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Choice procedures and power structure in social decisions

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Abstract In contrast to a social choice function, a social choice procedure is proposed which depends both on the way a set of alternatives is broken up into the subsets and the sequence in which each of these subsets is taken up for consideration. This article investigates the Arrow question in this generalized framework.

1 Introduction

The sensitivity of Arrow's well-known impossibility theorem (Arrow 1963) to the requirement of transitivity in social preference relation was first noticed by Sen (1969). The importance of transitive rationalization in economics lies in its role in the integrability controversy in demand theory. While Georgescu-Roegen (1936) established that the basic problem of integrability is the question of transitivity of the preference relation of a consumer, Hicks (1946) argued that the real issue underlying integrability is the question of path independence. This led Arrow to justify the requirement of transitivity in social preference relation to insure the independence of final choice from the way the alternatives are divided over pairs. He argued that it is "an important attribute of a genuinely democratic system capable of full adoption to varying environment" (Arrow 1959, p. 120).

Being curious about Arrow's justification of requiring transitivity in social preference relation, Plott (1973) proposed the framework of a social choice function which maps every pair made of a profile of individual preference orderings and a set of alternatives to a nonempty subset of the set presented for choice. In contrast, the Arrowian social welfare function maps a profile of individual preference orderings over a set of alternatives to a reflexive, complete, and transitive social preference relation. In this

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proposed framework, Plott described a two-stage choice process which is a mechanism of divide and conquer, where a set of alternatives is divided into subsets, a choice is made over each of these smaller sets, and then a final choice is made over the chosen alternatives in the first round. Since the final choice depends on the alternatives from the smaller sets, the final outcome crucially depends on the way alternatives are divided up for consideration. Path independence of this "divide and conquer" rule of making a decision means that the final choice is independent of the way in which the set of alternatives is initially divided up for consideration.[1] A close examination of Sen's well-known possibility theorem reveals that there exists a non-dictatorial social choice function which satisfies the other two Arrow conditions, namely, the weak Pareto principle and the independence of irrelevant of alternatives together with the requirement of path independence of two-stage choice process. However, the underlying power structure turns out to be oligarchic.[2]

An alternative choice process was proposed by Bandyopadhyay (1990a,b) in which choice proceeds in a sequence of a finite set of alternatives by considering the first two elements of the sequence to make a choice, then for every chosen element, compare with the third element of the sequence and take the union. Then for every chosen element of the earlier round, compare with the next element of the sequence and take the union, and so on, until all the alternatives have been considered. It was established that the path independence of such a choice process is equivalent to transitive rationalization.[3] This result in turn resolves Arrow's conjecture about the relation between a weak preference ordering and a path independent choice process.

A close observation of Bandyopadhyay's sequential choice process reveals that it allows some evidently inoptimal alternatives in the earlier round to be carried over in the next round for consideration. This led Bandyopadhyay (1998) to propose a refined sequential choice process in which choice proceeds in a sequence of nonempty subsets of a set of alternatives presented for choice such that the collection or union of all these subsets is the entire set. For a given sequence of subsets, the choice process begins by choosing from the first element of the sequence. Then every alternative chosen in the first round is compared with the alternatives of the next element of the path, and then collect all chosen alternatives. Now exclude all elements of second subset from the collected chosen alternatives that are rejected when the second subset is compared with any chosen element of the first round. The surviving alternatives are considered to be the winner in the second round. In the following stage, each of these winning alternatives of the second round is compared with the alternatives of the next element of the path. The process continues following the path until the last element of the path is considered. Note that the refinement in the proposed sequential choice process comes in its exclusion of inoptimal elements at each stage as choice proceeds along

[1] Note that a two-stage path independent choice process does not guarantee rational choice.

[2] Gibbard (1969) established that weakening transitive rationality to quasi-transitive rationality in social preference relation insures an oligarchic power structure if an aggregation rule which maps a profile of individual preference orderings to a reflexive, complete, and quasi-transitive social preference relation is required to satisfy the Arrowian conditions of weak Pareto principle and independence of irrelevant alternatives. See also Mas-Collel-Sonnenschein (1972).

