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A characterization of the single-peaked domain

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Abstract We provide in this paper two properties that are both necessary and sufficient to characterize the domain of single-peaked preference profiles. This characterization allows for a definition of single-peaked preference profiles without using an ad hoc underlying order of the alternatives and also sheds light on the structure of single-peaked profiles. Considering the larger domain of value-restricted preference profiles (Sen, Econometrica 34:491–499, 1966) we also provide necessary and sufficient conditions for a preference profile to be single-caved or group-separable. Our results show that for single-peaked, single-caved and group-separable profiles it is sufficient to restrict to profiles containing of either three individuals and three alternatives or two individuals and four alternatives.

1 Introduction

The notion of single-peaked preferences introduced by Black (1948, 1958) and Arrow (1951) is by now a key assumption in many economic models, ranging from Hotelling-Downs political competition models to most local public goods models. Agents in a given economy have single-peaked preferences if there exists a linear order of the set of alternatives such that any agent's preference relation along this ordering is either always strictly decreasing, always strictly increasing, or first strictly increasing and then strictly decreasing.[1] In many models this guarantees for instance the existence

[1] Preferences that are not strictly decreasing when deviating from the peak (but still decreasing) are called in the literature "single-plateaued" preferences.

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of an equilibrium, the existence of a Condorcet winner, or the existence of incentive-compatible mechanisms. More recently, Spector (2000) and DeMarzo et al. (2003) showed that repeated interactions may lead to unidimensional opinions, thereby providing additional rational for the use of single-peaked preferences.

While in many models the existence of such an underlying linear order is without loss of generality, there is no a priori reason to confine the use of single-peakedness to those models in which this underlying linear order has a straightforward interpretation. After all, single-peakedness is a much more valuable property than the existence of an (interpretable) underlying order of the alternatives. Yet, in order to extend the use of single-peakedness to general models in which there is no obvious a priori ordering of the alternatives we need to know under which conditions a preference profile is single-peaked.[2] We provide in this paper an answer to this question by identifying a pair of properties that are both necessary and sufficient for a preference profile to be single-peaked.

Beyond being an invitation to enjoy the properties offered by single-peakedness to environments different from the usual ones, our characterization result also sheds light on the structure of single-peaked preferences. A quick historical review of the literature can help to assess our contribution. When Arrow (1951) formalized the concept of single-peaked preferences his main objective was to provide a set of conditions ensuring the transitivity of the majority decision. Arrow's definition of single-peakedness involved the set of all alternatives. Inada (1964) then observed that Arrow's transitivity result can be obtained by requiring only single-peakedness for any triple of alternatives. This observation has a simple, yet powerful interpretation: for any triple of alternatives there is one that is not considered the worst by any individual. Although Inada already observed that this property does not necessarily imply single-peakedness over all alternatives his result still nourishes the temptation to intuit that single-peakedness for any triple and single-peakedness for all alternatives are two faces of the same coin.

Failure for a preference profile to be single-peaked mainly comes from the constraint imposed by the underlying linear ordering, i.e., alternatives must all fit in a one-dimensional space.[3] Single-peakedness can thus be lost when there are too many agents with too many different preferences, or when the set of alternatives is "too large." Indeed, whenever there are at most two agents and three alternatives any preference profile is single-peaked. Adding a third agent can create a conflict if the set of alternatives that are considered worst by at least one agent contains the three alternatives. Following Sen (1966) and Sen and Pattanaik (1969), we call a profile such that the set of worst alternatives has always a cardinality no greater than two *worst-restricted*. But single-peakedness may also be lost when adding a fourth alternative instead of a third agent. This situation, a bit more technical, can be summarized

[2] For any individual it is trivial to construct a linear ordering of the alternatives such that the preferences of this agent are single-peaked with respect to this ordering. The difficulty is to find a linear order that is common to all individuals.

[3] Single-peakedness can be defined on other underlying structures than that of linear orders. The notion of separability in Border and Jordan (1983) represents an n-dimensional version of single-peakedness, whereas Demange (1982) studies single-peaked preferences on a tree.

as follows. There are three alternatives such that the two individuals disagree on the relative ranking of them. Call the intermediate alternative the "pivot" for this triple. Then, single-peakedness requires that these agents should disagree on the relative ranking of a fourth alternative with respect to the pivot. Preference profiles satisfying this restriction are called α*-restricted*. Our main result (Theorem 1) states that these two properties that operate over very simple structures, worst-restriction and α-restriction, are the set of necessary and sufficient properties of a preference profile to be single-peaked.[4]

In addition to the characterization of the single-peaked domain we also provide a characterization of the domain of group-separable preferences, i.e., preference profiles such that the set of alternatives can be split in two sets, with the property that each individual prefers any alternative in one of the sets to any alternative in the other set. Such a characterization is mostly motivated by the relation between single-peaked and group-separable preferences profiles and goes back to the contributions of Sen (1966) and Sen and Pattanaik (1969).[5] Indeed, Sen and Pattanaik observed that Inada's condition over triples could be generalized by considering not only the worst alternatives but also the best or the "intermediate" alternatives. Requiring that for any triple of alternatives one is not considered as the most preferred by any individual (*best-restricted* profile) is a necessary condition to obtain single-cavedness. If instead we consider the intermediate alternative, i.e., neither the best nor the worst, we obtain a necessary condition (*medium-restricted* profile) for group-separability. While a characterization of single-caved profiles is easily obtained from our characterization of the single-peaked domain, group-separability is a bit more delicate to work with, mainly because such profiles do not come along with an intuitive topology of the alternatives (as opposed to single-peakedness that assumes that alternatives can be ordered on a one-dimensional space). Like single-peakedness, group separability is always obtained as long as there are no more than two individuals and three alternatives. Clearly, adding a third individual can make the preference profile not medium-restricted (and hence, not group-separable). But separability can also be lost when adding a fourth alternative instead of a third individual. Two conditions must be met for this to happen. First, the sets of worst and best alternatives of each individual must have an empty intersection. Second, both individuals' preferences must coincide on the relative ranking of the best and worst alternatives of one of the individuals, and yet hold opposite views with respect to the best and worst alternatives of the other individual. If such a configuration does not occur then we say that the profile is β*-restricted*. Our second result (Theorem 2) states that medium-restriction and β-restriction are the set of necessary and sufficient properties for a preference profile to be group-separable.

The paper is organized as follows. Notation and the main definitions are provided in Sect. 2. We present the characterization of the single-peaked and group-separable domains in Sect. 3 and 4, respectively. We conclude in Sect. 5. The main proofs are relegated to the Appendix.

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4 Note that the α-restriction is not related to the α-property in choice theory.

5 See also Dummett and Farquharson (1961) and Pattanaik (1970) for other accounts on the matter.

2 Preliminaries

Let X be a finite set of alternatives. We use the symbols $\subset$ and $\subseteq$ to denote strict and weak inclusion, respectively. For any two sets $A \subset B$, we denote by $\overline{A}_B$ the complementary set of A over the set B, i.e., $\overline{A}_B = \{x \in B : x \notin A\}$. If $B = X$ we simply write $\overline{A}$. The cardinality of a set A is denoted $|A|$.

Let N be a finite set of agents. A *preference profile* $P = (P_i)_{i \in N}$ describes the preference relations of agents, where each agent $i \in N$ is endowed with a connected, transitive, asymmetric preference relation P_i over X. The relation P_i is connected if for any two distinct alternatives x and x', we have either $x P_i x'$ or $x' P_i x$. It is transitive if for any x, x' and x'' such that $x P_i x'$ and $x' P_i x''$ we have $x P_i x''$. It is asymmetric if for any pair of alternatives x and x', if $x P_i x'$ then (*not* $x' P_i x$).

For any two alternatives x and x', the relation $x P_i x'$ means that agent i prefers alternative x to x'. Given two non-empty sets A and B and a preference relation P_i we use the shorthand $A P_i B$ to denote that for any $x \in A$ and any $x' \in B$, $x P_i x'$. For any preference relation P_i we denote by $\overline{P}_i$ the *reverse* relation, i.e., for any $x, x' \in X$, $x' \overline{P}_i x$ if and only if $x P_i x'$. Let $\overline{P} = (\overline{P}_i)_{i \in N}$.

