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Efficiency and stability in a model of wireless communication networks

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Abstract We introduce a model of (wireless communication) networks: a group of agents want to communicate with each other; an agent has his own position, chooses his costly communication range, and benefits from direct and indirect communications with other agents; any two agents can directly communicate if each agent is located within another agent's communication range; they can indirectly communicate if each agent is connected to another agent through a sequence of direct communications. Although efficiency and stability are not compatible in a general context, we identify interesting subclasses of problems where an efficient and stable network exists: the uniform interval model, the uniform circle model, and the communication favorable domain. We also investigate the consequence of allowing agents to relocate their positions. For certain networks, relocation-proofness is equivalent to stability.

1 Introduction

We introduce a model of (wireless communication) networks: a group of agents want to communicate with each other; an agent has his own position, chooses his costly

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communication range, and benefits from direct and indirect communications with other agents; any two agents can directly communicate if each agent is located within another agent's communication range; they can indirectly communicate if each agent is connected to another agent through a sequence of direct communications. Examples of such a situation are abundant in the real world: amateur radio, beacons (an ancient system of optical telegraph), etc.

Following Jackson and Wolinsky (1996), we investigate an existence of an efficient and stable network for the model. As in other models of networks, efficiency and stability are not compatible in a general context. However, we can identify interesting subclasses of problems where an efficient and stable network exists: the uniform interval model, in which agents are located uniformly on a unit interval; the uniform circle model, in which agents are located uniformly on a unit circle; and the communication favorable domain, in which the cost of setting a communication range is small compared with its benefits. We generalize the model by allowing agents to move their positions with relocation costs. As it turns out, for certain networks, relocation-proofness is equivalent to stability.

This paper belongs to a growing literature on the network formation in which agents' utilities depend on the structure of the network.[1] Compared with the existing network formation models, our model has at least three differences. Firstly, each agent is assigned with a position that plays a significant role in the formation of wireless communication networks. Since each agent's benefit and cost depend on his position, each agent plays a different role in the network formation process. We can also discuss how the possibility of relocating positions affects the process.

Secondly, any two agents enjoy the benefits of direct communications if each agent is located within another agent's communication range. Thus, each agent's decision on his communication range would affect all the other agents within his range. On the other hand, in the link-based network models, each agent's decision on a link would affect the agents connected through the link.

Thirdly, a wireless communication network cannot be fully described by a simple graph. Since two distinct networks may induce the same graph, we define a network as a pair of a communication range profile and a position profile.

The remainder of this paper is organized as follows. Section 2 introduces our model of networks, and the notions of efficiency and stability. Section 3 analyzes an existence of an efficient and stable network in two specific models, the uniform interval model and the uniform circle model. Section 4 shows that efficiency and stability are not compatible on the general domain and discusses how a positive result can be obtained by imposing a domain restriction. Section 5 generalizes the model by allowing agents to move their positions and investigates the implications. Concluding remarks follow in Sect. 6.

[1] See, for example Bala and Goyal (2000), Bloch and Dutta (2009), Bloch and Jackson (2007), Dutta and Mutuswami (1997), Dutta and Jackson (2000), Galeotti et al. (2006), Jackson and van den Nouweland (2005), Johnson and Gilles (2000), and for a survey Jackson (2005).

2 The model

Let $N \equiv \{1, \ldots, n\}$ where $n \geq 3$ be a set of agents. Each agent $i \in N$ has a *position* $p_i \in \mathbb{R}^K$ for $K \geq 1$. We assume that $p_i \neq p_j$ for all $i, j \in N$ with $i \neq j$. Let $p \equiv (p_i)_{i \in N}$ be a position profile, and $\mathcal{P}$ the set of all position profiles. Each agent $i \in N$ decides his *communication range* $r_i \in \mathbb{R}_+$. Let $r \equiv (r_i)_{i \in N}$ be a communication range profile, and $\mathcal{R}$ the set of all communication range profiles. A *wireless communication network,* or a *network,* is a pair $(r, p) \in \mathcal{R} \times \mathcal{P}$ of a communication range profile and a position profile. Let $\mathcal{W} \equiv \mathcal{R} \times \mathcal{P}$ be the set of all networks.

For all $p \in \mathcal{P}$ and all $i, j \in N$, the *metric distance between i and j* is $m(i, j; p) \equiv \|p_i - p_j\|$. Note that for all $p \in \mathcal{P}$ and all $i \in N$, $m(i, i; p) = 0$. For all $(r, p) \in \mathcal{W}$, let $g(r, p) \equiv \{\{i, j\} |\, m(i, j; p) \leq r_i \text{ and } m(i, j; p) \leq r_j\}$ be the graph induced by (r, p). For simplicity, $\{i, j\}$ is denoted by ij. If $ij \in g(r, p)$, agents i and j are (directly) *linked under* (r, p) and ij is called a *link*. For all $(r, p) \in \mathcal{W}$ and all $i, j \in N$ with $i \neq j$, a *path between i and j under* (r, p) is a sequence of agents $i_1, \ldots, i_J$ such that $i_k i_{k+1} \in g(r, p)$ for all $k \in \{1, \ldots, J-1\}$ with $i = i_1$ and $j = i_J$. For all $(r, p) \in \mathcal{W}$ and all $i, j \in N$ with $i \neq j$, the (geodesic) *distance between i and j under* (r, p), $d(i, j; r, p)$, is the number of links in a shortest path between i and j; if there is no path between i and j under (r, p), we set $d(i, j; r, p) = \infty$. We abuse a notation to denote a positive integer by d.