[3] See Bandyopadhyay (1998).

the path. The path independence of such a refined sequential choice process is equivalent to quasi-transitive rationalization. It is possible for a further refinement of this sequential choice process; the path independence of finer refined choice procedure is equivalent to acyclic rationalization.[4] Thus, in the light of social choice literature, the asymmetric power structure in social decision making depends on the adopted choice process. [5]

Some observations on the present state of the literature are in order. First, the outcome of the requirement of a path independent social choice function crucially depends on the description of a choice process. Second, all the choice processes that are discussed in the literature are based either on the way the entire set is divided into the subsets or on the sequence in which the successive subsets are taken up for consideration. Although we have described few choice processes, there are numerous possible ways one can think of a process to make a choice. So this observation demands a more general or abstract social choice process.

This article introduces a general structure in which primitive is not merely from a set the choice is made, but also the process that one is adopted to make the choice. In contrast to a social choice function, a social choice procedure is proposed which depends both on the way a set of alternatives is broken up into the subsets and the sequence in which each of these subsets is taken up for consideration. Specifically, for any given profile of individual preference orderings and for any given set of alternative social states, say A, a social choice procedure is a rule which, for every sequence of subsets of A that constitute the entire set, specifies a subset of the set A. This article investigates the Arrow question in this generalized framework.

2 Preliminaries

Let L be a society consisting of a finite N number of individuals. Every individual i in the society L has a preference relation which is a weak order on a finite set of alternative social states X. Let $[X]$ be the set of all possible non-empty subsets of X. An element of $[X]$ will be called an issue. We assume $|L| \geq 3$ and $|X| \geq 3$. A profile of individuals' preference orderings, called a situation, is a specification, for example, $s = (R_i)_{i \in L}$, of weak preference orderings R_i on X for each individual $i \in L$. Corresponding to R_i, P_i and I_i are the strict and indifference relations defined in the usual way. Let S be the set of all profiles of individual preference orderings.

Let $L_{xy} = \{i \in L | x P_i y\}$ and $L_{(xy)} = \{i \in L | x R_i y\}$ be the set of individuals in a society for whom the indicated preference holds, given the profile of individual preference orderings $s \in S$. Similarly, for another preference profile, say $s' \in L$, we use L'_{xy} and $L'_{(xy)}$, respectively, for the set for which the indicated preference holds.

A *social choice function* $C(.)$ is a rule which maps every issue $A \in [X]$ and every profile of individual preference orderings $s \in S$ to $[X]$ such that $\emptyset \neq C(s, A) \subseteq A$.

[4] See Bandyopadhyay (1998).

[5] For the relation between various transitive rationalization and power structure in the presence of other Arrow conditions, see Kelly (1978).

When the dependence on the preference orderings of individuals, i.e., a situation $s \in S$, is unambiguous, we will write social choice function as $C(A)$.

For any given situation s, a social choice function $C(.)$ generates a preference relation (called base relation by Georgescu-Roegen (1936)) R_b *on* X *as follows: for all* $x, y \in X$, xR_by *iff* $x \in C(\{x, y\})$. *Then,* xP_by *iff* $(xR_by$ *and* $\neg\, yR_bx)$; xI_by *iff* $(xR_by$ *and* $yR_bx)$.

For all $x_1, x_2, \ldots, x_n \in X$, we say that a binary relation Q over X is (i) *reflexive* if x_1Qx_1; (ii) *connected* if $x_1 \neq x_2 \rightarrow x_1Qx_2$ or x_2Qx_1; (iii) *acyclic* if $[(x_1Q^*x_2 \,\&\, x_2Q^*x_3 \,\&\, \ldots \,\&\, x_{n-1}Q^*x_n) \rightarrow x_1Qx_n]$, where Q^* is a sub-relation of Q such that $[x_1Q^*x_2 : (x_1Qx_2 \,\&\, \neg\, x_2Qx_1)]$; (iv) *quasi-transitive* if $[(x_1Q^*x_2 \,\&\, x_2Q^*x_3) \rightarrow x_1Q^*x_3]$; (v) *transitive* if $[(x_1Qx_2 \,\&\, x_2Qx_3) \rightarrow x_1Qx_3]$; (vi) a *weak order* iff it is reflexive, connected and transitive; (vii) *quasi-transitive order* iff it is reflexive, connected, and quasi-transitive; (viii) *suborder* iff it is reflexive, connected, and acyclic.

An element x in any set A is said to be a *best element* of A with respect to a binary relation Q iff xQy for all $y \in A$. The *set of best elements* in A with respect to Q, $M(A, Q) = \{x \in A | xQy$ for all $y \in A\}$.

