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Coalitionally strategy-proof social choice correspondences and the Pareto rule

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Abstract This paper studies coalitional strategy-proofness of social choice correspondences that map preference profiles into sets of alternatives. In particular, we focus on the Pareto rule, which associates the set of Pareto optimal alternatives with each preference profile, and examine whether or not there is a necessary connection between coalitional strategy-proofness and Pareto optimality. The definition of coalitional strategy-proofness is given on the basis of a max–min criterion. We show that the Pareto rule is coalitionally strategy-proof in this sense. Moreover, we prove that given an arbitrary social choice correspondence satisfying the coalitional strategy-proofness and nonimposition, all alternatives selected by the correspondence are Pareto optimal. These two results imply that the Pareto rule is the maximal correspondence in the class of coalitionally strategy-proof and nonimposed social choice correspondences.

1 Introduction

Gibbard (1973) and Satterthwaite (1975) give an influential result on a social choice function (SCF): every strategy-proof SCF for which the range has at least three elements is dictatorial. While an SCF associates an alternative with each preference profile, a social choice correspondence (SCC) associates a subset of alternatives with each preference profile. The question of whether there exists a strategy-proof social choice rule naturally arises also for SCCs. Unlike for SCFs, there exists a nondictatorial and strategy-proof SCC defined over an unrestricted domain. In fact, the Pareto rule, which assigns the set of Pareto optimal alternatives to each preference profile

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is strategy-proof in some senses (see Bandyopadhyay 1983c; Feldman 1979a; Kelly 1977; Nehring 2000; Rodriguez-Álvarez 2007).

This paper treats *coalitionally* strategy-proof SCCs with an unrestricted domain. In particular, we study to what extent the Pareto rule is coalitionally strategy-proof. We also investigate whether alternatives generated by a coalitionally strategy-proof SCC are necessarily Paretian.

While the definition of strategy-proofness for SCFs is not controversial in the literature, this is not the case for SCCs. There are many different notions of strategy-proofness for SCCs, because the outcome is allowed to be multi-valued. The notion of coalitional manipulation of SCCs is usually developed in the following two intuitive steps.[1] Let N and A denote a set of individuals and a set of alternatives, respectively. Step 1: For each individual i, one specifies a function f_i which for every ordering $\succsim_i$ of i, specifies exactly one ordering $\trianglerighteq_i$ over the different non-empty subsets of A. Step 2: One says that an SCC C is manipulable by a coalition S at a preference profile $\succsim$ iff there exists a preference profile $\succsim'$ such that $\succsim_{N\setminus S}=\succsim'_{N\setminus S}$ and for all $i \in S$, $C(\succsim') \vartriangleright_i C(\succsim)$, where for each $i \in S$, $\vartriangleright_i$ is the asymmetric factor of $\trianglerighteq_i = f_i(\succsim_i)$. Several definitions of coalitional manipulation arises from the fact that there are several ways of defining f_i in Step 1. For example, coalitional manipulability in a max–min sense can be captured in this two-step framework by simply specifying that for every individual $i \in N$, f_i is the max–min rule, so that for all non-empty subsets A' and A'' of A and for every possible ordering $\succsim_i$ of i, $A' \trianglerighteq_i A''$ iff a worst alternative in A' under $\succsim_i$ is at least as good as a worst alternative in A'' under $\succsim_i$.

There are a few approaches to defining strategy-proofness of SCCs, even if the definition is for individuals as well as for coalitions. One approach is to make explicit references to expected utilities, where probability measures over alternatives are given (see, e.g., Barberà et al. 2001; Benoît 2002; Ching and Zhou 2002; Duggan and Schwartz 2000; Feldman 1979a, 1980; Rodriguez-Álvarez 2007). Another approach is to define strategy-proofness in a non-probabilistic framework. Individuals are assumed to evaluate sets of alternatives based on their preference orders over alternatives by focusing on the best and/or the worst alternative in the sets (see, e.g., Bandyopadhyay 1983a,b,c,d; Barberà 1977a,b; Benoît 2002; Campbell and Kelly 2000; Demange 1987; Feldman 1979a,b; Gärdenfors 1976; Kelly 1977; Nehring 2000; Nishino et al. 1999; Pattanaik 1973, 1974, 1975, 1976; Rodriguez-Álvarez 2007, 2009; Sato 2008). This paper takes the latter approach and follows the line of research by Pattanaik (1973, 1974), Bandyopadhyay (1983a,b,c,d), and Nishino et al. (1999) that deal with coalitional non-manipulability of the multivalued social choice rules in a non-probabilistic framework.

