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On strategy-proof social choice correspondences

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Abstract We introduce two new concepts of strategy-proofness for social choice correspondences based on the theory of preferences over sets of alternatives under complete uncertainty. One is based on Pattanaik and Peleg (Soc Choice Welf 1:113–122, 1984) and the other is based on Bossert et al. (Econ Theory 16:295–312, 2000). We prove that there is no social choice correspondence satisfying anonymity, neutrality, a range condition, and either of our concepts of strategy-proofness.

1 Introduction

The Gibbard–Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975) states that strategy-proofness, nondictatorship, and resoluteness are inconsistent conditions for any social choice rule. Here, resoluteness means that only a single alternative is assigned to each preference profile. Consequently, the scope of the G-S theorem is restricted to social choice *functions*, but this is a "rather restrictive and unnatural assumption" (Gärdenfors 1976, p. 220). So, this paper undertakes to investigate what can be done with the G-S theorem when we deal with social choice *correspondences* (SCC) instead of social choice functions, i.e., drop the condition "resoluteness" in the G-S theorem.

Defining strategy-proofness for SCCs is not a trivial issue. Because agents have preferences over alternatives, we need preferences over sets of alternatives to define strategy-proofness for SCCs. Corresponding to each way of extending preferences over alternatives to preferences over sets of alternatives, many definitions are proposed in the literature. (Some of them are discussed in Sect. 4.)

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In this paper, we make use of results in the literature[1] on preferences over sets of alternatives under complete uncertainty. The theory of preferences over sets under complete uncertainty deals with situations where agents having preferences over alternatives know the set of possible outcomes, but they have no further information about the selection of the final outcome. Many ways to extend preferences over alternatives to preferences over sets of alternatives are axiomatically characterized. When the agents have no information about the probabilities of the final outcome from the value of an SCC, the situation is exactly the one supposed in the theory of preferences over sets under complete uncertainty. We define strategy-proofness for SCCs based on two prominent results of this theory: one is Pattanaik and Peleg (1984) and the other is Bossert et al. (2000).

We give two impossibility theorems. Theorem 3.1 shows that there is no SCC satisfying anonymity (symmetric treatment of agents), neutrality (symmetric treatment of alternatives), a range condition, and strategy-proofness based on the preferences characterized by Pattanaik and Peleg (1984). The same statement, Theorem 3.2, holds with the preferences characterized by Bossert et al. (2000). The theorems show that we cannot hope for a strategy-proof SCC satisfying standard conditions in social choice theory when the agents are completely uncertain about the final selection from the value of an SCC. Although there are many impossibility theorems for strategy-proof SCCs, either they use a relatively strong concept of strategy-proofness or they use nonstandard conditions to derive an impossibility. (See Sect. 4.)

The plan of the paper is the following. In Sect. 2, we give basic notation and definitions. Strategy-proofness for SCCs is defined and our main results are given in Sect. 3. In Sect. 4, we review some related literature. Section 5 is devoted to the proofs of the results.

2 Basic notation and definitions

Let $N = \{1, \ldots, n\}$ be a finite set of agents with $n \geq 3$, let X be a finite set of alternatives with $|X| \geq 3$, and let $\mathcal{X}$ be the set of all nonempty subsets of X. Let $\mathcal{L}$ denote the set of linear orders (complete, transitive, and antisymmetric binary relations) on X. Typically, a preference on X is given by a linear order, denoted by R or R_i, and a preference on $\mathcal{X}$ is given by a weak order (complete and transitive binary relation), denoted by $\succsim$ or $\succsim_i$. An element of $\mathcal{L}^N$ is called a **preference profile**, and generic notation for a preference profile is R_N, and its value at $i \in N$ is denoted by R_i. Given a preference profile R_N, an agent $i \in N$, and a linear order R'_i, let (R'_i, R_{-i}) denote the preference profile such that its value at i is R'_i and its value at $j \in N \setminus \{i\}$ is R_j. Given a linear order R on X, let P denote the strict relation induced by R. Given a weak order $\succsim$ on $\mathcal{X}$, let $\succ$ denote the strict relation and let $\sim$ denote the indifference relation induced by $\succsim$. For each linear order R on X and for each $A \in \mathcal{X}$, let $\max R|A$ denote the best element of A with respect to R, i.e., $(\max R|A, x) \in R$ for all $x \in A$; similarly, let $\min R|A$ denote the worst element of A with respect to R. We often drop

[1] The seminal work on preference under complete uncertainty is Kannai and Peleg (1984). See Barberà et al. (2004) for a recent survey.

R and write max A and min A when their meaning is clear. Also, when $A = X$, we simply write max R and min R instead of max $R|X$ and min $R|X$, respectively. For any statements A and B, $\begin{bmatrix} A \\ B \end{bmatrix}$ means "A and B".

A correspondence of $\mathcal{L}^N$ into X is called a **social choice correspondence** (SCC). We introduce standard conditions in social choice theory: anonymity requiring symmetric treatment of agents, neutrality requiring symmetric treatment of alternatives, and a range condition.