A social choice function $C(.)$ is said to be *transitive rational* iff there exists a weak order Q, defined on X, such that $C(s, A) = M(A, Q)$ for all $A \in [X]$. Q is then called the transitive rationalization of $C(.)$.[6] Similarly, a social choice function $C(.)$ is said to be *quasi-transitive rational* iff there exists a reflexive, connected and quasi-transitive relation Q, defined on X, such that $C(s, A) = M(A, Q)$ for all $A \in [X]$.

Now, following Richter (1966), given a choice function $C(.)$, we define the *revealed weak preference relation* V by the condition: xVy if and only if there exists $A \in [X]$ such that $x \in C(A)$ and $y \in A$. Whenever an alternative $x \in A \backslash C(A)$ we say x is *revealed inferior* in A. Clearly, $\neg\, xVy$ means that in the presence of y, x cannot be chosen; i.e., y globally blocks x, and we write yGx.

Corresponding to the above notion of revelation, for all $A \in [X]$, all *distinct* $x, y \in X$, and for $A \neq C(A)$ we define the following axioms.

Weak Axiom of Revealed Preference (WARP): $x \in A \backslash C(A)$ implies yGx for all $y \in C(A)$.[7]

Axiom of Quasi-Transitive Order (AQTO): $x \in A \backslash C(A)$ implies yGx for some $y \in C(A)$.[8]

In words, WARP requires that in the presence of a revealed preferred alternative, a revealed inferior alternative cannot be chosen. In other words, WARP says that every revealed inferior alternative in a set A is dominated or globally blocked by every revealed preferred alternative of A. AQTO requires that every revealed inferior alternative of a set A is globally blocked by some revealed preferred alternative of A.

Theorem A *Let* $C(.)$ *be a social choice function.*

[6] .Originally, Samuelson (1938) studied the rationalization question in the context of a demand function, i.e., a single-valued choice function.

[7] This condition is equivalent to Condition 4 (Arrow 1959) or Sen's (1971) conditions α and β together.

[8] This condition is equivalent to Sen's (1971) α and δ together.

(A.1) $C(.)$ is transitive rational iff it satisfies WARP;
(A.2) $C(.)$ is quasi-transitive rational iff it satisfies AQTO.

The first part of Theorem A is due to Arrow (1963), and the second part is due to Bandyopadhyay and Sengupta (1991).[9]

3 Choice process

3.1 Plott's two-stage process

In the published literature, it is Plott (1973) who described a two-stage choice process in which a set of alternatives is divided into smaller sets, a choice is made over each of these sets, and then a final choice is made from the set of alternatives chosen in the first round.[10] Path independence of this "divide and conquer" rule of making a decision means that the final choice is independent of the way in which the set of alternatives is initially divided up for consideration.

Formal description of choice procedures and path independence requires some additional notation. For $A \in [X]$, let $A_1, A_2, \ldots, A_n$ be a sequence of non-empty subsets of A such that $\cup_{i=1}^{n} A_i = A$.[11] Let $\langle A_1, A_2, \ldots, A_n \rangle$ denote an ordered set of subsets of A, and let $\Omega(A)$ be the set of all such ordered sets of subset of A.

Given a social choice function $C(.)$, the *two-stage process* is defined to be a function h which for every $A \in [X]$ and every $\langle A_1, A_2, \ldots, A_n \rangle \in \Omega(A)$ specifies a subset $h(\langle A_1, A_2, \ldots, A_n \rangle)$ of A such that $h(\langle A_1, A_2, \ldots, A_n \rangle) = C\left(\cup_{i=1}^{n} B_i\right)$ where, for $i \in \{1, 2, \ldots, n\}$, $B_i = C(A_i)$.

A choice function $C(.)$ satisfies *path independence in the two-stage process* (PI) iff, for all $A \in [X]$ and all sequences $\langle A_1, A_2, \ldots, A_n \rangle \in \Omega(A)$, $h(\langle A_1, A_2, \ldots, A_n \rangle) = h(\langle A \rangle)$.[12]

The condition PI says that the final choice would be independent of the way a given set of alternatives is initially divided into subsets for consideration.

3.2 Sertel-Van der Bellen's sequential process

Sertel and Van der Bellen (1980) introduced a choice procedure in which for any given sequence of non-empty subsets, a choice is made from the first element of the path; then the alternatives of the next element are compared with all alternatives chosen in the earlier round; and the process continues until the last element of the path is considered. We generalize this idea as described below.

