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Quasi-transitive and Suzumura consistent relations

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Abstract We examine properties of binary relations that complement quasi-transitivity and Suzumura consistency in the sense that they, in conjunction with the original axiom(s), are equivalent to transitivity. In general, the conjunction of quasi-transitivity and Suzumura consistency is weaker than transitivity but in the case of collective choice rules that satisfy further properties, this conjunction implies transitivity of the social relation. We prove this observation by characterizing the Pareto rule as the only collective choice rule such that collective preference relations are quasi-transitive and Suzumura consistent, and standard social choice axioms are satisfied.

1 Introduction

Arrow's (1951/1963) celebrated general possibility theorem depends crucially on three classes of assumptions. The first class is on the coherence postulate to be satisfied by social preference relations, which are to be constructed on the basis of individual preference orderings. The second class is on the ethical nature of a process or rule to be

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used for the purpose of constructing social preference relations. The third class is on the informational efficiency of a process or rule for the construction of social preferences. Arrow's own assumption belonging to the first class is that social preference relations should be reflexive, complete and transitive. His assumptions belonging to the second category are the weak Pareto principle, that is to say, the respect for unanimous individual strict preference, and the exclusion of dictatorial decision-making power in social choice. Finally, Arrow's assumption belonging to the third class is the axiom of independence of irrelevant alternatives.

In trying to find an escape route from Arrow's impasse, all three classes of Arrow's assumptions have been subjected to critical scrutiny. Within the first class of assumptions, it was Sen (1969) and Plott (1973) who initiated efforts to weaken Arrow's full transitivity assumption to quasi-transitivity or acyclicity in the context of social welfare functions, or to use path independence when considering social choice functions. Quasi-transitivity, which discards all other components of transitivity and retains only transitivity of strict preference, lies in between transitivity and acyclicity in logical strength. Unfortunately, just to replace transitivity with quasi-transitivity does not help us much in resolving Arrow's impossibility as long as we retain his other assumptions with slight strengthenings, as Mas-Colell and Sonnenschein (1972) and others have demonstrated.

If social preferences are assumed to be reflexive, complete and quasi-transitive, non-dictatorship is strengthened to anonymity and weak Pareto is strengthened to strong Pareto, the Pareto extension rule results; see Sen (1969, 1970, Theorem 5*3). The Pareto extension rule declares a strict social preference for an alternative x over an alternative y whenever everyone considers x at least as good as y and, for at least one individual, this preference is strict; in all other cases, social indifference results.

The Pareto rule is obtained if transitivity is used instead of quasi-transitivity and completeness is not imposed in the above-described result; see Weymark (1984). Unlike the Pareto extension rule, the Pareto rule declares two alternatives non-comparable (and not indifferent) in the absence of unanimity. Thus, both of these rules have the disadvantage that unanimity is required to establish a strict social preference between any two alternatives. This means that these two collective choice rules do not provide any guidance whenever there is a disagreement among the members of society regarding the ranking of two alternatives.

Another coherence postulate that also lies in between transitivity and acyclicity in logical strength was introduced by Suzumura (1976) under the name of consistency. To distinguish it from other properties carrying that label, we refer to it as Suzumura consistency. This property turns out to be more productive in providing possibilities for escaping from Arrow's impossibility, as Bossert and Suzumura (2008) have recently shown.

In this article, we examine the logical implications of assuming both quasi-transitivity and Suzumura consistency in the presence of the other Arrow axioms with slight strengthenings. What emerges is a full characterization of the Pareto rule by means of the combination of quasi-transitivity, Suzumura consistency, unrestricted domain, strong Pareto, anonymity and neutrality.

2 Relations and coherence properties

Suppose there is a set of alternatives X containing at least three elements, that is, $|X| \geq 3$ where $|X|$ denotes the cardinality of X. Let $R \subseteq X \times X$ be a (binary) relation. For simplicity, we write xRy instead of $(x, y) \in R$ and $\neg xRy$ instead of $(x, y) \notin R$. The *asymmetric factor* P of R is defined by

$$xPy \Leftrightarrow [xRy \text{ and } \neg yRx]$$

for all $x, y \in X$. The *symmetric factor* I of R is defined by

$$xIy \Leftrightarrow [xRy \text{ and } yRx]$$

for all $x, y \in X$. The *non-comparable factor* N of R is defined by

$$xNy \Leftrightarrow [\neg xRy \text{ and } \neg yRx]$$

for all $x, y \in X$.

