \{i\}} p_\ell\left(R^j\right) \leq W_\ell$. (The same argument applies to the opposite case.) Then, by *same-sideness* and $p_\ell\left(\bar{R}^i\right) = x_\ell^i = p_\ell\left(\tilde{R}^i\right)$, $x_\ell^i \leq f_\ell^i\left(\bar{R}^i, R^{-i}\right)$ and $x_\ell^i \leq f_\ell^i\left(\tilde{R}^i, R^{-i}\right)$. The proof is in two steps. □

Step 1. $f_\ell^i\left(\tilde{R}^i, R^{-i}\right) < f_\ell^i\left(\bar{R}^i, R^{-i}\right) + d$.

Proof There are two cases.

Case 1-1: $f_\ell^i\left(\bar{R}^i, R^{-i}\right) > W_\ell - d$.

In this case, $f_\ell^i\left(\bar{R}^i, R^{-i}\right) + d > W_\ell \geq f_\ell^i\left(\tilde{R}^i, R^{-i}\right)$.

Case 1-2: $f_\ell^i\left(\bar{R}^i, R^{-i}\right) \leq W_\ell - d$.

By contradiction, suppose that

$$f_\ell^i\left(\tilde{R}^i, R^{-i}\right) \geq f_\ell^i\left(\bar{R}^i, R^{-i}\right) + d. \tag{2}$$

Since $\tilde{R}^i \in \mathcal{R}^V\left(x_\ell^i, \tilde{x}_{-\ell}^i, d\right)$, by (a) of Definition 3, we have

$$f^i\left(\bar{R}^i, R^{-i}\right)\ \tilde{P}^i\ \left(f_\ell^i\left(\bar{R}^i, R^{-i}\right) + d,\ p_{-\ell}\left(\tilde{R}^i\right)\right).$$

By (2),

$$\left(f_\ell^i\left(\bar{R}^i, R^{-i}\right) + d,\ p_{-\ell}\left(\tilde{R}^i\right)\right)\ \tilde{R}^i\ f^i\left(\tilde{R}^i, R^{-i}\right).$$

This implies

$$f^i\left(\bar{R}^i, R^{-i}\right)\ \tilde{P}^i\ f^i\left(\tilde{R}^i, R^{-i}\right),$$

contradicting *strategy-proofness*. □

Step 2. $f_\ell^i\left(\bar{R}^i, R^{-i}\right) - d < f_\ell^i\left(\tilde{R}^i, R^{-i}\right)$.

Proof There are two cases.

Case 2-1: $f_\ell^i\left(\bar{R}^i, R^{-i}\right) < d$.

In this case, $f_\ell^i(\bar{R}^i, R^{-i}) - d < 0 \leq f_\ell^i(\tilde{R}^i, R^{-i})$.

Case 2-2: $f_\ell^i(\bar{R}^i, R^{-i}) \geq d$.

By contradiction, suppose that

$$f_\ell^i(\bar{R}^i, R^{-i}) - d \geq f_\ell^i(\tilde{R}^i, R^{-i}). \tag{3}$$

Since $\bar{R}^i \in \mathcal{R}^V(x_\ell^i, \bar{x}_{-\ell}^i, d)$, by (a) of Definition 3, we have

$$f^i(\tilde{R}^i, R^{-i}) \; \bar{P}^i \; \left(f_\ell^i(\tilde{R}^i, R^{-i}) + d,\; p_{-\ell}(\bar{R}^i)\right).$$

By (3),

$$\left(f_\ell^i(\tilde{R}^i, R^{-i}) + d,\; p_{-\ell}(\bar{R}^i)\right) \; \bar{R}^i \; f^i(\bar{R}^i, R^{-i}).$$

This implies

$$f^i(\tilde{R}^i, R^{-i}) \; \bar{P}^i \; f^i(\bar{R}^i, R^{-i}),$$

contradicting *strategy-proofness*. □

Lemma 5 says that for a *weakly symmetric* and *same-sided* rule, and for each commodity, when two agents have the same preference, if the preference is d-indifferent to all commodities except for that commodity, then their assignments of the commodity differ by at most d.

Lemma 5 *Let f be a weakly symmetric and same-sided rule. Let $\ell \in M$, $d \in \left(0, \frac{W_\ell}{2n}\right)$, $x_\ell \in X_\ell$, $x_{-\ell} \in X_{-\ell}$, $\tilde{R}^0 \in \mathcal{R}^V(x_\ell, x_{-\ell}, d)$, $i, j \in N$, $\tilde{R}^i = \tilde{R}^0 = \tilde{R}^j$, and $R^{-i,j} \in \mathcal{R}^{n-2}$. Then,*

$$f_\ell^i(\tilde{R}^{i,j}, R^{-i,j}) - d < f_\ell^j(\tilde{R}^{i,j}, R^{-i,j}) < f_\ell^i(\tilde{R}^{i,j}, R^{-i,j}) + d. \tag{4}$$

Proof Assume that $p_\ell(\tilde{R}^i) + p_\ell(\tilde{R}^j) + \sum_{k \in N \setminus \{i,j\}} p_\ell(R^k) \leq W_\ell$. (The same argument applies to the opposite case.) Then, by *same-sideness*, $x_\ell \leq f_\ell^i(\tilde{R}^{i,j}, R^{-i,j})$ and $x_\ell \leq f_\ell^j(\tilde{R}^{i,j}, R^{-i,j})$. The proof is in two steps.

Step 1. $f_\ell^j(\tilde{R}^{i,j}, R^{-i,j}) < f_\ell^i(\tilde{R}^{i,j}, R^{-i,j}) + d$.

Proof There are two cases.

Case 1-1: $f_\ell^i(\tilde{R}^{i,j}, R^{-i,j}) > W_\ell - d$.

In this case, $f_\ell^i(\tilde{R}^{i,j}, R^{-i,j}) + d > W_\ell \geq f_\ell^j(\tilde{R}^{i,j}, R^{-i,j})$.

Case 1-2: $f_\ell^i(\tilde{R}^{i,j}, R^{-i,j}) \leq W_\ell - d$.

By contradiction, suppose that

$$f_\ell^j\left(\tilde{R}^{i,j}, R^{-i,j}\right) \geq f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) + d. \tag{5}$$

Since $\tilde{R}^j \in \mathcal{R}^V\left(x_\ell, x_{-\ell}, d\right)$, by (a) of Definition 3, we have

$$f^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) \ \tilde{P}^j \ \left(f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) + d,\ p_{-\ell}\left(\tilde{R}^j\right)\right).$$

By (5),

$$\left(f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) + d,\ p_{-\ell}\left(\tilde{R}^j\right)\right) \ \tilde{R}^j \ f^j\left(\tilde{R}^{i,j}, R^{-i,j}\right).$$

This implies

$$f^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) \ \tilde{P}^j \ f^j\left(\tilde{R}^{i,j}, R^{-i,j}\right),$$

contradicting *weak symmetry*. □

Step 2. $f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) - d < f_\ell^j\left(\tilde{R}^{i,j}, R^{-i,j}\right)$.

Proof There are two cases.

Case 2-1: $f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) < d$.

In this case, $f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) - d < 0 \leq f_\ell^j\left(\tilde{R}^{i,j}, R^{-i,j}\right)$.

Case 2-2: $f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) \geq d$.

By contradiction, suppose that

$$f_\ell^i\left(\tilde{R}^{i,j}, R^{-i,j}\right) - d \geq f_\ell^j\left(\tilde{R}^{i,j}, R^{-i,j}\right). \tag{6}$$

Since $\tilde{R}^i \in \mathcal{R}^V\left(x_\ell, x_{-\ell}, d\right)$, by (a) of Definition 3, we have

$$f^j\left(\tilde{R}^{i,j}, R^{-i,j}\right) \ \tilde{P}^i \ \left(f_\ell^j\left(\tilde{R}^{i,j}, R^{-i,j}\right) + d,\ p_{-\ell}\left(\tilde{R}^i\right)\right).$$

By (6),

$$\left(f_\ell^j\left(\tilde{R}^{i,j}, R^{-i,j}\right) + d,\ p_{-\ell}\left(\tilde{R}^i\right)\right) \ \tilde{R}^i \ f^i\left(\tilde{R}^{i,j}, R^{-i,j}\right).$$

This implies

$$f^j\left(\tilde{R}^{i,j}, R^{-i,j}\right) \ \tilde{P}^i \ f^i\left(\tilde{R}^{i,j}, R^{-i,j}\right),$$

contradicting *weak symmetry*. □

The next lemma follows from Lemmas 4 and 5.

Lemma 6 *Let f be a strategy-proof, weakly symmetric, and same-sided rule. Let $\hat{N} \subsetneq N$, $i \in N \setminus \hat{N}$, and $K = N \setminus (\hat{N} \cup \{i\})$. Let $\ell \in M$, $d \in \left(0, \frac{W_\ell}{2n}\right)$, $x^i_\ell \in X_\ell$, $\bar{x}^i_{-\ell}, \tilde{x}^i_{-\ell} \in X_{-\ell}$, $\bar{R}^i \in \mathcal{R}^V\left(x^i_\ell, \bar{x}^i_{-\ell}, d\right)$, $\tilde{R}^i \in \mathcal{R}^V\left(x^i_\ell, \tilde{x}^i_{-\ell}, d\right)$, and $R^K \in \mathcal{R}^{|K|}$. For each $j \in \hat{N}$, let $\tilde{R}^j = \tilde{R}^i$. Then,*

$$f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i\right) - d < f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i\right) < f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i\right) + d, \text{ and} \tag{7}$$

$$\begin{aligned} &\text{for each } j \in \hat{N},\ f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i\right) - 2 \cdot d < f^j_\ell\left(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i\right) \\ &\quad < f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i\right) + 2 \cdot d. \end{aligned} \tag{8}$$

Proof Since $\bar{R}^i \in \mathcal{R}^V\left(x^i_\ell, \bar{x}^i_{-\ell}, d\right)$, $\tilde{R}^i \in \mathcal{R}^V\left(x^i_\ell, \tilde{x}^i_{-\ell}, d\right)$, and $\left(R^K, \tilde{R}^{\hat{N}}\right) \in \mathcal{R}^{n-1}$, Lemma 4 implies (7). Next, we show (8). Let $j \in \hat{N}$. Since $\tilde{R}^j = \tilde{R}^i$, $\tilde{R}^j \in \mathcal{R}^V\left(x^i_\ell, \tilde{x}^i_{-\ell}, d\right)$, and $\left(R^K, \tilde{R}^{\hat{N}\setminus\{j\}}\right) \in \mathcal{R}^{n-2}$, Lemma 5 implies that

$$f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i\right) - d < f^j_\ell\left(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i\right) < f^i_\ell\left(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i\right) + d.$$

Now, (8) follows from (7). □

Step IV: We show that f_ℓ coincides with the single-commodity uniform rule on the restricted domain $\bar{\mathcal{R}}^N\left(x_{-\ell}\right)$.