Given a social choice function $C(.)$, the *Sertel-Van der Bellen (SV) process* is defined to be a function h^a which for every $A \in [X]$ and every $\langle A_1, A_2, \ldots, A_n \rangle \in \Omega(A)$

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9 See also Sen (1969), Schwartz (1976).

10 In an unpublished paper it was Afriat (1967) who first introduced a similar choice process.

11 This is a finite ordered cover of a set of alternatives. It was called *path* by Sertel and Van der Bellen (1980).

12 Note that what is relevant for the final outcome is the way a given set of alternatives is divided up for consideration, not the sequence in which the smaller sets are taken up for consideration.

specifies a subset $h^a(\langle A_1, A_2, \ldots, A_n\rangle)$ of A such that $h^a(\langle A_1, A_2, \ldots, A_n\rangle) = J^n$ where, $J^1 = C(A_1)$ and for every $i \in \{2, 3, \ldots n\}$, $J^i = C\left(J^{i-1} \cup A_i\right)$. Note that $J^n = C(J^n - 1 \cup A_n)$ is the terminal choice.[13]

This is a generalization of the procedure originally introduced by Sertel and Van der Bellen in the sense that they restricted the subsets A_i's to a single element set. Note, the *SV* procedure requires that in each stage all alternatives chosen in the earlier round are compared together with the alternatives of the next element of the path. The process continues following the path until the last element of the path is considered.

A social choice function $C(.)$ satisfies *path independence in SV process* (SVPI) iff, for all $A \in [X]$ and all sequences $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, $h^a(\langle A_1, A_2, \ldots, A_n\rangle) = h^a(\langle A\rangle)$.

It was shown in Bandyopadhyay (1990a,b) that the path independence with respect to the two-stage choice process and path independence with respect to the Sertel–Van der Bellen process are equivalent.

3.3 Sequential choice process

We now introduce an alternative procedure in which choice proceeds in a sequence of non-empty subsets, a choice is made from the first element of the path; then compare every surviving alternative in the earlier round with the next element of the path and collect all the chosen alternatives; and the process continues until the last element of the path is considered. In other words, for a set of alternatives A and for any given sequence, $\langle A_1, A_2, ..., A_n\rangle$ in $\Omega(A)$, a choice is made from the first element of the sequence, A_1. Let the set of chosen elements in the first round, $C(A_1)$, be T^1. Then the next element of the sequence, A_2, is compared with every alternative, a, in the set T^1. We collect all alternatives chosen by comparing each element of T^1 together with A_2. Let the set of chosen alternatives in the second round, $\cup_a \in T^1 C(\{a\} \cup A_2)$ be T^2. The process continues following the sequence until the last element of the sequence is considered. This is a generalization of a choice process originally introduced by Bandyopadhyay (1988). In contrast, the *SV* choice process describes a procedure in which the entire set of previously chosen elements is to be compared together with the alternatives of the next element of the sequence.

For a social choice function $C(.)$, the *sequential choice process* is defined to be a function g which for every $A \in [X]$ and every $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, specifies a subset $g(\langle A_1, A_2, \ldots, A_n\rangle)$ of A such that $g(\langle A_1, A_2, \ldots, A_n\rangle) = T^n$ where, $T^1 = C(A_1)$ and for $i \in \{2, 3, \ldots, n\}$, $T^i = \cup_{a \in T^{i-1}} C(\{a\} \cup A_i)$. Note that $T^n = \cup_{a \in T^{n-1}} C(\{a\} \cup A_n)$ is the terminal choice.[14]

[13] This is a generalization of the usual (Afriat–Plott–Sertel & Van der Bellen) formula-tion of path independence. It is rather similar to a notion called *search symmetry* (Afriat (1967)): one gets the same outcome whichever of certain paths one follows through a sequence of subsets of a set A of alternatives. This is also known as *path invariance* (Sertel and Van der Bellen (1980)). In Sertel & Van der Bellen, a choice set could be empty.

[14] This is a generalization of a process introduced in Bandyopadhyay (1988) where the smaller sets A_i's were restricted to the single element sets.

For a single-valued social choice function, for any sequence $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, both the SV choice process and the sequential choice process will yield identical outcome, i.e., $h^a(\langle A_1, A_2, \ldots, A_n\rangle) = g(\langle A_1, A_2, \ldots, A_n\rangle)$.

A social choice function $C(.)$ satisfies *path independence in the sequential choice process* (PI′) iff, for all $A \in [X]$ and all sequences $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, $g(\langle A_1, A_2, \ldots, A_n\rangle) = g(\langle A\rangle)$.