Definition 2.1 An SCC F is said to satisfy

(i) **anonymity** if for every preference profile R_N, and for each permutation σ of N, $F(R_N) = F(R_N^\sigma)$, where R_N^σ denotes the preference profile defined by for each $i \in N$, $R_i^\sigma = R_{\sigma(i)}$.

(ii) **neutrality** if for each preference profile R_N, and for each permutation ρ of X, $\rho(F(R_N)) = F(\rho(R_N))$, where $\rho(R_N)$ denotes the preference profile defined by for each $i \in N$,

$$\rho(R_i) = \{(x, y) \in X^2 \mid (\rho^{-1}(x), \rho^{-1}(y)) \in R_i\}.$$

(iii) the **range condition** if there exist $x \in X$ and $R_N \in \mathcal{L}^N$ such that $F(R_N) = \{x\}$.

3 Strategy-proofness and results

In this section, we define strategy-proofness for SCCs. For that purpose, we need preferences on $\mathcal{X}$. We define a class of *plausible* weak orders on $\mathcal{X}$ associated with each linear order on X by means of an **extension rule**, a correspondence of $\mathcal{L}$ into the set of weak orders on $\mathcal{X}$. For an extension rule E and a linear order R, $E(R)$ is interpreted as the set of admissible or plausible preferences on $\mathcal{X}$ with respect to a preference R on X.

Definition 3.1 For each extension rule E, an SCC F is said to be E **strategy-proof** if for every preference profile R_N, for every $i \in N$, and for every $R'_i \in \mathcal{L}$,

$$F(R_N) \succsim_i F(R'_i, R_{-i}), \quad \forall \succsim_i \in E(R_i). \tag{3.1}$$

There are many conceivable extension rules. Whether an extension rule is plausible or not depends on the context under consideration. An important assumption in this paper is that every agent knows that the set of possible outcomes is $F(R_N)$, but he has no information about the probabilities of those outcomes. Thus, the agents are *completely uncertain* about the final selection. The literature on preferences over sets of alternatives under complete uncertainty presents a variety of plausible preferences over sets associated with each preference over basic alternatives, and it is natural to apply those results to our research. We can categorize those preferences over sets into two groups: given a preference R over alternatives, the first group consists of preferences such that a set A is indifferent to the set $\{\max R|A, \min R|A\}$, and the second group consists of preferences such that a set A is not necessarily indifferent to the set $\{\max R|A, \min R|A\}$. The preferences in the first group are consistent with the notion of *limited rationality* according to which an agent confronted with a complex decision problem often simplifies the problem by focusing on salient features of the problem, and hence those preferences deserve to be considered. (See Bossert et al. 2000 for a discussion of this point.) However, in this paper, we consider the preferences in the second group. (Strategy-proofness based on preferences in the first group is considered by Sato 2007.) We deal with two prominent extension rules characterized by Pattanaik and Peleg (1984) and Bossert et al. (2000), respectively.

Definition 3.2 (Pattanaik and Peleg 1984) For each linear order R on X, let $E^{PP}(R)$ be the set consisting of two weak orders $\succsim^{\text{bot}}$ and $\succsim^{\text{top}}$ on $\mathcal{X}$ defined below. $\succsim^{\text{bot}}$ uses the worst alternative with respect to R as the primary criterion for ranking elements of $\mathcal{X}$, but when the worst alternatives in A and B are identical, the worst alternatives in the reduced sets $A \setminus \{\min A\}$ and $B \setminus \{\min B\}$ are considered. By proceeding in this manner, if all of the alternatives in B are eliminated while some alternatives of A remain, then $A \succsim^{\text{bot}} B$. $\succsim^{\text{top}}$ is the dual of $\succsim^{\text{bot}}$. Formally,

- assume $a_k P a_{k-1} P \cdots P a_2 P a_1$ and $b_h P b_{h-1} \cdots P b_2 P b_1$, then

$$A \succsim^{\text{bot}} B \iff \begin{cases} \exists l \ \text{ such that } \begin{bmatrix} a_l P b_l \\ \forall m < l, \quad a_m = b_m \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} k \geq h \\ \forall m \leq h, \quad a_m = b_m \end{bmatrix}, \end{cases}$$

- assume $a_1 P a_2 P \cdots P a_{k-1} P a_k$ and $b_1 P b_2 P \cdots P b_{h-1} P b_h$, then

$$A \succsim^{\text{top}} B \iff \begin{cases} \exists l \ \text{ such that } \begin{bmatrix} a_l P b_l \\ \forall m < l, \quad a_m = b_m \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} k \leq h \\ \forall m \leq k, \quad a_m = b_m \end{bmatrix}. \end{cases}$$

This extension rule E^{PP} is called the **PP extension rule** (after Pattanaik and Peleg).