Lemma 7 *Let f be a strategy-proof, unanimous, weakly symmetric, and nonbossy rule. Then, for each $\ell \in M$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N\left(x_{-\ell}\right)$, and each $i \in N$, $f^i(R) = U^i(R)$.*

Proof Let $\ell \in M$. By *same-sideness* (Lemma 1), for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N\left(x_{-\ell}\right)$, each $i \in N$, and each $\ell' \neq \ell$, we have $f^i_{\ell'}(R) = x^i_{\ell'} = U^i_{\ell'}(R)$. Thus we only show that for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N\left(x_{-\ell}\right)$, and each $i \in N$, $f^i_\ell(R) = U^i_\ell(R)$.

Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N\left(x_{-\ell}\right)$ be such that $\sum_{i \in N} p_\ell\left(R^i\right) = W_\ell$. Then, by the definition of U, for each $i \in N$, $U^i_\ell(R) = p_\ell\left(R^i\right)$. By *unanimity*, for each $i \in N$, we have $f^i_\ell(R) = p_\ell\left(R^i\right)$. Thus, for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N\left(x_{-\ell}\right)$ such that $\sum_{i \in N} p_\ell\left(R^i\right) = W_\ell$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.

Next, we show that for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N\left(x_{-\ell}\right)$ such that $\sum_{i \in N} p_\ell\left(R^i\right) < W_\ell$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.

We introduce some notation. Given a preference profile $R \in \mathcal{R}^n$, let $\pi(R) \equiv \left(\pi_1(R), \ldots, \pi_n(R)\right)$ be a permutation on N such that $p_\ell\left(R^{\pi_1(R)}\right) \geq \cdots \geq p_\ell\left(R^{\pi_n(R)}\right)$. We simply write $\pi_1, \ldots, \pi_n$ when we can omit R as an argument without confusion. Let $\mathcal{R}^N_\ell(0)$ be the set of preference profiles R such that $\sum_{i \in N} p_\ell\left(R^i\right) < W_\ell$

and for each $i \in N$, $p_\ell(R^i) \leq \frac{W_\ell}{n}$. Given $k \in \{1, \ldots, n-1\}$, let $\mathcal{R}_\ell^N(k)$ be the set of preference profiles R such that $\sum_{i \in N} p_\ell(R^i) < W_\ell$ and

$$\begin{aligned}
p_\ell(R^{\pi_1}) &> \frac{W_\ell}{n},\\
p_\ell(R^{\pi_2}) &> \frac{W_\ell - p_\ell(R^{\pi_1})}{n-1},\\
&\ldots\\
p_\ell(R^{\pi_k}) &> \frac{W_\ell - \sum_{i=1}^{k-1} p_\ell(R^{\pi_i})}{n-k+1}, \text{ and}\\
p_\ell(R^{\pi_j}) &\leq \frac{W_\ell - \sum_{i=1}^{k} p_\ell(R^{\pi_i})}{n-k} \text{ for each } j \in \{k+1, \ldots, n\}.
\end{aligned}$$

Note that $\bigcup_{k=0}^{n-1} \mathcal{R}_\ell^N(k) = \{R \in \mathcal{R}^n \mid \sum_{i \in N} p_\ell(R^i) < W_\ell\}$. For each $k \in \{0, 1, \ldots, n-1\}$, let $\mathcal{R}_\ell^N(k, 0)$ be the subdomain of $\mathcal{R}_\ell^N(k)$ such that for each $j \in \{k+1, \ldots, n\}$, $p_\ell(R^{\pi_j}) = 0$.

For each $k \in \{0, 1, \ldots, n-1\}$ and each $x_{-\ell} \in Z_{-\ell}$, let $\bar{\mathcal{R}}^N(k, x_{-\ell}) \equiv \mathcal{R}_\ell^N(k) \cap \bar{\mathcal{R}}^N(x_{-\ell})$. Note that for each $x_{-\ell} \in Z_{-\ell}$, $\bigcup_{k=0}^{n-1} \bar{\mathcal{R}}^N(k, x_{-\ell}) = \{R \in \bar{\mathcal{R}}^N(x_{-\ell}) \mid \sum_{i \in N} p_\ell(R^i) < W_\ell\}$.

By induction, we will show that for each $k \in \{0, 1, \ldots, n-1\}$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$, and each $i \in N$, we have $f_\ell^i(R) = U_\ell^i(R)$. □

Step 1. For each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(0, x_{-\ell})$, and each $i \in N$, we have $f_\ell^i(R) = U_\ell^i(R)$.

Proof Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(0, x_{-\ell})$. Then, by the definition of the uniform rule U, for each $i \in N$, $U_\ell^i(R) = \frac{W_\ell}{n}$. We show that for each $i \in N$, $f_\ell^i(R) = \frac{W_\ell}{n}$. Let $\hat{R} \in \bar{\mathcal{R}}^N(x_{-\ell})$ be such that for each $i \in N$, $p_\ell(\hat{R}^i) = 0$. If for each $i \in N$, $f_\ell^i(\hat{R}) = \frac{W_\ell}{n}$, then, by *group uncompromisingness* (Lemma 3-ii), $f(\hat{R}) = f(R)$. Thus, we only have to show that for each $i \in N$, $f_\ell^i(\hat{R}) = \frac{W_\ell}{n}$.

By contradiction, suppose that there is $j \in N$ such that $f_\ell^j(\hat{R}) > \frac{W_\ell}{n}$. Without loss of generality, let $j = 1$. Denote $e_1 \equiv f_\ell^1(\hat{R}) - \frac{W_\ell}{n}$, and for each $k \in N \setminus \{1\}$, let $e_k \equiv \frac{e_{k-1}}{n-k+2}$. Then, (i) for each $k \in N \setminus \{1\}$, $e_k > 0$ and $e_k < e_{k-1}$, (ii) for each $k \in \{2, 3, \ldots, n\}$,

$$e_k = \frac{e_1}{\prod_{t=0}^{k-2}(n-t)},$$

and (iii) for each $k \in \{2, 3, \ldots, n\}$,

$$\frac{e_{k-1}}{n-k+1} - e_k = \frac{e_1}{\prod_{t=0}^{k-1}(n-t)} \geq \frac{e_1}{n!}. \tag{9}$$

Let $d \equiv \frac{e_1}{2 \cdot n!}$. For each $i \in N$, let $\bar{R}^i \in \mathcal{R}^V\big(0, p_{-\ell}\big(R^i\big), d\big)$, and $\bar{R} \equiv \big(\bar{R}^1, \ldots, \bar{R}^n\big)$. Then, by *peak-onlyness* (Lemma 2), $f\big(\bar{R}\big) = f\big(\hat{R}\big)$.

Let $\tilde{x}_{-\ell} \in X_{-\ell}$, $\tilde{R}_0 \in \mathcal{R}^V\big(0, \tilde{x}_{-\ell}, d\big)$, and $\tilde{R}_0^1 \equiv \tilde{R}_0$. □

Step 1-1. $f_\ell^1\big(\tilde{R}_0^1, \bar{R}^{-1}\big) \geq \frac{W_\ell}{n} + e_1 = f_\ell^1\big(\hat{R}\big)$.

Proof Suppose on the contrary that $f_\ell^1\big(\tilde{R}_0^1, \bar{R}^{-1}\big) < \frac{W_\ell}{n} + e_1$. Since $f\big(\bar{R}\big) = f\big(\hat{R}\big)$, we have $f_\ell^1\big(\bar{R}\big) = \frac{W_\ell}{n} + e_1$. Let $\check{R}^1 \in \mathcal{R}$ be such that $p\big(\check{R}^1\big) = p\big(\bar{R}^1\big)$ and $f^1\big(\tilde{R}_0^1, \bar{R}^{-1}\big) \;\check{P}^1\; f^1\big(\bar{R}\big)$. Then, by *peak-onlyness* (Lemma 2), $f^1\big(\check{R}^1, \bar{R}^{-1}\big) = f^1\big(\bar{R}\big)$. Thus, $f^1\big(\tilde{R}_0^1, \bar{R}^{-1}\big) \;\check{P}^1\; f^1\big(\check{R}^1, \bar{R}^{-1}\big)$, contradicting *strategy-proofness*. □

Given $\hat{N} \subset N$, let $\tilde{R}_0^{\hat{N}}$ be such that for each $i \in \hat{N}$, $\tilde{R}_0^i = \tilde{R}_0$.

Step 1-2. For each $k \in \{1, 2, \ldots, n\}$,

(a) *if k is even, then there is $\hat{N} \subset N$ such that $|\hat{N}| = k$ and for each $i \in \hat{N}$, we have $f_\ell^i\big(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}\big) \leq \frac{W_\ell}{n} - e_k$, and*

(b) *if k is odd, then there is $\hat{N} \subset N$ such that $|\hat{N}| = k$ and for each $i \in \hat{N}$, we have $f_\ell^i\big(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}\big) \geq \frac{W_\ell}{n} + e_k$.*

Proof The proof proceeds by induction on k. Let $k \in \{1, 2, \ldots, n\}$. When $k = 1$, by Step 1-1, we have already proven that (b) holds. Assume that $k \geq 2$.