We are ready to introduce examples of networks. For all $p \in \mathcal{P}$, let (r^0, p) be the *empty network* such that for all $i \in N, r_i^0 = 0$; for all $p \in \mathcal{P}$, let $(r^{\max}, p)$ be the *maximal network* such that for all $i \in N, r_i^{\max} = \max_{j \neq i} m(i, j; p)$; for all $p \in \mathcal{P}$, let $(r^{\min}, p)$ be the *minimal communication range network* or *minimal network* such that for all $i \in N, r_i^{\min} = \min_{j \neq i} m(i, j; p)$ if there is an agent $j \neq i$ such that $\min_{k \neq i} m(i, k; p) = m(i, j; p) = \min_{k \neq j} m(j, k; p)$, or $r_i^{\min} = 0$ otherwise.

Let $b : \mathbb{N} \to \mathbb{R}_+$ be the *benefit function* that associates with each $d \in \mathbb{N}$ a nonnegative real value. We assume that b is strictly decreasing and $\lim_{d \to \infty} b(d) = 0$. Let $c : \mathbb{R}_+ \to \mathbb{R}_+$ be the *communication cost function* or *cost function* that associates with each communication range r_i a nonnegative real value. We assume that c is strictly increasing, continuous, concave, and $c(0) = 0$.

For all $(r, p) \in \mathcal{W}$ and all $i \in N$, the *utility* of i under (r, p) is defined to be his benefit minus his cost, that is,

$$u_i(r, p) \equiv \sum_{j \neq i} b(d(i, j; r, p)) - c(r_i).$$

For all $(r, p) \in \mathcal{W}$, let $v(r, p) \equiv \sum_{i \in N} u_i(r, p)$ be the *value of the network* (r, p).

Next we introduce two axioms, which will be the main interests of this paper. *Efficiency* requires that a network maximize the sum of individual utilities for a given position profile.

Definition A network $(r, p) \in \mathcal{W}$ is *efficient* if for all $(r', p) \in \mathcal{W}$, $v(r, p) \geq v(r', p)$.

Stability requires that no agent benefit by adjusting his communication range alone. For all $i \in N$ and all $r \in \mathcal{R}$, let $r_{-i} \equiv (r_j)_{j \in N \setminus \{i\}}$.

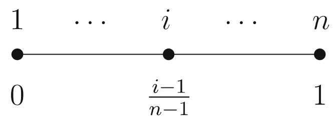

Fig. 1 The uniform interval model

Definition A network $(r, p) \in \mathcal{W}$ is *stable* if for all $i \in N$ and all $(r', p) \in \mathcal{W}$ such that $r_{-i} = r'_{-i}$, $u_i(r, p) \geq u_i(r', p)$.

For all $p \in \mathcal{P}$ and all $i \in N$, let $CR_i(p) \equiv \{q \in \mathbb{R}_+ |$ for some $j \in N, q = m(i, j; p)\}$ be the set of the metric distances from agent i to all the agents (including himself).

Remark 1 Under our formulation of the network, if an increase in an agent's communication range does not change the induced graph, the agent becomes worse off. As a consequence, if there is $i \in N$ such that $r_i \notin CR_i(p)$, then $(r, p) \in \mathcal{W}$ is neither stable nor efficient.

3 Two specific models of position profiles

We introduce two models of position profiles: the *uniform interval model*[2] and the *uniform circle model*. In the uniform interval model, agents are assumed to be located uniformly on a unit interval, and in the uniform circle model, on a unit circle. We present examples showing that efficiency and stability can be compatible in these models.

3.1 The uniform interval model

In the uniform interval model, agents are located uniformly on a unit interval [0, 1]. Formally, as in Fig. 1, let the position profile $p^- \equiv (p_i^-)_{i \in N}$ be such that for all $i \in N$, $p_i^- = \frac{i-1}{n-1}$.

We introduce a parameterized family of networks for the uniform interval model, which contains the empty, the minimal, and the maximal networks. For all $\alpha \in \{0, 1, \ldots, n-1\}$, (r^α, p^-) is a network under which any pair of agents are directly linked if the metric distance between them is at most $\frac{\alpha}{n-1}$. Formally, for all $\alpha \in \{0, 1, \ldots, n-1\}$, let $r^\alpha \equiv (r_i^\alpha)_{i \in N}$ be such that

$$r_i^\alpha = \begin{cases} \min\left\{\frac{\alpha}{n-1}, \frac{n-i}{n-1}\right\}, & \text{if } i \leq \frac{n}{2}, \\ \min\left\{\frac{\alpha}{n-1}, \frac{i-1}{n-1}\right\}, & \text{if } i > \frac{n}{2}. \end{cases}$$

If $\alpha = 1$, $r^\alpha = r^{\min}$ and if $\alpha = n-1$, $r^\alpha = r^{\max}$.

We present examples of stable networks.

[2] This model was first studied by Johnson and Gilles (2000) in the network formation literature.