The objective of this paper is twofold. Our first objective is to examine in what senses the Pareto rule is coalitionally strategy-proof. Many people find Pareto optimality a desirable property of socially chosen alternatives. We explore properties of the Pareto rule. While notions of manipulation based on a max–min sense are widely discussed, see, e.g., pioneering works by Pattanaik (1973, 1974, 1975, 1976), Bandyopadhyay (1983c) has shown that the Pareto rule is coalitionally strategy-proof

[1] This two-step framework is due to an anonymous referee.

in a max–max sense but *not* in a max–min sense. A coalition S can manipulate an outcome[2] in a max–min sense if every member of S prefers the worst alternative for him in the "false" outcome obtained after a deviation rather than the worst alternative for him in the "true" outcome. This paper introduces a new notion of coalitional strategy-proofness in a max–min sense, which is weaker than Bandyopadhyay (1983c), and shows that the Pareto rule is coalitionally strategy-proof in this sense. Suppose that there is a focal alternative in the "true" outcome that everyone in coalition S agrees to eliminate. Note that on a max–min criterion, everyone cares only the worst alternative in an outcome. So, for this agreement to be meaningful, the "false" outcome should not include any alternative worse than the focal alternative for any member in S. Informally speaking, we say that "a coalition S can manipulate the outcome in a strong max–min sense" if there is a deviation that yields a false outcome as described above. It should be noted that the notion of coalitional manipulation in a strong max–min sense is not developed based on the standard two-step framework described above. This new notion of coalitional strategy-proofness parallels that of Demange (1987). She considers non-manipulation by optimistic people while our notion is based on pessimism. Our result extends a result for the individual strategy-proofness in a max–min sense by Feldman (1979a) to the coalitional strategy-proofness.

Our second objective is to investigate the relation between coalitional strategy-proofness and Pareto optimality. "Nonimposition" is a natural requirement that every alternative can be an outcome of an SCC as a singleton. This assumption rules out trivial SCCs. Consider the following question: if an SCC is coalitionally strategy-proof and nonimposed, is it contained in the Pareto rule? Under a max–min definition, the answer to the above question is "yes". Thus, non-manipulability implies Pareto optimality. This together with our first result in turn implies that the Pareto rule is the maximal coalitionally strategy-proof SCC. We see by means of an example that if the max–max criterion is used for the definition of manipulability, then the argument above is invalid. Also, we confirm through examples that coalitional strategy-proofness under the max–min criteria and nonimposition are independent.

This paper is organized as follows. In Sect. 2, we formally describe notation and definitions for our social choice problem. In Sect. 3, we investigate the coalitional strategy-proofness of the Pareto rule. In Sect. 4, we examine the relation between the coalitionally strategy-proof SCCs and the Pareto rule. In Sect. 5, we give some concluding remarks.

2 Notation and definitions

Let A be a finite set of alternatives ($|A| \geq 2$), and $N = \{1, \ldots, n\}$ ($n \geq 2$) a set of individuals in society. Each individual $i \in N$ has a *preference order* on A, which is denoted by $\succsim_i$. For all $i \in N$, $\succsim_i$ is a binary relation on A satisfying completeness ($a \succsim_i b$ or $b \succsim_i a$ holds for any pair of alternatives $a, b \in A$) and transitivity (if $a \succsim_i b$ and $b \succsim_i c$, then $a \succsim_i c$ holds for all $a, b, c \in A$). For all $i \in N$, $\succ_i$ and

[2] We are using the term "outcome" to mean the set of chosen alternatives specified by the SCC for the preference profile under consideration.

$\sim_i$ are respectively the asymmetric (strict preference) and symmetric (indifference) components of $\succsim_i$. Let $\mathcal{R}$ denote the set of all preference orders on A. Let us denote by $\succsim = (\succsim_1, \succsim_2, \ldots, \succsim_n)$ a *preference profile* of N and by $\mathcal{R}^n$ the set of all preference profiles. A social choice correspondence (SCC) C associates a non-empty subset of A with each preference profile $\succsim \in \mathcal{R}^n$, i.e., $C : \mathcal{R}^n \to 2^A \setminus \{\emptyset\}$.