If R is interpreted as a *weak preference relation,* that is, xRy means that x is considered at least as good as y, then P and I are the *strict preference relation* and the *indifference relation* corresponding to R.

The following two properties are what we refer to as *richness* properties because they require certain pairs of alternatives to be in a relation.

Reflexivity: For all $x \in X$,

$$xRx.$$

Completeness: For all $x, y \in X$ such that $x \neq y$,

$$xRy \text{ or } yRx.$$

The following three properties of *transitivity, quasi-transitivity* and *Suzumura consistency* are *coherence* properties because they demand that, if certain pairs are in R, then other pairs must be in R as well (as is the case for transitivity and quasi-transitivity) or other pairs cannot be in R (as is the case for Suzumura consistency).

Transitivity: For all $x, y, z \in X$,

$$[xRy \text{ and } yRz] \Rightarrow xRz.$$

The next coherence property requires that the asymmetric factor P of R be transitive.

Quasi-transitivity: For all $x, y, z \in X$,

$$[xPy \text{ and } yPz] \Rightarrow xPz.$$

The *transitive closure* $tc(R)$ of a relation R is the smallest transitive relation containing R. That is, for all $x, y \in X$,

$$x\, tc(R)\, y \Leftrightarrow [\exists K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that } x = x^0 \text{ and } x^{k-1} R x^k\ \forall k \in \{1, \ldots, K\} \text{ and } x^K = y].$$

Clearly, xRy implies $x\, tc(R)\, y$ for all $x, y \in X$ because the case $K = 1$ is included in the definition of the transitive closure.

Suzumura consistency rules out the existence of preference cycles with at least one strict preference.

Suzumura consistency: For all $x, y \in X$,

$$x\, tc(R)\, y \;\Rightarrow\; \neg\, yPx.$$

Transitivity implies quasi-transitivity and Suzumura consistency. If R is reflexive and complete, transitivity and Suzumura consistency are equivalent, whereas transitivity remains stronger than quasi-transitivity. Without further properties, quasi-transitivity and Suzumura consistency are independent and their conjunction does not imply transitivity. To see that this is the case, consider the following examples. For each of them, we consider a three-element set of alternatives $X = \{x, y, z\}$.

Example 1 Let xIy, yIz and zPx. This relation is quasi-transitive and not Suzumura consistent.

Example 2 Let xPy and yPz. This relation is Suzumura consistent and not quasi-transitive.

Example 3 Let xIy and yIz. This relation is quasi-transitive and Suzumura consistent and not transitive.

An *ordering* is a reflexive, complete and transitive relation. If R is an ordering, there is no ambiguity in using chains of individual preferences involving more than two alternatives; for instance, $xPyPz$ means that x is better than y which, in turn, is better than z and, by the transitivity of R, x is better than z. The set of all orderings on X is denoted by $\mathcal{R}$.

As mentioned earlier, quasi-transitivity and Suzumura consistency are independent properties that are implied by transitivity. For the sake of providing a comprehensive treatment, we now identify the precise conditions that need to be added to one or both of the weaker properties in order to arrive at a conjunction that is equivalent to transitivity.

First, consider quasi-transitivity. The following condition *QT-complementarity* is what is needed to arrive at a conjunction that is equivalent to transitivity.

QT-complementarity: For all $x, y, z \in X$,

$$[xRy \text{ and } yRz] \;\Rightarrow\; [xRz \;\text{ or }\; (\neg\, yRx \text{ and } \neg\, zRy)].$$

Next, we provide a complementary condition to Suzumura consistency. Recall that we do not impose the richness properties of reflexivity and completeness in this section in order to identify minimally necessary complements with respect to transitivity.

SC-complementarity: For all $x, y, z \in X$,

$$[xRy \text{ and } yRz] \Rightarrow [xRz \quad \text{or} \quad (z\, tc(R)\, x \text{ and } \neg\, yRx) \\ \text{or} \quad (z\, tc(R)\, x \text{ and } \neg\, zRy)].$$

Finally, consider the case where R satisfies both quasi-transitivity and Suzumura consistency. The required complementary axiom is defined as follows.

