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Networks and outside options

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Abstract We introduce and characterize a component efficient value for TU games with a cooperation structure which in contrast to the Myerson (Math Oper Res 2:225–229, 1977) value accounts for outside options. It is based on the idea that the distribution of the worth within a component should be consistent with some connected graph which reflects the outside options of the component's players.

1 Introduction

One right-glove holder, R, and one left-glove holder, ℓ, actually sell their pair of gloves which is worth 1 via some agent, $A1$. How should R, ℓ, and $A1$ split the proceeds? Would this split change if there were a second agent $A2$? In order to answer this kind of questions, Myerson (1977), Borm et al. (1992), and Hamiache (1999) consider values for TU games with a cooperation structure, i.e., an undirected graph on the player set (henceforth, CO-games and CO-values). In the following, we focus on the Myerson value as the most eminent one.

Our leading example then corresponds to a TU game with 3 (or 4) players, R, ℓ, $A1$, (and $A2$), where the worth of a coalition is 1 if it contains a matching pair, i.e., the players R and ℓ, and is 0 if it does not so. The fact that R and ℓ sell their pair via $A1$ then can be modelled by the following graphs:

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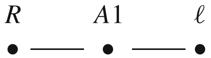
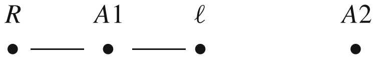


(1.1)



(1.2)

In both cases, the Myerson value μ assigns the same payoffs to R, ℓ, and $A1$, $\mu_R = \mu_\ell = \mu_{A1} = \frac{1}{3}$. Though A is a Null player, he obtains a positive payoff, what fits nicely with our intuitions on his role in this transaction—he actually facilitates the sale. Yet, a bit unintuitively, the share of $A1$ is not affected by the presence of the *potential* competitor $A2$. Even if R and ℓ agreed to sell their pair of gloves via $A1$, as modelled by (1.1) or (1.2), they could use the presence of $A2$ as an argument to lower $A1$'s share. Thus, the Myerson value does not account for the "outside option" of R and ℓ to sell their pair of gloves via $A2$. Outside options, however, may be important (von Neumann and Morgenstern 1944, p. 36 ; Maschler 1992, p. 595).

The Myerson value as well as the values considered by Borm et al. (1992) and Hamiache (1999) share this neglect of outside options with the Aumann and Drèze (1974) value (henceforth AD-value) for TU-games with a coalition structure, i.e., a partition of the player set (henceforth CS-games). Recently, Wiese (2007) and Casajus (2008) introduce the outside-option value and the χ-value in order to remedy this peculiarity of the AD-value. Experiments within the framework of gloves games provide some support for the predictive power of the χ-value (Pfau 2007). Hence, it seems to be worthwhile to look for a CO-value which generalizes this concept.

In this paper, we introduce and axiomatize the "graph-χ-value", $\chi^\#$, which extends the χ-value to CO-games and thus accounts for outside options. To achieve this, we restrict the Myerson fairness axiom to situations without outside options or where outside options are not affected. An outside-option consistency axiom determines how players within the same component assess their outside options. It turns out that the $\chi^\#$-value coincides with χ-value for internal completely connected components. For our leading example, we obtain the following payoffs: If $A2$ is not present then the payoffs are as for the Myerson value. But in presence of $A2$, the payoff of $A1$ decreases. In particular, we have $\chi^\#_R = \chi^\#_\ell = \frac{4}{9}$ and $\chi^\#_{A1} = \frac{1}{9}$, which indicates that the $\chi^\#$-value rewards outside options without neglecting the role of player $A1$.

The plan of this paper is as follows: The next section provides basic definitions and notation. In the third section, we discuss several axioms for CO-games with respect to outside options. The $\chi^\#$-value is introduced and axiomatized in the fourth section. The fifth section demonstrates the difference between $\chi^\#$-value and the Myerson value concerning the stability of networks. Some remarks conclude the paper.

2 Basic definitions and notation

Let $\mathcal{U}$ be an infinite set. A (TU) game is a pair (N, v) consisting of a non-empty and finite set $N \subseteq \mathcal{U}$, the player set, and a function $v \in V(N) := \{f : 2^N \to \mathbb{R} | v(\emptyset) = 0\}$, the coalition function. In general, we consider the set of all TU games, possibly equipped with some additional structure. Subsets of N are called coalitions; $v(K)$ is called the worth of $K \subseteq N$. For $v \in V(N)$, $v|_{N'}$ denotes the restriction of v to $N' \subseteq N$. For $\emptyset \neq T \subseteq N$, set (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise. A game is called superadditive iff $v(K \cup K') \geq v(K) + v(K')$ for all $K, K' \subseteq N$ such that $K \cap K' = \emptyset$. A player i is called a Null player iff $v(K) = v(K \cup \{i\})$ for all $K \subseteq N \backslash \{i\}$.