An alternative a is *Pareto optimal* if there exists no alternative b such that $b \succsim_i a$ for all $i \in N$ and $b \succ_j a$ for some $j \in N$. Given any $\succsim \in \mathcal{R}^n$, the set of all Pareto optimal alternatives is denoted by $\mathrm{PAR}(\succsim)$. The *Pareto rule* is an SCC, C, such that, for all $\succsim \in \mathcal{R}^n$, $C(\succsim) = \mathrm{PAR}(\succsim)$.

Three types of definitions of strategy-proofness are given below.

Definition 1 (max–max CSP) We say that an SCC C is *coalitionally manipulable in a max–max sense* by $S \subseteq N$ at $\succsim \in \mathcal{R}^n$ if there exists a profile $\succsim' = (\succsim_{N\setminus S}, \succsim'_S) \in \mathcal{R}^n$ such that for all $i \in S$, [for some alternative $a' \in C(\succsim')$, $a' \succ_i a$ for all $a \in C(\succsim)$]. When C is not coalitionally manipulable in a max–max sense by any $S \subseteq N$ at any $\succsim \in \mathcal{R}^n$, it is *coalitionally strategy-proof in a max–max sense* (max–max CSP).

Definition 2 (max–min CSP) We say that an SCC C is *coalitionally manipulable in a max–min sense* by $S \subseteq N$ at $\succsim \in \mathcal{R}^n$ if there exists a profile $\succsim' = (\succsim_{N\setminus S}, \succsim'_S) \in \mathcal{R}^n$ such that for all $i \in S$, [for some alternative $a \in C(\succsim)$, $a' \succ_i a$ for all $a' \in C(\succsim')$]. When C is not coalitionally manipulable in a max–min sense by any $S \subseteq N$ at any $\succsim \in \mathcal{R}^n$, it is *coalitionally strategy-proof in a max–min sense* (max–min CSP).

Definition 3 (s-max-min CSP) We say that an SCC C is *coalitionally manipulable in a strong max–min sense* by $S \subseteq N$ at $\succsim \in \mathcal{R}^n$ if there exist a profile $\succsim' = (\succsim_{N\setminus S}, \succsim'_S) \in \mathcal{R}^n$ and an alternative $a \in C(\succsim)$ such that for all $i \in S$, $a' \succ_i a$ for all $a' \in C(\succsim')$. When C is not coalitionally manipulable in a strong max–min sense by any $S \subseteq N$ at any $\succsim \in \mathcal{R}^n$, it is *coalitionally strategy-proof in a strong max–min sense* (s-max-min CSP).

A coalition manipulates an SCC at a profile if its members, by misrepresenting their preferences, can secure a set of alternatives that they prefer to the set of alternatives chosen when they are honest. Definitions 1 and 2 are taken from Bandyopadhyay (1983c). Under Definition 1, if a set of alternatives includes the highest-ranked alternative for each member of coalition, then it is a most preferred outcome for the coalition. On the other hand, according to Definition 2, a set of alternatives is evaluated by focusing on the worst alternative in the set. Thus, among all the subsets of A, the most preferred sets for an individual are the sets including only the alternatives highest-ranked by the individual. Pattanaik (1973, 1974) consider a refinement of the max–min criterion. S-max-min CSP in Definition 3 is a pessimistic version of the coalitional strategy-proofness discussed by Demange (1987). In Definition 3, we say that a coalition S can manipulate an outcome if there is a focal alternative a in the "true" outcome and all members of S prefer every alternative in the "false" outcome obtained after their deviation to a. The coalition has a *consensus* on which alternative to be eliminated based on a max–min criterion.

Other definitions akin to Definitions 1, 2, and 3 are possible. Gärdenfors (1979) gives a survey of various definitions of strategy-proofness.

3 Coalitional strategy-proofness of the Pareto rule

In this section, we investigate coalitional strategy-proofness of the Pareto rule, PAR$(\cdot)$. Bandyopadhyay (1983c) shows that the Pareto rule is max–max CSP, but is *not* max–min CSP. We show that the Pareto rule is well-behaved to the extent that it is s-max-min CSP, which is a weak version of max–min CSP.