Proposition 1 *In the uniform interval model, if* $\sum_{d=1}^{n-1} b(d) \geq c(\frac{1}{n-1})$*, then* $(r^{\min}, p^-)$ *is stable.*

Proof Let $(r^{\min}, p^-)$ be given. By Remark 1, any agent is worse off whenever he increases his communication range. Since $\sum_{d=1}^{n-1} b(d) \geq c(\frac{1}{n-1})$, no agent can be better off by decreasing his communication range. Therefore, $(r^{\min}, p^-)$ is stable. □

Proposition 2 *In the uniform interval model, if* $b(1) - b(2) \geq c(\frac{1}{n-1})$*, then for all* $\alpha \in \{0, 1, \ldots, n-1\}$, (r^α, p^-) *is stable.*

Proof For any $\alpha \in \{0, 1, \ldots, n-1\}$, let (r^α, p^-) be given. By Remark 1, any agent is worse off whenever he increases his communication range. Now suppose that agent i decreases his communication range from r_i^α to r_i'. By concavity of c, the cost abatement is at most $(\alpha - \beta)c(\frac{1}{n-1})$ where β is the largest integer such that $\frac{\beta}{n-1} \leq r_i'$. On the other hand, the benefit decrement is at least $(\alpha - \beta)(b(1) - b(2))$. Since $b(1) - b(2) \geq c(\frac{1}{n-1})$, agent i cannot be better off by decreasing his communication range. Altogether, for all $\alpha \in \{0, 1, \ldots, n-1\}$, (r^α, p^-) is stable. □

Examples of efficient networks follow.

Proposition 3 *In the uniform interval model, if* $b(1)-b(2) \geq c(\frac{1}{n-1})$*, then* $(r^{\max}, p^-)$ *is efficient.*

Proof By Remark 1, we consider only the networks $(r, p^-) \in \mathcal{W}$ such that $r \leq r^{\max}$. Suppose that (r, p^-) is not maximal. Since there is $i \in N$ such that $r_i < r_i^{\max}$, there is at least one pair of agents who are not directly linked under (r, p^-). Among pairs of agents not directly linked, let i and j be a pair who are closest to each other in the metric distance. Let $r_i' = \max\{r_i, m(i, j; p^-)\}$ and $r_j' = \max\{r_j, m(i, j; p^-)\}$. Then, i and j are directly linked under $(r_i', r_j', r_{-ij}, p^-)$ where $r_{-ij} \equiv (r_k)_{k \in N \setminus ij}$.

We claim that for $k = i, j$, $r_k' - r_k \leq \frac{1}{n-1}$. Suppose, without loss of generality, that $r_i' - r_i > \frac{1}{n-1}$. Then, there is $k \in N$ such that $r_i < m(i, k; p^-) < r_i'$. Since $r_i' - r_i > 0$ implies $r_i' = m(i, j; p^-)$, we have $m(i, k; p^-) < m(i, j; p^-)$. Since $r_i < m(i, k; p^-)$, i and k are not directly linked under (r, p^-). Altogether, we have a contradiction to the choice of i and j.

We compare $v(r, p^-)$ with $v(r_i', r_j', r_{-ij}, p^-)$. For i and j, by concavity of c, the cost increment is at most $c(\frac{1}{n-1})$ while the benefit increment is at least $b(1) - b(2)$. Since $b(1) - b(2) \geq c(\frac{1}{n-1})$, i and j are weakly better off from the direct communication, which implies that $v(r_i', r_j', r_{-ij}, p^-) \geq v(r, p^-)$. By iterating the argument, we conclude that $(r^{\max}, p^-)$ is efficient. □

Proposition 4 *In the uniform interval model, if* $2b(1)+(2n-5)b(2) \leq c(1)-c(\frac{n-2}{n-1})$*, then* (r^0, p^-) *is efficient.*

Proof By Remark 1, we consider only the networks $(r, p^-) \in \mathcal{W}$ such that $r \leq r^{\max}$. Suppose that (r, p^-) is not empty. If $g(r, p^-) = \emptyset$, by Remark 1, $v(r, p^-) <$

$v(r^0, p^-)$. Otherwise, let $S = \{k \in N | \max_{jk \in g(r,p^-)} m(j,k; p^-) = q^*\}$ where $q^* = \max_{ij \in g(r,p^-)} m(i,j; p^-)$. Note that $|S| \geq 2$. Let $r'_S = (r'_k)_{k \in S}$ be such that for each $k \in S$, $r'_k = q^* - \frac{1}{n-1}$. Let $r_{-S} = (r_i)_{i \in N \setminus S}$. Note that each agent in S loses at most two links from this adjustment. Also, for each $k \in S$, $r_k - r'_k \geq \frac{1}{n-1}$.

We compare $v(r, p^-)$ with $v(r'_S, r_{-S}, p^-)$. For each $k \in S$, by concavity of c, the cost abatement is at least $c(1) - c\left(\frac{n-2}{n-1}\right)$ while the benefit decrement is at most $2b(1) + (n-3)b(2)$. For each $i \in N \setminus S$, the benefit decrement is at most $|S|b(2)$. Thus, the total benefit decrement is at most $|S|(2b(1) + (n-3)b(2)) + (n - |S|)|S|b(2)$ and the total cost abatement is at least $|S|\left(c(1) - c\left(\frac{n-2}{n-1}\right)\right)$. Since $2b(1) + (n-3)b(2) + (n - |S|)b(2) \leq c(1) - c\left(\frac{n-2}{n-1}\right)$, we have $v(r'_S, r_{-S}, p^-) \geq v(r, p^-)$. By iterating the argument, we conclude that (r^0, p^-) is efficient. □

Remark 2 If c is strictly concave, $(r^{\max}, p^-)$ in Proposition 3 is uniquely efficient. However, the concavity of c is not sufficient to prove its uniqueness. The same remark holds for (r^0, p^-) in Proposition 4.