A value φ assigns payoff vectors $\varphi(N, v) \in \mathbb{R}^N$ to all games. Set $\Sigma(N) := \{\sigma : N \to \{1, \ldots, |N|\} \,|\sigma \text{is bijective}\}$. For $i \in N$ and $\sigma \in \Sigma(N)$, let $K_i(\sigma) := \{j \in N | \sigma(j) \leq \sigma(i)\}$ and $MC_i(\sigma, v) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley (1953) value Sh is given by

$$\mathrm{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i(\sigma, v), \quad i \in N. \tag{2.1}$$

Let $\mathbb{P}(N)$ denote the set of all partitions of N. A coalition structure for (N, v) is some $\mathcal{P} \in \mathbb{P}(N)$; $\mathcal{P}(i)$ denotes the component containing player i; $\mathcal{P}' \in \mathbb{P}(N)$ is finer than $\mathcal{P} \in \mathbb{P}(N)$ if $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ for all $i \in N$. A CS-game is a triple $(N, v, \mathcal{P})$, where (N, v) is a game and $\mathcal{P} \in \mathbb{P}(N)$; a CS-value φ assigns payoff vectors $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$ to all CS-games. For $K \subseteq N$, we denote $\sum_{i \in K} \varphi_i(N, v, \cdot)$ by $\varphi_K(N, v, \cdot)$. The AD-value (Aumann and Drèze 1974) is given by $\mathrm{AD}_i(N, v, \mathcal{P}) = \mathrm{Sh}_i(\mathcal{P}(i), v|_{\mathcal{P}(i)})$, $i \in N$; the χ-value (Casajus 2008) is defined by

$$\chi_i(N, v, \mathcal{P}) := \mathrm{Sh}_i(N, v) + \frac{v(\mathcal{P}(i)) - \mathrm{Sh}_{\mathcal{P}(i)}(N, v)}{|\mathcal{P}(i)|}, \quad i \in N. \tag{2.2}$$

A cooperation structure for (N, v) is an undirected graph (N, L), $L \subseteq L^N := \{\{i, j\} \,| i, j \in N, i \neq j\}$; a typical element of L is written as ij; $L + ij := L \cup \{ij\}$, $L - ij := L \backslash \{ij\}$. Given any graph (N, L), N splits into (maximal connected) components the set of which is denoted by $\mathcal{C}(N, L) \in \mathbb{P}(N)$; $C_i(N, L) \in \mathcal{C}(N, L)$ denotes the component containing $i \in N$. Set $L|_S = \{\{i, j\} \in L | i, j \in S\}$, $S \subseteq N$. For $\mathcal{P} \in \mathbb{P}(N)$, set $L^{\mathcal{P}} := \bigcup_{P \in \mathcal{P}} L^P$, which implies $\mathcal{C}(N, L) = \mathcal{P}$. For $K, K' \subseteq N$, $K \cap K' = \emptyset$, we set $[K, K'] := \{ij | i \in K, j \in K'\}$. A CO-game is a triple (N, v, L), where (N, v) is a game and $L \subseteq L^N$; a CO-value assigns payoff vectors $\varphi(N, v, L) \in \mathbb{R}^N$ to all CO-games. The Myerson (1977) value μ is defined by

$$\mu(N, v, L) := \mathrm{Sh}\left(N, v^L\right), \quad v^L(K) := \sum_{S \in \mathcal{C}(K, L|_K)} v(S), \quad K \subseteq N. \tag{2.3}$$

3 Axioms for CO-values

In this section, we consider several axioms for CO-values with respect to outside options.

Axiom 3.1 (Component efficiency, CE). *For all* $C \in \mathcal{C}(N, L)$,

$$\varphi_C(N, v, L) = v(C).$$

Axiom 3.2 (Fairness, F). *For all* $ij \in L$, *we have*

$$\varphi_i(N, v, L) - \varphi_j(N, v, L) = \varphi_i(N, v, L - ij) - \varphi_j(N, v, L - ij).$$

CE and **F** are the original axioms that characterize the Myerson value. **CE** indicates that the connected components $C \in \mathcal{C}(N, L)$ are the productive units. The very nice fairness axiom **F** has strong consequences far beyond pure fairness considerations. In particular, van den Nouweland (1993, pp. 28–29) shows that μ satisfies the following axiom.

Axiom 3.3 (Component decomposability, CD). *For all* $i \in C \in \mathcal{C}(N, L)$,

$$\varphi_i(N, v, L) = \varphi_i(C, v|_C, L|_C).$$

Hence, the payoffs within a component $C \in \mathcal{C}(N, L)$ are not affected by the players outside, neither from their actual cooperation structure $L|_{N \backslash C}$ nor from the potential contributions of players in C to coalitions containing players from $N \backslash C$. Therefore, the Myerson value cannot account for outside options. It shares this property with the AD-value for CS-games. In fact, μ and AD coincide for completely connected components, i.e., $\mathrm{AD}(N, v, \mathcal{P}) = \mu\left(N, v, L^{\mathcal{P}}\right)$ (Myerson 1977).

Therefore, one could argue that **F** is too strong and one could think of restricting **F** to those situations were outside options are not involved: (i) Removing a link ij does not split a component, i.e., outside options do not change. (ii) ij is removed from a connected graph, i.e., from a cooperation structure which lacks outside options. This idea is captured by the following two axioms.