QT–SC-complementarity: For all $x, y, z \in X$,

$$[xRy \text{ and } yRz] \Rightarrow [xRz \quad \text{or} \quad (\neg\, yRx \text{ and } \neg\, zRy) \\ \text{or} \quad (z\, tc(R)\, x \text{ and } \neg\, yRx) \\ \text{or} \quad (z\, tc(R)\, x \text{ and } \neg\, zRy)].$$

We obtain the following result.

Theorem 1 *Let R be a relation on X. The following properties are equivalent.*

(1) Transitivity;
(2) Quasi-transitivity and QT-complementarity;
(3) Suzumura consistency and SC-complementarity;
(4) Quasi-transitivity and Suzumura consistency and QT–SC-complementarity.

Proof That (1) implies (2), (3) and (4) is immediate.

To establish that (2) implies (1), suppose that R satisfies quasi-transitivity and QT-complementarity, and let $x, y \in X$ be such that xRy and yRz. By QT-complementarity, it follows that xRz or $[\neg yRx$ and $\neg zRy]$.

If xRz, we are done.
If $\neg yRx$ and $\neg zRy$, we have xPy and yPz, and quasi-transitivity implies xPz and thus xRz.

Now we prove that (3) implies (1). Suppose that R satisfies Suzumura consistency and SC-complementarity, and let $x, y \in X$ be such that xRy and yRz. By SC-complementarity, it follows that xRz or $[z\, tc(R)\, x$ and $\neg yRx]$ or $[z\, tc(R)\, x$ and $\neg zRy]$.

If xRz, we are done.
If $z\, tc(R)\, x$ and $\neg yRx$, it follows that $y\, tc(R)\, x$ and xPy, contradicting Suzumura consistency. Thus, this case cannot occur.
If $z\, tc(R)\, x$ and $\neg zRy$, it follows that $z\, tc(R)\, y$ and yPz, contradicting Suzumura consistency. Thus, this case cannot occur either.

Finally, we show that (4) implies (1). Suppose that R satisfies quasi-transitivity, Suzumura consistency and QT–SC-complementarity, and let $x, y \in X$ be such that xRy and yRz. By QT–SC-complementarity, it follows that xRz or $[\neg yRx$ and $\neg zRy]$ or $[z\, tc(R)\, x$ and $\neg yRx]$ or $[z\, tc(R)\, x$ and $\neg zRy]$.

If $x R z$, we are done.
If $\neg y R x$ and $\neg z R y$, we have $x P y$ and $y P z$, and quasi-transitivity implies $x P z$ and thus $x R z$.
If $z\ tc(R)\ x$ and $\neg y R x$, it follows that $y\ tc(R)\ x$ and $x P y$, contradicting Suzumura consistency. Thus, this case cannot occur.
If $z\ tc(R)\ x$ and $\neg z R y$, it follows that $z\ tc(R)\ y$ and $y P z$, contradicting Suzumura consistency. Thus, this case cannot occur either and the proof is complete. □

Clearly, transitivity implies QT-complementarity and SC-complementarity, each of which, in turn, implies QT–SC-complementarity. None of the axioms QT-complementarity, SC-complementarity and QT–SC-complementarity by itself implies transitivity. This can be demonstrated by means of examples analogous to Examples 1, 2 and 3.

An interesting weakening of Suzumura consistency is obtained if the scope of the axiom is restricted to triples.

Triple Suzumura consistency: For all $x, y, z \in X$,

$$[x R y \text{ and } y R z] \Rightarrow \neg z P x.$$

A related but weaker condition was previously introduced by Blair et al. (1976), namely, the property of *triple acyclicity* which can be obtained from triple Suzumura consistency by replacing the two instances of weak preference R in the above implication with strict preferences P.

The complementary property of triple Suzumura consistency with respect to transitivity is given by the following simplified version of SC-complementarity.

TSC-complementarity: For all $x, y, z \in X$,

$$[x R y \text{ and } y R z] \Rightarrow \neg x N z.$$

We can also identify a minimal property that needs to be added to the conjunction of triple Suzumura consistency and quasi-transitivity in order to arrive at a property that is equivalent to transitivity.

QT–TSC-complementarity: For all $x, y, z \in X$,

$$[x R y \text{ and } y R z] \Rightarrow [\neg x N z \text{ or } (\neg y R x \text{ and } \neg z R y)].$$

The following theorem results immediately from the definitions of the requisite properties. Thus, we do not provide a formal proof.