We can make the following observations for the uniform interval model. Since the empty network is always stable, under the assumption of Proposition 4, it is both efficient and stable. On the other hand, from Propositions 2 and 3, if $b(1)-b(2) \geq c(\frac{1}{n-1})$, the maximal network is both efficient and stable.

3.2 The uniform circle model

In the uniform circle model, agents are located uniformly on a unit circle in the counter clockwise direction. Formally, as in Fig. 2, let the position profile $p^o \equiv (p_i^o)_{i \in N}$ be such that for all $i \in N$, $p_i^o = \left(\cos \frac{2(i-1)\pi}{n}, \sin \frac{2(i-1)\pi}{n}\right)$. Furthermore, to simplify our analysis, we will assume that n is odd.

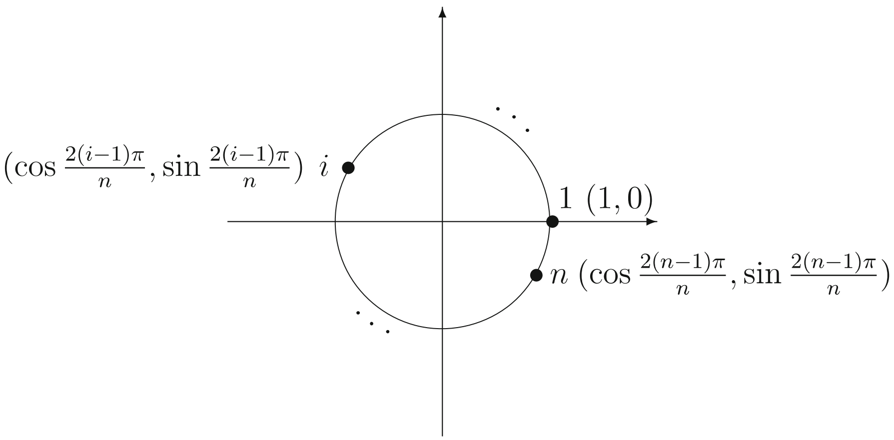

Fig. 2 The uniform circle model

We introduce a parameterized family of networks for the uniform circle model, which contains the empty, the minimal, and the maximal networks. For all $\alpha \in \left\{0, 1, \ldots, \frac{n-1}{2}\right\}$, let $\rho^\alpha \equiv \sqrt{2 - 2\cos\frac{2\alpha\pi}{n}}$. Note that ρ^α is strictly increasing and strictly concave in α. For all $\alpha \in \left\{0, 1, \ldots, \frac{n-1}{2}\right\}$, (r^α, p^o) is a network under which any pair of agents are directly linked if the metric distance between them is at most ρ^α. Formally, for all $\alpha \in \left\{0, 1, \ldots, \frac{n-1}{2}\right\}$, let r^α be such that for all $i \in N$, $r_i^\alpha = \rho^\alpha$. If $\alpha = 1$, $r^\alpha = r^{\min}$. On the other hand, if $\alpha = \frac{n-1}{2}$, $r^\alpha = r^{\max}$.

We discuss examples of stable and efficient networks. Since the proofs are similar to those in the uniform interval model except the calculation of costs and benefits, they are omitted.

Proposition 5 *In the uniform circle model:*

(i) *If* $2\sum_{d=1}^{(n-1)/2} b(d) \geq c(\rho^1)$*, then* $(r^{\min}, p^o)$ *is stable.*
(ii) *If* $2b(1) - 2b(2) \geq c(\rho^1)$*, then for all* $\alpha \in \{0, 1, \ldots, \frac{n-1}{2}\}$, (r^α, p^o) *is stable.*
(iii) *If* $2b(1) - 2b(2) \geq c(\rho^1)$*, then* $(r^{\max}, p^o)$ *is efficient.*
(iv) *If* $2b(1) + (2n - 5)b(2) \leq c(\rho^{(n-1)/2}) - c(\rho^{(n-3)/2})$*, then* (r^0, p^o) *is efficient.*

We can make the following observations for the uniform circle model. Since the empty network is always stable, under the assumption of Proposition 5 (iv), it is both efficient and stable. On the other hand, from Proposition 5 (ii) and (iii), if $2b(1) - 2b(2) \geq c(\rho^1)$, the maximal network is both efficient and stable.

4 General model of position profiles

As shown in the previous section, efficiency and stability are compatible if the position profile is appropriately specified. We investigate whether such a result can be generalized to more general domain of position profiles. However, the answer is negative: we can easily find a position profile in which an efficient and stable network does not exist.

Proposition 6 *If c is unbounded, there exists a position profile $p \in \mathcal{P}$ such that any efficient network is not stable.*

Proof Let $N = \{1, 2, 3\}$. Suppose that c is unbounded. We divide into two cases.