Axiom 3.4 (Weak fairness 1, WF1). *If* $j \in C_i(N, L - ij)$ *then*

$$\varphi_i(N, v, L) - \varphi_i(N, v, L - ij) = \varphi_j(N, v, L) - \varphi_j(N, v, L - ij).$$

Axiom 3.5 (Weak fairness 2, WF2) *If* L *is connected on* N *and* $ij \in L$ *then*

$$\begin{aligned} &\varphi_i(N, v, L) - \varphi_i\left(C_i(N, L - ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\ &\quad = \varphi_j(N, v, L) - \varphi_j\left(C_j(N, L - ij), v|_{C_j(N,L-ij)} L|_{C_j(N,L-ij)}\right). \end{aligned}$$

Slightly adapting the Myerson (1977) proof, one shows that the gap concerning uniqueness after relaxing **F** can be closed by invoking **CD**; more specifically, μ is characterized by **CE**, **CD**, and **WF2**. Of course, it is clear that **CD** must be weakened in order to make a CO-value sensitive to outside options. Consider some relaxation of **CD**, which is equivalent to **CD** in presence of **CE**: For all $i, j \in C \in \mathcal{C}(N, L)$,

$$\varphi_i(N, v, L) - \varphi_j(N, v, L) = \varphi_i(C, v|_C, L|_C) - \varphi_j(C, v|_C, L|_C). \qquad (3.1)$$

In (3.1), the payoff differences within C are determined via some reference game, the restriction to C. In the following, we suggest a modification of this reference game that allows for outside options.

We start from the splitting axiom, the central axiom of the characterization of the χ-value for CS-games (Casajus 2008), which determines how outside options are evaluated.

Axiom 3.6 (Splitting, SP). *If $\mathcal{P}'$ is finer than $\mathcal{P}$ then for all $i, j \in P \in \mathcal{P}'$, we have*

$$\varphi_i(N, v, \mathcal{P}) - \varphi_i\left(N, v, \mathcal{P}'\right) = \varphi_j(N, v, \mathcal{P}) - \varphi_j\left(N, v, \mathcal{P}'\right).$$

In essence, **SP** says that players of some component who decide to leave the component together should gain or lose to the same extent. This can be justified, for example, within the Hart and Kurz (1983) approach on coalition formation. In order to form a (separating) coalition, *all* players involved have to agree on forming this particular coalition. In this respect, these players are symmetric and should be treated equally. Compare with **F**: *All* players involved in the creation of the link ij, i and j, are affected in the same way; van den Nouweland et al. (1992) apply an axiom like **F** to hyperlinks which may contain more than two players.

Since all partitions are finer than $\{N\}$, **SP** is equivalent to

$$\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, v, \{N\}) - \varphi_j(N, v, \{N\}) \tag{3.2}$$

holding for all $i, j \in P \in \mathcal{P}$, which is closer to (3.1). In order to account for outside options, the χ-value employs the trivial partition $\{N\}$ as the reference partition. The crucial property of $\{N\}$ is that all players in $P \in \mathcal{P}$ are "connected" to all players in $N \backslash P$. The intended interpretation is that the players in a component P are allowed to refer to their productive potential within any coalition when they bargain on the distribution of $v(P)$.

In general, there are not only *productive outside options* in CO-games, i.e., non-zero marginal contributions to coalitions that contain players outside ones own component. In addition, even a Null player i from (N, v) (and therefore a player without productive outside options) may be a non-Null player in $\left(N, v^L\right)$ for some $L \subseteq L^N$. In particular, i may have non-zero marginal contributions in $\left(N, v^L\right)$ to coalitions containing outsiders. These potential gains/losses from linking outsiders may be called *linking outside options*. Fix $C \in \mathcal{C}(N, L)$. In order to reflect all productive outside options of $i \in C$, the player set of the reference game must be N. Further, it is quite intuitive to require that the reference graph R for player $i \in C$ keeps link structure within C, i.e., $R|_C = L|_C$. When the potential of $i \in C$ to link $j, k \in N \backslash C$ is to be modelled, one has to consider the links ij and ik. For symmetry and comprehensiveness reasons, one thus has $[C, N \backslash C] \subseteq R$. Since

$$L(i, N) := L|_C \cup [C, N \backslash C] \tag{3.3}$$

is connected, the reference game $(N, v, L(i, N))$ also embodies all productive outside options under a component efficient CO-value. It is clear that $L(i, N)$ is the minimal graph for such a reference game.

But what about the links within $N \backslash C$, in particular, those in $L|_{N \backslash C}$? If $j, k \in N \backslash C$ were linked in the reference graph, then this graph could not reflect the potential to

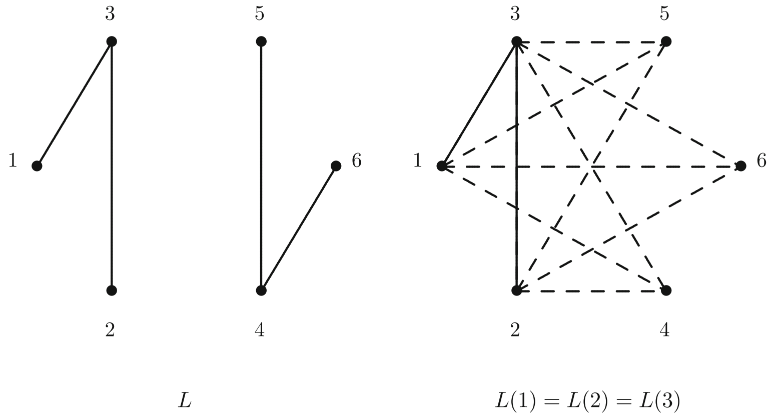


Fig. 1 A lower outside option graph

link j and k. Hence, $L(i, N)$ models the outside options of the players in C in the broadest sense. Moreover, one could argue that outside options come into play when the graph already has been formed and remains fixed. But then, the players are not necessarily restricted to refer to the actual link structure when they bargain on the distribution off $v(C)$. Finally, it turns out (see Lemma 4.5 and Theorem 4.6) that it does not matter whether one employs $L(i, N)$ which we call the *lower outside-option graph* (LOOG) or

$$L^+(i, N) := L|_C \cup [C, N\backslash C] \cup L^{N\backslash C} \tag{3.4}$$

which we call the *upper outside-option graph* (UOOG) or any graph between the LOOG and the UOOG. When there is no danger of confusion, we write $L(i)$ or $L^+(i)$.