For any set $A \subseteq X$, let $w(A, P_i)$ be the least preferred alternative in A according to the preference relation P_i, i.e., $w(A, P_i) = \{x \in A : \text{ for each } x' \neq x,\ x' P_i x\}$ and let $W(A, P) = \cup_{i \in N}\{w(A, P_i)\}$. Similarly, let $b(A, P_i)$ be the most preferred alternative in A according to the preference relation P_i, i.e., $b(A, P_i) = \{x \in A : \text{ for each } x' \neq x,\ x P_i x'\}$ and $B(A, P) = \cup_{i \in N}\{b(A, P_i)\}$.[6] Finally, if A contains 3 alternatives, let $m(A, P_i)$ denote the alternative in A which is neither $b(A, P_i)$ nor $w(A, P_i)$. Let $M(A, P) = \cup_{h \in N}\{m(A, P_i)\}$.

Definition 1 A profile P is *worst-restricted* (resp. *best-restricted*, *medium-restricted*) for a triple of alternatives, $x, x', x'' \in X$, if $|W(\{x, x', x''\}, P)| \leq 2$ (resp. $|B(\{x, x', x''\}, P)| \leq 2$, $|M(\{x, x', x''\}, P)| \leq 2$). A profile P is *worst-restricted* (resp. *best-restricted*, *medium-restricted*) if it is *worst-restricted* (resp. *best-restricted*, *medium-restricted*) for any triple of alternatives in X.

Value-restricted preferences (where for any triple “value” can be either “worst”, “best” or “medium” and for different triples it can take different meanings) were introduced by Sen (1966). In words, a preference profile over a triple of alternatives is a value-restricted profile if there is an alternative that is never the most preferred (resp. least preferred or “medium”-preferred) for any individual. The fact that preference profiles over any triple of alternatives satisfy at least one value-restriction allows the method of majority decision to be a social welfare function satisfying Arrow’s conditions (and in particular, being transitive). Notice that imposing all the triples of alternatives to respect the same restriction, we obviously obtain profiles satisfying value-restriction. Let $\mathscr{P}^{\text{worst}}$, $\mathscr{P}^{\text{best}}$, and $\mathscr{P}^{\text{medium}}$ denote the set of preference profiles that are worst-restricted, best-restricted and medium-restricted, respectively.

[6] Notice that given our assumptions, the worst and the best alternative for each individual exist and are unique.

Table 1 Admissible orientations (example)

L_1:	a	b	c	d
L_2:	b	a	c	d
L_3:	c	a	b	d
L_4:	d	a	b	c
L_5:	d	c	b	a
L_6:	d	c	a	b
L_7:	d	b	a	c
L_8:	c	b	a	d

3 Single-peaked preference profiles

For any $A \subseteq X$, let $\mathscr{L}_A$ denote the set of all (strict) linear orders over A, i.e., the set of connected, transitive, and asymmetric binary relations. We denote by L_A (when clear, we use L) a generic element of $\mathscr{L}_A$.[7] For any two alternatives x and x', the relation xLx' means that in the ordering L alternative x ranks before alternative x'. We say that two alternatives x and x' are *consecutive* in L if xLx' (resp. $x'Lx$) and there does not exist another alternative x'' such that $xLx''Lx'$ (resp. $x'Lx''Lx$). The fact that k alternatives are consecutive in L for $k \geq 3$ is defined similarly.

Given a preference relation P_i and a set $A \subseteq X$, a linear order L over A is an *admissible orientation of A with respect to P_i* if for any triple of alternatives of A, $b(A, P_i)$, x' and x'' such that either $b(A, P_i)Lx'Lx''$ or $x''Lx'Lb(A, P_i)$, we have $x'P_ix''$.

Consider an individual i with the following preference relation over alternatives a, b, c, and d: $aP_ibP_icP_id$. This preference relation admits eight different admissible orientations. Table 1 depicts all admissible orientations for individual i for these four alternatives, where, for instance, the admissible orientation L_3 is $cL_3aL_3bL_3d$.

We denote by $\mathscr{L}^*_A(P_i)$ the set of all admissible orientations of A with respect to P_i. Additionally, $\mathscr{L}^*_A(P) = \cap_{i \in N} \mathscr{L}^*_A(P_i)$. Clearly, for each individual $i \in N$, $\mathscr{L}^*_X(P_i) \neq \emptyset$ —e.g., $P_i \in \mathscr{L}^*_X(P_i)$. However, it is not always possible to guarantee the existence of an ordering L over X such that $L \in \mathscr{L}^*_X(P)$.[8]

Definition 2 A preference profile P is *single-peaked* if $\mathscr{L}^*_X(P) \neq \emptyset$.

The set of single-peaked preference profiles is denoted by $\mathscr{P}^{\text{SP}}$. Observe that if a profile $P \notin \mathscr{P}^{\text{SP}}$, then for any ordering L over X there exists three consecutive alternatives in L, say x, x' and x'' such that $xLx'Lx''$ and an $i \in N$ such that $w(\{x, x', x''\}, P_i) = x'$.

It is well known that if a profile $P \in \mathscr{P}^{\text{SP}}$ then P is worst-restricted. Yet, as Inada (1964) showed through an example, the converse is not true. The existence of

[7] Note also that any preference relation P_i is, in our context, also a linear order. However, to add clarity we will not use the terminology "linear ordering" to define a preference relation.

[8] Our definition of single-peakedness clearly corresponds to our aim of characterizing classical preference domains. There is also the notion of single-peaked choice, that can be explored in the works of Moulin (1984) or Bossert and Peters (2006).

an admissible orientation for any triple does not guarantee the existence of an admissible orientation for set X. Inada's example includes a set of four alternatives and a set of four individuals. Our characterization of single-peakedness is based on Inada's example, but considering only four alternatives and two individuals. Indeed, two of the preferences in Inada's example are redundant as a source of non-single-peakedness. The other two do not hold the following property:

Definition 3 A preference profile P is α-restricted if there do not exist two agents, $i, j \in N$, and four alternatives x, x', x'', x''' such that

(i) They are opposite in a triple: $x P_i x' P_i x'''$, $x''' P_j x' P_j x$. In this case alternative x' is called the *pivot*.
(ii) They coincide in ranking: $x'' P_i x'$, $x'' P_j x'$.

If a profile is not α-restricted then there are two individuals $i, j \in N$ and four alternatives who "agree" to set alternatives x and x''' as the end points of a common admissible orientation. The "disagreement" arises however with alternatives x' and x''. Let $\mathscr{P}^\alpha$ denote the set of preference profiles that are α-restricted.

Notice that in Definition 3 the case $x' P_i x''$, $x' P_j x''$ does not have to be excluded explicitly. If for either i or j the alternative x'' is the least preferred (among x, x', x'' and x'''), then it is easy to find an admissible orientation (e.g., simply by putting x'' either the first or the last of the orientation). If, on the contrary, the alternative x'' is not the least preferred by any of the two individuals, then there does not exist an admissible orientation for individuals i and j. However, a quicker way consists of observing that we are back to conditions (i) and (ii) by permuting x' and x'' (i.e., alternative x'' is now the pivot). Hence, we do not need to explicitly rule out the case $x' P_i x''$, $x' P_j x''$.

As the following examples show, it is easy to check that if a preference profile fails to satisfy either worst-restriction or α-restriction, then it cannot be single-peaked.

Example 1 Consider three individuals, i, j, k and l and four alternatives, a, b, c and d. The following table depicts these indivuals preferences over these four alternatives.

P_i	P_j	P_k	P_l
a	a	a	d
b	b	d	b
c	d	b	a
d	c	c	c

An easy way to check for failure of worst-restriction is to see whether there are three distinct alternatives with the same "preference level," i.e., all these alternatives are the k-th preferred, for some $k \geq 3$. This is obviously not the case for the fourth preferred alternatives. Considering the third preferred alternative, we notice that there are three individuals, j, k, and l, whose third preferred alternative is d, b, and a, respectively. It suffices now to check that for j, k and l the alternatives $\{a, b\}$, $\{a, d\}$, and $\{d, b\}$ lie above d, b, and a in their preferences, respectively. Since this is the case in this example, we can conclude that the profile does not satisfy worst-restriction.