3.4 Refined sequential choice process

In our description of the sequential choice process we have the following observations. Consider a sequence $\langle A_1, \ldots, A_{i+1}, \ldots, A_n\rangle$. Suppose $C(A_1) = \{x, y\}$. Also suppose $C(\{x\} \cup A_2) = \{x\}$ and $C(\{y\} \cup A_2) = \{z\}$. Clearly, z is rejected in the set $\{x\} \cup A_2$. However, following the sequential choice procedure, once again z is to be compared with A_3. Clearly, the sequential choice procedure allows a defeated alternative to be carried over to the next round for further consideration. Since Chernoff (1954) showed that a rational choice set never contains an element which is rejected in any smaller set comparison, the sequential choice procedure clearly allows some inefficiency in making a decision. We now propose a refinement of our sequential choice procedure in which every chosen alternative of the previous round is compared with the next element of the sequence, and whenever an alternative of the new element of the sequence is defeated by one of the chosen alternatives of the earlier round then that alternative must be excluded.

For a social choice function $C(.)$, *a refined sequential choice process* is defined to be a function g^r which for every $A \in [X]$ and every $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, specifies a subset $g^r(\langle A_1, A_2, \ldots, A_n\rangle)$ of A such that $g^r(\langle A_1, A_2, \ldots, A_n\rangle) = T_r^n$ where, $T_r^1 = C(A_1)$ and for $i \in \{2, 3, \ldots, n\}$, $T_r^i = \cup_{a \in T_r^{i-1}} C(\{a\} \cup A_i) \setminus R^i$ where, $R^i = \{x \in A_i | x \notin C(\{y\} \cup A_i)\}$ for some $y \in T_r^{i-1}$. Note that $T_r^n = \cup_{a \in T^{n-1}r} C(\{a\} \cup A_n) \setminus R^n$ is the terminal choice.

The elements in the set R^i are inoptimal elements of A_i when compared with the chosen elements of the previous, $(i-1)$th, round. The refined sequential choice process differs from the earlier sequential choice process in the sense that it excludes the inoptimal elements R^i at each stage of the process.

A social choice function $C(.)$ satisfies *path independence in the refined sequential choice process* (PI″) iff, for all $A \in [X]$ and all sequences $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, $g^r(\langle A_1, A_2, \ldots, A_n\rangle) = g^r(\langle A\rangle)$.

Theorem B *(Bandyopadhyay and Sengupta (1991); Bandyopadhyay (1998)). (1) WARP ↔ PI′; (2) AQTO ↔ PI″.*

Clearly, Theorems A and B together implies that a social choice function $C(.)$ is transitive (respectively, quasi-transitive) rational if and only if it satisfies PI′ (respectively, PI″). As a consequence, a social choice function satisfying the weak Pareto principle and the condition of independence of irrelevant alternatives is dictatorial (respectively, oligarchical) if it is required to satisfy the PI′ (respectively, PI″). Thus, the asymmetric power structure in a social choice crucially depends on the choice process that is being adopted to make a decision. In real life there are numerous ways one can describe a choice process.

4 Choice procedures

Given an ordered subsets of $A \in [X]$, $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, it is obvious that if a chooser has to choose according to the order then she will possibly choose first from A_1 (i.e., $C(A_1)$), however, the final outcome depends on the adopted process. For example, one can compare $C(A_1)$ with A_2, or for each element of $C(A_1)$ with A_2, or for each element of $C(A_1)$ with each element of A_2, or as it is suggested by Plott. Furthermore, after choosing at the second round what a chooser would do with the rejected element in the earlier round when it is confronting A_3 once again depends the description of the adopted process. So it is of interest to investigate the class of choice processes that are associated with a particular power structure. So we introduce below a notion of an abstract social choice process.

A *social choice procedure*, $C^{\mathrm{P}}(.)$, is a rule which for every profile of individual preference orderings $s \in S$ and for every issue $A \in [X]$, every integer $n \in \{1, 2, \ldots |A|\}$ and for every n-element of sequence of subsets of A, $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$ specifies a non-empty subset of A,(i.e., $\emptyset \neq C^{\mathrm{P}}(s, \langle A_1, A_2, \ldots, A_n\rangle) \subseteq A$ such that $C^{\mathrm{P}}(s, \langle A_1\rangle) = H$. Whenever a situation s $\in S$ is unambiguous, we will write social choice procedure as $C^{\mathrm{P}}(\langle A_1, A_2, \ldots, A_n\rangle)$. Clearly, for $n = 1$, a social choice procedure is reduced a social choice function.