Figure 1 provides an example of a LOOG. On the left side, we have the link set L of some graph. On the right one, the outside-option graph $L(1)$ for player 1 is given where the original links kept are drawn as solid lines while the additional links are drawn as dashed lines. The component of player 1 in L comprises the players in $\{1, 2, 3\}$. Player 1's outside-option graph hence does not contain the link 12 which is missing in L. Further, all original links among players outside 1's component, i.e., in the set $\{4, 5, 6\}$, have been removed. Finally, the players in $\{1, 2, 3\}$ are completely connected with those outside (dashed lines).

Since the LOOG seems to capture outside options in a broadest sense, i.e., the productive as well as the linking outside options with respect to all outsiders, we employ this graph in our axiomatization of the $\chi^{\#}$-value. Using the LOOG, the idea of (3.2) can be expressed for CO-games as follows.

Axiom 3.7 (Outside-option consistency, OO). *If* $i, j \in C \in \mathcal{C}(N, L)$ *then*

$$\varphi_i(N, v, L) - \varphi_j(N, v, L) = \varphi_i(N, v, L(i)) - \varphi_j(N, v, L(j)).$$

4 A χ-value for cooperation structure games

In this section, we show that some of the axioms advocated in the previous section already characterize a CO-value. The proof is prepared by two lemmas.

Lemma 4.1 **OO** *and* **WF2** *imply* **WF1**.

Proof For $j \in C_i(N, L - ij)$, (3.3) implies that $L(i) - ij = (L - ij)(i)$ is connected. We then have

$$\begin{aligned}\varphi_i(N, v, L) - \varphi_j(N, v, L) &\overset{\textbf{OO}}{=} \varphi_i(N, v, L(i)) - \varphi_j(N, v, L(i)) \\ &\overset{\textbf{WF2}}{=} \varphi_i(N, v, L(i) - ij) - \varphi_j(N, v, L(i) - ij) \\ &= \varphi_i(N, v, (L - ij)(i)) - \varphi_j(N, v, (L - ij)(i)) \\ &\overset{\textbf{OO}}{=} \varphi_i(N, v, L - ij) - \varphi_j(N, v, L - ij)\end{aligned}$$

which proves the claim. □

Lemma 4.2 *If φ satisfies* **CE**, **WF1**, *and* **WF2** *then it coincides with μ on all connected graphs.*

This result is in line with our intention to model outside options. Connected graphs lack outside options. Therefore, one could argue that all arguments in favor of μ apply in these situations.

Proof μ is characterized by **CE** and **F** where the latter strengthens **WF1** and together with **CD** implies **WF2**. We mimic the Myerson (1977) proof of uniqueness. Let φ and $\bar{\varphi}$ both satisfy **CE**, **WF1**, and **WF2**. Suppose N is a minimal player set such that φ and $\bar{\varphi}$ differ on a connected graph. By **CE**, N contains at least two players. Further, suppose L is a minimal connected graph on N such that $\varphi \neq \bar{\varphi}$. If $j \in C_i(N, L - ij)$ then **WF1** and the minimality of L imply $\varphi_i(N, v, L) - \varphi_j(N, v, L) = \bar{\varphi}_i(N, v, L) - \bar{\varphi}_j(N, v, L)$. And if $j \notin C_i(N, L - ij)$ then **WF2** and again the minimality of N imply $\varphi_i(N, v, L) - \varphi_j(N, v, L) = \bar{\varphi}_i(N, v, L) - \bar{\varphi}_j(N, v, L)$. Since L is connected, we have $\varphi_i(N, v, L) - \bar{\varphi}_i(N, v, L) = \Delta$ for some Δ and all $i \in N$. **CE** then implies $\Delta = 0$. Contradiction. □

Theorem 4.3 *There is a unique CO-value that satisfies* **CE**, **WF2**, *and* **OO**.

Proof Suppose φ satisfies **CE**, **WF2**, and **OO**. By **OO**, we have $L(i) = L(j)$ and

$$\varphi_i(N, v, L) - \varphi_i(N, v, L(i)) = \varphi_j(N, v, L) - \varphi_j(N, v, L(i))$$

for $i, j \in C \in \mathcal{C}(N, L)$. Summing up over C combined with **CE** gives

$$|C|(\varphi_i(N, v, L) - \varphi_i(N, v, L(i))) = v(C) - \varphi_C(N, v, L(j)).$$

Since $L(i)$ is connected, Lemmas 4.1 and 4.2 imply

$$\varphi_i(N, v, L) = \mu_i(N, v, L(i)) + \frac{v(C) - \mu_C(N, v, L(i))}{|C|}. \tag{4.1}$$

Hence, φ is unique.