On the contrary, note that we cannot find any pair of individuals and four alternatives such that the profile fails to satisfy α-restriction. More precisely, notice that we can find two individuals, i and l, and three alternatives a, b, and d that satisfy condition (i) of the definition α-restriction, where the pivot is b. However, the fourth alternative c, is not preferred to alternative b by *both* individuals. That is, these two individuals and four alternatives do not comply with condition (ii) of Definition 3.

Example 2 Consider four individuals, i, j, k, and l and four alternatives, a, b, c, and d. The following table depicts these individuals' preferences over these four alternatives.

P_i	P_j	P_k	P_l
a	a	a	a
b	b	d	c
c	d	b	b
d	c	c	d

The above profile clearly satisfies worst-restriction, yet it does not satisfy α-restriction. A quick way to check failure of α-restriction consists of identifying a possible "pivot" by looking at an alternative that is the k-th preferred by two individuals ($k \geq 3$). This is the case for instance with individuals k and l and alternative b (which would be the "pivot"). Alternatives d and c are ranked differently with respect to b by individuals k and l. That is, for individuals k and l the preference relation over b, c, and d are opposite. To complete our checking we must ensure that there is a fourth alternative that is preferred by both individuals to the pivot. This is the case for alternative a, so we can conclude that the above profile fails to satisfy α-restiction.

The main result of this section shows that in fact worst-restriction and α-restriction are also sufficient to characterize single-peakedness.[9]

Theorem 1 $\mathscr{P}^{\text{SP}} = \mathscr{P}^{\text{worst}} \cap \mathscr{P}^{\alpha}$.

The proof of Theorem 1 consists of showing that for any profile $P \in \mathscr{P}^{\text{worst}} \cap \mathscr{P}^{\alpha}$ we can construct a linear order L such that $L \in \mathscr{L}_X^*(P)$. To this end, we consider the collection of sets $\{W(X, P), W(\overline{W(X, P)}, P), \ldots\}$. Since $P \in \mathscr{P}^{\text{worst}}$, the cardinality of all these sets is at most 2. We then construct a linear order L such that the alternatives in $W(X, P)$ are located at the extreme ends of L, and then completing L with the alternatives in $\overline{W(X, P)}$. Since the arguments used in the the construction of L slightly differ whether the sets in $\{W(X, P), W(\overline{W(X, P)}, P), \ldots\}$ contain one or two alternatives, we first consider the sets containing two alternatives and then complete L by incorporating the singleton sets.

Observe that if $A \subset X$ and $\mathscr{L}_A^*(P) = \emptyset$, then $\mathscr{L}_X^*(P) = \emptyset$. Similarly, if for any coalition $S \subset N$ we have $\cap_{i \in S} \mathscr{L}_A^*(P_i) = \emptyset$, then $\mathscr{L}_A^*(P) = \emptyset$. Theorem 1 shows that some basic converse statement also holds. More precisely, if $\mathscr{L}_X^*(P) = \emptyset$ then there exist minimal sets A, S of alternatives and individuals (with minimal dimensions, 3 and 3, or 4 and 2) such that $\cap_{i \in S} \mathscr{L}_A^*(P_i) = \emptyset$.

Another preference domain, similar to single-peakedness, is the class of *single-caved* preference profiles.[10]

[9] Notice that Examples 1 and 2 clearly show that worst-restriction and α-restriction are independent properties.

[10] Some authors call such preferences *single dipped*, e.g., Klaus et al. (1997).

Definition 4 A preference profile P is *single-caved* if $\overline{P}$ is single-peaked, i.e., $\mathscr{L}_X^*(\overline{P}) \neq \emptyset$.

Let $\mathscr{P}^{\text{SC}}$ be the set of single-caved profiles. The set $\mathscr{P}^{\text{SC}}$ can also be characterized directly as the set of preference profiles that are best-restricted and $\overline{\alpha}$-restricted, where this latter property is the α-restriction in which we invert individuals' preferences. That is, a profile P is $\overline{\alpha}$-restricted if there do not exist two individuals $i, j \in N$ and four alternatives, x, x', x'', and x''' such that (a) $x P_i x' P_i x'''$, $x''' P_j x' P_j x$, and (b) $x' P_i x''$ and $x' P_j x''$. Let $\mathscr{P}^{\overline{\alpha}}$ denote the set of preference profiles that are $\overline{\alpha}$-restricted. We then obtain the following characterization of the domain of single-caved preference profiles.

Corollary 1 $\mathscr{P}^{\text{SC}} = \mathscr{P}^{\text{best}} \cap \mathscr{P}^{\overline{\alpha}}$.

Proof That $\mathscr{P}^{\text{SC}} \subseteq \mathscr{P}^{\text{best}} \cap \mathscr{P}^{\overline{\alpha}}$ is obvious. To show that $\mathscr{P}^{\text{best}} \cap \mathscr{P}^{\overline{\alpha}} \subseteq \mathscr{P}^{\text{SC}}$ consider a profile $P \in \mathscr{P}^{\text{best}} \cap \mathscr{P}^{\overline{\alpha}}$. It is easy to see that $P \in \mathscr{P}^{\text{best}}$ implies $\overline{P} \in \mathscr{P}^{\text{worst}}$ and $P \in \mathscr{P}^{\overline{\alpha}}$ implies $\overline{P} \in \mathscr{P}^{\alpha}$. So, $\overline{P} \in \mathscr{P}^{\text{SP}}$ and thus $P \in \mathscr{P}^{\text{SC}}$. □

4 Group-separable preference profiles

Given a preference relation P_i and a set $A \subseteq X$, a non-empty set $E \subset A$ is a *separation of A with respect to P_i* if either $E P_i \overline{A}_E$ or $\overline{A}_E P_i E$. We denote by $\mathscr{S}_A(P_i)$ the set of all separations of A with respect to P_i. Additionally, $\mathscr{S}_A(P) = \cap_{i \in N} \mathscr{S}_A(P_i)$. Clearly, for each individual $i \in N$, $\mathscr{S}_A(P_i) \neq \emptyset$. For instance, $\{b(A, P_i)\} \in \mathscr{S}_A(P_i)$ and $\{w(A, P_i)\} \in \mathscr{S}_A(P_i)$. However, it is not always possible to guarantee, for any $A \subseteq X$, the existence of a separation E such that $E \in \mathscr{S}_A(P)$.

Definition 5 A preference profile P is *group-separable* if for any $A \subseteq X$, $\mathscr{S}_A(P) \neq \emptyset$.

The set of group-separable preference profiles is denoted by $\mathscr{P}^{\text{GS}}$. It is easy to see that if a profile is group-separable, then it is also medium-restricted (each group of three alternatives is separable). In a first analysis by Inada (1964), the author announces that these two ideas are equivalent. However, as Inada points out (1969), this is not the case. The existence of a separation for any triple does not guarantee the existence of a separation for set X. We present here another source of non-group-separability that considers a group of four alternatives and only two individuals.

Definition 6 A preference profile P is *β-restricted* if there do not exist $i, j \in N$ and four alternatives x, x', x'' and x''' such that

$$x P_i x' P_i x'' P_i x''' \quad \text{and} \quad x' P_j x''' P_j x P_j x''. \tag{1}$$

Let $\mathscr{P}^{\beta}$ be the set of preference profiles that are β-restricted. The conditions depicted in Eq. 1 can be re-stated as follows. First, individuals i and j's respective best and worst alternatives must differ, i.e., $\{w(A, P_i), b(A, P_i)\} \cap \{w(A, P_j), b(A, P_j)\} = \emptyset$ where $A = \{x, x', x'', x'''\}$. Second, the preference of both individuals must coincide with respect to the best and worst alternatives of one of them and differ with respect of the best and worst alternatives of the other. That is, $b(A, P_j) P_i w(A, P_j)$ and $w(A, P_i) P_j b(A, P_i)$. The combination of these two requirements gives Eq. 1.

Example 3 Consider four individuals, i, j, k, and l and four alternatives, a, b, c, and d. The following table depicts these indivuals preferences over these four alternatives.