Case (a): k is even.

Our induction hypothesis is that there is $\bar{N} \subset N$ such that $|\bar{N}| = k - 1$, and for each $i \in \bar{N}$, $f_\ell^i\big(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\big) \geq \frac{W_\ell}{n} + e_{k-1}$.

Suppose that for each $j \in N \setminus \bar{N}$, $f_\ell^j\big(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\big) > \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1}$. Then,

$$
\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i\big(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\big) \quad \text{(by \textit{feasibility})} \\
&> (k-1) \cdot \left(\frac{W_\ell}{n} + e_{k-1}\right) + (n-k+1) \cdot \left(\frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1}\right) \\
&= W_\ell + (k-2) \cdot e_{k-1} \\
&\geq W_\ell, \quad (\text{by } k \geq 2 \text{ and } e_{k-1} > 0)
\end{aligned}
$$

which is a contradiction.

Thus, there is $j \in N \setminus \bar{N}$ such that $f_\ell^j\big(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\big) \leq \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1}$. Let $\hat{N} \equiv \{j\} \cup \bar{N}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$
\begin{aligned}
f_\ell^i\big(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}\big) &< f_\ell^j\big(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\big) + 2 \cdot d \quad \text{(by Lemma 6)} \\
&\leq \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1} + 2 \cdot d \\
&= \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1} + \frac{e_1}{n!} \quad \left(\text{by } d = \frac{e_1}{2 \cdot n!}\right) \\
&\leq \frac{W_\ell}{n} - e_k. \quad (\text{by } k \geq 2 \text{ and (9)})
\end{aligned}
$$

Case (b): k is odd.

Our induction hypothesis is that there is $\bar{N} \subset N$ such that $|\bar{N}| = k - 1$, and for each $i \in \bar{N}$, $f_\ell^i\left(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\right) \leq \frac{W_\ell}{n} - e_{k-1}$.

Suppose that for each $j \in N \setminus \bar{N}$, $f_\ell^j\left(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\right) < \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1}$. Then,

$$\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i\left(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\right) \quad \text{(by \textit{feasibility})} \\
&< (k-1) \cdot \left(\frac{W_\ell}{n} - e_{k-1}\right) + (n-k+1) \cdot \left(\frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1}\right) \\
&= W_\ell - (k-2) \cdot e_{k-1} \\
&\leq W_\ell, \quad \text{(by } k \geq 2 \text{ and } e_{k-1} > 0\text{)}
\end{aligned}$$

which is a contradiction.

Thus, there is $j \in N \setminus \bar{N}$ such that $f_\ell^j\left(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}\right) \geq \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1}$. Let $\hat{N} \equiv \{j\} \cup \bar{N}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$\begin{aligned}
f_\ell^i\left(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}\right) &> f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) - 2 \cdot d \quad \text{(by Lemma 6)} \\
&\geq \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1} - 2 \cdot d \\
&= \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1} - \frac{e_1}{n!} \quad \left(\text{by } d = \frac{e_1}{2 \cdot n!}\right) \\
&\geq \frac{W_\ell}{n} + e_k. \quad \text{(by } k \geq 2 \text{ and (9))}
\end{aligned}$$

Thus, Step 1-2 holds. □

Step 1-3. *We derive a contradiction to conclude that for each* $i \in N$, $f_\ell^i\left(\hat{R}\right) = \frac{W_\ell}{n}$.
Case 1: n is even.

Then, by Case (a) of Step 1-2, for each $i \in N$, $f_\ell^i\left(\tilde{R}_0^N\right) \leq \frac{W_\ell}{n} - e_n$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i\left(\tilde{R}_0^N\right) \quad \text{(by \textit{feasibility})} \\
&\leq n \cdot \left(\frac{W_\ell}{n} - e_n\right) \\
&< W_\ell, \quad \text{(by } e_n > 0\text{)}
\end{aligned}$$

which is a contradiction.

Case 2: n is odd.

Then, by Case (b) of Step 1-2, for each $i \in N$, $f_\ell^i(\tilde{R}_0^N) \geq \frac{W_\ell}{n} + e_n$. Thus,

$$\begin{aligned} W_\ell &= \sum_{i \in N} f_\ell^i(\tilde{R}_0^N) \quad \text{(by } \textit{feasibility}\text{)} \\ &\geq n \cdot \left(\frac{W_\ell}{n} + e_n\right) \\ &> W_\ell, \quad \text{(by } e_n > 0) \end{aligned}$$

which is a contradiction.

Therefore, for each $i \in N$, we have $f_\ell^i(\hat{R}) = \frac{W_\ell}{n}$.

Step 2. Let $k \in \{0, 1, \ldots, n-2\}$. *Assume that for each* $h \in \{0, 1, \ldots, k\}$, *each* $\hat{x}_{-\ell} \in Z_{-\ell}$, *each* $\bar{R} \in \bar{\mathcal{R}}^N(h, \hat{x}_{-\ell})$, *and each* $i \in N$, *we have* $f_\ell^i(\bar{R}) = U_\ell^i(\bar{R})$. *Then, for each* $R \in \mathcal{R}_\ell^N(k+1, 0)$ *and each* $i \in \{\pi_1, \ldots, \pi_{k+1}\}$, *we have* $f_\ell^i(R) = p_\ell(R^i)$.

Proof Let $R \in \mathcal{R}_\ell^N(k+1, 0)$. Without loss of generality, assume that agents are indexed so that $p_\ell(R^1) \geq \cdots \geq p_\ell(R^{k+1})$. Let $\bar{K} \equiv \{1, 2, \ldots, k+1\}$. □

Step 2-1. Let $\hat{x}_{-\ell} \in Z_{-\ell}$ *and* $\hat{R} \in \mathcal{R}^N(\hat{x}_{-\ell})$ *be such that for each* $i \in N$, $p_\ell(\hat{R}^i) = p_\ell(R^i)$. *Then, for each* $i \in \bar{K}$, *we have* $f_\ell^i(\hat{R}) = p_\ell(\hat{R}^i)$.

Proof Let $i \in \bar{K}$. By contradiction, suppose that $f_\ell^i(\hat{R}) \neq p_\ell(\hat{R}^i)$. By *same-sideness* (Lemma 1), $f_\ell^i(\hat{R}) > p_\ell(\hat{R}^i)$.

Let $\tilde{R}_0^i \in \mathcal{R}$ be such that $p_\ell(\tilde{R}_0^i) = 0$ and $p_{-\ell}(\tilde{R}_0^i) = p_{-\ell}(\hat{R}^i)$. Then, by *own uncompromisingness* (Lemma 3-i), $f_\ell^i(\tilde{R}_0^i, \hat{R}^{-i}) = f_\ell^i(\hat{R})$. Note that for some $h \in \{0, 1, \ldots, k\}$, we have $(\tilde{R}_0^i, \hat{R}^{-i}) \in \bar{\mathcal{R}}^N(h, \hat{x}_{-\ell})$.

By the assumption of Step 2, $f_\ell^i(\tilde{R}_0^i, \hat{R}^{-i}) = U_\ell^i(\tilde{R}_0^i, \hat{R}^{-i})$. Also, since $\hat{R} \in \mathcal{R}_\ell^N(k+1, 0)$, by the definition of the uniform rule U, we have $U_\ell^i(\hat{R}) = p_\ell(\hat{R}^i)$. Since $p_\ell(\tilde{R}_0^i) = 0 < p_\ell(\hat{R}^i)$, by the definition of U, we get $U_\ell^i(\tilde{R}_0^i, \hat{R}^{-i}) \leq U_\ell^i(\hat{R})$. Thus,

$$f_\ell^i(\tilde{R}_0^i, \hat{R}^{-i}) = U_\ell^i(\tilde{R}_0^i, \hat{R}^{-i}) \leq U_\ell^i(\hat{R}) = p_\ell(\hat{R}_i) < f_\ell^i(\hat{R}) = f_\ell^i(\tilde{R}_0^i, \hat{R}^{-i}).$$

This is a contradiction. □

Step 2-2. For each $i \in \bar{K}$, *we have* $f_\ell^i(R) = p_\ell(R^i)$.

Proof Let $\bar{R} \in \mathcal{R}^n$ be such that for each $i \in N$, (i) $p_\ell(\bar{R}^i) = p_\ell(R^i)$, (ii) $p_{-\ell}(\bar{R}^i) = f_{-\ell}^i(R)$, and (iii) $UC(\bar{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\bar{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$.

Then, by *strategy-proofness*, $f^1(\bar{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\bar{R}^1, R^{-1}) = f(R)$. Repeating this argument for $i = 2, \ldots, n$, we have $f(\bar{R}) = f(R)$.

By *feasibility*, $f_{-\ell}(R) \in Z_{-\ell}$. Since for each $i \in N$, $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, we have $\bar{R} \in \mathcal{R}^N(f_{-\ell}(R))$. By Step 2-1 and for each $i \in N$, $p_\ell(\bar{R}^i) = p_\ell(R^i)$, it follows that for each $i \in \bar{K}$, $f^i_\ell(\bar{R}) = p_\ell(\bar{R}^i) = p_\ell(R^i)$. Since $f(\bar{R}) = f(R)$, for each $i \in \bar{K}$, we have $f^i_\ell(R) = p_\ell(R^i)$. □

Step 3. For each $k \in \{1, 2, \ldots, n-2\}$*, each* $x_{-\ell} \in Z_{-\ell}$*, each* $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$*, and each* $i \in N$*, we have* $f^i_\ell(R) = U^i_\ell(R)$.