For any $A \in [X]$, a social choice procedure, $C^{\mathrm{P}}(.)$, is said to be *path independent* iff for all $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, $C^{\mathrm{P}}(\langle A_1, A_2, \ldots, A_n\rangle) = C^{\mathrm{P}}(\langle A\rangle)$.

For any $A \in [X]$, a social choice procedure, $C^{\mathrm{P}}(.)$, is said to be *upper path independent* iff for all $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, $C^{\mathrm{P}}(\langle A\rangle) \subseteq C^{P}(\langle A_1, A_2, \ldots, A_n\rangle)$.

For any $A \in [X]$, a social choice procedure, $C^{\mathrm{P}}(.)$, is said to be *lower path independent* iff for all $\langle A_1, A_2, \ldots, A_n\rangle \in \Omega(A)$, $C^{\mathrm{P}}(\langle A_1, A_2, \ldots, A_n\rangle) \subseteq C^{\mathrm{P}}(\langle A\rangle)$.[15]

The conditions which the social choice procedure may be required to satisfy for all pairs of alternative social states, $x, y \in A \subseteq X$ and for all situations $s, s' \in S$ are:

Independence (IN): $[L_{(xy)} = L'_{(xy)}$ and $L_{(yx)} = L'_{(yx)}] \rightarrow C(s, \langle\{x, y\}\rangle) = C\left(s', \langle\{x, y\}\rangle\right)$.

Weak Pareto optimality (PO) $|L_{xy}| = N \rightarrow y \notin C(\langle A\rangle)$.

Absence of veto (AV) $|L_{xy}| = N - 1 \rightarrow y \notin C(\langle A\rangle)$.

Strict monotonicity (SM) $[L_{xy} \subseteq L'_{yx} \& L'_{yx} \subseteq L_{yx}$ and if at least one of these is a proper subset$] \rightarrow [x \in C(s, \langle\{x, y\}\rangle) \rightarrow \{x\} = C\left(s', \langle\{x, y\}\rangle\right)]$.

These properties are familiar in the literature and do not need any discussion. Under the restriction that the issue A contains exactly two elements, our Pareto optimality condition is known in the literature as the weak Pareto principle (WP).

We now introduce certain conditions which are concerned with the distribution of "power" among individuals in a society. For a distinct pair of alternatives $x, y \in X$, and for all $s \in S$, a set of individuals $\not{L} \subseteq L$ is

[15] Factorizing the path independence of two stage choice procedure, one can obtain $h(\langle A_1, A_2, \ldots, A_n\rangle) \subseteq h(\langle A\rangle)$ what Ferejohn and Grether (1977) called weak path independence.

Decisive for x against y $[x P_i y$ forall $i \in \text{Ł}] \rightarrow \{x\} = C(\langle\{x, y\}\rangle)$.
Quasi-decisive for x against y $[x P_i y$ forall $i \in \text{Ł}] \rightarrow x \in C(\langle\{x, y\}\rangle)$.

We utilize the notion of decisiveness and quasi-decisiveness to introduce the possible power structure of a unique group of individuals in making a social decision.

Dictatorship There exists an individual $i \in L$ who is decisive for every pair of alternatives.
Oligarchic There exists a unique coalition of individuals Ł, $\text{Ł} \subseteq L$, who is decisive for every pair of alternatives and every individual $i \in \text{Ł}$ is quasi-decisive for every pair of alternatives.
Weak-dictatorship There exists an individual $i \in L$ who is quasi-decisive for every pair of alternatives.
Almost-dictatorship A weak-dictator $i \in L$ is said to be almost-dictator iff for all $x, y \in X$ and all $s \in S$, $[x P_i y \;\&\; x R_j y$ forsome $j \neq i \in L] \rightarrow \{x\} = C(\langle\{x, y\}\rangle)$.

Throughout this article, we assume that the social choice procedure satisfies the independence condition.

Theorem 1 *Let $C^{\mathrm{P}}(.)$ be a path independent choice procedure.*

(1.1) If $C^{\mathrm{P}}(.)$ satisfies PO then there exists a unique oligarchy.
(2.2) If single-valued $C^{\mathrm{P}}(.)$ satisfies PO then there exists a dictator.
(3.3) If $C^{\mathrm{P}}(.)$ satisfies PO and SM then there exists an almost dictator.