P_i	P_j	P_k	P_l
a	d	b	a
b	c	d	b
c	b	a	c
d	a	c	d

Routine examination shows that the above profile is medium-restricted. Notice however that the preferences P_i and P_k fail to satisfy the requirement of β-resriction.

On the contrary, the profile depicted in Example 1 satisfies β-restriction (with individuals i, j, and l with alternatives a, c, and d) but fails to satisfy medium-restriction.[11]

It is not difficult to see that any group-separable profile must be β-restricted. With the previous analysis, any group-separable profile must be both medium-restricted and β-restricted. We conclude our results by showing that this pair of properties constitutes a characterization of group-separable profiles.

Theorem 2 $\mathscr{P}^{\text{GS}} = \mathscr{P}^{\text{medium}} \cap \mathscr{P}^{\beta}$.

The proof of Theorem 2 consists first of showing that if a profile $P \in \mathscr{P}^{\text{medium}} \cap \mathscr{P}^{\beta}$ then P restricted to the set $B(X, P) \cup W(X, P)$ is group-separable.[12] Given a separation set E and its complementary F, it may be that for some individual i the set E (or F) is not connected, i.e., there exist $x, x' \in E$ and $x'' \notin E$ such that $x P_i x'' P_i x'$. The second and main step of the proof consists of showing that $E \cup \{x''\} \in \mathscr{S}_{E \cup F \cup \{x''\}}(P)$. Continuing this way with all other alternatives we eventually obtain a separation of X for the profile P, provided $P \in \mathscr{P}^{\text{medium}} \cap \mathscr{P}^{\beta}$.

5 Conclusion

Our results first show that single-peakedness (or single-cavedness or group separability) can be lost just because of the existence of two individuals and four alternatives, or three individuals and three alternatives. What is perhaps more surprising is that we are able to endow the reverse statement with a precise meaning: if a profile is

[11] So Examples 1 and 3 show that medium-restriction and β-restriction are independent properties.

[12] We start by observing that if $P \in \mathscr{P}^{\text{GS}}$ then any profile $\hat{P}$ such that $\hat{P}_i \in \{P_i, \overline{P}_i\}$ is also group-separable. It follows that for any pair of individuals $i, j \in N$, we can assume without loss of generality that $b(X, P_i) P_j w(X, P_i)$.

not single-peaked (single-caved or group-separable, respectively) then there must be a violation of that property for a set of three preferences over three alternatives or a set of two preferences over four alternatives.

Absent an exact characterization of either single-peaked or group-separable preferences, these preference domains usually (if not only) appear in models in which they can be considered as “natural” assumptions like for instance Hotelling or Downsian types of models. It is our contention that the characterizations we offer can open the way to the use of those domains in other contexts. As far as we know, Bartholdi and Trick (1986) is the only paper that identifies single-peakedness without using an a priori linear order of the alternatives to be ranked. Their approach differs from ours, however, since they do not provide a set of properties that characterize single-peakedness, as we do. Instead, they provide an algorithm to find admissible orientations and show that given a preference profile single-peakedness can be stated in polynomial time.[13]

The identification of which sets of preferences allow this kind of representation constitutes one of the main problems of, among other fields, the design of strategy-proof social choice functions.[14] Inspired by classical results (e.g., Moulin 1980), a huge amount of literature has discussed two of its main features. First, the generalization of the notion of single-peakedness to non-linear structures, in the search of strategy-proof social choice functions. Second, the deep analysis of voting by committees (see Barberà et al. (1991)) show that single-peakedness is the *seed* of non-manipulability in several domains. In a recent paper, Nehring and Puppe (2006) show the relevance of this approach.

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Appendix

In order to prove Theorem 1 we introduce the following definition. Given an ordering L over a set A, two consecutive alternatives $x, x' \in A$ such that xLx' and an alternative $x'' \notin A$, we say that we *augment the ordering* L *with the relation* $xLx''Lx'$ to mean that now L is an ordering over $A \cup \{x''\}$ such that for any alternative $\hat{x}$ such that $\hat{x}Lx$ (resp. $x'L\hat{x}$) we have $\hat{x}Lx''$ (resp. $x''L\hat{x}$). Augmenting a linear by incorporating two alternatives is defined similarly.

[13] To proceed they transform the problem of finding a linear ordering into a matrix problem and apply an algorithm originally proposed by Booth and Lueker (1976) —see also Knoblauch (2010). Trick (1989) proposes an algorithm to check single-peakedness on a tree.

[14] Although related, the existing literature has mainly focused on the size of the “Condorcet Domain”. See Fishburn (2002) and the references therein.

Proof of Theorem 1 That $\mathscr{P}^{\mathrm{SP}} \subseteq \mathscr{P}^{\alpha} \cap \mathscr{P}^{\mathrm{worst}}$ is obvious. We now show that $\mathscr{P}^{\alpha} \cap \mathscr{P}^{\mathrm{worst}} \subseteq \mathscr{P}^{\mathrm{SP}}$. To this end, let $P \in \mathscr{P}^{\alpha} \cap \mathscr{P}^{\mathrm{worst}}$. Let $X_1, \ldots, X_k$ be the collection of sets of alternatives such that $X_1 = W(X, P)$, and for each $1 \leq h \leq k$, $X_h = W(\overline{\cup_{j<h} X_j}, P)$. Let $Y = \{\cup_h X_t : |X_t| = 2\}$ and $Z = \{\cup_h X_t : |X_t| = 1\}$. Since $P \in \mathscr{P}^{\mathrm{worst}}$, $Y \cup Z = X$. To prove that $\mathscr{P}^{\alpha} \cap \mathscr{P}^{\mathrm{worst}} \subseteq \mathscr{P}^{\mathrm{SP}}$ we show that we can construct an ordering L such that $L \in \mathscr{L}^*_Y(P)$ and then we show that L can be augmented to incorporate alternatives in Z such that $L \in \mathscr{L}^*_{Y \cup Z}(P)$.

Step 1 Construction of L such that $L \in \mathscr{L}^*_Y(P)$.

Let $Y_1, \ldots, Y_y$ be the collection of sets of alternatives such that $Y_1 = W(Y, P)$, and for each $1 \leq h \leq y$, $Y_h = W(\overline{(\cup_{j<h} Y_j)}_Y, P)$.[15]

Let $Y_1 = \{e_1, f_1\}$, and let L be the linear ordering over Y_1 such that $e_1 L f_1$. Clearly, $L \in \mathscr{L}^*_{Y_1}(P)$. We now proceed by induction to show that we can augment L to include also a relation over all alternatives in Y. To this end, suppose that for some $t \geq 1$ we can construct two sets, $E_t = \{e_1, \ldots, e_t\}$ and $F_t = \{f_1, \ldots, f_t\}$ and a linear order L such that $e_1 L \ldots L e_t L f_t L \ldots L f_1$, and $L \in \mathscr{L}^*_{\cup_{h \leq t} Y_h}(P)$, where for each $h \leq t$, $Y_h = \{e_h, f_h\}$.