Theorem 2 *Let R be a relation on X. The following properties are equivalent.*

(1) Transitivity;
(2) Triple Suzumura consistency and TSC-complementarity;
(3) Quasi-transitivity and triple Suzumura consistency and QT–TSC-complementarity.

Sen (1969, 1970, Chapter 1*) decomposed transitivity into the combination of quasi-transitivity and PI-transitivity, where a relation R is PI-transitive if and only if, for all $x, y, z \in X$, $[xPy$ and $yIz]$ implies $\neg zPx$. This decomposition only applies *if the relation R is complete*. It deserves emphasis that our decompositions in Theorems 1 and 2 do not presuppose completeness.

3 Collective choice rules

Now we consider coherence properties in collective choice problems. The (finite) population is $\{1, \ldots, n\}$ with $n \in \mathbb{N} \setminus \{1\}$. The set of all orderings on X is $\mathcal{R}$ and the n-fold Cartesian product of $\mathcal{R}$ is $\mathcal{R}^n$. The set of all binary relations on X is denoted by $\mathcal{B}$ and $\mathcal{T}$ is the set of all reflexive and transitive relations on X. Analogously, the set of all reflexive and Suzumura consistent relations on X is denoted by $\mathcal{C}$ and the set of all reflexive and quasi-transitive relations on X is $\mathcal{Q}$. A (preference) profile is an n-tuple $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{R}^n$. When considering two profiles such as $\mathbf{R}$ and $\mathbf{R}'$, it is understood that $\mathbf{R}$ is given by the n-tuple $(R_1, \ldots, R_n)$ and $\mathbf{R}'$ is given by $(R'_1, \ldots, R'_n)$.

A *collective choice rule* is a mapping $f: \mathcal{D} \to \mathcal{B}$ where $\mathcal{D} \subseteq \mathcal{R}^n$ is the *domain* of this function, assumed to be non-empty. A *transitive collective choice rule* is a collective choice rule f such that $f(\mathbf{R}) \in \mathcal{T}$ for all $\mathbf{R} \in \mathcal{D}$. Analogously, a *quasi-transitive collective choice rule* is a collective choice rule f such that $f(\mathbf{R}) \in \mathcal{Q}$ for all $\mathbf{R} \in \mathcal{D}$, a *Suzumura consistent collective choice rule* is a collective choice rule f such that $f(\mathbf{R}) \in \mathcal{C}$ for all $\mathbf{R} \in \mathcal{D}$, and a *quasi-transitive and Suzumura consistent collective choice rule* is a collective choice rule f such that $f(\mathbf{R}) \in \mathcal{Q} \cap \mathcal{C}$ for all $\mathbf{R} \in \mathcal{D}$. For each profile $\mathbf{R} \in \mathcal{D}$, $R = f(\mathbf{R})$ is the social preference corresponding to $\mathbf{R}$, and P and I are the strict preference relation and the indifference relation corresponding to R. In analogy with individual preference relations, when considering two profiles $\mathbf{R}$ and $\mathbf{R}'$, we write $R = f(\mathbf{R})$ and $R' = f(\mathbf{R}')$ for the resulting social relations.

An example of a transitive (and, thus, quasi-transitive and Suzumura consistent) collective choice rule is the *Pareto rule* $f^p: \mathcal{R}^n \to \mathcal{B}$ defined by $R^p = f^p(\mathbf{R})$, where

$$xR^p y \;\Leftrightarrow\; [xR_i y \;\forall i \in \{1, \ldots, n\}]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^n$.

We use $B(x, y; \mathbf{R})$ to denote the set of individuals such that $x \in X$ is better than $y \in X$ in the profile $\mathbf{R} \in \mathcal{R}^n$, that is, for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^n$, $B(x, y; \mathbf{R}) = \{i \in \{1, \ldots, n\} \mid xP_i y\}$.

The following axioms are standard in the literature on Arrovian social choice theory.

Unrestricted domain: $\mathcal{D} = \mathcal{R}^n$.

Strong Pareto: For all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{D}$,

(i) $xR_i y \;\forall i \in \{1, \ldots, n\} \;\Rightarrow\; xRy$;
(ii) $[xR_i y \;\forall i \in \{1, \ldots, n\}$ and $\exists j \in \{1, \ldots, n\}$ such that $xP_j y] \;\Rightarrow\; xPy$.