Case 1 $b(1) < 3b(2)$. By the assumptions on c, there exist $x, y \in \mathbb{R}_{++}$ such that $c(x) < \min\{b(2), 3b(2) - b(1)\}$ and $c(y) = b(1) + b(2)$. Let $p \in \mathcal{P}$ be such that $p_1 = (0, 0)$, $p_2 = (x, 0)$, and $p_3 = (x + y, 0)$. Since $c(x) < b(2)$ and $c(y) = b(1) + b(2)$, we have $x < y$, which implies that $r^{\min} = (x, x, 0)$ and $r^{\max} = (x + y, y, x + y)$. Let $r^* = (x, y, y)$ and $r' = (0, y, y)$. By Remark 1, we need to consider only the following 5 networks for efficiency: (r^0, p), $(r^{\min}, p)$, (r', p), (r^*, p), and $(r^{\max}, p)$. Note that $v(r^0, p) = 0$, $v(r^{\min}, p) = 2b(1) - 2c(x)$, $v(r', p) = 2b(1) - 2c(y)$, $v(r^*, p) = 4b(1) + 2b(2) - c(x) - 2c(y)$, and $v(r^{\max}, p) = 6b(1) - c(y) - 2c(x + y)$. Since $v(r', p) < v(r^{\min}, p)$, (r', p) is not efficient. Now we show that (r^*, p) is uniquely efficient. Since $0 < c(x) < \min\{b(2), 3b(2) - b(1)\}$ and $c(y) = b(1) + b(2)$, we have

$$\begin{aligned}
v(r^*, p) - v(r^0, p) &= 4b(1) + 2b(2) - c(x) - 2c(y) \\
&> 4b(1) + 2b(2) - b(2) - 2b(1) - 2b(2) \\
&= 2b(1) - b(2) \\
&> 0, \\
v(r^*, p) - v(r^{\min}, p) &= 2b(1) + 2b(2) + c(x) - 2c(y) \\
&= 2b(1) + 2b(2) + c(x) - 2b(1) - 2b(2) \\
&= c(x) \\
&> 0, \text{ and} \\
v(r^*, p) - v(r^{\max}, p) &= 2c(x+y) - c(y) - c(x) - 2b(1) + 2b(2) \\
&> 2c(x+y) - c(y) + b(1) - 3b(2) - 2b(1) + 2b(2) \\
&= 2c(x+y) - c(y) - b(1) - b(2) \\
&> c(y) - b(1) - b(2) \\
&= 0.
\end{aligned}$$

Altogether, (r^*, p) is uniquely efficient. However, by setting $r_2'' = x$, we have $u_2(r^*, p) = 2b(1) - c(y) < b(1) - c(x) = u_2(r_2'', r_{-2}^*, p)$, which implies that (r^*, p) is not stable.

Case 2 $3b(2) \leq b(1)$. By the assumptions on c, there exist $x, y \in \mathbb{R}_{++}$ such that $c(x) < b(2), c(y) = b(1)$, and $c(x+y) < \frac{3}{2}b(1) - b(2)$. Let $p \in \mathcal{P}$ be such that $p_1 = (0, 0)$, $p_2 = (x, 0)$, and $p_3 = (x + y, 0)$. Since $c(x) < b(2)$ and $c(y) = b(1)$, we have $x < y$, which implies that $r^{\min} = (x, x, 0)$ and $r^{\max} = (x + y, y, x + y)$. Let $r^* = (x, y, y)$ and $r' = (0, y, y)$. Similarly to Case 1, we can show that $(r^{\max}, p)$ is uniquely efficient. However, by setting $r_1'' = x$, we have $u_1(r^{\max}, p) = 2b(1) - c(x+y) < b(1) + b(2) - c(x) = u_1(r_1'', r_{-1}^{\max}, p)$, which implies that $(r^{\max}, p)$ is not stable.

This example is easily extended to the models with more than three agents by assigning to each agent $i \in \{4, \ldots, n\}$, $p_i = (x, iz)$ where $z \in \mathbb{R}_{++}$ is chosen to satisfy $c(z) > n(n-1)b(1)$. Also, it is obvious that the result can be extended to the positions in a more than two-dimensional space. □

Since it is natural to require that the cost function be unbounded, Proposition 6 shows that it is not easy for a network to satisfy both efficiency and stability in a general environment. One way-out is to impose a restriction on the position profiles as we did in the previous section. Another possibility is to impose a restriction on the cost and benefit functions as in the below.

Definition A position profile $p \in \mathcal{P}$ is *communication favorable* with respect to b and c if $c(\max_{i \in N}(\max_{j \neq i} m(i, j; p))) < b(1) - b(2)$.

Since this condition requires that the maximum communication cost between agents be less than the additional benefit increased by a direct communication, each agent has an incentive to form the maximal network. Let $\mathcal{P}_f(b, c)$ be the set of all communication favorable position profiles with respect to b and c.

It is not surprising that the maximal network is the only efficient one on the communication favorable domain.

Proposition 7 *For all $p \in \mathcal{P}_f(b, c)$, $(r^{\max}, p)$ is uniquely efficient.*

Proof Let $p \in \mathcal{P}_f(b, c)$. Suppose that (r, p) is not maximal. Then, any pair of agents $i, j \in N$ who are not linked are better off by increasing their communication ranges to $m(i, j; p)$, which increases the value of the network. By Remark 1, among the networks that induce the complete graph, only the maximal network maximizes the value of the network. Altogether, we conclude that $(r^{\max}, p)$ is the only efficient network. □

To show that at least one efficient network is stable on this restricted domain of position profiles, we check whether the maximal network is stable. In fact, there are many stable networks on this domain. We show the result by introducing two axioms. The *minimal range property* requires that each agent should choose the minimal range to induce a given graph.

Definition A network $(r, p) \in \mathcal{W}$ satisfies the *minimal range property* if for all $(r', p) \in \mathcal{W}$ such that $g(r', p) = g(r, p)$, $r \leq r'$.