Let $Y_{t+1} = \{x, x'\}$. We now show that there exists $\hat{x}, \tilde{x} \in Y_{t+1}$ such that $\tilde{x} \neq \hat{x}$ and L can be augmented with the relations $e_t L \hat{x} L \tilde{x} L f_t$. To this end, let $S_1 = \{x, x'\} \backslash W(\{e_t, x, x'\}, P)$ and $S_2 = \{x, x'\} \backslash W(\{f_t, x, x'\}, P)$. We first show that $S_1, S_2 \neq \emptyset$. By construction, there exists $l > 1$ such that $X_l = \{e_t, f_t\}$. So, $\{e_t, f_t\} = W(\overline{\cup_{h<l} X_h}, P)$ and thus, since $x, x' \in \overline{\cup_{h<l} X_h}$, $e_t \in W(\{e_t, x, x'\}, P)$ and $f_t \in W(\{f_t, x, x'\}, P)$. Since $P \in \mathscr{P}^{\mathrm{worst}}$, $|W(\{e_t, x, x'\}, P)| \leq 2$ and $|W(\{f_t, x, x'\}, P))| \leq 2$. So, $S_1, S_2 \neq \emptyset$. We claim that there exist $\hat{x} \in S_1$ and $\tilde{x} \in S_2$ such that $\hat{x} \neq \tilde{x}$. To see this, suppose on the contrary that there does not exist such a pair of alternatives. The only possibility is then $S_1 = S_2$ and $|S_1| = 1$. Without loss of generality, suppose $x' \notin S_1, S_2$. So, and thus there exists $i \in N$ such that $x' = w(\{e_t, x, x'\}, P_i)$. So, $e_t P_i x'$ and $x P_i x'$. Given that $\{e_t, f_t\} = W(\overline{\cup_{h<l} X_h}, P)$ it must be $x' P_i f_t$. Similarly, since $x' \notin S_2$ there exists $j \in N$ such that $x' = w(\{x, x', f_t\}, P_j)$. So, $x P_j x'$ and $f_t P_j x'$. Like for individual i, it can be shown that $x' P_j e_t$. Since $e_t P_i x'$ and $x' P_j e_t$, $i \neq j$. So, (P_i, P_j) is not α-restricted when considering the alternatives e_t, x, x', f_t (where the pivot is x'), so P is not α-restricted, a contradiction. Hence, there exists $\hat{x} \in S_1$ and $\tilde{x} \in S_2$ such that $\hat{x} \neq \tilde{x}$. Without loss of generality, let $\hat{x} = x$ and $\tilde{x} = x'$. We now set $e_{t+1} = x$ and $f_{t+1} = x'$, and augment the ordering L with the following relations

$$e_t L e_{t+1} L f_{t+1} L f_t. \tag{A.1}$$

We claim that $L \in \mathscr{L}^*_{\cup_{h \leq t+1} Y_h}(P)$. To see this, suppose on the contrary that $L \notin \mathscr{L}^*_{\cup_{h \leq t+1} Y_h}(P)$. So, there exists $i \in N$ and three alternatives consecutive in L, say x, x' and x'', such that $x L x' L x''$ and $w(\{x, x', x''\}, P_i) = x'$. Suppose $x' = e_h$, where $h \leq t - 1$. So, $\{x, x', x''\} \subset \{e_1, \ldots, e_t\}$, and thus $L \notin \mathscr{L}^*_{\cup_{h \leq t} Y_h}(P)$, a contradiction

15 Note that the sets $Y_1, Y_2, \ldots$ are ordered as the the sets X_t, i.e., if there are two sets Y_h and $Y_{h'}$ such that $h < h'$ then there exists two sets X_t and $X_{t'}$ such that $X_t = Y_h$, $X_{t'} = Y_{h'}$ and $t < t'$.

with the induction hypothesis. Similarly, we have $x' \neq f_h$ for $h \leq t-1$. So, $x' \in \{e_t, e_{t+1}, f_{t+1}, f_t\}$. Suppose first that $x' = e_{t+1}$. So, $x' = w(\{e_t, e_{t+1}, f_{t+1}\}, P_i)$ and thus $x' \in W(\{e_t, e_{t+1}, f_{t+1}\}, P)$. Hence, $e_{t+1} \notin S_1$, a contradiction. So, $x' \neq e_{t+1}$. Similarly, it can be shown that $x' = f_{t+1}$ implies $f_{t+1} \notin S_2$, a contradiction. So, $x' \in \{e_t, f_t\}$. Without loss of generality, suppose that $x' = e_t$. So, $x = e_{t-1}$ and $x'' = e_{t+1}$ and thus $e_{t-1} P_i e_t$ and $e_{t+1} P_i e_t$. Since $L \in \mathscr{L}^*_{\cup_{h \leq t} Y_h}(P)$, we have

$$e_{t-1} P_i e_t P_i f_t. \tag{A.2}$$

Since $Y_t = \{e_t, f_t\} = W((\overline{\cup_{h<t} Y_h})_Y, P)$, there exists $j \in N$ such that $e_t = w((\overline{\cup_{h<t} Y_h})_Y, P_j)$. Since $e_t, e_{t+1}, f_t \in (\overline{\cup_{h<t} Y_h})_Y$, $e_{t+1} P_j e_t$ and $f_t P_j e_t$. So, $j \neq i$. Since $L \in \mathscr{L}^*_{\cup_{h \leq t} Y_h}(P)$, $f_t P_j e_t$ implies $e_t P_i e_{t-1}$. So, we have

$$f_t P_j e_t P_i e_{t-1} \quad \text{and} \quad e_{t+1} P_j e_t. \tag{A.3}$$

So, (P_i, P_j) is not α-restricted when considering the alternatives $e_{t-1}, e_t, e_{t+1}, f_t$ (with e_t as the pivot). So, P is not α-restricted, a contradiction. Hence $L \in \mathscr{L}^*_{\cup_{h \leq t+1} Y_h}(P)$.

Step 2 Construction of L such that $L \in \mathscr{L}^*_{Y \cup Z}(P)$.

Let $\{c_0^1, \ldots, c_0^{p_0}\}$ be the set of all alternatives (if any) such that for $1 \leq h \leq p_0$, $X_h = Z_h = \{c_0^h\}$. (So, notice that $X_{p_0+1} = Y_1$.) Similarly, for $1 \leq h \leq y$, let $\{c_h^1, \ldots, c_h^{p_h}\}$ be the set of all alternatives (if any) such that there exists t such that $X_t = Y_h$, $X_{t+j} = \{c_h^j\}$ for $1 \leq j \leq p_h$, and if $h < y$, $X_{t+p_h+1} = Y_{h+1}$. Clearly, $\bigcup_{h \leq y} \bigcup_{j \leq p_j} \{c_h^j\} = Z$. We now show that we can augment the ordering L obtained in Step 1 by including the elements $(c_0^1, \ldots, c_y^{p_y})$.

Consider first c_y^1 and augment L with the relations $e_y L c_y^1 L f_y$. We claim that $L \in \mathscr{L}^*_{Y \cup \{c_y^1\}}(P)$. To this end, suppose that $L \notin \mathscr{L}^*_{Y \cup \{c_y^1\}}(P)$. So, there exists an individual $i \in N$ and three consecutive alternatives in L, x, x' and x'' such that $x L x' L x''$ and $w(\{x, x', x''\}, P_i) = x'$. It is easy to check that $x' \in \{e_y, c_y^1, f_y\}$. Suppose first that $x' = c_y^1$. So, $c_y^1 = w(\{e_y, c_y^1, f_y\}, P_i)$. Let t such that $X_t = \{e_y, f_y\}$. Hence, $\overline{\cup_{h<t} X_t} = \{e_y, f_y\} \cup_{h=1,\ldots,p_y} \{c_y^h\}$. By construction, $c_y^h P_j c_k^1$ for all $1 < h \leq p_y$ and each individual $j \in N$. So, $c_y^1 = w(\{e_y, c_y^1, f_y\}, P_i)$ implies $c_y^1 = w(\overline{\cup_{h<t} X_t}, P_i)$ and thus $c_y^1 \in W(\overline{\cup_{h<t} X_t}, P)$. So, $c_y^1 \in X_t$, a contradiction. Hence, $x' \in \{e_y, f_y\}$. Without loss of generality, suppose $x' = e_y$. So, $x = e_{y-1}$ and $x'' = c_y^1$. Since $L \in \mathscr{L}^*_Y(P)$, $e_{y-1} P_i e_y$ implies $e_y P_i f_y$. So, we have

$$e_{y-1} P_i e_y P_i f_y \quad \text{and} \quad c_y^1 P_i e_y. \tag{A.4}$$

Since $e_y = w(\overline{\cup_{h<t} X_t}, P_i)$, there exists $j \neq i$ such that $e_y = w(\overline{\cup_{h<t} X_t}, P_j)$. So, $f_y P_j e_y$. Since $L \in \mathscr{L}^*_Y(P)$, we have $e_y P_j e_{y-1}$. Since $c_y^1 \notin X_t$, $c_y^1 P_j e_y$. So, we have

$$f_y P_i e_y P_i e_{y-1} \quad \text{and} \quad c_y^1 P_i e_y. \tag{A.5}$$

So, (P_i, P_j) is not α-restricted when considering the alternatives e_{y-1}, e_y, c_y^1, f_y (with e_y as the pivot). So, P is not α-restricted, a contradiction. Hence, $L \in \mathscr{L}^*_{Y\cup\{c_y^1\}}(P)$.