Proof The proof proceeds by induction on k. Let $k \in \{1, 2, \ldots, n-2\}$. Assume that

(A) *For each* $h \in \{0, 1, \ldots, k-1\}$*, each* $\hat{x}_{-\ell} \in Z_{-\ell}$*, each* $\bar{R} \in \bar{\mathcal{R}}^N(h, \hat{x}_{-\ell})$*, and each* $i \in N$*, we have* $f^i_\ell(\bar{R}) = U^i_\ell(\bar{R})$.
By Step 1, we have already proven that (A) holds when $k = 1$. We will prove that

(B) *For each* $x_{-\ell} \in Z_{-\ell}$*, each* $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$*, and each* $i \in N$*, we have* $f^i_\ell(R) = U^i_\ell(R)$.

Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$. Without loss of generality, we may assume that agents are indexed so that $p_\ell(R^1) \geq \cdots \geq p_\ell(R^n)$. Let $K \equiv \{1, 2, \ldots, k\}$.

Then, by the definition of the uniform rule U, for each $i \in K$, $U^i_\ell(R) = p_\ell(R^i)$, and for each $j \in N \setminus K$, $U^j_\ell(R) = \frac{W_\ell - \sum_{i=1}^k p_\ell(R^i)}{n-k} = \lambda_\ell(R)$. Note that, since $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$ implies $\sum_{i \in N} p_\ell(R^i) < W_\ell$, we have $\lambda_\ell(R) > 0$. We show that for each $i \in N$, $f^i_\ell(R) = U^i_\ell(R)$.

Let $\hat{R}^{-K} \in \mathcal{R}^{|-K|}$ be such that for each $i \in N \setminus K$, $p_\ell(\hat{R}^i) = 0$ and $p_{-\ell}(\hat{R}^i) = p_{-\ell}(R^i)$. Then, since the uniform rule U is *group uncompromising*, for each $i \in N$, $U^i_\ell(R^K, \hat{R}^{-K}) = U^i_\ell(R)$. If for each $i \in N$, $f^i_\ell(R^K, \hat{R}^{-K}) = U^i_\ell(R^K, \hat{R}^{-K})$, then, by *group uncompromisingness* (Lemma 3-ii), $f(R) = f(R^K, \hat{R}^{-K})$. Thus, we only have to show for each $i \in N$, $f^i_\ell(R^K, \hat{R}^{-K}) = U^i_\ell(R^K, \hat{R}^{-K})$.

By (A) and Step 2, we have already proven that for each $i \in K$, $f^i_\ell(R^K, \hat{R}^{-K}) = p_\ell(R^i)$. Thus, we only show that for each $i \in N \setminus K$, $f^i_\ell(R^K, \hat{R}^{-K}) = \lambda_\ell(R)$.

By contradiction, suppose that there is $j \in N \setminus K$ such that $f^j_\ell(R^K, \hat{R}^{-K}) > \lambda_\ell(R)$. Without loss of generality, assume that $j = k+1$. Note that, since for each $i \in K$, $f^i_\ell(R^K, \hat{R}^{-K}) = p_\ell(R^i)$, and for each $i \in \{k+2, \ldots, n\}$, $f^i_\ell(R^K, \hat{R}^{-K}) \geq 0$, by feasibility, $f^{k+1}_\ell(R^K, \hat{R}^{-K}) \leq W_\ell - \sum_{i=1}^k p_\ell(R^i)$.

Let $e_1 \equiv f^{k+1}_\ell(R^K, \hat{R}^{-K}) - \lambda_\ell(R)$, and for each $h \in \{2, 3, \ldots, n-k\}$, let $e_h \equiv \frac{e_{h-1}}{n-k-h+2}$. Then, (i) for each $h \in \{2, 3, \ldots, n-k\}$, $e_h > 0$ and $e_h < e_{h-1}$, (ii) for each $h \in \{2, 3, \ldots, n-k\}$,

$$e_h = \frac{e_1}{\prod_{t=0}^{h-2}(n-k-t)},$$

and (iii) for each $h \in \{2, 3, \ldots, n-k\}$,

$$\frac{e_{h-1}}{n-k-h+1} - e_h = \frac{e_1}{\prod_{t=0}^{h-1}(n-k-t)} \geq \frac{e_1}{(n-k)!}. \tag{10}$$

Let $d \equiv \frac{e_1}{2\cdot(n-k)!}$. For each $i \in N \setminus K$, let $\bar{R}^i \in \mathcal{R}^V\big(0, p_{-\ell}\big(\hat{R}^i\big), d\big)$, and $\bar{R}^{-K} \equiv \big(\bar{R}^{k+1}, \ldots, \bar{R}^n\big)$. Then, by *peak-onlyness* (Lemma 2), $f\big(R^K, \bar{R}^{-K}\big) = f\big(R^K, \hat{R}^{-K}\big)$.

Let $\tilde{x}_{-\ell} \in Z_{-\ell}$, $\tilde{R}_0 \in \mathcal{R}_0^V\big(0, \tilde{x}_{-\ell}, d\big)$, and $\tilde{R}_0^{k+1} \equiv \tilde{R}_0$. □

Step 3-1. $f_\ell^{k+1}\big(R^K, \tilde{R}_0^{k+1}, \bar{R}^{-K\cup\{k+1\}}\big) \geq \lambda_\ell\big(R\big) + e_1 = f_\ell^{k+1}\big(R^K, \hat{R}^{-K}\big)$.

Proof The proof is similar to Step 1-1. By contradiction, suppose not. Since $f\big(R^K, \bar{R}^{-K}\big) = f\big(R^K, \hat{R}^{-K}\big)$, we have $f_\ell^{k+1}\big(R^K, \bar{R}^{-K}\big) = \lambda_\ell\big(R\big) + e_1$. Let $\check{R}^{k+1} \in \mathcal{R}$ be such that $p\big(\check{R}^{k+1}\big)=p\big(\bar{R}^{k+1}\big)$ and $f^{k+1}\big(R^K, \tilde{R}_0^{k+1}, \bar{R}^{-K\cup\{k+1\}}\big)\ \check{P}^{k+1}\ f^{k+1}\big(R^K, \bar{R}^{-K}\big)$. Then, by *peak-onlyness* (Lemma 2), $f^{k+1}\big(R^K, \check{R}^{k+1}, \bar{R}^{-K\cup\{k+1\}}\big) = f^{k+1}\big(R^K, \bar{R}^{-K}\big)$. Thus, $f^{k+1}\big(R^K, \tilde{R}_0^{k+1}, \bar{R}^{-K\cup\{k+1\}}\big)\ \check{P}^{k+1}\ f^{k+1}\Big(R^K, \check{R}^{k+1}, \bar{R}^{-K\cup\{k+1\}}\Big)$, contradicting *strategy-proofness*. □

Given $\hat{N} \subset N \setminus K$, let $\tilde{R}_0^{\hat{N}}$ be such that for each $i \in \hat{N}$, $\tilde{R}_0^i = \tilde{R}_0$.

Step 3-2. For each $h \in \{1, 2, \ldots, n-k\}$,

(a) *if h is even, then there is $\hat{N} \subset N \setminus K$ such that $|\hat{N}| = h$ and for each $i \in \hat{N}$, we have $f_\ell^i\big(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}\big) \leq \lambda_\ell(R) - e_h$, and*

(b) *if h is odd, then there is $\hat{N} \subset N \setminus K$ such that $|\hat{N}| = h$ and for each $i \in \hat{N}$, we have $f_\ell^i\big(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}\big) \geq \lambda_\ell\big(R\big) + e_h$.*

Proof The proof is similar to Step 1-2, and proceeds by induction on h. Let $h \in \{1, 2, \ldots, n-k\}$. When $h = 1$, by Step 3-1, we have already proven that (b) holds. Assume that $h \geq 2$.

Case (a): h is even.

Our induction hypothesis is that there is $\bar{N} \subset N \setminus K$ such that $|\bar{N}| = h - 1$ and for each $i \in \bar{N}$, $f_\ell^i\big(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\big) \geq \lambda_\ell\big(R\big) + e_{h-1}$.

Suppose that for each $j \in N\setminus\big(K\cup\bar{N}\big)$, $f_\ell^j\big(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\big) > \lambda_\ell\big(R\big) - \frac{e_{h-1}}{n-k-h+1}$. By (A) and Step 2, for each $i \in K$, $f_\ell^i\big(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\big) = p_\ell\big(R^i\big)$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i\in N} f_\ell^i\big(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\big) \quad \big(\text{by } \textit{feasibility}\big)\\
&> \sum_{i\in K} p_\ell\big(R^i\big) + \big(h-1\big)\cdot\big(\lambda_\ell\big(R\big) + e_{h-1}\big)\\
&\quad +(n-k-h+1)\cdot\left(\lambda_\ell(R) - \frac{e_{h-1}}{n-k-h+1}\right)\\
&= W_\ell + (h-2)\cdot e_{h-1} \quad \Big(\text{by } \sum_{i\in K} p_\ell\big(R^i\big) + (n-k)\cdot\lambda_\ell(R) = W_\ell\Big)\\
&\geq W_\ell, \quad (\text{by } h \geq 2 \text{ and } e_{h-1} > 0)
\end{aligned}$$

which is a contradiction. Thus, there is $j \in N \setminus \big(K \cup \bar{N}\big)$ such that $f_\ell^j\Big(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\Big) \leq \lambda_\ell\big(R\big) - \frac{e_{h-1}}{n-k-h+1}$.

Let $\hat{N} \equiv \bar{N} \cup \{j\}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$\begin{aligned} f_\ell^i\left(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}\right) &< f_\ell^j\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) + 2\cdot d \quad \text{(by Lemma 6)} \\ &\leq \lambda_\ell\left(R\right) - \frac{e_{h-1}}{n-k-h+1} + 2\cdot d \\ &= \lambda_\ell\left(R\right) - \frac{e_{h-1}}{n-k-h+1} + \frac{e_1}{(n-k)!} \quad \left(\text{by } d = \frac{e_1}{2\cdot(n-k)!}\right) \\ &\leq \lambda_\ell(R) - e_h. \quad \text{(by } h \geq 2 \text{ and (10))} \end{aligned}$$

Case (b): h is odd.