Definition 3.3 (Bossert et al. 2000) For each linear order R on X, let $E^{BPX}(R)$ denote the set consisting of two weak orders $\succsim^{\min}$ and $\succsim^{\max}$. The basic ideas of these weak orders are similar to those of $\succsim^{\text{bot}}$ and $\succsim^{\text{top}}$ in $E^{PP}(R)$. $\succsim^{\min}$ also uses the worst alternatives with respect to R as the primary criterion for ranking elements of $\mathcal{X}$, but $\succsim^{\min}$ uses the best alternatives as the secondary criterion. When the worst alternatives in A and B are identical, $\succsim^{\min}$ considers the best alternatives and moves to the next step only if both the best and the worst alternatives of A and B are identical. In such cases, $\succsim^{\min}$ removes both the best and the worst alternatives from A and B, and considers the remaining sets $A \setminus \{\max A, \min A\}$ and $B \setminus \{\max B, \min B\}$ in the same manner. $\succsim^{\max}$ is the dual of $\succsim^{\min}$. Formally,

- assume $a_k P a_{k-1} P \cdots P a_2 P a_1$ and $b_h P b_{h-1} \cdots P b_2 P b_1$, then

$$A \succsim^{\min} B \iff \begin{cases} \exists l \;\text{ such that } \begin{bmatrix} a_l P b_l \;\text{ or } \begin{bmatrix} a_l = b_l \\ a_{k-l+1} P b_{k-l+1} \end{bmatrix} \\ a_m = b_m \;\; \forall m < l, \;\; \forall m > k-l+1 \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} k \geq h \\ \forall m \leq h/2, \begin{bmatrix} a_m = b_m \\ a_{k-m+1} = b_{h-m+1} \end{bmatrix} \end{bmatrix} \end{cases}$$

- assume $a_1 P a_2 P \cdots P a_{k-1} P a_k$ and $b_1 P b_2 P \cdots P b_{h-1} P b_h$, then

$$A \succsim^{\max} B \iff \text{the same statement as that of the right-hand side of } \succsim^{\min}.$$

This extension rule is called the **BPX extension rule** (after Bossert, Pattanaik, and Xu).

Example 3.1 Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $x_1 P x_2 P x_3 P x_4 P x_5$, and consider rankings on $\mathcal{A} = \{\{x_3\}, \{x_1, x_5\}, \{x_2, x_3, x_5\}, \{x_1, x_4, x_5\}\}$. Then,

- $\{x_3\} \succ^{\text{bot}} \{x_1, x_5\} \succ^{\text{bot}} \{x_2, x_3, x_5\} \succ^{\text{bot}} \{x_1, x_4, x_5\}$, and
- $\{x_3\} \succ^{\min} \{x_1, x_4, x_5\} \succ^{\min} \{x_1, x_5\} \succ^{\min} \{x_2, x_3, x_5\}$.

Thus, $\succsim^{\text{bot}}$ and $\succsim^{\min}$ differ from each other in ranking the sets in $\mathcal{A}$. However, on $\mathcal{A}$, $\succsim^{\text{top}}$ and $\succsim^{\max}$ are equal to the following $\succsim$:

- $\{x_1, x_4, x_5\} \succ \{x_1, x_5\} \succ \{x_2, x_3, x_5\} \succ \{x_3\}$.

Thus, $(\{x_1, x_4, x_5\}, \{x_2, x_3, x_5\}) \notin \left(\succsim^{\text{bot}} \cap \succsim^{\text{top}}\right)$ whereas $(\{x_1, x_4, x_5\}, \{x_2, x_3, x_5\}) \in \left(\succsim^{\min} \cap \succsim^{\max}\right)$.

Based on the introduced extension rules E^{PP} and E^{BPX}, E^{PP} strategy-proofness and E^{BPX} strategy-proofness are defined (Definition 3.1). The following two theorems show that those strategy-proofness cannot be achieved by SCCs satisfying anonymity, neutrality, and the range condition.

Theorem 3.1 *There is no SCC satisfying anonymity, neutrality, the range condition, and E^{PP} strategy-proofness.*

Theorem 3.2 *There is no SCC satisfying anonymity, neutrality, the range condition, and E^{BPX} strategy-proofness.*

One might have a reservation about the validity of E^{PP} strategy-proofness and E^{BPX} strategy-proofness because the extension rules E^{PP} and E^{BPX} are rather restrictive. A natural inquiry is "Certainly, an agent with a preference R on X might have the preferences over $\mathcal{X}$ specified by E^{PP} and E^{BPX}, but he might have other different weak orders on $\mathcal{X}$. Do the conclusions of Theorems 3.1 and 3.2 remain the same in such cases?" As formally stated in the next corollary, the impossibilities of Theorems 3.1 and 3.2 are unchanged, because the more possible weak orders on $\mathcal{X}$ are associated with $R \in \mathcal{X}$, the more possibilities for manipulation there are.