Anonymity: For all bijections $\rho\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$R_i = R'_{\rho(i)} \quad \forall i \in \{1,\ldots,n\} \;\Rightarrow\; R = R'$$

where $R = f(\mathbf{R})$ and $R' = f(\mathbf{R}')$.

Neutrality: For all $x, y, x', y' \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$\begin{aligned}&[xR_i y \Leftrightarrow x'R'_i y' \text{ and } yR_i x \Leftrightarrow y'R'_i x'] \;\; \forall i \in \{1,\ldots,n\}\\ \Rightarrow\; &[xRy \Leftrightarrow x'R'y' \text{ and } yRx \Leftrightarrow y'R'x']\end{aligned}$$

where $R = f(\mathbf{R})$ and $R' = f(\mathbf{R}')$.

In Bossert and Suzumura (2008), we identified all Suzumura consistent collective choice rules satisfying the above axioms. We state this result as a first step toward the characterization theorem of this article. To define the corresponding rules, let

$$\mathcal{S} = \left\{(w,\ell) \in \{0,\ldots,n\}^2 \mid |X|\ell < w + \ell \leq n\right\} \cup \left\{(0,0)\right\}$$

and, furthermore, define

$$\Sigma = \left\{S \subseteq \mathcal{S} \mid (w,0) \in S \;\forall w \in \{0,\ldots,n\}\right\}.$$

For $S \in \Sigma$, define the *S-rule* $f^S\colon \mathcal{R}^n \to \mathcal{B}$ by $R^S = f^S(\mathbf{R})$, where

$$xR^S y \;\Leftrightarrow\; [\exists (w,\ell) \in S \text{ such that } |B(x,y;\mathbf{R})| = w \text{ and } |B(y,x;\mathbf{R})| = \ell]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^n$. The set S specifies the pairs of numbers of agents who have to consider an alternative x better (respectively worse) than an alternative y in order to obtain a weak preference of x over y according to the profile under consideration. Clearly, because only the number of individuals matters and not their identities, the resulting rule is anonymous. Analogously, neutrality is satisfied because these numbers do not depend on the alternatives to be ranked. Strong Pareto follows from the requirement that the pairs $(w,0)$ be in S in the definition of Σ. Reflexivity of the social relation follows from the reflexivity of the individual preferences and the observation that $(0,0) \in S$ for all $S \in \Sigma$. As shown in Bossert and Suzumura (2008, Theorem 1), the social relation R^S is Suzumura consistent due to the restrictions imposed on the pairs (w,ℓ) in the definition of $\mathcal{S}$. Conversely, the S-rules are the only Suzumura consistent collective choice rules satisfying our four axioms. Thus, we obtain the following result.

Theorem 3 (Bossert and Suzumura 2008) *A Suzumura consistent collective choice rule f satisfies unrestricted domain, strong Pareto, anonymity and neutrality if and only if there exists $S \in \Sigma$ such that $f = f^S$.*

Clearly, the Pareto rule is the special case of an S-rule that is obtained for

$$S = \{(w,0) \mid w \in \{0,\ldots,n\}\}.$$

If $|X| \geq n$, this is the only S-rule. This is the case because only pairs (w, ℓ) where $\ell = 0$ are in S in the presence of this inequality. To see this, suppose, to the contrary, that there exists $(w, \ell) \in S$ such that $\ell > 0$. Because $(w, \ell) \in S$, it follows that $n \geq w + \ell > |X|\ell > 0$. Combined with $|X| \geq n$, this implies $n > n\ell$ which is impossible if $\ell > 0$. Thus, if $|X| \geq n$, transitivity is implied by the conjunction of Suzumura consistency and the axioms employed in Theorem 3. However, if $|X| < n$, the Pareto rule is not the only S-rule. For example, consider the collective choice rule f^S corresponding to the set $S = \{(w, 0) \mid w \in \{0, \ldots, n\}\} \cup \{(n-1, 1)\}$. For $(w, \ell) = (n-1, 1)$, we have $n = n - 1 + 1 = w + \ell = n \cdot 1 > |X|\ell$ and, thus, the relevant inequalities are satisfied.