Our induction hypothesis is that there is $\bar{N} \subset N \setminus K$ such that $|\bar{N}| = h-1$ and for each $i \in \bar{N}$, $f_\ell^i\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) \leq \lambda_\ell(R) - e_{h-1}$.

Suppose that for each $j \in N\setminus\left(K\cup\bar{N}\right)$, $f_\ell^j\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) < \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1}$. By (A) and Step 2, for each $i \in K$, $f_\ell^i\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) = p_\ell\left(R^i\right)$. Thus,

$$\begin{aligned} W_\ell &= \sum_{i\in N} f_\ell^i\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) \quad \text{(by } \textit{feasibility}\text{)} \\ &< \sum_{i\in K} p_\ell\left(R^j\right) + (h-1)\cdot\left(\lambda_\ell(R) - e_{h-1}\right) \\ &\quad +(n-k-h+1)\cdot\left(\lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1}\right) \\ &= W_\ell - (h-2)\cdot e_{h-1} \quad \left(\text{by } \sum_{i\in K} p_\ell\left(R^i\right) + (n-k)\cdot\lambda_\ell(R) = W_\ell\right) \\ &\leq W_\ell, \quad \text{(by } h \geq 2 \text{ and } e_{h-1} > 0) \end{aligned}$$

which is a contradiction. Thus, there is $j \in N\setminus(K\cup\bar{N})$ such that $f_\ell^j\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) \geq \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1}$.

Let $\hat{N} \equiv \bar{N} \cup \{j\}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$\begin{aligned} f_\ell^i\left(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}\right) &> f_\ell^j\left(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}\right) - 2\cdot d \quad \text{(by Lemma 6)} \\ &\geq \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1} - 2\cdot d \\ &= \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1} - \frac{e_1}{(n-k)!} \quad \left(\text{by } d = \frac{e_1}{2\cdot(n-k)!}\right) \\ &\geq \lambda_\ell(R) + e_h. \quad \text{(by } h \geq 2 \text{ and (10))} \end{aligned}$$

Thus, Step 3-2 holds.

Step 3-3. We derive a contradiction to conclude that for each $i \in N \setminus K$, $f_\ell^i\left(R^K, \hat{R}^{-K}\right) = \lambda_\ell(R)$.

By (A) and Step 2, for each $i \in K$, $f_\ell^i\left(R^K, \tilde{R}_0^{-K}\right) = p_\ell\left(R^i\right)$.

Case 1: $n-k$ is even.

Then, by Case (a) of Step 3-1, for each $i \in N \setminus K$, we have $f_\ell^i\left(R^K, \tilde{R}_0^{-K}\right) \leq \lambda_\ell(R) - e_{n-k}$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i\left(R^K, \tilde{R}_0^{-K}\right) \quad (\text{by } \textit{feasibility}) \\
&\leq \sum_{i \in K} p_\ell\left(R^i\right) + (n-k) \cdot \lambda_\ell(R) - (n-k) \cdot e_{n-k} \\
&= W_\ell - (n-k) \cdot e_{n-k} \quad \left(\text{by } \sum_{i \in K} p_\ell\left(R^i\right) + (n-k) \cdot \lambda_\ell(R) = W_\ell\right) \\
&< W_\ell, \quad (\text{by } n-k \geq 2 \text{ and } e_{n-k} > 0)
\end{aligned}$$

which is a contradiction.

Case 2: $n-k$ is odd.

Then, by Case (b) of Step 3-1, for each $i \in N \setminus K$, we have $f_\ell^i\left(R^K, \tilde{R}_0^{-K}\right) \geq \lambda_\ell(R) + e_{n-k}$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i\left(R^K, \tilde{R}_0^{-K}\right) \quad (\text{by } \textit{feasibility}) \\
&\geq \sum_{i \in K} p_\ell\left(R^i\right) + (n-k) \cdot \lambda_\ell(R) + (n-k) \cdot e_{n-k} \\
&= W_\ell + (n-k) \cdot e_{n-k} \quad \left(\text{by } \sum_{i \in K} p_\ell\left(R^i\right) + (n-k) \cdot \lambda_\ell(R) = W_\ell\right) \\
&> W_\ell, \quad (\text{by } n-k \geq 2 \text{ and } e_{n-k} > 0)
\end{aligned}$$

which is a contradiction.

Therefore, for each $i \in N \setminus K$, we have $f_\ell^i\left(R^K, \hat{R}^{-K}\right) = \lambda_\ell(R)$. □

Step 4. For each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(n-1, x_{-\ell})$, and each $i \in N$, we have $f_\ell^i(R) = U_\ell^i(R)$.

Proof Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(n-1, x_{-\ell})$. Without loss of generality, we may assume that $p_\ell(R^1) \geq \cdots \geq p_\ell\left(R^n\right)$. Then, by the definition of the uniform rule U, for each $i \in N \setminus \{n\}$, $U_\ell^i(R) = p_\ell\left(R^i\right)$, and $U_\ell^n(R) = W_\ell - \sum_{i=1}^{n-1} p_\ell\left(R^i\right)$. Let $\hat{R}^n \in \mathcal{R}$ be such that $p_\ell\left(\hat{R}^n\right) = 0$ and $p_{-\ell}\left(\hat{R}^n\right) = p_{-\ell}\left(R^n\right)$. By Steps 1 and 3, when $k = n-2$, we have already proven that the assumption of Step 2 holds. Thus Step 2 implies that for each $i \in N \setminus \{n\}$, $f_\ell^i\left(\hat{R}^n, R^{-n}\right) = p_\ell\left(R^i\right)$. By *feasibility*, $f_\ell^n\left(\hat{R}^n, R^{-n}\right) = W_\ell - \sum_{i=1}^{n-1} p_\ell\left(R^i\right)$. Since $\sum_{i \in N} p_\ell\left(R^i\right) < W_\ell$, we have $W_\ell - \sum_{i=1}^{n-1} p_\ell\left(R^i\right) > 0$. Thus $p_\ell\left(\hat{R}^n\right) < f_\ell\left(\hat{R}^n, R^{-n}\right)$. By *own uncompromisingness* (Lemma 3-i), $f(R) = f\left(\hat{R}^n, R^{-n}\right)$. Thus, for each $i \in N$, we have $f_\ell^i(R) = U_\ell^i(R)$.

In the same way, we can also show that for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ such that $\sum_{i \in N} p_\ell(R^i) > W_\ell$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.

We have completed the proof of Lemma 7. □

Step V: We complete the proof of Theorem 1 by extending the result of Step IV.

Let f be a *strategy-proof*, *unanimous*, *weakly symmetric*, and *nonbossy* rule. We show that for each $R \in \mathcal{R}^n$ and each $\ell \in M$, $f_\ell(R) = U_\ell(R)$. Let $R \in \mathcal{R}^n$ and $\ell \in M$. Let $\bar{R} \in \mathcal{R}^n$ be such that for each $i \in N$, (i) $p_\ell(\bar{R}^i) = p_\ell(R^i)$, (ii) $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, and (iii) $UC(\bar{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\bar{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$. By *strategy-proofness*, $f^1(\bar{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\bar{R}^1, R^{-1}) = f(R)$. Repeating this argument for $i = 2, \ldots, n$, we have $f(\bar{R}) = f(R)$.

By *feasibility*, $f_{-\ell}(R) \in Z_{-\ell}$. Since for each $i \in N$, $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, we have $\bar{R} \in \bar{\mathcal{R}}^N(f_{-\ell}(R))$. Thus, Lemma 7 implies $f_\ell(\bar{R}) = U_\ell(\bar{R})$. Since the uniform rule is *peak-only*, $U_\ell(\bar{R}) = U_\ell(R)$. Thus, $f_\ell(\bar{R}) = U_\ell(R)$. Since $f(\bar{R}) = f(R)$, we obtain $f_\ell(R) = U_\ell(R)$.

4 Concluding remarks

We considered the problem of allocating several infinitely divisible commodities among agents with continuous, strictly convex, and separable preferences. We established that on this class of preferences, a rule satisfies *strategy-proofness*, *unanimity*, *weak symmetry*, and *nonbossiness* if and only if it is the uniform rule. This result extends to the class of continuous, strictly convex, and multidimensional single-peaked preferences. We conclude by commenting on future research.

As we discussed in Remark 2, the *only if* part of Theorem 1 fails if we drop any of the first three axioms. However, it is an open question whether the uniqueness part of Theorem 1 holds without *nonbossiness*. By applying the proof techniques that we developed in the previous version of this article,[32] Adachi (2010) characterized the uniform rule by means of *strategy-proofness*, *same-sideness*, and *no-envy* in the multiple-commodity model with more than two agents. His axioms do not include *nonbossiness*. However, since *unanimity* and *weak symmetry* are weaker than *same-sideness* and *no-envy* respectively, our result is independent of his, and the question we raised above remains open.

In the one-commodity case, since the uniform rule is *nonbossy*, *strategy-proofness*, *same-sideness*, and *weak symmetry* imply *nonbossiness* (Sprumont 1991; Ching 1994). Moreover, *effective pairwise strategy-proofness* and *unanimity* imply *nonbossiness* (Serizawa 2006).[33] Therefore, it is also an open question whether these logical relationships for the one-commodity case extend to the multiple-commodity case.

[32] The previous version was published as a Discussion Paper. See Morimoto et al. (2010).

[33] *Effective pairwise strategy-proofness* requires that rules are strategy-proof and that no pair of agents can increase the welfare of any agent of the pair without decreasing the welfare of the other member of the pair, and neither member of the pair has an incentive to betray his partner. Serizawa (2006) characterized the uniform rule by *effective pairwise strategy-proofness*, *unanimity*, and *weak symmetry*.