Theorem 1 *PAR$(\cdot)$ is s-max-min CSP.*

Proof Suppose by contradiction that PAR$(\cdot)$ is not s-max-min CSP at some $\succsim \in \mathcal{R}^n$. There are a coalition $S \subseteq N$, a preference profile $\succsim'_S \in \mathcal{R}^s$, and an alternative $a \in$ PAR$(\succsim)$ such that

$$b \succ_i a \quad \text{for all } b \in \text{PAR}(\succsim') \text{ and all } i \in S, \tag{1}$$

where $\succsim' = (\succsim_{N\setminus S}, \succsim'_S)$. Since $a \notin$ PAR$(\succsim')$, there is an alternative $c \in$ PAR$(\succsim')$ such that $c \succsim'_i a$ for all $i \in N$ and $c \succ'_j a$ for some $j \in N$. This implies that

$$c \succsim_i a \quad \text{for all } i \in N\setminus S, \tag{2}$$

because $\succsim'_i = \succsim_i$ for $i \in N\setminus S$. Since $c \in$ PAR$(\succsim')$,

$$c \succ_i a \quad \text{for all } i \in S \tag{3}$$

by (1). (2) and (3), however, contradict $a \in$ PAR$(\succsim)$. □

Feldman (1979a) proves that the Pareto rule is individually strategy-proof in a max–min sense. Theorem 1 gives a coalitional version of his result. The max–min criterion is too strong for PAR to be coalitionally strategy-proof, as shown in Bandyopadhyay (1983c). Theorem 1, however, shows that PAR behaves moderately well in the sense of s-max-min CSP.

4 Pareto property of coalitionally strategy-proof SCCs

In this section, we consider the following question: given an arbitrary SCC that is coalitionally strategy-proof, how is it related to the Pareto rule? This parallels the question of Feldman (1979a) for individually strategy-proof SCCs.

We assume hereafter that individual preferences are strict. Specifically, preference orders are antisymmetric (for all $a, b \in A$, $a \succsim b$ and $b \succsim a$ imply that $b = a$). We denote strict preference by $\succ$. The binary relation $\succ$ is called a *linear order* (for a discussion of linear orders, see Peleg 1984). Let us denote by $\mathcal{L}$ the set of all linear orders on A and by $\mathcal{L}^n$ its n-ary Cartesian product. Let $\bar{a}(\succ_i)$ represents the best alternative among A under $\succ_i$. Similarly, let us denote by $\underline{a}(\succ_i)$ the worst alternative among A under $\succ_i$.

Also, in order to avoid trivial SCCs, we make an assumption of "nonimposition" for SCCs. This says that the range of an SCC includes all singleton sets of alternatives. It is a weak form of voters' sovereignty.

Definition 4 (NI) An SCC C is *nonimposed* (NI) if for every $a \in A$, there exists $\succ \in \mathcal{L}^n$ such that $C(\succ) = \{a\}$.

Given a coalitionally strategy-proof and NI SCC C, we ask whether or not

$$C(\succ) \subseteq \text{PAR}(\succ) \text{ holds for all } \succ \in \mathcal{L}^n. \tag{4}$$

Property (4) implies that all alternatives chosen by a coalitionally strategy-proof and NI SCC are Pareto optimal.

We show that (4) is true when coalitional strategy-proofness is defined in the s-max-min sense, which implies that it is true in the max–min sense. Feldman (1979a) states that (4) holds for $|N| \leq 3$ and $|A| \leq 3$ for the "individual" version of strategy-proofness in Definition 3. We prove a coalitional version of his statement for general $|N|$ and $|A|$.