Corollary 3.1 *Let E be any extension rule such that either*

(i) $E^{PP}(R) \subset E(R)$ *for all* $R \in \mathcal{L}$*, or*
(ii) $E^{BPX}(R) \subset E(R)$ *for all* $R \in \mathcal{L}$*, holds.*

Then, there is no SCC satisfying anonymity, neutrality, the range condition, and E strategy-proofness.

Now, we show that the conditions anonymity, neutrality, and the range condition are essential for our impossibility theorem. First, dictatorship satisfies all of the conditions but anonymity. Next, a constant SCC such that $F(R_N) = \{x\}$ for some $x \in X$ and for every preference profile R_N satisfies all of the conditions but neutrality. Lastly, an SCC such that $F(R_N) = X$ for every preference profile R_N satisfies all of the conditions but the range condition.

Finally, we note that even if any kind of binary relation over $\mathcal{X}$ (such as quasi-orders) is admitted as the value of an extension rule, then the statement of Corollary 3.1 holds as long as either of the two conditions in Corollary 3.1 is satisfied.[2]

4 Related literature

Much has been written about strategy-proof SCCs (Gärdenfors 1976; Feldman 1979; Duggan and Schwartz 2000; Barberà et al. 2001; Benoît 2002; Ching and Zhou 2002, among others) However, there is no agreement on what is the best definition of strategy-proofness for SCCs, and there are many different definitions of strategy-proofness.[3] In this section, we review selected related papers. An important point is that the more preferences over sets are permitted, the easier we obtain impossibility results.

Ching and Zhou (2002) and Barberà et al. (2001) give impossibility results with relatively large sets of preferences on $\mathcal{X}$. The concept of strategy-proofness by Ching and Zhou (2002) is stronger than ours.

Ching and Zhou (2002) (CZ strategy-proofness): An SCC F is strategy-proof if for any preference profile R_N, for any $i \in N$, and for any $R'_i \in \mathcal{L}$,

(i) aR_ib for all $a \in F(R_N)\setminus F(R'_i, R_{-i})$ and for all $b \in F(R'_i, R_{-i})$, and
(ii) aR_ib for all $a \in F(R_N)$ and for all $b \in F(R'_i, R_{-i})\setminus F(R_N)$.

This definition of strategy-proofness assumes diverse preferences on $\mathcal{X}$, and it is stronger than E^{PP} and E^{BPX} strategy-proofness. To see this point, let $X = \{x_1, x_2, x_3\}$ and suppose that there exist an SCC F, a preference profile R_N, and R'_i such that

$$x_1 P_i x_2 P_i x_3, \quad F(R_N) = \{x_1, x_3\}, \quad \text{and} \quad F(R'_i, R_{-i}) = \{x_2, x_3\}.$$

Then, $x_3 \in F(R_N)$ and $x_2 \in F(R'_i, R_{-i})\setminus F(R_N)$, which violates the second condition of CZ strategy-proofness. In other words, a preference $\succsim_i$ such that $\{x_2, x_3\} \succ_i \{x_1, x_3\}$

[2] This fact was pointed out by an anonymous Associate Editor.

[3] Gärdenfors (1979) studies different definitions of manipulation given by Pattanaik (1973, 1975), Gärdenfors (1976), Kelly (1977), Fishburn (1978).

belongs to the set of admissible preferences on $\mathcal{X}$ associated with R_i. Clearly, in our extension rules, $\succsim_i \notin E^{PP}(R_i) \cup E^{BPX}(R_i)$, while it can be seen that every preference in $E^{PP}(R_i) \cup E^{BPX}(R_i)$ is admissible in the sense of Ching and Zhou (2002).

Barberà et al. (2001) present two kinds of preferences on $\mathcal{X}$. Their first concept of strategy-proofness is the same as that of CZ strategy-proofness. The second definition is based on the following extension rule: For each $R \in \mathcal{L}$, let $E^{BDS}(R)$ be the set consisting of $\succsim$ such that for any $A, B \in \mathcal{X}$,

$$A \succsim B \iff \sum_{a \in A} \frac{u(a)}{|A|} \geq \sum_{b \in B} \frac{u(b)}{|B|}$$

for some numerical representation u of R. Consequently, $\bigcup_{R \in \mathcal{L}} E^{BDS}(R)$ is considered to be the set of admissible preferences on $\mathcal{X}$. In defining $\bigcup_{R \in \mathcal{L}} E^{BDS}(R)$, Barberà et al. (2001) assume that the alternatives in each set are mutually compatible, and there is no uncertainty. Barberà et al. (2001) show that the second concept of strategy-proofness and unanimity lead to dictatorial or bi-dictatorial rules. Because there are many possible numerical representations of R, many preferences on $\mathcal{X}$ are admitted. For example, the following preference that is neither in $\bigcup_{R \in \mathcal{L}} E^{PP}(R)$ nor in $\bigcup_{R \in \mathcal{L}} E^{BPX}(R)$ falls within $\bigcup_{R \in \mathcal{L}} E^{BDS}(R)$:

$$\{x_1\} \succ_i \{x_2\} \succ_i \{x_3\} \succ_i \{x_4\} \succ_i \{x_5\} \quad \text{and} \quad \{x_1, x_3, x_4, x_5\} \succ_i \{x_1, x_2, x_5\}.$$

(Let $u(x_1) = 12$, $u(x_2) = 9$, $u(x_3) = 8$, $u(x_4) = 4$, and $u(x_5) = -120$.) On the other hand, because $\succsim^{\text{bot}} \notin \bigcup_{R \in \mathcal{L}} E^{BDS}(R)$, the second concept of strategy-proofness of Barberà et al. (2001) is not strictly stronger than ours. Thus, their second concept of strategy-proofness and ours are independent from each other.