As we discussed in Sect. 1, in contrast to the one-commodity model, *strategy-proofness* is not compatible with *Pareto-efficiency* under distributional properties such as *weak symmetry* in the multiple-commodity model. Thus, we gave up *Pareto-efficiency*, and imposed the weaker property of *unanimity*. Another interesting research direction is to relax *strategy-proofness*. One such research is to investigate desirable allocation rules by employing the property of *Maskin monotonicity* (for example, see Thomson 2010) instead of *strategy-proofness*.

It is worthwhile to comment on whether our characterization results of the uniform rule hold on smaller domains. For example, the domain of continuous, strictly convex, and additive symmetric single-peaked preferences is an interesting subclass of our preference domain.[34] In the one-commodity case, Mizobuchi and Serizawa (2006) show that the uniform rule is a unique rule satisfying *strategy-proofness*, *same-sideness*, and *weak symmetry* on the symmetric single-peaked domain.[35] One might wonder whether our proof techniques can be applied to obtain the same result as Mizobuchi and Serizawa (2006) in the multiple-commodity case. However, our proof techniques cannot be applied on the domain of continuous, strictly convex, and additive symmetric single-peaked preferences since our proofs employ asymmetric preferences. Another way to obtain the same result would be to extend the proof techniques of Mizobuchi and Serizawa (2006) to the multiple-commodity case. In Introduction, we discussed the difficulty of characterizing *strategy-proof* rules in the model with multiple agents and commodities. The same reason makes it difficult to extend the proof techniques of Mizobuchi and Serizawa (2006) to the multiple-commodity case. It is still an open question whether our results hold on smaller domains.

Appendix A

In Appendix A, we show the existence of strict Maskin monotonic transformations used in the proofs of Corollary 1, Case 1 and 2 of Lemma 1, Step 2-2 of Lemma 7, and Step V in the proof of Theorem 2.

The following fact shows the existence of strict Maskin monotonic transformations.

Fact A *Let $R^i \in \mathcal{R}$, $x \in X \setminus \{p(R^i)\}$, and $y \in X \setminus \{x\}$ be such that for each $\ell \in M$, either $p_\ell(R^i) \geq y_\ell \geq x_\ell$ or $p_\ell(R^i) \leq y_\ell \leq x_\ell$. Then, there is $\hat{R}^i \in \mathcal{R}$ such that $p(\hat{R}^i) = y$ and $\hat{R}^i$ is a strict Maskin monotonic transformation of R^i at x.*

Proof of Fact A We can construct such a preference by composing several preferences with the peak point y (see Fig. 5). We only prove this fact for the two-commodity case. Let $M = \{1, 2\}$. Applying the similar arguments to the two-commodity case, we can show that such a transformation exists for the more than two-commodity case.

Note that, there is a preference relation R^i_e on $\mathbb{R}^m$ such that (i) R^i_e is continuous, strictly convex, and separable, and (ii) R^i_e is equivalent to R^i on X, that is, for each

[34] A preference R^i is *additive symmetric single-peaked* if there are symmetric single-peaked functions $\{v_\ell : X_\ell \to \mathbb{R}\}_{\ell \in M}$ such that for each $x, y \in X$, $x\ R^i\ y$ if and only if $\sum_{\ell \in M} v_\ell(x_\ell) \geq \sum_{\ell \in M} v_\ell(y_\ell)$.

[35] More precisely, they establish the characterization result of the uniform rule on a minimally rich domain. The domain of symmetric single-peaked preferences is a minimally rich domain.

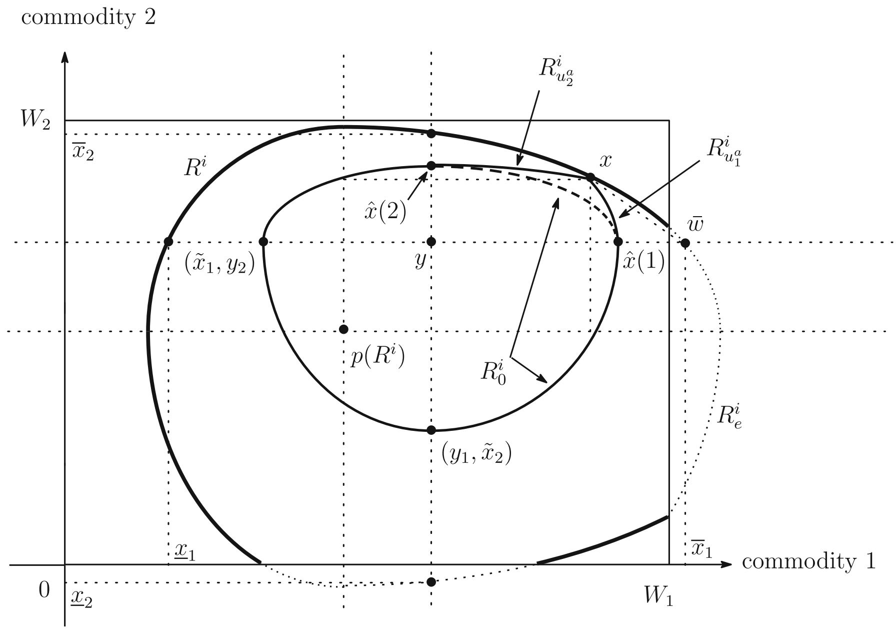

Fig. 5 Illustration of constructing a strict Maskin monotonic transformation of R^i at x in Fact A

$\hat{x}, \hat{y} \in X$, $\hat{x}\ R^i_e\ \hat{y}$ if and only if $\hat{x}\ R^i\ \hat{y}$. Assume that there is a preference relation $\hat{R}^i_e$ on $\mathbb{R}^m$ that is a strict Maskin monotonic transformation of R^i_e at x with peak y. Consider the restriction $\hat{R}^i$ of $\hat{R}^i_e$ to X, that is, for each $\hat{x}, \hat{y} \in X$, $\hat{x}\ \hat{R}^i\ \hat{y}$ if and only if $\hat{x}\ \hat{R}^i_e\ \hat{y}$. Then, $\hat{R}^i$ is a strict Maskin monotonic transformation of R^i at x defined over X. Thus, we only show the existence of a strict Maskin monotonic transformation $\hat{R}^i_e$ of R^i_e at x with peak y.

Without loss of generality, assume that for each $\ell \in M$, $p_\ell(R^i) \leq x_\ell$. Then, for each $\ell \in M$, $p_\ell(R^i) \leq y_\ell \leq x_\ell$. The proof of Fact A consists of two steps.

Step 1. Construction of $\hat{R}^i_e$.

Let

$$\overline{x}_1 \equiv \max\left\{x_1' \in \mathbb{R} \mid \left(x_1', y_2\right)\ I^i_e\ x\right\}, \quad \overline{x}_2 \equiv \max\left\{x_2' \in \mathbb{R} \mid \left(y_1, x_2'\right)\ I^i_e\ x\right\},$$
$$\underline{x}_1 \equiv \min\left\{x_1' \in \mathbb{R} \mid \left(x_1', y_2\right)\ I^i_e\ x\right\}, \quad \underline{x}_2 \equiv \min\left\{x_2' \in \mathbb{R} \mid \left(y_1, x_2'\right)\ I^i_e\ x\right\},$$
$$\overline{d}_1 \equiv \overline{x}_1 - x_1, \quad \overline{d}_2 \equiv \overline{x}_2 - x_2,$$
$$a^{\overline{d}_1} \equiv \begin{cases} \frac{\left(x_2 - y_2\right)^2}{2 \cdot \overline{d}_1 \cdot \left(x_1 - y_1\right)} & \text{if } \overline{d}_1 > 0, \\ 0 & \text{if } \overline{d}_1 = 0, \end{cases} \quad a^{\overline{d}_2} \equiv \begin{cases} \frac{\left(x_1 - y_1\right)^2}{2 \cdot \overline{d}_2 \cdot \left(x_2 - y_2\right)} & \text{if } \overline{d}_2 > 0, \\ 0 & \text{if } \overline{d}_2 = 0. \end{cases}$$

Let $a > \max\{a^{\overline{d}_1}, a^{\overline{d}_2}\}$. Note that $a > 0$. Let $R^i_{u_1^a}$ and $R^i_{u_2^a}$ be preferences represented by the separable and quadratic utility functions u_1^a and u_2^a with peak y and parameter

a defined as follows, respectively: for each $\hat{x} \in X$,

$$u_1^a(\hat{x}) \equiv -a \cdot (\hat{x}_1 - y_1)^2 - (\hat{x}_2 - y_2)^2, \text{ and } u_2^a(\hat{x}) \equiv -(\hat{x}_1 - y_1)^2 - a \cdot (\hat{x}_2 - y_2)^2.$$

Note that, for large $a > 0$, the preference $R^i_{u^a_\ell}$ is almost indifferent to all commodity except for commodity ℓ. Let $B^+(y) \equiv \{x' \in \mathbb{R}^m \mid y_1 \leq x'_1 \text{ and } y_2 \leq x'_2\}$, and $\bar{U}^+(x) \equiv UC(R^i_{u^a_1}, x) \cap UC(R^i_{u^a_2}, x) \cap B^+(y)$.

Let $\tilde{x_1} \in (\underline{x}_1, y_1)$, $\tilde{x_2} \in (\underline{x}_2, y_2)$, and let $\hat{x}(1), \hat{x}(2) \in \mathbb{R}^m_+$ be such that

$$u_1^a(\hat{x}(1)) = u_1^a(x), \hat{x}_1(1) \geq y_1, \hat{x}_2(1) = y_2,$$
$$\text{and } u_2^a(\hat{x}(2)) = u_2^a(x), \hat{x}_1(2) = y_1, \hat{x}_2(2) \geq y_2.$$

Finally, let R^i_0 be a continuous, strictly convex, and separable preference on $\mathbb{R}^m$ such that $p(R^i_0) = y$, $UC(R^i_0, \hat{x}(1)) \subset SUC(R^i_e, x)$, $UC(R^i_0, \hat{x}(1)) \cap B^+(y) \subset \bar{U}^+(x)$, and $\hat{x}(1)\ I^i_0\ \hat{x}(2)\ I^i_0\ (\tilde{x}_1, y_2)\ I^i_0\ (y_1, \tilde{x}_2)$.[36]

Let $\hat{R}^i_e$ be a continuous, strictly convex, and separable preference on $\mathbb{R}^m$ such that $p(\hat{R}^i_e) = y$ and $UC(\hat{R}^i_e, x) \equiv \bar{U}^+(x) \cup UC(R^i_0, \hat{x}(1))$.