Consider now c_y^2. We claim that we can augment L either with the relations $e_y L c_y^1 L c_y^2 L f_y$ or with the relations $e_y L c_y^2 L c_y^1 L f_y$. To see this, suppose that augmenting L with the relations $e_y L c_y^1 L c_y^2 L f_y$ implies $L \notin \mathscr{L}^*_{Y\cup\{c_y^1, c_y^2\}}(P)$ and that augmenting L with the relation $e_y L c_y^2 L c_y^1 L f_y$ also implies $L \notin \mathscr{L}^*_{Y\cup\{c_y^1, c_y^2\}}(P)$. If we cannot augment L with the relations $e_y L c_y^1 L c_y^2 L f_y$ then there exist three consecutive alternatives in L, x, x' and x'' such that $xLx'Lx''$ and $W(\{x, x', x''\}, P) = \{x'\}$. A reasoning similar to the previous one shows that $x' \in \{c_y^1, c_y^2\}$. Since by construction, $c_y^2 P_h c_y^1$ for each individual $h \in N$, we have $x' = c_y^1$. So, there exists $i \in N$ such that $e_y P_i c_y^1$ and $c_y^2 P_i c_y^1$. Since $L \in \mathscr{L}^*_{Y\cup\{c_y^1\}}(P)$, $e_y P_i c_y^1$ implies $c_y^1 P f_y$. So, we have

$$e_y P_i c_y^1 P_i f_y \quad \text{and} \quad c_y^2 P_i c_y^1. \tag{A.6}$$

Similarly, if we cannot augment L with the relations $e_y L c_y^2 L c_y^1 L f_y$ then we can repeat the above reasoning and deduce that there exists $j \in N$ such that

$$f_y P_j c_y^1 P_j e_y \quad \text{and} \quad c_y^2 P_j c_y^1. \tag{A.7}$$

Clearly, $j \neq i$ and (P_i, P_j) is not α-restricted when considering the alternatives e_y, c_y^1, c_y^2, f_y (with c_y^1 as the pivot). So, P is not α-restricted, a contradiction. Hence, $L \in \mathscr{L}^*_{Y\cup\{c_y^1, c_y^2\}}(P)$. It suffices now to repeat the same arguments with alternatives $c_y^3, \ldots, c_y^{p_y}$ with the following rule: If L is augmented with the relations $e_y L c_y^1 L c_y^2 L f_y$ (resp. $e_y L c_y^2 L c_y^1 L f_y$) then L has to be augmented with the relations $c_y^2 L c_y^3 L \ldots L c_y^{p_y} L f_y$ (resp. $e_y L c_y^{p_y} L \ldots L c_y^1 L f_y$). The same arguments as the ones developed above show that $L \in \mathscr{L}^*_{Y\cup_{h=1,\ldots,p_y}\{c_y^h\}}(P)$. For $1 \leq h < k$ a similar reasoning can be used to show that we can augment L with either $e_h c_h^1 \ldots c_h^{p_h} e_{h+1}$ (but not $e_h c_h^{p_h} \ldots c_h^1 e_{h+1}$) or with $f_{h+1} c_h^{p_h} \ldots c_h^1 f_h$ (but not $f_{h+1} c_h^1 \ldots c_h^{p_h} f_h$). This concludes the proof. □

In order to prove Theorem 2 we introduce the following notation. For any profile $P \in \mathscr{P}$ let $I(P)$ be the set of profiles such that $P' \in I(P)$ if for each $i \in N$, $P'_i \in \{P_i, \overline{P}_i\}$.

Proof of Theorem 2 That $\mathscr{P}^{\mathrm{GS}} \subseteq \mathscr{P}^{\beta} \cap \mathscr{P}^{\mathrm{medium}}$ is obvious. We now show that $\mathscr{P}^{\beta} \cap \mathscr{P}^{\mathrm{medium}} \subseteq \mathscr{P}^{\mathrm{GS}}$. Let $P \in \mathscr{P}^{\beta} \cap \mathscr{P}^{\mathrm{medium}}$. Without loss of generality, we can consider $P' \in I(P)$ such that for each $h \in N$, $b_1 P'_h w_1$. With a slight abuse of notation, let $P' = P$.

Step 1 There exist two sets E and F such that $B(X, P) \cup W(X, P) = E \cup F$ and $E \in \mathscr{S}_{E\cup F}(P)$.

Table 2 Case 2.1

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6
w_k	w_k	w_k	b_1	b_1	b_1
b_k	b_1	b_1	w_1	w_k	w_k
b_1	b_k	w_1	w_k	b_k	w_1
w_1	w_1	b_k	b_k	w_1	b_k

For any individual $i \in N$, let (with some abuse of notation) $b_i = b(X, P_i)$ and $w_i = w(X, P_i)$. Let $E = \{b_1\}$ and $F = \{w_1\}$. Clearly, $E \in \mathscr{S}_{E \cup F}(P)$. Consider now any $1 < k < n$ and suppose that $E \in \mathscr{S}_{E \cup F}(P)$, where $E = \cup_{h<k}\{b_h\}$ and $F = \cup_{h<k}\{w_h\}$. Consider now individual k and the alternatives b_k and w_k. Since $E \in \mathscr{S}_{E_{k-1} \cup F_{k-1}}(P)$, we have $E P_k F$. Let $E' = E \cup \{b_k\}$ and $F' = F \cup \{w_k\}$. So, $E' P_k F'$ and thus $E' \in \mathscr{S}_{E' \cup F'}(P_{i_k})$. To prove *Step 1* by induction, it suffices to show that $E' \in \mathscr{S}_{E' \cup F'}(P)$. We distinguish between the following cases:

Case 1. $b_k \in E$ and $w_k \in F$. So, $E' = E$ and $F' = F$ and thus we obviously have $E' \in \mathscr{S}_{E' \cup F'}(P)$.

Case 2. $b_k \notin E$ *and* $w_k \notin F$. Suppose that $E' \notin \mathscr{S}_{E' \cup F'}(P)$. So, there exists i such that $E' \notin \mathscr{S}_{E' \cup F'}(P_i)$. Hence, $i \neq k$. Since $E \in \mathscr{S}_{E \cup F}(P_i)$ and $b_1 P_h w_1$ for each $h \in N$, $E P_i F$. So, there exist $x' \in E'$ and $y' \in F'$ such that $y' P_i x'$. By the induction hypothesis, notice that $x' \in E$ implies $y' = w_k$ and $y' \in F$ implies $x' = b_k$. We distinguish between the following subcases. [*Case* 2.1.]

Case 2.1. $x' = b_k$ *and* $y' = w_k$. So, $w_k P_i b_k$. Since $b_k \notin E$, $b_k \neq b_1$ and thus $b_k P_k b_1$. Similarly, since $w_k \notin F$, $w_k \neq w_1$ and thus $w_1 P_k w_k$. So, we have

$$b_k P_k b_1 P_k w_1 P_k w_k. \tag{A.8}$$

By construction, and since $b_k \neq b_1$ and $w_k \neq b_1$ we have $b_1 P_1 \{b_k, w_k\} P_1 w_1$. Suppose that $w_k P_1 b_k$. So, (P_1, P_k) is not β-restricted, and thus $P \notin \mathscr{P}^\beta$, a contradiction. Hence, $b_k P_1 w_k$ and thus

$$b_1 P_1 b_k P_1 w_k P_1 w_1. \tag{A.9}$$

Since $w_k P_i b_k$, $i \neq 1$. Combining $w_k P_i b_k$ and $b_1 P_i w_1$ the six possible configurations for P_i are given in Table 2.