Benoît's (2002) concept of strategy-proofness is weaker than ours. However, Benoît's impossibility result needs a condition called *near unanimity*. Near unanimity requires that when all but one agent have a common maximal singleton, then the singleton must be chosen at the profile. Duggan and Schwartz (2000), whose concept of strategy-proofness is also weaker than ours, impose a similar condition to near unanimity, called *residual resoluteness*, to derive their impossibility theorem. Barberà et al. (2001, p. 393) remark on near unanimity as follows:

> A less innocuous assumption made on social choice rules is that they satisfy the property of "near unanimity". ... For instance, the rule which selects the union (over individuals) of maximal elements violates this property.

Moreover, the rule selecting the set of Pareto-efficient alternatives violates near unanimity and residual resoluteness. Benoît (2002) and Duggan and Schwartz (2000) exclude such SCCs by near unanimity and residual resoluteness, respectively, and not by strategy-proofness. In other words, their impossibility theorems depend on nonstandard conditions in social choice theory.

Gärdenfors (1976) considers strategy-proof SCCs with anonymity and neutrality. With a rather weak concept of strategy-proofness, Gärdenfors gives an impossibility result when there are three alternatives and three agents, and agents' preferences over alternatives are weak orders. Also, with linear preferences, Gärdenfors shows that there exists an SCC satisfying strategy-proofness, anonymity, neutrality, and the Condorcet criterion.[4] At the same time, Gärdenfors (1976, p.226) conjectures that any strategy-proof SCC satisfying some standard conditions such as neutrality and anonymity "should be too undecisive to be of practical interest". Feldman (1979) also considers strategy-proof social decision functions (SDF) with anonymity and neutrality. An SDF is a function of $\mathcal{X} \times \mathcal{L}^N$ into $\mathcal{X}$, and it is a more general concept than an SCC. (For each SDF C, $C(X, \cdot)$ is an SCC in our sense.) Feldman (1979) gives a series of interesting results. However, three conditions used in most of his theorems are rather restrictive. The first condition is $|X| \geq n$. In many situations, this requirement is difficult to satisfy, notably, in elections. The second and the third ones are *contraction consistency* and *expansion consistency*.[5] The plurality rule and the Borda rule, both are famous and widely prevalent in real societies, violate contraction consistency and expansion consistency, respectively.

Thus, our contribution is that we derive impossibility theorems with (i) weak concepts of strategy-proofness based on prominent results in the literature on preferences over sets of alternatives under complete uncertainty, and (ii) standard conditions (anonymity, neutrality, and the range condition) in social choice theory.

5 Proofs

Lemma 5.1 *If an SCC F satisfies either E^{PP} strategy-proofness or E^{BPX} strategy-proofness, then for every preference profile R_N, for any $i \in N$, for any $R_i' \in \mathcal{L}$,*

$$(\max R_i|F(R_N), \max R_i|F(R_i', R_{-i})) \in R_i \tag{5.1}$$

and

$$(\min R_i|F(R_N), \min R_i|F(R_i', R_{-i})) \in R_i. \tag{5.2}$$

[4] An SCC F is said to satisfy the Condorcet criterion if $F(R_N) = \{x\}$ whenever x is the Condorcet winner with respect to R_N.

[5] A SDF C satisfies contraction consistency if for any preference profile R_N and for any $A \in \mathcal{X}$,

$$x \in C(A, R_N) \Rightarrow \forall y \in A, \quad x \in C(\{x, y\}, R_N).$$

A SDF C satisfies expansion consistency if for preference profile R_N and for any $A \in \mathcal{X}$,

$$\begin{bmatrix} x \in A \\ x \in C(\{x, y\}, R_N), \ \ \forall y \in A \end{bmatrix} \Rightarrow x \in C(A, R_N).$$

Proof First, assume that F is E^{PP} strategy-proof. If (5.1) fails to hold, then $F(R'_i, R_{-i}) \succ_i^{\text{top}} F(R_N)$, which is a contradiction. Also, if (5.2) fails to hold, then $F(R'_i, R_{-i}) \succ_i^{\text{bot}} F(R_N)$, which is a contradiction.