Intruding-proofness requires that no agent can communicate with another agent by his unilateral deviation.

Definition A network $(r, p) \in \mathcal{W}$ is *intruding-proof* if for all $i, j \in N$, $m(i, j; p) \leq r_i$ implies $m(i, j; p) \leq r_j$.

As it turns out, on our communication favorable domain, *stability* is equivalent to the *minimal range property* and *intruding-proofness* together.

Proposition 8 *On the communication favorable domain, a network is stable if and only if it satisfies the minimal range property and intruding-proofness.*

Proof Let $p \in \mathcal{P}_f(b, c)$. First, we show that stability implies the minimal range property. Suppose that $(r, p) \in \mathcal{W}$ does not satisfy the minimal range property. Then, there is $i \in N$ who is better off by decreasing his communication range as long as the induced graph remains unchanged. Therefore, (r, p) is not stable, a contradiction.

Next, we show that stability implies intruding-proofness. Suppose that (r, p) is not intruding-proof. Then, there is $i \in N$ who can communicate with another agent by increasing his communication range. Since $p \in \mathcal{P}_f(b, c)$, he is better off by doing so. Therefore, (r, p) is not stable, a contradiction.

Finally, we show that the minimal range property and intruding-proofness together imply stability. Let $(r, p) \in \mathcal{W}$ satisfy the two properties. By the minimal range property, any decrease in each agent's communication range results in losing at least one link in $g(r, p)$. Since $p \in \mathcal{P}_f(b, c)$, this decrease makes him worse off. By intruding-proofness, any increase in each agent's communication range cannot change $g(r, p)$. Altogether, we conclude that (r, p) is stable. □

Since it can easily be shown that the maximal network satisfies the *minimal range property* and *intruding-proofness*, we establish the following corollary.

Corollary 1 *On the domain of communication favorable position profiles, the maximal network is stable.*

Therefore, the maximal network is both efficient and stable on the domain of communication favorable position profiles. Next, we investigate whether its uniqueness can be maintained without imposing efficiency. Since the empty network is always stable, the answer is negative. Moreover, our next example shows the possibility of having many stable networks.

Example 1 A stable network may not be unique. Let $N = \{1, 2, 3\}$. Let $p_1 = (0, 0)$, $p_2 = (0, 2)$, and $p_3 = (1, 2)$. Then, $\max_{i \in N}(\max_{j \neq i} m(i, j; p)) = m(1, 3; p) = \sqrt{5} \approx 2.24$. Assume that $b(d) = 5(.5)^d$ and $c(r_i) = \frac{1}{2}r_i$. Since $c(m(1, 3; p)) = 1.12 < 1.25 = b(1) - b(2)$, p is communication favorable with respect to b and c. Since (r^0, p), $(r^{\min}, p)$, and $(r^{\max}, p)$ satisfy the minimal range property and intruding-proofness, all of them are stable.

On the other hand, if we strengthen our stability notion to *strong stability* (Dutta and Mutuswami 1997; Jackson and van den Nouweland 2005), the maximal network emerges as the only strongly stable network on the communication favorable domain. *Strong stability* requires a coalition to deviate if each member of the coalition is (strictly) better off. For all $S \subseteq N$ and all $r \in \mathcal{R}$, let $r_S \equiv (r_i)_{i \in S}$ and $r_{-S} \equiv (r_i)_{i \in N \setminus S}$.

Definition A network $(r, p) \in \mathcal{W}$ is *strongly stable* if for all $S \subseteq N$ and all $(r', p) \in \mathcal{W}$ such that $r_{-S} = r'_{-S}$, there exists $i \in S$ such that $u_i(r, p) \geq u_i(r', p)$.

Proposition 9 *On the communication favorable domain, the maximal network is uniquely strongly stable.*

Proof Let $p \in \mathcal{P}_f(b, c)$. It is easy to show that $(r^{\max}, p)$ is strongly stable. Now suppose that (r, p) is not maximal. Then, there is $S \subseteq N$ such that for all $i \in S$, $r_i \neq r_i^{\max}$. Let $r'_S = r_S^{\max}$. If $r_i > r_i^{\max}$, by Remark 1, i is better off by setting $r'_i = r_i^{\max}$. If $r_i < r_i^{\max}$, i can communicate with at least one additional agent by setting $r'_i = r_i^{\max}$. Since $p \in \mathcal{P}_f(b, c)$, i is better off by doing so. Therefore, (r, p) is not strongly stable. Altogether, we conclude that $(r^{\max}, p)$ is uniquely strongly stable. □

A stronger version of strong stability, *strong stability 2*, which is proposed by Jackson and van den Nouweland (2005), requires a coalition to deviate if each member of the coalition is weakly better off and at least one of them (strictly) better off. Since the maximal network satisfies *strong stability 2* on the communication favorable domain, we can impose this requirement instead.

5 Relocation-proofness of a network

We generalize the model by allowing each agent to change his position with relocation cost. The process of network formation with position mobility is as follows: (i) each agent is located on a position, (ii) each agent chooses his communication range, (iii) each agent chooses whether to relocate his position and adjust his communication range, and (iv) the process continues until no agent has an incentive to change his position or communication range.

Let $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ be the *relocation cost function* that associates with each relocation metric distance $||p'_i - p_i||$ a nonnegative real value. We assume that ℓ is strictly increasing and $\ell(0) = 0$. Furthermore, we require that moving an agent's position be at least as costly as adjusting his communication range, that is, for all $x \in \mathbb{R}_+$, $\ell(x) \geq c(x)$.