Theorem 2 *Suppose that an SCC C is s-max-min CSP and NI. Then,*

$$C(\succ) \subseteq PAR(\succ) \quad \textit{for all } \succ \in \mathcal{L}^n.$$

Proof To the contrary, suppose that there exist a preference profile $\succ \in \mathcal{L}^n$ and an alternative $a \in A$ such that

$$a \in C(\succ) \tag{5}$$

and

$$a \notin \text{PAR}(\succ). \tag{6}$$

Equation (6) implies that there exists an alternative $b \in A$ such that

$$b \succ_i a \quad \text{for all } i \in N. \tag{7}$$

Since C satisfies NI, there exists a profile $\succ^b \in \mathcal{L}^n$ such that

$$C(\succ^b) = \{b\}. \tag{8}$$

Thus, from (5), (7), and (8), C is coalitionally manipulable in a strong max–min sense by N at $\succ$, which is a contradiction. Therefore, for all $\succ \in \mathcal{L}^n$, $a \in C(\succ)$ implies $a \in \text{PAR}(\succ)$. □

When individual preferences are not strict, Theorem 2 does not hold. A simple counterexample is the indeterminate dictator rule (see, e.g., Nehring 2000) which assigns to each preference profile the set of alternatives best for at least one of the individuals. This rule is s-max-min CSP and NI. For example, consider the following preference profile $\succsim \in \mathcal{R}^n$ such that $a_1 \succ_1 a_2 \succsim_1 a$ for all $a \in A \backslash \{a_1, a_2\}$, and $a_1 \sim_i a_2 \succ_i a$ for all $a \in A \backslash \{a_1, a_2\}$ and all individuals $i \neq 1$. The indeterminate dictator rule assigns $\{a_1, a_2\}$ to this profile. However, $\text{PAR}(\succsim) = \{a_1\}$.

Corollary 1 *Suppose that an SCC C is max–min CSP and NI. Then,*

$$C(\succ) \subseteq PAR(\succ) \quad \textit{for all } \succ \in \mathcal{L}^n.$$

Property (4) fails for a max–max CSP SCC, as shown in the next example.

Example 1 Let $N = \{1, 2\}$ and $A = \{a_1, a_2, a_3\}$. Define an SCC C as follows:

$$C(\succ) = \begin{cases} \{a_1, a_2\} & \text{for } \succ \in \mathcal{L}^n \text{ such that } a_1 \succ_1 a_2 \succ_1 a_3 \\ \{\bar{a}(\succ_1)\} & \text{otherwise} \end{cases}.$$

In this example, C is max–max CSP and NI. However, if $a_1 \succ_i a_2 \succ_i a_3$ for all $i \in N$, then $C(\succ) = \{a_1, a_2\}$ and $\text{PAR}(\succ) = \{a_1\}$, which violates (4).

Next, we confirm that the key properties used so far are independent.

Proposition 1 (1) *S-max-min CSP and NI are independent.* (2) *Max–min CSP and NI are independent.*

Proof We observe through examples that the above proposition holds. Each example below indicates that one property does not imply the other property.

Consider the SCC C that assigns to each preference profile the set A of alternatives. It is easily verified that C is max–min CSP. Therefore, C is s-max-min CSP. However, C is not NI.

Consider Example 1 again. The SCC C is NI, but it is not s-max-min CSP. For $\succ \in \mathcal{L}^n$ such that $a_1 \succ_1 a_2 \succ_1 a_3$, $C(\succ) = \{a_1, a_2\}$. However, if individual 1 misrepresents his preference to $a_1 \succ'_1 a_3 \succ'_1 a_2$, we obtain $C(\succ') = \{a_1\}$, where $\succ' = (\succ'_1, \succ_{N\setminus\{1\}})$. Thus, C is manipulable in a strong max–min sense by individual 1 at $\succ$. Hence, C is not s-max-min CSP. Thus, C is not max–min CSP. □

It is clear that PAR is NI. Therefore, it follows from Theorems 1 and 2 that:

Theorem 3 *$PAR(\cdot)$ is the maximal correspondence satisfying s-max-min CSP and NI.*

5 Concluding remarks

This paper derives a possibility result on coalitionally strategy-proof SCCs: there exists a nondictatorial SCC satisfying coalitional strategy-proofness. This provides one way to overcome the negative conclusion of Gibbard–Satterthwaite theorem.

The set of all Pareto optimal alternatives is shown to be the maximal correspondence in the class of the SCCs satisfying coalitional strategy-proofness in the s-max-min sense and nonimposition. Selectivity is an important attribute of SCCs. The characterization of minimal nondictatorial coalitionally strategy-proof SCCs in the class is an open problem.

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