Next, assume that F is E^{BPX} strategy-proof. If (5.1) fails to hold, then $F(R'_i, R_{-i}) \succ_i^{\max} F(R_N)$, which is a contradiction. Also, if (5.2) fails to hold, then $F(R'_i, R_{-i}) \succ_i^{\min} F(R_N)$, which is a contradiction. □

Lemma 5.2 *Let F be an SCC satisfying either E^{PP} or E^{BPX} strategy-proofness and let x be any element of X. If $F(R_N) = \{x\}$ for some preference profile R_N, then for any preference profile R'_N such that $\{y \in X | (y, x) \in R'_i\} \subset \{y \in X \mid (y, x) \in R_i\}$ for all $i \in N$, we have $F(R'_N) = \{x\}$.*

Proof We prove this statement by induction. Let k be any integer such that $0 \leq k \leq n-1$. Assume $F(R'_{\{1,\ldots,k\}}, R_{-\{1,\ldots,k\}}) = \{x\}$. We prove that $F(R'_{\{1,\ldots,k,k+1\}}, R_{-\{1,\ldots,k,k+1\}}) = \{x\}$. For notational simplicity, let $K = \{1, \ldots, k\}$ and let $K_{+1} = \{1, \ldots, k+1\}$. Suppose on the contrary that $F(R'_{K_{+1}}, R_{-K_{+1}}) \neq \{x\}$. Let v be any element of $F(R'_{K_{+1}}, R_{-K_{+1}})$ with $v \neq x$.

Case 1: $v P_{k+1} x$. In this case, we have

$$(\max R_{k+1}|F(R'_{K_{+1}}, R_{-K_{+1}}), \max R_{k+1}|F(R'_K, R_{-K})) \in P_{k+1},$$

which is a contradiction with Lemma 5.1.
Case 2: $x P_{k+1} v$. In this case, $x P'_{k+1} v$ holds. Thus

$$(\min R'_{k+1}|F(R'_K, R_{-K}), \min R'_{k+1}|F(R'_{K_{+1}}, R_{-K_{+1}})) \in P'_{k+1},$$

which is a contradiction with Lemma 5.1. Therefore, $F(R'_{K_{+1}}, R_{-K_{+1}}) = \{x\}$. By induction on k, we can conclude that $F(R'_N) = \{x\}$. □

Definition 5.1 An SCC F is said to satisfy **unanimity** if for any $x \in X$ and for any preference profile R_N such that $\max R_i = x$ for all $i \in N$, $F(R_N) = \{x\}$.

Lemma 5.3 *If an SCC F satisfies E^{PP} or E^{BPX} strategy-proofness, neutrality, and the range condition, then F satisfies unanimity.*

Proof Let x be any element of X and let R_N be any preference profile such that $\max R_i = x$ for all $i \in N$. By the range condition, there exist $y \in X$ and a preference profile R'_N such that $F(R'_N) = \{y\}$. Let $R''_N = R'_N(x, y)$, where $R'_N(x, y)$ denotes the preference profile obtained by interchanging x and y at R_N. By neutrality, $F(R''_N) = \{x\}$. Let R^*_N be the preference profile defined by for each $i \in N$, $R^*_i = R''_i(\max R''_i, x)$, where $R''_i(\max R''_i, x)$ denotes the linear order obtained by interchanging $\max R''_i$ and x at R''_i. Note that at R^*_N, every agent puts x at the top of his preference. By Lemma 5.2, $F(R^*_N) = \{x\}$. Because $\{z \in X \mid (z, x) \in R_i\} = \{x\} = \{z \in X \mid (z, x) \in R^*_i\}$ for all $i \in N$, by Lemma 5.2, we have $F(R_N) = \{x\}$. □

Table 1 Construction of the profile R'_N in Lemma 5.3

$R_N : x \notin F(R_N), y \in F(R_N)$ $\Longrightarrow$ $R'_N : \{y\} = F(R'_N)$

Agent	Best	Worst
i	x	$\cdots$
$N\backslash\{i\}$	$\cdots$	$\cdots$

Agent	Best			Worst
i	x	y	$\cdots$	
$N\backslash\{i\}$	y	$\cdots$	$\cdots$	x

Lemma 5.4 *Let F be an E^{PP} or E^{BPX} strategy-proof SCC satisfying unanimity and let R_N be any preference profile such that* $\max R_i \notin F(R_N)$ *for some $i \in N$. Let* $\max R_i = x$ *and let y be any element of $F(R_N)$. Then,*

- *lift y to the second place at R_i, and*
- *at each R_j $(j \in N\backslash\{i\})$, lift y to the top and take x to the bottom.*

Let R'_N be the resulting preference profile. Then, $F(R'_N) = \{y\}$. (See Table 1.)