Chernoff (1954) showed that a chosen alternative from a set A cannot be rational if that element is rejected in any subset of A. He introduced the following property for rational choice.

A social choice procedure is said to satisfy the Chernoff condition (CC) iff for all $A_1, A_2 \in [A_1 \cup A_2]$ and all $x \in A_1$, if $x \notin C^{\mathrm{P}}(\langle A_1\rangle)$, then $x \notin C^{\mathrm{P}}(\langle A_1 \cup A_2\rangle)$.

Theorem 2 *Let $C^{\mathrm{P}}(.)$ be a choice procedure. Then the upper path independence condition is equivalent to CC.*

Sen (1977) identified the requirement of Chernoff condition is the reason behind the Arrow-Gibbard impossibility theorems.[16] *However, the consequence of retaining the lower path independence condition is stated below.*

Theorem 3 *There exists a non-oligarchic social choice procedure that satisfies WP and lower path independence.*

Theorem 4 *Let $C^{\mathrm{P}}(.)$ be a lower path independent choice procedure.*

(4.1) *There is no $C^{\mathrm{P}}(.)$ which satisfies AV.*
(4.2) *For $|X|\rangle N$, if $C^{\mathrm{P}}(.)$ satisfies PO then there exists an individual who is quasi-decisive over at least* $(|X| - N + 1)\,(|X| - 1)$ *pairs of alternatives.*
(4.3) *If $C^{P}(.)$ satisfies PO and SM then there exists an almost dictator.*

Our results essentially show that the asymmetric power structure remains intact when one invokes path independence or lower path independence of an abstract social choice procedure. This suggests that the connection between the requirement of path independence and asymmetric power structure is more robust than a mere adoption of a particular choice process.

16 See Bandyopadhyay (1984, 1985, 1986).

5 Proofs

To prove Theorem 1 we will use the following lemma.

Lemma 1 *Let $C^{\mathrm{P}}(\langle.\rangle)$ be a path independent social choice procedure. Then the base relation R_b is quasi-transitive.*

Proof Let $A = \{x, y, z\}$. Suppose xP_by and yP_bz. If zR_bx, then for $\langle\{x, y\}, \{z\}\rangle \in \Omega(A)$, $z \in C^{\mathrm{P}}(\langle\{x, y\}, \{z\}\rangle)$, whereas for $\langle\{y, z\}, \{x\}\rangle \in \Omega(A)$, $\{x\} = C^{\mathrm{P}}(\langle\{y, z\}, \{x\}\rangle)$, a contradiction that $C^{\mathrm{P}}(\langle.\rangle)$ is a path independent choice procedure.

The rest of the proof of (1.1) follows from Gibbard (1969). For a single valued social choice procedure, it is immediate that the base relation R_b is transitive and thus the rest of the proof of (1.2) follows from Arrow (1963). Following Bordes and Salles (1978), the lemma below completes the proof of (1.3).

Lemma 2 *If a social choice procedure $C^{\mathrm{P}}(\langle.\rangle)$ satisfies SM and if there is a weak dictator then he is a unique almost dictator.*

Proof of Theorem 2 We first show that CC implies upper path independence. Suppose not. Then for some sequence $\langle A_1, A_2\rangle \in \Omega(A)$, $x \in C^{\mathrm{P}}(\langle A\rangle)$ and $x \notin C^{\mathrm{P}}(\langle A_1, A_2\rangle)$. For $x \in A_1$, there are two possibilities: (i) $x \notin C^{\mathrm{P}}(\langle A_1, \rangle)$ and (ii) $x \in C^{\mathrm{P}}(\langle A_1, \rangle)$. If (i) holds, then the contradiction is immediate. If (ii) holds, then, given $x \in C^{\mathrm{P}}(\langle A\rangle)$ and $x \notin C^{\mathrm{P}}(\langle A_1, A_2\rangle)$, $x \notin C^{\mathrm{P}}(\langle B\rangle)$ for some $B \subseteq A$. Once again contradicting CC. Following the same argument, the contradiction is immediate for $x \notin A_1$.