By construction, the value given by (4.1) satisfies **CE**. If $C_i(N, L) = N$ then $L(i) = L(j) = L$ by (3.3), and therefore

$$\begin{aligned}
&\varphi_i(N, v, L) - \varphi_j(N, v, L) \\
&\quad = \mu_i(N, v, L) - \mu_j(N, v, L) \\
&\quad = \mu_i(N, v, L - ij) - \mu_j(N, v, L - ij) \\
&\quad = \mu_i\left(C_i(N, L - ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\
&\qquad - \mu_j\left(C_j(N, L - ij), v|_{C_j(N,L-ij)} L|_{C_j(N,L-ij)}\right) \\
&\quad = \varphi_i\left(C_i(N, L - ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\
&\qquad - \varphi_j\left(C_j(N, L - ij), v|_{C_j(N,L-ij)} L|_{C_j(N,L-ij)}\right)
\end{aligned}$$

by (4.1), μ satisfying **F**, μ satisfying **CD**, and again (4.1) together with $L|_{C_k(N,L-ij)}$ being connected on $C_k(N, L - ij)$, for $k = i, j$. Hence, φ satisfies **WF2**. If $j \in C_i(N, L)$ then $L(i) = L(j)$ by (3.3) and therefore

$$\begin{aligned}
\varphi_i(N, v, L) - \varphi_j(N, v, L) &= \mu_i(N, v, L(i)) - \mu_j(N, v, L(j)) \\
&= \varphi_i(N, v, L(i)) - \varphi_j(N, v, L(j))
\end{aligned}$$

by (4.1) and $L(i)$ being connected on N, and Lemma 4.2 which shows that φ satisfies **OO**. □

The CO-value defined by (4.1) is called "the graph-χ-value" and we denote it by $\chi^{\#}$, where the symbol # is intended to indicate a graph. It employs the Myerson value of the outside-option graph $L(i)$ as a yardstick to distribute the payoff *within* the component $C_i(N, L)$. The players within a component depart from their outside-option payoffs $\mu_i(N, v, L(i))$ and then compare the worth of the coalition with the sum of the outside-option payoffs; the difference, positive or negative, is distributed equally. By (4.1) and (3.3) or Lemma 4.2, $\chi^{\#}$ coincides with the Myerson value if a game lacks outside options, i.e., if the graph is connected.

It is easy to see that the characterization of $\chi^{\#}$ is non-redundant. μ satisfies **CE** and **WF2**, but not **OO**. Further, χ applied to $\mathcal{C}(N, L)$ satisfies **CE** and **OO**, but not **WF2**. Finally, the CO-value φ given by $\varphi_i(N, v, L) = \chi_i^{\#}(N, v, L) + 1$ obviously satisfies **WF2** and **OO**, but not **CE**.

In Sect. 3, we also suggest the UOOG. It is easy to check that Lemma 4.1 and 4.3 remain true if one replaces **OO** by **OO**$^+$ where $L(i)$ is replaced by $L^+(i)$. Denote the resulting CO-value by χ^+ which is given by replacing the $L(j)$ in (4.1) by $L^+(j)$. Interestingly, $\chi^{\#}$ and χ^+ coincide. We show this by proving that $\chi^{\#}$ and χ^+ satisfy the following strong version of **OO**.

Axiom 4.4 (Strong outside option consistency, SOO) *If* $i, j \in C \in \mathcal{C}(N, L)$, $L|_C = L'|_C$, *and* $[C, N \backslash C] \subseteq L'$ *then*

$$\varphi_i(N, v, L) - \varphi_j(N, v, L) = \varphi_i\left(N, v, L'\right) - \varphi_j\left(N, v, L'\right).$$

The outside-option graphs L' in **SOO** all coincide with the original graph on the component C, i.e., they express the same inside options as the original graph. Further, they all reflect comprehensive and symmetric productive outside options via the links in $[C, N\backslash C]$. The difference between two such outside-option graphs lies on the link set $L^{N\backslash C}$, the links in $N\backslash C$. The more links from $L^{N\backslash C}$ the graph L' contains the less linking outside option are modelled. If $L^{N\backslash C} \subseteq L'$, i.e., $L' = L^{+}(i)$, then any subset of $N\backslash C$ is connected internally. Hence, L' does not reflect linking outside options. Vice versa, if $L^{N\backslash C} \cap L' = \emptyset$, i.e., $L' = L(i)$, then the players in $N\backslash C$ are connected only via players in C, i.e., L' also reflects comprehensive linking outside options. Yet, as the following lemma and theorem reveal, this difference cancels out since **SOO** applies the same outside-option graph to both players involved.

Lemma 4.5 *$\chi^{\#}$ and χ^{+} satisfy* **SOO**.

Proof Consider $j \in C_i(N, L)$ and let L' be as in **SOO**. Since L' is connected, $\chi^{\#} = \mu$ on connected graphs, and by (4.1), the claim on $\chi^{\#}$ is equivalent to

$$\mu_i(N, v, L(i)) - \mu_j(N, v, L(i)) = \mu_i\left(N, v, L'\right) - \mu_j\left(N, v, L'\right) \tag{4.2}$$

By (3.3) and (3.4), replacing $L(i)$ by $L^{+}(i)$, etc., the proof also runs through for χ^{+}.