Once rules other than the Pareto rule are available, transitivity is no longer guaranteed (but, of course, all S-rules are Suzumura consistent as established in Bossert and Suzumura (2008, Theorem 1)). For example, suppose $X = \{x, y, z\}$, $n = 4$, $S = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (3, 1)\}$ and consider the profile $\mathbf{R}$ defined by

$$\begin{aligned} &x P_1 y P_1 z, \\ &x P_2 y P_2 z, \\ &z P_3 x P_3 y, \\ &y P_4 z P_4 x. \end{aligned}$$

According to $R^S = f^S(\mathbf{R})$, we have $x P^S y$ and $y P^S z$ because $|B(x, y; \mathbf{R})| = |B(y, z; \mathbf{R})| = 3$ and $|B(y, x; \mathbf{R})| = |B(z, y; \mathbf{R})| = 1$. But $|B(x, z; \mathbf{R})| = |B(z, x; \mathbf{R})| = 2$ and, thus, $\neg x R^S z$ so that R^S is not transitive (not even quasi-transitive). However, f^S satisfies all of our axioms and always generates reflexive and Suzumura consistent social relations.

The *Pareto extension rule* $f^e \colon \mathcal{R}^n \to \mathcal{B}$ is defined by $R^e = f^e(\mathbf{R})$, where

$$x R^e y \;\Leftrightarrow\; \neg\, y P^p x$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^n$. The relation $R^e = f^e(\mathbf{R})$ is quasi-transitive, reflexive and complete for all $\mathbf{R} \in \mathcal{R}^n$. However, R^e is not necessarily Suzumura consistent (and, thus, not necessarily transitive).

The two examples mentioned above establish that Suzumura consistency of a social relation is not implied by its quasi-transitivity and, conversely, Suzumura consistency does not imply quasi-transitivity. Thus, requiring the social relation to possess both of these coherence properties leads to a problem that has not been addressed in the earlier literature.

Sen (1969, 1970, Theorem 5*3) characterized the Pareto extension rule by weakening the transitivity of the social ranking to quasi-transitivity while retaining the completeness assumption. Weymark (1984, Theorem 3) characterized the Pareto rule by imposing transitivity but not completeness on the social relation. Our new result stated below establishes that the conjunction of quasi-transitivity and Suzumura consistency implies transitivity in the presence of our axioms even if the social relation is not complete and, thus, we obtain an alternative characterization of the Pareto rule.

Sen's (1969, 1970) and Weymark's (1984) results are valid even without the full force of neutrality—its well-known weakening *independence of irrelevant alternatives* is sufficient for their theorems. This property is defined as follows.

Independence of irrelevant alternatives: For all $x, y \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$\begin{aligned} & [xR_i y \Leftrightarrow xR'_i y \text{ and } yR_i x \Leftrightarrow yR'_i x] \forall i \in \{1, \ldots, n\} \\ \Rightarrow \; & [xRy \Leftrightarrow xR'y \text{ and } yRx \Leftrightarrow yR'x] \end{aligned}$$

where $R = f(\mathbf{R})$ and $R' = f(\mathbf{R}')$.

If social preferences are assumed to be quasi-transitive and Suzumura consistent and unrestricted domain, strong Pareto and anonymity are satisfied, independence of irrelevant alternatives is not sufficiently strong to characterize the Pareto rule. Suppose $x^0, y^0 \in X$ are two distinct alternatives and define a collective choice rule by letting

$$xRy \;\Leftrightarrow\; [xR^p y \;\text{ or } (\neg\, xR^p y \text{ and } \neg\, yR^p x \text{ and } \{x, y\} = \{x^0, y^0\})]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^n$. This is a quasi-transitive and Suzumura consistent (but not transitive) collective choice rule satisfying unrestricted domain, strong Pareto, anonymity and independence of irrelevant alternatives that differs from the Pareto rule. Neutrality is not satisfied and, thus, independence of irrelevant alternatives is not sufficient to imply neutrality in our setting. Transitivity of the social relation or the combination of quasi-transitivity and completeness, on the other hand, guarantee neutrality if added to the remaining axioms and the independence condition.

The following theorem is the main result of this article.

Theorem 4 *A quasi-transitive and Suzumura consistent collective choice rule f satisfies unrestricted domain, strong Pareto, anonymity and neutrality if and only if $f = f^p$.*

Proof That f^p is a quasi-transitive and Suzumura consistent collective choice rule that satisfies the axioms of the theorem statement is immediate.