Now we show that upper path independence implies CC. Suppose not. Then for $A, B \in [X]$, where B ⊆ A, suppose $x \notin C^{\mathrm{P}}(\langle B\rangle)$, $x \in B$; however, $x \in C^{\mathrm{P}}(\langle A\rangle)$. Since $x \notin C^{\mathrm{P}}(\langle B\rangle)$, then for a path $\langle B, A/B\rangle \in \Omega(A)$, $x \notin C^{\mathrm{P}}(\langle B, A/B\rangle)$. Given $x \in C^{\mathrm{P}}(\langle A\rangle)$, by upper path independence, $x \in C^{\mathrm{P}}(\langle B, A/B\rangle)$, a contradiction.

The proof of Theorem 3 is very similar to Bordes (1976) and therefore is omitted.

Proof of Theorem 4 We first prove (4.1). Given AV, for all distinct $a, b \in X$, there exists at least one coalition which is "almost decisive" for a against b, i.e., a set of individuals L' such that aP_ib for all $i \in L'$ and bP_ja for all $j \in (L - L')$ implies aP_bb. Compare all almost decisive sets, and let L^* be the smallest almost decisive coalition for x against y. It is clear that $|L^*| \geq 2$.

Construct a situation s as follows:

$$\begin{array}{ll} \text{For all } i \in (L^* - \{h\}) & xP_{\mathrm{i}}yP_{\mathrm{i}}z, \\ \text{for all } j \in (L - L^*) & yP_{\mathrm{j}}zP_{\mathrm{j}}x, \\ \text{and} & zP_{\mathrm{h}}xP_{\mathrm{h}}y. \end{array} \tag{1}$$

Since L^* is an almost decisive coalition for x against y, xP_by. By AV, yP_bz. Since L^* is the smallest almost decisive coalition, we must have zP_bx. Now, by lower path independence, for $\langle\{x, y\}, \{z\}\rangle \in \Omega(A)$, $C^P(\langle\{x, y\}, \{z\}\rangle) \subseteq C^{\mathrm{P}}(\langle A\rangle)$, i.e., $z \in C^{\mathrm{P}}(\langle A\rangle)$, which contradicts AV.

To prove (4.2) we use the following two lemmas.

Lemma 3 *Let $C^{\mathrm{P}}(\langle.\rangle)$ be a lower path independent social choice procedure. Then for any $A \in [X]$, the base relation P_b is either acyclic or $C^{\mathrm{P}}(\langle A\rangle) = A$.*

Proof Let $A = \{x, y, z\}$. Suppose not. Suppose $C^{\mathrm{P}}(\langle A\rangle) \neq A$. Without loss of generality suppose $z \notin C^{\mathrm{P}}(\langle A\rangle)$. Now, suppose xP_by and yP_bz. If zP_bx, then for $\langle\{x, y\}, \{z\}\rangle \in \Omega(A)$, $\{z\} = C^{\mathrm{P}}(\langle\{x, y\}, \{z\}\rangle)$, a contradiction that $C^{\mathrm{P}}(\langle.\rangle)$ is a lower path independent choice procedure.

Lemma 4 *Let $C^{\mathrm{P}}(\langle.\rangle)$ be a social choice procedure which satisfies WP. If its base relation P_b is acyclic then for $|X|\rangle N$, there exists an individual who is quasi-decisive over at least $(|X| - N + 1)(|X| - 1)$ pairs of alternatives.*

The proof is left to the reader since it is very similar to Blair and Pollak (1982).

Now to complete the proof of (4.2) one needs to consider a situation s that resulted in a cycle that contains a Pareto-dominated alternative. By Lemma 3, all of the alternatives in the cycle, including the Pareto-dominated one, are chosen from the set of alternatives involved in the cycle. This contradicts PO. The proof is complete by Lemma 4.

To prove (4.3) we consider the following lemma originally introduced by Mas-Collel and Sonnenschein (1972).

Lemma 5 *If a social choice procedure $C^{\mathrm{P}}(\langle.\rangle)$ satisfies WP and SM then there exists a weak dictator or else a cycle of social strict preference in the base relation.*

To complete the proof consider an issue A that contains three or more alternatives. First, restrict attention to the set of all but one of the alternatives. If PO and SM are satisfied, then by Lemma 5, either there exist a weak dictator or else a cycle in P_b on the restricted set of alternatives. The remaining alternative can be inserted into the profile so that it is both Pareto-dominated and it appears in the cycle. Once again, by Lemma 3, all of the alternatives in the cycle, including the Pareto-dominated one, are chosen from the set of alternatives involved in the cycle. This contradicts PO. The rest of the proof is immediate from Lemma 2.

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