It is easy now to notice that for $l = 1, 2, 4, 6$, the profiles $(P_1, P_k, P_i^l) \notin \mathscr{P}^{\text{medium}}$ with the alternatives $\{b_k, w_1, w_k\}$, $\{b_1, w_1, w_k\}$, $\{b_1, b_k, w_1\}$, and $\{b_1, b_k, w_k\}$, respectively. If $l = 3, 5$ then the profiles (P_1, P_i^l) or (P_k, P_i^l) are not in $\mathscr{P}^\beta$ considering the alternatives (b_1, b_k, w_1, w_k) and (b_1, b_k, w_1, w_k), respectively. Hence, for $l = 1, \ldots, 6$, we have $P \notin \mathscr{P}^{\text{medium}} \cap \mathscr{P}^\beta$, a contradiction. So, $E' \in \mathscr{S}_{E' \cup F'}(P)$.

Case 2.2 $x' = b_k$ and $y' \neq w_k$. So, $y' \in F$. Suppose first that $i = 1$. Since $b_k \notin E \cup F$, $b_k \neq w_1$ and thus $b_k P_1 w_1$. Since $y' \in F$, $y' \neq b_1$, and thus $b_1 P_1 y'$. Finally, since $y' P_i b_k$ we then have

Table 3 Case 2.2

$P^1_{h^*}$	$P^2_{h^*}$	$P^3_{h^*}$
b_k	b_1	b_1
b_1	b_k	w_1
w_1	w_1	b_k
y'	y'	y'

$$b_1 P_1 y' P_1 b_k P_1 w_1. \tag{A.10}$$

Since $b_k \notin E \cup F, b_k P_k b_1$. Since $E' \in \mathscr{S}(E' \cup F', P_k), b_1 P_k \{w_1, y'\}$. Suppose first that $w_1 P_k y'$. So, we have $b_k P_k b_1 P_k w_1 P_k y'$. It follows that $(P_1, P_k) \notin \mathscr{P}^\beta$ where P_1 is given by Eq. A.10. So, $P \notin \mathscr{P}^\beta$, a contradiction. Thus we have

$$b_k P_k b_1 P_k y' P_k w_1. \tag{A.11}$$

Since $y' \in F$ and $y' \neq w_1$, there exists an individual j with $1 < j < k$ such that either $y' = b_j$ or $y' = w_j$. Without loss of generality, suppose $y' = w_j$. Since $y' \neq w_1, w_1 P_j y'$. We then have three possibilities (depicted in Table 3),Using P_1 and P_k given by Eqs. A.10 and A.11, it is easy to check that for $l = 1, 2, 3$ the profiles $(P_1, P^l_j, P_k) \notin \mathscr{P}^{\text{medium}}$-contradiction with alternatives $\{b_k, w_1, y'\}, \{b_1, b_k, y'\}, \{b_1, b_k, w_1\}$, respectively. So, $P \notin \mathscr{P}^{\text{medium}}$, a contradiction. Hence, $h^* \neq 1$, i.e., $E' \in \mathscr{S}_{E' \cup F'}(P_1)$. It follows then that $b_1 P_1 b_k P_1 y'$. By construction $E' \in \mathscr{S}_{E' \cup F'}(P_k)$. Since $b_k \neq b_1$, we have $b_k P_k b_1 P_k y'$. By the definition of $i, y' P_i b_k$. Since $E \in \mathscr{S}_{E \cup F}(P_i)$ and $E P_i F, b_1 P_i y'$. So, $b_1 P_{h^*} y' P_{h^*} b_k$. This implies that $(P_1, P_{h^*}, P_k) \notin \mathscr{P}^{\text{medium}}$, and thus $P \notin \mathscr{P}^{\text{medium}}$, a contradiction. So, $E' \in \mathscr{S}_{E' \cup F'}(P)$.

Case 2.3 *$x' \neq b_k$ and $y' = w_k$.* This case is the symmetric of *Case 2.2*, the proof is therefore omitted.

Case 3 *$b_k \notin E$ and $w_k \in F$.* Suppose $E' \notin \mathscr{S}_{E' \cup F'}(P)$. So, there exists i such that $E' \notin \mathscr{S}_{E' \cup F'}(P_i)$. Hence, $i \neq k$. Since $E \in \mathscr{S}_{E \cup F}(P_i), E P_i F$. So, there exist $x' \in E'$ and $y' \in F'$ such that $y' P_i x'$. Since $w_k \in F$ and since $E \in \mathscr{S}_{E \cup F}(P_i), x' = b_k$ and $y' \in F$. We can then repeat the argument developed in *Case 2.2.*, obtaining then either $P \notin \mathscr{P}^{\text{medium}}$ or $P \notin \mathscr{P}^\beta$. So, $E' \in \mathscr{S}_{E' \cup F'}(P)$.

Case 4 *$b_k \in E$ and $w_k \notin F$.* This case is the symmetric of *Case 3* using the same argument as in *Case 2.3*, the proof is therefore omitted.

We then conclude that $\cup_{h \leq k} \{b_h\} \in \mathscr{S}_{\cup_{h \leq k}\{b_{i_h}\} \cup_{h \leq k}\{w_{i_h}\}}(P)$. By induction, we then have that $B(X, P) \in \mathscr{S}_{B(X,P) \cup W(X,P)}(P)$. Let $E_1 = B(X, P)$ and $F_1 = W(X, P)$.

Step 2 There exist $E^* \subset X$ such that $E^* \in \mathscr{S}_X(P)$.

Table 4 Step 2

P_i	P_j	P_l	$P_{l'}$
$b_{l'}$	$b_{l'}$		
x	e_i	e_i	$b_{l'}$
e_i	f_j	x	x
f_j	x	w_l	f_j
w_l	w_l		

For each $h \in N$, let $e_h = w(E_1, P_h)$, $f_h = b(F_1, P_h)$, $H_h(E_1) = \{x \in \overline{E_1} : x P_h e_h\}$ and $H_h(F_1) = \{x \in \overline{F_1} : f_h P_h x\}$. Suppose $\cup_{h \in N}(H_h(E_1) \cup H_h(F_1)) = \emptyset$. Let $E^* = X \backslash F_1 = \overline{F_1}$. Clearly, $E^* \in \mathscr{S}_X(P)$, and thus we are done. Suppose then that $\cup_{h \in N} H_h(E_1) \cup H_h(F_1) \neq \emptyset$. Without loss of generality suppose that there exists $i \in N$ such that $H_i(E_1) \neq \emptyset$, and let $x = b(H_i(E_1), P_i)$ —the case when $H_i(F_1) \neq \emptyset$ is symmetric defining $x = w(H_i(F_1), P_i)$.

Since $E_1 \in \mathscr{S}_{E_1 \cup F_1}(P)$, $x \notin F_1$. We show that $E_1 \cup \{x\} \in \mathscr{S}_{E_1 \cup F_1 \cup \{x\}}(P)$. To this end, suppose $E_1 \cup \{x\} \notin \mathscr{S}_{E_1 \cup F_1 \cup \{x\}}(P)$. So, there exists $j \in N$ such that $E_1 \cup \{x\} \notin \mathscr{S}_{E_1 \cup F_1 \cup \{x\}}(P_j)$. Clearly, $E_1 \cup \{x\} \in \mathscr{S}_{E_1 \cup F_1 \cup \{x\}}(P_i)$. So, $i \neq j$, and there exists $x' \in F_1$ such that $x' P_j x$. Without loss of generality, let $x' = f_j$. Since $x \notin F_1$, we have $f_j P_j x P_j w_j$. Since $e_i \in E_1$ and $f_j \in F_1$ we have

$$b_i P_i x P_i e_i P_i f_j \quad \text{and} \quad e_i P_j f_j P_j x P_j w_j. \tag{A.12}$$

Suppose first $e_i = b_j$. So, $b_j = e_i P_j b_i$. Since $b_i \in E_1$ and $f_j \in F_1$, $b_i P_j f_j$. So, we have $e_i P_j b_i P_j f_j P_j x$. We then have $(P_i, P_j) \notin \mathscr{P}^\beta$ and thus $P \notin \mathscr{P}^\beta$, a contradiction. So, $e_i \neq b_j$ and thus there exists $l \in N, l \neq i, j$ such that $e_i = b_l$. Using a similar argument it can be shown that we have $f_j \neq w_i$, which implies that there exists $l' \in N, l' \neq i, j$ such that $f_j = w_{l'}$. We now show that $l \neq l'$. Suppose on the contrary that $l = l'$. So, $b_l = e_i P_l x P_l f_j = w_{l'}$. It follows that $(P_i, P_j, P_l) \notin \mathscr{P}^{\text{medium}}$ and thus $P \notin \mathscr{P}^{\text{medium}}$ where (P_i, P_j) is the profile given by Eq. A.12, a contradiction. So, $l \neq l'$.