An agent relocates his position or changes his communication range if it is beneficial to him. *Relocation-proofness* requires that each agent should not have such an incentive. For all $i \in N$ and all $p \in \mathcal{P}$, let $p_{-i} \equiv (p_j)_{j \in N \setminus \{i\}}$.

Definition A network $(r, p) \in \mathcal{W}$ is *relocation-proof* if for all $i \in N$ and all $(r', p') \in \mathcal{W}$ such that $r_{-i} = r'_{-i}$ and $p_{-i} = p'_{-i}$,

$$u_i(r, p) \geq u_i(r', p') - \ell(||p'_i - p_i||).$$

It is easy to check that *relocation-proofness* implies *stability.*

Our next example shows that an agent may have an incentive to relocate his position in a stable network even though he incurs the relocation cost.

Example 2 A stable network may not be relocation-proof. Let $N = \{1, 2, 3, 4, 5\}$. Let $p_1 = (0, 0)$, $p_2 = (1, 1)$, $p_3 = (2, 1)$, $p_4 = (2, 2)$, and $p_5 = (1, 2)$. Assume that $b(1) = 0.9$, $b(2) = 0.1$, and $c(r_i) = r_i$. Let $r = (0, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2})$, $r' = (\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2})$, and $r'' = (0, 1, \sqrt{2}, \sqrt{2}, \sqrt{2})$. Since $u_1(r, p) > u_1(r', p)$, agent 1 does not have an incentive to deviate. Since $u_2(r, p) > u_2(r'', p)$, neither does agent 2. Symmetric arguments apply to agents 3, 4, and 5. Altogether, we conclude that (r, p) is stable.

Now we show that the stable network (r, p) is not relocation-proof. Let $\ell(||p'_i - p_i||) = \frac{5}{4}||p'_i - p_i||$. As in Fig. 3, if agent 1 relocates his position to $p'_1 = \left(\frac{3}{2}, \frac{3}{2}\right)$ and changes his communication range to $r'_1 = \frac{\sqrt{2}}{2}$, his utility goes up: $u_1(r, p) = 0 < 0.24 = u_1(r'_1, r_{-1}, p'_1, p_{-1}) - \ell(||p'_1 - p_1||)$.

We investigate whether it is possible to identify the relation between stability and relocation-proofness in a general context. Of course, it is easy to check these two concepts are equivalent for the empty network. We extend the relation for the maximal network.

Proposition 10 *The maximal network is relocation-proof if and only if it is stable.*

Proof It is enough to show that a stable maximal network is relocation-proof. Suppose that $(r^{\max}, p)$ is stable, but not relocation-proof. That is, there exist $i \in N$, $r'_i \in \mathbb{R}_+$ and $p'_i \in \mathbb{R}^K$ such that $u_i(r'_i, r^{\max}_{-i}, p'_i, p_{-i}) - \ell(||p'_i - p_i||) > u_i(r^{\max}, p)$. As in Fig. 4, let $k \in N$ be such that $m(i, k; p'_i, p_{-i}) \leq r'_i$ and that for all $j \in N$ with $m(i, j; p'_i, p_{-i}) \leq r'_i$, $m(i, k; p) \geq m(i, j; p)$. Let $r''_i = m(i, k; p)$. Since other

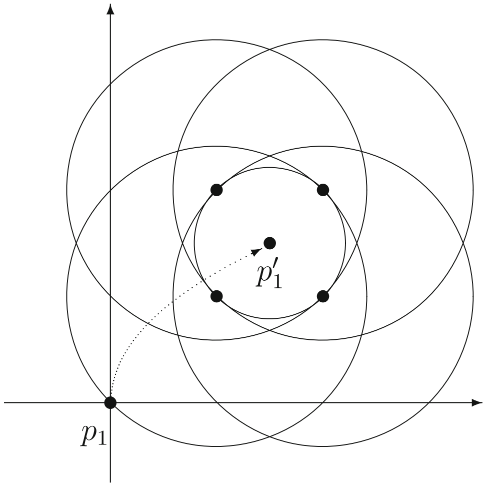

Fig. 3 A stable network may not be relocation-proof

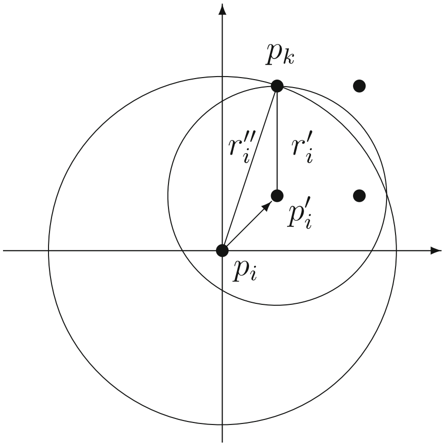

Fig. 4 How to choose k and r''_i in the proof of Proposition 10. Let $k \in N$ be an agent located within r'_i from p'_i but farthest from p_i, and $r''_i = m(i, k; p)$

agents' communication ranges are maximal and for all $j \in N$, $m(i, j; p'_i, p_{-i}) \leq r'_i$ implies $m(i, j; p) \leq r''_i$, we obtain $g(r'_i, r^{\max}_{-i}, p'_i, p_{-i}) \subseteq g(r''_i, r^{\max}_{-i}, p)$. Therefore,

$$\sum_{j \neq i} b(d(i, j; r''_i, r^{\max}_{-i}, p)) \geq \sum_{j \neq i} b(d(i, j; r'_i, r^{\max}_{-i}, p'_i, p_{-i})). \quad (1)$$