Conversely, suppose f is a quasi-transitive and Suzumura consistent collective choice rule that satisfies the axioms of the theorem statement. By Theorem 3, f is an S-rule for some $S \in \Sigma$. It is sufficient to show that $\ell = 0$ for all $(w, \ell) \in S$. By way of contradiction, suppose this is not the case, that is, there exists $(w, \ell) \in S$ such that $\ell > 0$. Define

$$w^* = \min\{w \mid \exists \ell > 0 \text{ such that } (w, \ell) \in S\}.$$

By assumption, w^* is well-defined and by its definition, there exists $\ell^* > 0$ such that $(w^*, \ell^*) \in S$. By definition of Σ, $w^* > 2\ell^* \geq 2$. By unrestricted domain and because X contains at least three elements, we can choose three alternatives $x, y, z \in X$ and a profile $\mathbf{R} \in \mathcal{R}^n$ such that

$$\begin{aligned} & xP_i yP_i z \;\forall i \in \{1, \ldots, w^* - \ell^*\}, \\ & zP_i xP_i y \;\forall i \in \{w^* - \ell^* + 1, \ldots, w^*\}, \\ & yP_i zP_i x \;\forall i \in \{w^* + 1, \ldots, w^* + \ell^*\} \end{aligned}$$

and, if $n > w^* + \ell^*$,

$$xI_iyI_iz\ \forall i \in \{w^* + \ell^* + 1, \dots, n\}.$$

We have $|B(x, y; \mathbf{R})| = w^*$ and $|B(y, x; \mathbf{R})| = \ell^*$ and, furthermore, $|B(y, z; \mathbf{R})| = w^*$ and $|B(z, y; \mathbf{R})| = \ell^*$. Therefore, because $(w^*, \ell^*) \in S$, xPy and yPz. The quasi-transitivity of R implies xPz. Because $|B(x, z; \mathbf{R})| = w^* - \ell^*$ and $|B(z, x; \mathbf{R})| = 2\ell^*$, it follows from the definition of an S-rule that $(w^* - \ell^*, 2\ell^*) \in S$. This contradicts the minimality of w^* because $\ell^* > 0$. □

4 Concluding remarks

Quasi-transitivity and Suzumura consistency are two properties of a not necessarily complete and reflexive relation, which are weaker in logical strength than transitivity and stronger in logical strength than acyclicity. In general, even the conjunction of quasi-transitivity and Suzumura consistency still falls short of implying transitivity. The first part of this article is concerned with the identification of the conditions under which the logical gap between quasi-transitivity, Suzumura consistency, and the conjunction of these properties, on the one hand, and transitivity, on the other hand, can be exactly bridged with no logical gap or redundancy. For this purpose, we identify three concepts of complementarity, namely, QT-complementarity, SC-complementarity, and QT–SC-complementarity, such that each one of the conjunctions [quasi-transitivity and QT-complementarity], [Suzumura consistency and SC-complementarity], and [quasi-transitivity and Suzumura consistency and QT–SC-complementarity] turns out to be logically equivalent to transitivity. The same exercise is conducted for the weaker concept of triple Suzumura consistency.

The second part of this article is concerned with the role of quasi-transitivity and Suzumura consistency in the Arrovian social choice theory. It is well known how quasi-transitivity and Suzumura consistency fare in this context, respectively, in isolation, but their conjunct role has been left unexplored. The main result of this article, namely Theorem 4, shows that a quasi-transitive and Suzumura consistent collective choice rule f satisfies unrestricted domain, strong Pareto, anonymity and neutrality if and only if f is the Pareto rule. This theorem makes use of the anonymity axiom. Clearly, without this property, further collective choice rules become available and their definitions can be formulated in terms of various forms of decisive coalition structures; see, for instance, Kirman and Sondermann (1972); Hansson (1976) and Bossert and Suzumura (2010a, Chapter 10) for results involving transitive or quasi-transitive social preferences. The approach employed in these contributions also allows for infinite populations to be considered. Bossert and Suzumura (2010b) provide some results when social relations are acyclical or Suzumura consistent. It remains to be explored if the setting outlined there can be adapted so as to accommodate the combination of quasi-transitivity and Suzumura consistency.

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