Consider $\sigma, \rho \in \Sigma(N)$ such that $\sigma(i) > \sigma(j), \sigma(i) = \rho(j), \sigma(j) = \rho(i)$, and $\sigma(k) = \rho(k)$ for all $k \in N\backslash\{i, j\}$. In order to show (4.2), by (2.1) and (2.3) it suffices to prove

$$\begin{aligned} &MC_i\left(\sigma, v^{L(i)}\right) + MC_i\left(\rho, v^{L(i)}\right) - \left(MC_j\left(\sigma, v^{L(i)}\right) + MC_j\left(\rho, v^{L(i)}\right)\right) \\ &\quad = MC_i\left(\sigma, v^{L'}\right) + MC_i\left(\rho, v^{L'}\right) - \left(MC_j\left(\sigma, v^{L'}\right) + MC_j\left(\rho, v^{L'}\right)\right). \end{aligned} \tag{4.3}$$

If $K \subseteq C_i(N, L)$ then $L(i)|_K = L|_K = L'|_K$ and if $j \in K \nsubseteq C_i(N, L)$ then $C(K, L(i)|_K) = \{K\} = C\left(K, L'|_K\right)$ by (3.3) and our assumption on L'. Hence by (2.3), $v^{L(i)}(K) = v^{L'}(K)$ for all $K \subseteq N$ such that $i \in K$ or $j \in K$. By our choice of σ and ρ, this already implies

$$MC_i\left(\sigma, v^{L(i)}\right) = MC_i\left(\sigma, v^{L'}\right) \quad \text{and} \quad MC_j\left(\rho, v^{L(i)}\right) = MC_j\left(\rho, v^{L'}\right). \tag{4.4}$$

Further, we have $K_i(\rho)\backslash\{i\} = K_j(\sigma)\backslash\{j\}$ and therefore

$$\begin{aligned} MC_i\left(\rho, v^{L(i)}\right) - MC_j\left(\sigma, v^{L(i)}\right) &= v^{L(i)}\left(K_i(\rho)\right) - v^{L(i)}\left(K_j(\sigma)\right) \\ &= v^{L'}\left(K_i(\rho)\right) - v^{L'}\left(K_j(\sigma)\right) \\ &= MC_i\left(\rho, v^{L'}\right) - MC_j\left(\sigma, v^{L'}\right) \end{aligned} \tag{4.5}$$

where the second equation follows from the arguments above. Then, (4.4) and (4.5) together imply (4.3). □

By Theorem 4.3 and since **SOO** implies both **OO** and **OO**$^{+}$, there is a unique CO-value that satisfies **CE**, **WF2**, and **SOO**. Since $\chi^{\#}$ and χ^{+} are characterized by

CE, **WF2**, and **OO** or **OO** $^+$, respectively, by Lemma 4.5, this CO-value coincides with $\chi^{\#} = \chi^{+}$.

Theorem 4.6 $\chi^{\#} = \chi^{+}$ *is the unique* CO-*value that satisfies* **CE**, **WF2**, *and* **SOO**.

If a connected component $C \in \mathcal{C}(N, L)$ is completely connected internally, i.e., $L|_C = L^C$, then $L^+(i) = L^N$ for $i \in C$ by (3.4). Since $\mu\left(N, v, L^N\right) = \text{Sh}(N, v)$ (Myerson 1977), Theorem 4.6, (4.1), and (2.2) imply the following Theorem where part (ii) is immediate from part (i). Since L^N is connected, we also have $\chi^{\#}\left(N, v, L^N\right) = \text{Sh}(N, v)$. Hence, $\chi^{\#}$ generalizes the Shapley value and the χ-value, the latter justifying its name "graph-χ-value".

Corollary 4.7 *(i) If* $i \in C \in \mathcal{C}(N, L)$, $L|_C = L^C$, *and* $\mathcal{P}(i) = C$ *then* $\chi^{\#}_i(N, v, L) = \chi_i(N, v, \mathcal{P})$. *(ii)* $\chi^{\#}\left(N, v, L^{\mathcal{P}}\right) = \chi(N, v, \mathcal{P})$.

5 Stable networks

Dutta et al. (1998) study network formation in superadditive TU games by the following network formation game (φ-NFG) which was formally introduced by Myerson (1991, p. 448). Given a TU game (N, v) and a CO-value φ, we consider the strategic form game Γ^{φ}. The player set is N and player $i \in N$ has the strategy set $S_i := \{s_i | s_i \subseteq N \backslash \{i\}\}$. Any strategy profile $s = (s_i)_{i \in N} \in S := \prod_{i \in N} S_i$ induces a graph $L(s) := \left\{ij \in L^N | i \in s_j \wedge j \in s_i\right\}$. The players' payoffs are given by φ, i.e., $u^{\varphi}_i(s) := \varphi_i(N, v, L(s))$. Dutta et al. (1998) consider a class of CO-values including μ and then apply some solution concepts to these games: the Nash equilibrium, the undominated Nash equilibrium (UNE), and the coalition-proof Nash equilibrium (CPNE). In order to illustrate the difference between the Myerson value and the $\chi^{\#}$-value, we focus on the Nash equilibrium and the CPNE.