Step 2. $\hat{R}^i_e$ *is a strict Maskin monotonic transformation of* R^i_e *at* x.

Since $UC(R^i_0, \hat{x}(1)) \subset SUC(R^i_e, x)$, for each $z \in UC(\hat{R}^i_e, x) \setminus \bar{U}^+(x)$, we have $z\ P^i_e\ x$. Next, we show that $UC(\hat{R}^i_e, x) \cap LC(R^i_e, x) \cap B^+(y) = \{x\}$. By contradiction, suppose that there is $z \neq x$ such that $z \in UC(\hat{R}^i_e, x) \cap B^+(y)$ and $z \in LC(R^i_e, x) \cap B^+(y)$. If for each $\ell \in M$, $y_\ell \leq z_\ell \leq x_\ell$, then $z\ P^i_e\ x$, which contradicts $z \in LC(R^i_e, x)$. Thus, there is $\ell \in M$ such that $z_\ell > x_\ell$. Without loss of generality, assume that $\ell \equiv 1$. Let $\bar{w} \equiv (x_1 + \bar{d}_1, y_2)$, and for each $\bar{a} \in [0, 1)$, let $z^{\bar{a}} \equiv \bar{a} \cdot x + (1 - \bar{a}) \cdot \bar{w}$. Let $\bar{a} \in [0, 1)$. Then, by $a > \max\{a^{\bar{d}_1}, a^{\bar{d}_2}\}$,

$$\begin{aligned}
u_1^a(z^{\bar{a}}) &= -a\left((x_1 - y_1) + (1 - \bar{a})\bar{d}_1\right)^2 - \bar{a}(x_2 - y_2)^2 \\
&= -a(x_1 - y_1)^2 - 2a(1 - \bar{a})\bar{d}_1(x_1 - y_1) - a(1 - \bar{a})^2\bar{d}_1^2 - \bar{a}(x_2 - y_2)^2 \\
&\leq -a(x_1 - y_1)^2 - 2a(1 - \bar{a})\bar{d}_1(x_1 - y_1) \\
&\quad -\bar{a}(x_2 - y_2)^2 \quad \text{by } a > 0, \bar{a} < 1, \bar{d}_\ell \geq 0 \\
&< -a(x_1 - y_1)^2 - \frac{(x_2 - y_2)^2}{2 \cdot \bar{d}_1 \cdot (x_1 - y_1)} \cdot 2(1 - \bar{a})\bar{d}_1(x_1 - y_1) \\
&\quad -\bar{a}(x_2 - y_2)^2 \quad \text{by } a > a^{\bar{d}_1} \\
&= u_1^a(x).
\end{aligned}$$

36 $SUC(R^i, y)$ ($SLC(R^i, y)$) denotes the strict upper (lower) contour set of R^i at y, that is, $SUC(R^i, y) \equiv \{x \in X \mid x\ P^i\ y\}$.

Since $z \in UC\big(\hat{R}_e^i, x\big)$, $u_1^a(z) \geq u_1^a(x) > u_1^a\big(z^{\bar{a}}\big)$. Thus, for each $\bar{a} \in [0, 1)$, $u_1^a(z) > u_1^a(z^{\bar{a}})$. Since $z \in LC\big(R_e^i, x\big)$, $x\ R_e^i\ z$. By strictly convexity of R_e^i, for each $\bar{a} \in [0, 1)$, $z^{\bar{a}}\ R_e^i\ z$. Let $\bar{z} \in X$ be such that $\bar{z}_2 \equiv z_2$ and there is $\hat{a} \in [0, 1)$ such that $\bar{z} \equiv z^{\hat{a}}$. Then, $u_1^a(z) > u_1^a(\bar{z})$ implies $\bar{z}_1 > z_1$. Thus, $z\ P_e^i\ \bar{z}\ R_e^i\ z$, which is a contradiction. □

Appendix B

As we noted in Remark 4, in the two-agent case, the domain $\mathcal{R}^2$ of continuous, strictly convex, and separable preferences does not satisfy *indirect sequential inclusion* introduced by Barberà et al. (2010). In Appendix B, we give an example illustrating the violation of *indirect sequential inclusion* of our preference domain.

First, we introduce the condition on preference profile, called "sequential inclusion". Given a preference relation R^i on X, let $\hat{R}(R^i)$ be a preference relation on X such that for each $y, z \in X$, $y\ \hat{R}(R^i)\ z$ if and only if $(W - y)\ R^i\ (W - z)$, and $\hat{P}(R^i)$ and $\hat{I}(R^i)$ be the strict and indifferent preference relation associated with $\hat{R}(R^i)$, respectively.

Sequential inclusion: A preference profile (R^1, R^2) in a domain $\hat{\mathcal{R}}^2$ satisfies *sequential inclusion* if for each $y, z \in X$, if $y\ P^1\ z$ and $y\ \hat{P}(R^2)\ z$, then either $LC(R^1, z) \subset SLC(\hat{R}(R^2), y)$ or $LC(\hat{R}(R^2), z) \subset SLC(R^1, y)$.

The following is the definition of *indirect sequential inclusion* defined on a domain $\hat{\mathcal{R}}^2$.

Indirect sequential inclusion: For each $(R^1, R^2) \in \hat{\mathcal{R}}^2$,

(a) (R^1, R^2) satisfies *sequential inclusion*, or
(b) for each $y, z \in X$, there is $(\hat{R}^1, \hat{R}^2) \in \hat{\mathcal{R}}^2$ such that
  (b-1): $\hat{R}^1$ and $\hat{R}^2$ are the strict Maskin monotonic transformations of R^1 and R^2 at z and $(W - z)$, respectively.
  (b-2): $y\ \hat{P}^1\ z$ and $y\ \hat{P}(\hat{R}^2)\ z$, and
  (b-3): either $LC(\hat{R}^1, z) \subset SLC(\hat{R}(\hat{R}^2), y)$ or $LC(\hat{R}(\hat{R}^2), z) \subset SLC(\hat{R}^1, y)$.

The following example says that our preference domain $\mathcal{R}^2$ does not satisfy *indirect sequential inclusion*.

Example 7 (See Fig. 6) Let $N \equiv \{1, 2\}$, $M \equiv \{1, 2\}$, $y \equiv W$, and $z \equiv (0, 0)$. Let $\big(R^1, R^2\big) \in \mathcal{R}^2$ be such that $p\big(R^1\big) \equiv (0, W_2) \equiv p\big(R^2\big)$, (1) $y\ I^1\ \big(0, \frac{W_2}{2}\big)\ P^1\ \big(W_1, \frac{W_2}{2}\big)\ I^1\ z$, and (2) $y\ \hat{I}\big(R^2\big)\ \big(\frac{W_1}{2}, 0\big)\ \hat{P}\big(R^2\big)\ \big(\frac{W_1}{2}, W_2\big)\ \hat{I}\big(R^2\big)\ z$. We show that the profile $\big(R^1, R^2\big)$ violates the requirement (a) and (b) of *indirect sequential inclusion*. The proof consists of two steps.

Step 1. The preference profile (R^1, R^2) does not satisfy sequential inclusion.

Note that $y\ P^1\ z$ and $y\ \hat{P}\big(R^2\big)\ z$. Since $p\big(\hat{R}\big(R^2\big)\big) = \big(W_1, 0\big)$, $\big(W_1, 0\big) \notin SLC\big(\hat{R}\big(R^2\big), y\big)$. However, $\big(W_1, 0\big) \in LC\big(R^1, z\big)$. Thus, $LC(R^1, z)$ is not included in $SLC(\hat{R}(R^2), y)$. Also, since $p\big(R^1\big) = \big(0, W_2\big)$, $\big(0, W_2\big) \notin SLC\big(R^1, y\big)$. However, $(0, W_2) \in LC\big(\hat{R}\big(R^2\big), z\big)$. Thus, $LC\big(\hat{R}\big(R^2\big), z\big)$ is not included in $SLC\big(R^1, y\big)$. Therefore, the preference profile $\big(R^1, R^2\big)$ does not satisfy *sequential inclusion*.

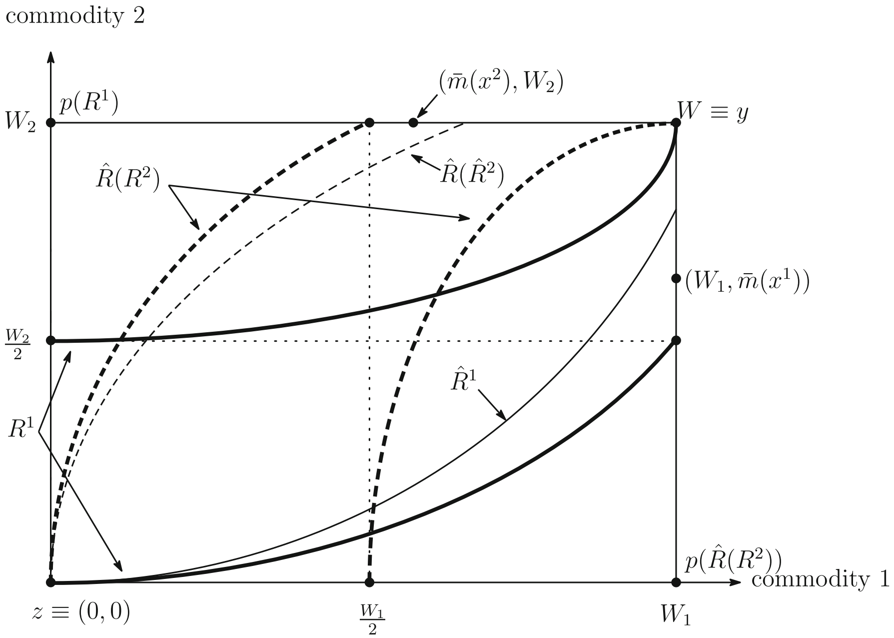

Fig. 6 Illustration of (R^1, R^2) and $(\hat{R}^1, \hat{R}^2)$ in Example 7

Step 2. There is no $(\hat{R}^1, \hat{R}^2) \in \mathcal{R}^2$ satisfying conditions (b-1), (b-2), and (b-3).