Consider now individuals i, j and l and the alternatives x, e_i, w_l. Clearly, since by construction $e_i = b_l$, $m(\{e_i, w_l, x\}, P_l) = x$. Since $w_l \in F_1$, $e_i P_i w_l$ and thus using Eq. A.12 we have $m(\{e_i, w_l, x\}, P_i) = e_i$. Since $P \in \mathscr{P}^{\text{medium}}$, we have $m(\{e_i, w_l, x\}, P_j) \neq w_l$. Since $w_l \in F_1$, $e_i P_j w_l$, and thus since $P \in \mathscr{P}^{\text{medium}}$ we have $x P_j w_l$, for otherwise we would have an $m(\{e_i, w_l, x\}, P_j) = w_l$, i.e., (P_i, P_j, P_l) would not be medium-restricted. A similar argument with individuals i, j, and l' and alternatives $b_{l'}, x, f_j$ yields $b_{l'} P_i x$. So, we have

$$b_{l'}, P_i x P_i e_i P_i \{f_j, w_l\} \quad \text{and} \quad \{b_{l'}, e_i\} P_j f_j P_j x P_j w_l\,, \tag{A.13}$$

where $e_i P_i \{f_j, w_l\}$ reads as $e_i P_i f_j$ and $e_i P_j w_l$ but no relation is specified for f_j and w_l, and similarly for $\{b_{l'}, e_i\} P_j f_j$. Note that Eq. A.13 implies $f_j \neq w_l$ and $e_i \neq b_{l'}$. Consider individuals i and j and alternatives e_i, f_j, x, w_l. Since $P \in \mathscr{P}^\beta$, we have $f_j P_i w_l$. Similarly, the alternatives $e_i, f_j, x, b_{l'}$ and the fact that $P \in \mathscr{P}^{\beta}$ yield $b_{l'} P_j e_i$. This yields the profile $(P_i, P_j, P_l, P_{l'})$ depicted in Table 4. Consider now the individuals i, l and l' and the alternatives $e_i, b_{l'}, x$. Clearly, $m(\{e_i, b_{l'}, x\}, P_i) = x$. From the definitions of b_l and $b_{l'}$ and the fact that $e_i = b_l$, we have $m(\{e_i, b_{l'}, x\}, P_{l'}) \in \{x, e_i\}$ and $m(\{e_i, b_{l'}, x\}, P_l) \in \{x, b_{l'}\}$. Since $P \in \mathscr{P}^{\text{medium}}$, $m(\{e_i, b_{l'}, x\}, P_{l'}) = e_i$ implies $m(\{e_i, b_{l'}, x\}, P_l) = x$. Similarly, $m(\{e_i, b_{l'}, x\}, P_l) = b_{l'}$ implies $m(\{e_i, b_{l'}, x\}, P_{l'}) = x$. Suppose first $m(\{e_i, b_{l'}, x\}, P_l) = x$. Since $b_{l'} \in E$, we have $e_i P_l x P_l b_{l'} P_l w_l$. Observe now that $m(\{b_{l'}, f_j, x\}, P_i) = x$, $m(\{b_{l'}, f_j, x\}, P_j) = f_j$. Since $b_{l'} \in E$ and $f_j \in F$, we have $m(\{b_{l'}, f_j, x\}, P_l) = b_{l'}$. So, $P \notin \mathscr{P}^{\text{medium}}$, a contradiction. So, $m(\{e_i, b_{l'}, x\}, P_l) = b_{l'}$ and thus $m(\{e_i, b_{l'}, x\}, P_{l'}) = x$. Repeating the above reasoning yields again $P \notin \mathscr{P}^{\text{medium}}$. Let $E_2 = E_1 \cup \{x\}$ and $F_2 = F_1$. So, we have $E_2 \in \mathscr{S}_{E_2 \cup F_2}(P)$.

Suppose now that we have defined E_t and F_t such that for each $2 \leq t' \leq t$, we have either

- $E_{t'} = E_{t'-1} \cup \{x\}$ with $x = b(H_i(E_{t'-1}), P_i)$ for some $i \in N$ where $H_i(E_{t'-1}) = \{x \in \overline{E_{t'-1}} : x P_h w(E_{t'-1}, P_i)\}$, and $F_{t'} = F_{t'-1}$.
- $F_{t'} = F_{t'-1} \cup \{x\}$ with $x = w(H_i(F_{t'-1}), P_i)$ for some $i \in N$ where $H_i(F_{t'-1}) = \{x \in \overline{F_{t'-1}} : b(F_{t'-1}, P_i) P_h x\}$, and $E_{t'} = E_{t'-1}$.

and $E_t \in \mathscr{S}_{E_t \cup F_t}(P)$.

For each $h \in N$, define the sets $H_h(E_t)$ and $H_h(F_t)$ as the sets $H_h(E_{t'})$ and $H_h(F_{t'})$, $t' < t$. Without loss of generality, suppose that there exists $i \in N$ such that $H_i(E_t) \neq \emptyset$ and let $x = b(H_i(E_t), P_i)$ —the case when $H_i(F_t) \neq \emptyset$ is symmetric defining $x = w(H_i(F_t), P_i)$. We show that $E_t \cup \{x\} \in \mathscr{S}_{E_t \cup \{x\} \cup F_t}(P)$. To this, suppose on the contrary that $E_t \cup \{x\} \notin \mathscr{S}_{E_t \cup \{x\} \cup F_t}(P)$.

So, there exists $j \in N$ such that $E_t \cup \{x\} \notin \mathscr{S}_{E_t \cup F_t \cup \{x\}}(P_j)$. Clearly, $E_t \cup \{x\} \in \mathscr{S}_{E_t \cup F_t \cup \{x\}}(P_i)$. So, $i \neq j$, and there exists $x' \in F_t$ such that $x' P_j x$. For each $h \in N$, let $e_h^t = w(E_t, P_h)$ and $f_h^t = b(F_t, P_h)$. Without loss of generality, let $x' = f_j^t$. Since $x \notin F_t$, we have $f_j^t P_j x P_j w_j$. Since $e_i^t \in E_t$ and $f_j^t \in F_t$ we have

$$b_i P_i x P_i e_i^t P_i f_j^t \text{ and } e_i^t P_j f_j^t P_j x P_j w_j. \tag{A.14}$$

If $e_i^t = b_l$ for some $l \neq i$ and $f_j^t = w_{l'}$ for some $l' \neq j$, it suffices to repeat the arguments developed after Eq. A.12. Suppose then that there does not exist $l \neq i$ such that $e_i^t = b_l$. So, there exists $t' < t$ and $l \neq i$ such that $e_i^t = b(H_l(E_{t'}), P_l)$. Since $x \notin E_t$, $x \notin E_{t'}$. So, $e_i^t P_l x$. By construction, $x P_l w_l$ and thus we have

$$e_i^t P_l x P_l w_l. \tag{A.15}$$

Similarly, if there does not exist $l' \neq j$ such that $f_j^t = w_{l'}$ then there exists $t' < t$ and $l' \neq i$ such that $f_j^t = w(H_{l'}(F_{t'}), P_{l'})$. Since $x \notin F_t$, $x \notin F_{t'}$. So, we have

$$b_{l'} P_{l'} x P_{l'} f_j^t. \tag{A.16}$$

Combining the profiles depicted in Eqs. A.14, A.15, A.16 yield a profile similar to that of Table 4. It is readily verified that the same arguments go through, which allows us to conclude that $E_t \cup \{x\} \in \mathscr{S}_{E_t \cup \{x\} \cup F_t}(P)$. Since the number of alternatives is finite, there exists T such that $E_T \in \mathscr{S}_{E_T \cup \overline{F_T}}(P)$ and for each $h \in N$, $H_h(E_T) = H_h(F_T) = \emptyset$. It suffices then to define $E^* = \overline{F_T}$ and we have $E^* \in \mathscr{S}_X(P)$. □

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