Proof First, we prove $x \notin F(R'_{N\backslash\{i\}}, R_i)$. We prove this claim by induction with the induction base $x \notin F(R_N)$. Let k be any integer with $0 \leq k \leq |N\backslash\{i\}| - 1$ and assume that there exists $K \subset N\backslash\{i\}$ such that $|K| = k$ and $x \notin F(R'_K, R_{-K})$. Our goal is to prove that there exists $K' \subset N\backslash\{i\}$ such that $|K'| = k + 1$ and $x \notin F(R'_{K'}, R_{-K'})$. Let j be any element of $(N\backslash\{i\})\backslash K$. We claim that $K' = K \cup \{j\}$ is a desired set of agents. Suppose on the contrary that $x \in F(R'_{K'}, R_{-K'})$ holds. Note that

- $\min R'_j|F(R'_K, R_{-K}) \neq x = \min R'_j$, and
- $\min R'_j|F(R'_{K'}, R_{-K'}) = x = \min R'_j$.

That is, $(\min R'_j|F(R'_K, R_{-K}), \min R'_j|F(R'_{K'}, R_{-K'})) \in P'_j$, which is a contradiction with Lemma 5.1. Thus, K' is such that $|K'| = k+1$ and $x \notin F(R'_{K'}, R_{-K'})$. By induction on k, there exists $K'' \subset (N\backslash\{i\})$ with $|K''| = |N\backslash\{i\}|$ and $x \notin F(R'_{K''}, R_{-K''})$. Because $|K''| = |N\backslash\{i\}|$ implies $K'' = N\backslash\{i\}$, we have $x \notin F(R'_{N\backslash\{i\}}, R_i)$.

Next, we prove $F(R'_N) = \{y\}$. First, we show that $x \notin F(R'_N)$. Suppose on the contrary that $x \in F(R'_N)$. Then,

- by the first part of the proof, $\max R_i|F(R'_{N\backslash\{i\}}, R_i) \neq x = \max R_i$, and
- $\max R_i|F(R'_{N\backslash\{i\}}, R'_i) = x = \max R_i$.

Thus, $(\max R_i|F(R'_{N\backslash\{i\}}, R'_i), \max R_i|F(R'_{N\backslash\{i\}}, R_i)) \in P_i$, which is a contradiction with Lemma 5.1. Therefore, $x \notin F(R'_N)$. We now show that $F(R'_N) = \{y\}$. Suppose on the contrary that $F(R'_N) \neq \{y\}$. Then, there exists $v \in X\backslash\{x, y\}$ such that $v \in F(R'_N)$. Because F satisfies unanimity, $F(R'_i(x, y), R'_{N\backslash\{i\}}) = \{y\}$. Because $(y, v) \in P'_i$, it follows that $(\min R'_i|F(R'_i(x, y), R'_{N\backslash\{i\}}), \min R'_i|F(R'_N)) \in P'_i$, which is a contradiction. □

Lemma 5.5 *Let F be an E^{PP} or E^{BPX} strategy-proof SCC satisfying neutrality and unanimity. If there exists a preference profile R_N and an agent $i \in N$ such that* $\max R_i \notin F(R_N)$*, then, for any $y \in F(R_N)$, for any preference profile R^*_N such that* $\max R^*_j = y$ *for all $j \in N\backslash\{i\}$, we have $F(R^*_N) = \{y\}$.*

Proof Let y be an element of $F(R_N)$ and let $x = \max R_i$. Let R'_N denote the preference profile constructed in Lemma 5.4. Then Lemma 5.4 shows that $F(R'_N) = \{y\}$.

First, we show that for any $R''_i \in \mathcal{L}$, we have $F(R''_i, R'_{-i}) = \{y\}$. Suppose on the contrary that there exists $\hat{R}_i \in \mathcal{L}$ such that $F(\hat{R}_i, R'_{-i}) \neq \{y\}$. Let v be any element of $F(\hat{R}_i, R'_{-i})$ with $v \neq y$. There are two cases to consider.

Case 1: $v = x$. In this case,

- $\max R'_i | F(\hat{R}_i, R'_{-i}) = x = \max R'_i$, and
- $\max R'_i | F(R'_N) = y \neq \max R'_i$.

Therefore, $(\max R'_i | F(\hat{R}_i, R'_{-i}), \max R'_i | F(R'_N)) \in P'_i$, which is a contradiction with Lemma 5.1.
Case 2: $v \neq x$. By the neutrality of F, $F(R'_N(x, v)) = \{y\}$. Because $\max R'_j(x, v) = \max R'_j = y$ for all $j \in N \backslash \{i\}$, Lemma 5.2 shows that $F(R'_i(x, v), R'_{-i}) = \{y\}$. Note that

- $\max R'_i(x, v) | F(\hat{R}_i, R'_{-i}) = v = \max R'_i(x, v)$ and
- $\max R'_i(x, v) | F(R'_i(x, v), R'_{-i}) = y \neq \max R'_i(x, v)$.

Therefore, $(\max R'_i(x, v) | F(\hat{R}_i, R'_{-i}), \max R'_i(x, v) | F(R'_i(x, v), R'_{-i})) \in P'_i(x, v)$, which is a contradiction.