On the other hand, by the triangle inequality,

$$\begin{aligned} r'_i + ||p'_i - p_i|| &\geq m(i, k; p'_i, p_{-i}) + ||p'_i - p_i|| \\ &= ||p_k - p'_i|| + ||p'_i - p_i|| \\ &\geq ||p_k - p_i|| \\ &= m(i, k; p) \\ &= r''_i. \end{aligned}$$

By concavity of c, $c(r_i') + c(||p_i' - p_i||) \geq c(r_i'')$, which implies that $c(r_i') + \ell(||p_i' - p_i||) \geq c(r_i'')$. Together with (1),

$$\begin{aligned} u_i(r_i'', r_{-i}^{\max}, p) &= \sum_{j \neq i} b(d(i, j; r_i'', r_{-i}^{\max}, p)) - c(r_i'') \\ &\geq \sum_{j \neq i} b(d(i, j; r_i', r_{-i}^{\max}, p_i', p_{-i})) - c(r_i') - \ell(||p_i' - p_i||) \\ &= u_i(r_i', r_{-i}^{\max}, p_i', p_{-i}) - \ell(||p_i' - p_i||), \end{aligned}$$

which contradicts to the stability of $(r^{\max}, p)$. □

In the uniform interval model, from Propositions 2 and 10, if $b(1) - b(2) \geq c(\frac{1}{n-1})$, the maximal network is relocation-proof. Also, in the uniform circle model, from Propositions 5 and 10, if $2b(1) - 2b(2) \geq c(\rho^1)$, the maximal network is relocation-proof.

For the minimal network, we can establish its relocation-proofness only for the two specific models.

Proposition 11 *In the uniform interval model, if* $\sum_{d=1}^{n-1} b(d) \geq c(\frac{1}{n-1})$ *and* $\ell(\frac{1}{n-1}) - c(\frac{1}{n-1}) \geq b(1) - b(2) + (n-3)(b(2) - b(n-1))$*, then* $(r^{\min}, p^-)$ *is relocation-proof.*

Proof Let $(r^{\min}, p^-)$ be given. Suppose that agent i moves to $p_i' \in \mathbb{R}$ and sets $r_i' \in \mathbb{R}_+$. We divide into two cases.

Case 1 $||p_i' - p_i^-|| < \frac{1}{n-1}$. If $r_i' < \min_{j \neq i} m(i, j; p_i', p_{-i}^-)$, agent i incurs costs only. Since $\sum_{d=1}^{n-1} b(d) \geq c(\frac{1}{n-1})$, he cannot be better off. If $r_i' \geq \min_{j \neq i} m(i, j; p_i', p_{-i}^-)$, then $g(r_i', r_{-i}^{\min}, p_i', p_{-i}^-) \subseteq g(r^{\min}, p^-)$, so that agent i's benefit does not increase. Moreover, moving i's position by $||p_i' - p_i^-||$ is at least as costly as adjusting his communication range by the same length. Therefore, agent i cannot be better off.

Case 2 $||p_i' - p_i^-|| \geq \frac{1}{n-1}$. Then, the benefit increment is at most $b(1) - b(2) + (n-3)(b(2) - b(n-1))$ whichever $r_i' \in \mathbb{R}_+$ agent i chooses. On the other hand, the cost increment is at least $\ell\left(\frac{1}{n-1}\right) - c\left(\frac{1}{n-1}\right)$. Since $\ell\left(\frac{1}{n-1}\right) - c\left(\frac{1}{n-1}\right) \geq b(1) - b(2) + (n-3)(b(2) - b(n-1))$, agent i cannot be better off.

Altogether, $(r^{\min}, p^-)$ is relocation-proof. □

We state without proof a related result for the uniform circle model.

Proposition 12 *In the uniform circle model with at least seven agents,*[3] *if* $2\sum_{d=1}^{(n-1)/2} b(d) \geq c(\rho^1)$ *and* $\ell(\rho^1) - c(\rho^1) \geq b(1) - b(2) + (n-3)\left(b(2) - b\left(\frac{n-1}{2}\right)\right)$*, then* $(r^{\min}, p^o)$ *is relocation-proof.*

[3] With at least seven agents, ρ^1 is less than the radius for the uniform circle model.

6 Concluding remarks

This paper presents a model of wireless communication networks. In the general model, we observe the tension between efficiency and stability as in Jackson and Wolinsky (1996). We partially resolve the tension in two ways: by appropriately specifying agents' positions and by restricting the cost and benefit of communications. We generalize the model by allowing agents to relocate their positions and show that relocation-proofness is equivalent to stability for the maximal and the empty networks.

Alternatively, we can analyze the model by taking a cooperative game theoretic approach: first, define the worth of each coalition using a value function and then, consider its distribution using an allocation rule. This approach can be applied to our model since the communication benefit is generated by an individual decision through mutual communications. In our work in progress, we study such variations of wireless communication networks.

Another interesting question is to analyze a directed model of wireless communication networks. In our model, the benefits are generated only by mutual communications. However, for a model of strategic advertising and broadcasting, it is reasonable to assume that the benefits are generated by unilateral access to other agents. We hope to address the issue in our future research.

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