By contradiction, suppose that there is $(\hat{R}^1, \hat{R}^2) \in \mathcal{R}^2$ satisfying (b-1), (b-2) and (b-3).

First, we show that the inclusion $LC(\hat{R}^1, z) \subset SLC(\hat{R}(\hat{R}^2), y)$ does not hold. By (b-1), (b-2), and $(W_1, \frac{W_2}{2})\ I^1\ z$, there is $x^1 \in [0, W_2]$ such that $(W_1, x^1)\ \hat{I}^1\ z$ and $\frac{W_2}{2} < x^1$. Note that (b-2) implies $x^1 < W_2$. Let $\bar{m}(x^1) \equiv \frac{1}{2} \cdot (\frac{W_2}{2} + x^1)$. Then, $(W_1, \bar{m}(x^1)) \in LC(\hat{R}^1, z)$. Since $p(R^2) = (0, W_2)$ and (b-1), $p_2(\hat{R}^2) = W_2$, and so, $p_2(\hat{R}(\hat{R}^2)) = 0$. Thus, $p_1(\hat{R}(\hat{R}^2)) \leq W_1 = y_1$ and $p_2(\hat{R}(\hat{R}^2)) = 0 < \bar{m}(x^1) < W_2 = y_2$. Then, since $(W_1, \bar{m}(x^1))\ \hat{P}(\hat{R}^2)\ y$, we have $(W_1, \bar{m}(x^1)) \notin SLC(\hat{R}(\hat{R}^2), y)$. Thus, the inclusion $LC(\hat{R}^1, z) \subset SLC(\hat{R}(\hat{R}^2), y)$ does not hold.

Next, we show that the inclusion $LC(\hat{R}(\hat{R}^2), z) \subset SLC(\hat{R}^1, y)$ does not hold. By (b-1), (b-2), and $(\frac{W_1}{2}, W_2)\ \hat{I}(R^2)\ z$, there is $x^2 \in [0, W_1]$ such that $(x^2, W_2)\ \hat{I}(\hat{R}^2)\ z$ and $\frac{W_1}{2} < x^2$. Note that (b-2) implies $x^2 < W_1$. Let $\bar{m}(x^2) \equiv \frac{1}{2} \cdot (\frac{W_1}{2} + x^2)$. Then $(\bar{m}(x^2), W_2) \in LC(\hat{R}(\hat{R}^2), z)$. Since $p(R^1) = (0, W_2)$ and (b-1), $p_1(\hat{R}^1) = 0$. Thus, $p_1(\hat{R}^1) = 0 < \bar{m}(x^2) < W_1 = y_1$ and $p_2(\hat{R}^1) \leq W_2 = y_2$. Then, since $(\bar{m}(x^2), W_2)\ \hat{P}^1\ y$, we have $(\bar{m}(x^2), W_2) \notin SLC(\hat{R}^1, y)$. Thus, the inclusion $LC(\hat{R}(\hat{R}^2), z) \subset SLC(\hat{R}^1, y)$ also does not hold. This is a contradiction to (b-3).

Acknowledgments We would like to thank Hans Peters for his valuable ideas at the initial stage of this research. We are also very grateful to William Thomson for his many detailed and helpful comments. We appreciate the careful reading of the associate editor and the anonymous referees, and their detailed and useful comments. We also acknowledge the useful comments by Masaki Aoyagi, Lars Ehlers, Yuji Fujinaka, Kazuhiko Hashimoto, Kentaro Hatsumi, Shinji Ohseto, Hiroki Saitoh, Yves Sprumont, Shohei Takagi, Atsushi Tsuneki, and participants at the 2008 Spring Annual Meeting of the Japanese Economic Association in Tohoku University, and the Ninth International Meeting of the Society for Social Choice and Welfare in Concordia University. The paper was partially written while Morimoto was visiting the Department of Economics at the University of Rochester. He appreciates the good research environment of the department, and acknowledges financial support from the Research Fellowships for Young Scientists of the Japan Society for the Promotion of Science. All remaining errors are our own.

References

Adachi T (2010) The uniform rule with several commodities: a generalization of Sprumont's characterization. J Math Econ 46:952–964

Amorós P (2002) Single-peaked preferences with several commodities. Soc Choice Welf 19:57–67

Anno H, Sasaki H (2009) Second-best efficiency of allocation rules: strategy-proofness and single-peaked preferences with multiple commodities. Working Paper, Waseda University

Barberà S, Jackson MO (1995) Strategy-proof exchange. Econometrica 51:51–87

Barberà S, Gul F, Stacchetti E (1993) Generalized median voter schemes and committees. J Econ Theory 61:262–289

Barberà S, Jackson MO, Neme A (1997) Strategy-proof allotment rules. Games Econ Behav 18:1–21

Barberà S, Massó J, Serizawa S (1998) Strategy-proof voting on compact ranges. Games Econ Behav 25:272–291

Barberà S, Berga D, Moreno B (2010) Individual versus group strategy-proofness: when do they coincide? J Econ Theory 145:1648–1674

Bénassy JP (1982) The economics of market disequilibrium. Academic Press, New York

Bogomolnaia A, Deb R, Ehlers L (2005) Strategy-proof assignment on the full preference domain. J Econ Theory 123:161–186

Border KC, Jordan JS (1983) Straightforward elections, unanimity and phantom voters. Rev Econ Stud 50:153–170

Ching S (1994) An alternative characterization of the uniform rule. Soc Choice Welf 11:131–136

Ching S, Serizawa S (1998) A maximal domain for the existence of strategy-proof rules. J Econ Theory 78:157–166

Ehlers L, Klaus B (2003) Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. Soc Choice Welf 21:265–280

Fleurbaey M, Maniquet F (1997) Implementability and horizontal equity imply no-envy. Econometrica 65:1215–1219

Foley D (1967) Resource allocation and the public sector. Yale Econ Essays 7:45–98

Hatfield JW (2009) Strategy-proof, efficient, and nonbossy quota allocations. Soc Choice Welf 33:505–515

Le Breton M, Sen A (1999) Separable preferences, strategyproofness, and decomposability. Econometrica 67:605–628

Le Breton M, Zaporozhets V (2009) On the equivalence of coalitional and individual strategy-proofness properties. Soc Choice Welf 33:287–309

Maskin E (1999) Nash equilibrium and welfare optimality. Rev Econ Stud 66:23–38

Massó J, Neme A (2001) Maximal domain of preferences in the division problem. Games Econ Behav 37:367–387

Massó J, Neme A (2007) Bribe-proof rules in the division problem. Games Econ Behav 61:331–343

Miyagawa E (2001) House allocation with transfers. J Econ Theory 100:329–355

Mizobuchi H, Serizawa S (2006) Maximal domain for strategy-proof rules in allotment economies. Soc Choice Welf 27:195–210

Morimoto S, Serizawa S, Ching S (2010) A characterization of the uniform rule with several commodities and agents. Osaka University, ISER discussion paper no. 769

Moulin H (1980) On strategy-proofness and single-peakedness. Public Choice 35:437–456

Otten G, Peters H, Volij O (1996) Two characterizations of the uniform rule for division problems with single-peaked preferences. Econ Theory 7:291–306

Pápai S (2000) Strategyproof assignment by hierarchical exchange. Econometrica 68:1403–1433

Sasaki H (2003) Limitation of efficiency: strategy-proofness and single-peaked preferences with many commodities. Working paper, Rice University

Satterthwaite MA, Sonnenschein H (1981) Strategy-proof allocation mechanisms at differentiable points. Rev of Econ Stud 48:587–597

Schummer J (2000) Eliciting preferences to assign positions and compensation. Games Econ Behav 30:293–318

Serizawa S (1996) Strategy-proof and individually rational social choice functions for public good economies. Econ Theory 7:501–512

Serizawa S (2002) Inefficiency of strategy-proof rules for pure exchange economies. J Econ Theory 106:219–241

Serizawa S (2006) Pairwise strategy-proofness and self-enforcing manipulation. Soc Choice Welf 26:305–331

Sprumont Y (1991) The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica 59:509–519

Svensson L-G, Larsson B (2002) Strategy-proof and nonbossy allocation of indivisible goods and money. Econ Theory 20:483–502

Thomson W (1990) The consistency principle. In: Ichiishi T, Neyman A, Tauman Y (eds) Game theory and applications, proceedings of the 1987 international conference, Ohio State University, Columbus, Ohio. Academic Press, pp 187–215

Thomson W (1994a) Consistent solutions to the problem of fair division when preferences are single-peaked. J Econ Theory 63:219–245

Thomson W (1994b) Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Soc Choice Welf 11:205–223

Thomson W (1995) Population-monotonic solutions to the problem of fair division when preferences are single-peaked. Econ Theory 5:229–246

Thomson W (1997) The replacement principle in economies with single-peaked preferences. J Econ Theory 76:145–168

Thomson W (2004) Consistency and its converse: an elementary introduction. University of Rochester, Mimeo

Thomson W (2010) Implementation of solutions to the problem of fair division when preferences are single-peaked. Rev Econ Des 14:1–15