Bernheim et al. (1987) define the CPNE inductively: For all $T \subseteq N$ and $s_{N \backslash T} \in S_{N \backslash T} := \prod_{i \in N \backslash T} S_i$, the game $\Gamma^{\varphi}\left(s_{N \backslash T}\right)$ consists of the player set T, the strategy sets S_i, $i \in T$, and the payoff functions $u^{\varphi}_i\left[s_{N \backslash T}\right]$, $i \in T$ where $u^{\varphi}_i\left[s_{N \backslash T}\right](s_T) := u^{\varphi}_i\left(s_T, s_{N \backslash T}\right)$ for all $i \in T$ and $s_T \in \prod_T S_i$. For $|N| = 1$, a strategy profile $s^* \in S$ is a CPNE if $u^{\varphi}_i\left(s^*_i\right)$ maximizes u^{φ}_i over S. For $|N| > 1$, a strategy profile $s^* \in S$ is called *self-enforcing* if for all $T \subsetneq N$, s^*_T is a CPNE of $\Gamma^{\varphi}(s^*_{N \backslash T})$. A strategy profile s^* is a CPNE if it is self-enforcing and if there is no self-enforcing strategy profile $s \in S$ such that $u^{\varphi}_i(s) > u^{\varphi}_i(s^*)$ for all $i \in N$.

Dutta et al. (1998, Proposition 1) show that any network can be supported by a Nash equilibrium of the μ-NFG. This may not be the case in the $\chi^{\#}$-NFG. Consider the game (N, u_T), $N = \{1, 2, 3\}$, $T = \{1, 2\}$. We then have $\chi^{\#}_3(N, u_T, \{23\}) = -\frac{1}{4}$. Yet, the Null player 3 can avoid the negative payoff by playing $s_3 = \emptyset$, which results in $C_3(N, L(s)) = \{3\}$ and $\chi^{\#}_3\left(N, u_{\{1,2\}}, L(s)\right) = 0$. Hence, the network $\{23\}$ cannot emerge in a Nash equilibrium of the $\chi^{\#}$-NFG, i.e., the Nash equilibrium already allows for predictions on which networks prevail.

In the example above, the Null player 3 obtains a negative payoff. On the one hand, the discussion of the Null player axiom in Casajus (2008, Sect. 3) reveals that this is unavoidable when a component efficient solution concept recognizes outside options.

Table 1 Payoffs for the example

L	$\mu(N, v, L)$	$\chi^{\#}(N, v, L)$
$\emptyset$	(0, 0, 0)	(0, 0, 0)
{12}	(18, 18, 0)	(24, 12, 0)
{13}	(12, 0, 12)	(21, 0, 3)
{23}	(0, 0, 0)	(0, 3, −3)
{12, 13}	(22, 10, 4)	(22, 10, 4)
{12, 23}	(18, 18, 0)	(18, 18, 0)
{13, 23}	(16, 4, 16)	(16, 4, 16)
L^N	(22, 10, 4)	(22, 10, 4)

Since any player i can enforce the singleton component $\{i\}$ by playing $s_i = \emptyset$, on the other hand, $v(\{i\})$ constitutes an upper bound for his payoffs in an equilibrium network.

Remark 5.1 If s^* is a Nash equilibrium of $\Gamma^{\chi^{\#}}$ on (N, v), then $\chi_i^{\#}(N, v, L(s^*)) \geq v(\{i\})$ for all $i \in N$.

Therefore, it is not too odd for a Null player to obtain a negative payoff in some networks—such networks do not evolve.

Stronger solution concepts, UNE and CPNE, yield more clear cut general results. In particular, Dutta et al. (1998, Theorems 1 and 2) show that the complete network may arise from a UNE or a CPNE of the μ-NFG. Moreover, any UNE or CPNE of the μ-NFG leads to the same payoffs as the complete network. The latter may not be the case in the $\chi^{\#}$-NFG.

The following example illuminates the difference between the Myerson value and the $\chi^{\#}$-value concerning network formation. Consider the TU game (N, v), $N = \{1, 2, 3\}$, $v(N) = v(\{1, 2\}) = 36$, $v(\{1, 3\}) = 24$, and else $v(S) = 0$. It is easy to check that this game is superadditive but not convex. Straightforward calculations give the payoffs listed in Table 1. For connected networks (the bottom four rows) the μ-payoffs and the $\chi^{\#}$-payoffs coincide since there are no outside options. The second to fourth row show that $\chi^{\#}$ accounts for outside options while μ does not so. The Myerson value equally splits the payoffs of any two-player coalition among its members. Yet, if just player 1 and 2 formed a link, for example, the $\chi^{\#}$-value rewards player 1's outside option to create the worth of 24 together with player 3—player 1 obtains a much higher payoff than player 2, $\chi_1^{\#} = 24 > 12 = \chi_2^{\#}$. Similar for the other one-link networks.

By Dutta et al. (1998, Theorem 2), the complete network L^N can be supported by the CPNE of the μ-NFG where all players wish to form all links. In our example, one easily checks that this is the unique such network. In the $\chi^{\#}$-NFG, besides the complete network, however, the network {12} is stable in this sense. Note that the resulting partition $\{\{1, 2\}, \{3\}\}$ is the unique χ-stable coalition structure (Casajus 2008).

In the following, we frequently refer to the $\chi^{\#}$-payoffs in Table 1 without mentioning this explicitly. The network {12} can be supported by the CPNE $s^* = (\{2\}, \{1\}, \emptyset)$. Obviously, there are no profitable one-player deviations. Thus, (s_1^*, s_2^*), (s_1^*, s_3^*), and (s_2^*, s_3^*) are self-enforcing in $\Gamma^{\chi^{\#}}(s_3^*)$, $\Gamma^{\chi^{\#}}(s_2^*)$, and $\Gamma^{\chi^{\#}}(s_1^*)$, respectively. In

$\Gamma^{\chi^{\#}}\left(s_1^*\right)$, there are two other self-enforcing strategy profiles involving the strategies $s_2 = \{1, 3\}$ and $s_3 = \{2\}$ or $s_3 = \{2, 3\}$. The resulting networks are $\{12\}$ and $\{12, 23\}$, respectively, which both result in a zero payoff for player 3. Hence, $\left(s_2^*, s_3^*\right)$ is a CPNE in $\Gamma^{\chi^{\#}}\left(s_1^*\right)$. Moreover, in $\Gamma^{\chi^{\#}}\left(s_2^*\right)$, just the links 12 and 13 can be formed. Since player 1 strictly prefers the network $\{12\}$ and since he can enforce it, $\left(s_1^*, s_3^*\right)$ is self-enforcing and any other self-enforcing strategy profile also gives the network $\{12\}$. Hence, $\left(s_1^*, s_3^*\right)$ a CPNE in $\Gamma^{\chi^{\#}}\left(s_2^*\right)$. In $\Gamma^{\chi^{\#}}\left(s_3^*\right)$, the players 1 and 2 just can form the link 12 or not but both prefer to do so. Therefore, $\left(s_1^*, s_2^*\right)$ is self-enforcing and all self-enforcing strategy profiles lead to the network $\{12\}$. Hence, $\left(s_1^*, s_2^*\right)$ is a CPNE in $\Gamma^{\chi^{\#}}\left(s_3^*\right)$. Since player 1 strictly prefers the graph $\{12\}$ over all other graphs, s^* is a CPNE of the $\chi^{\#}$-NFG.

One important thing about s^* is that player 3 does not wish to form a link with player 2. At first glance, this seems to be odd. But since player 1 does not wish to form a link with player 3, there is—in principle—the possibility that player 2 just wishes to form a link with player 3. In this case, player 3 prefers to be isolated and to obtain a zero payoff since under the network $\{23\}$ his payoff were negative. Moreover, player 3 does not gain by forming the link 23. Hence, if there were (even very small) costs for establishing links as studied by Slikker and van den Nouweland (2000), then player 3 would prefer not to form this link.

However, the fact that players 1 and 2 both gain by deviating from the complete network does not prevent it from being supported by a CPNE. Let $\bar{s}$ denote the unique strategy profile that creates L^N. Obviously, there are no profitable one-player deviations. Thus, $(\bar{s}_1, \bar{s}_2)$, $(\bar{s}_1, \bar{s}_3)$, and $(\bar{s}_2, \bar{s}_3)$ are self-enforcing in $\Gamma^{\chi^{\#}}(\bar{s}_3)$, $\Gamma^{\chi^{\#}}(\bar{s}_2)$, and $\Gamma^{\chi^{\#}}(\bar{s}_1)$, respectively. Moreover, in $\Gamma^{\chi^{\#}}(\bar{s}_3)$, there is no other such strategy profile. In particular, player 2 can profitably deviate from $(s_1, s_2) = (\{2\}, \{1\})$ by choosing $\bar{s}_2$. Therefore, $(\bar{s}_1, \bar{s}_2)$ is a CPNE in $\Gamma^{\chi^{\#}}(\bar{s}_3)$. In $\Gamma^{\chi^{\#}}(\bar{s}_2)$, there is one other self-enforceable strategy combination, $(s_1, s_3) = (\{2\}, \{2\})$, but which is dominated by $(\bar{s}_1, \bar{s}_3)$. Hence, $(\bar{s}_1, \bar{s}_3)$ is a CPNE in $\Gamma^{\chi^{\#}}(\bar{s}_2)$. In $\Gamma^{\chi^{\#}}(\bar{s}_1)$, again, there is one other self-enforceable strategy combination, $(s_2, s_3) = (\{1\}, \{1\})$, but which gives the same payoffs as $(\bar{s}_1, \bar{s}_3)$. Hence, $(\bar{s}_2, \bar{s}_3)$ is CPNE in $\Gamma^{\chi^{\#}}(\bar{s}_1)$. Since there is no other network where all players gain, $\bar{s}$ is a CPNE.

6 Conclusion

In this paper, we introduce and advocate a CO-value, $\chi^{\#}$, which combines the ideas underlying the Myerson value and the χ-value. In contrast to the Myerson value, this value accounts for the outside options of the players. This way, $\chi^{\#}$ may recognize, for example, the potential competition between linking agents. In Sect. 5, we demonstrate that network formation under the $\chi^{\#}$-value and under the Myerson value, respectively, may lead to different networks. Moreover, this difference seems be related to χ-stability. Hence, further research on stability under the $\chi^{\#}$-value, both in general and in specific applications, and on their relation to χ-stability and to stability under the Myerson value seems to be worthwhile. van den Nouweland et al. (1992) extend the Myerson value to the class of TU games with a conference structure (hypergraph on the player set) (henceforth CF-value) which we call the Myerson CF-value. Since the characterization of the Myerson CF-value is analogous to that of the Myerson value, slightly adapting the arguments of this paper and those of van den Nouweland et al. (1992), it is hardly more than a finger exercise to extend our CO-value into a CF-value with analogous properties.

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