Second, we prove that for any preference profile R^*_N such that $\max R^*_j = y$ for all $j \in N \backslash \{i\}$, we have $F(R^*_N) = \{y\}$. By the first part of the proof, $F(R^*_i, R'_{-i}) = \{y\}$. Because $F(R^*_i, R'_{-i}) = \{\max R'_j\} = \{\max R^*_j\}$ for all $j \in N \backslash \{i\}$, Lemma 5.2 shows that $F(R^*_N) = \{y\}$. □

Remark 5.1 Let F be an E^{PP} or E^{BPX} strategy-proof SCC satisfying neutrality and unanimity. If there exists a preference profile R_N and an agent $i \in N$ such that $\max R_i \notin F(R_N)$, then, for any $z \in X$, for any preference profile R'_N such that $|\{j \in N \mid \max R'_j = z\}| \geq n - 1$, we have $F(R'_N) = \{z\}$.

Proof Let y be any element of $F(R_N)$. First, we show that for any R^*_N such that $|\{j \in N \mid \max R^*_j = y\}| \geq n - 1$, we have $F(R^*_N) = \{y\}$. If $|\{j \in N \mid \max R^*_j = y\}| = n$, then $F(R^*_N) = \{y\}$ directly follows from unanimity. So, suppose that $|\{j \in N \mid \max R^*_j = y\}| = n - 1$. Let h be the element of $N \backslash \{j \in N \mid \max R^*_j = y\}$. Let σ be the permutation of N interchanging i and h. Then, by anonymity, $\max R^\sigma_h = \max R_i \notin F(R_N) = F(R^\sigma_N)$. Because $y \in F(R^\sigma_N)$, Lemma 5.5 implies $F(R^*_N) = \{y\}$.

Second, let z be any element of X and let R'_N be any preference profile such that $|\{j \in N \mid \max R'_j = z\}| \geq n - 1$. Then, $F(R'_N) = \{z\}$ follows from above argument and neutrality. □

Lemma 5.6 *Let F be an E^{PP} or E^{BPX} strategy-proof SCC satisfying neutrality and unanimity. Then for every $i \in N$ and for every preference profile R_N, $\max R_i \in F(R_N)$ holds.*

Proof We prove this Lemma with the aid of the impossibility theorem of Benoît (2002). An alternative proof can be found in Sato (2007).

Table 2 The profile R_N in the proof of Theorem 3.1

Agent	Best			Worst
1	$x_{\|X\|}$	x_1	$\cdots$	x_2
$N\backslash\{1\}$	x_1	x_2	$\cdots$	$x_{\|X\|}$

In the following, we describe a slightly weaker version of the impossibility theorem by Benoît to avoid unnecessary complexity. For the precise statement of the theorem, see the original article. For each linear order R on X, a weak order $\succsim$ on $\mathcal{X}$ is said to satisfy the **B condition** if for any $A \in \mathcal{X}$,

(i) $A \neq \{\max A\} \Rightarrow \{\max A\} \succ A$,

(ii) $\begin{bmatrix} A \neq \max A \\ A \neq \{\max A, \max(A\backslash\{\max A\})\} \end{bmatrix} \Rightarrow \{\max A, \max(A\backslash\{\max A\})\} \succ A$,

(iii) $\begin{bmatrix} A \neq \max A \\ A \neq \{\max A, \max(A\backslash\{\max A\})\} \\ A \neq \max(A\backslash\{\max A\}) \end{bmatrix} \Rightarrow \max(A\backslash\{\max A\}) \succ A$, and

(iv) $A \neq \min A \Rightarrow A \succ \min A$.

Then, Benoît's impossibility theorem (his Theorem 1) shows that there is no SCC F such that

(i) F is E strategy-proof, where the extension rule E is such that for each $R \in \mathcal{L}$, any $\succsim \in E(R)$ satisfies the B condition, and
(ii) for any $x \in X$ and for any R_N such that $|\{i \in N \mid \max R_i = x\}| \geq n-1$, $F(R_N) = \{x\}$.

We call this the **B theorem**. (Remember that this is a weaker version of, not the precise statement of, Benoît's theorem.)

Now, we prove our Lemma. Let F be an E^{PP} or E^{BPX} strategy-proof SCC satisfying neutrality and unanimity. Suppose on the contrary that there exists $i \in N$ such that $\max R_i \notin F(R_N)$. Then, by Remark 5.1, the second condition of the B theorem is satisfied. It can be seen that both E^{PP} and E^{BPX} satisfy the first condition. Thus, F satisfies both conditions of the B theorem, which is a contradiction. □

Proof of Theorem 3.1 Suppose on the contrary that there exists an E^{PP} strategy-proof SCC F satisfying anonymity, neutrality, and the range condition. Let $X = \{x_1, x_2, \ldots, x_{|X|}\}$ and let R_N a preference profile described in Table 2.

Then, by Lemma 5.6, $\{x_1, x_{|X|}\} \subset F(R_N)$.

Now, we show that $x_2 \in F(R_N)$. Suppose on the contrary that $x_2 \notin F(R_N)$. Let R_2' be a linear order such that $\max R_2' = x_2$. Then, $\{x_1, x_2, x_{|X|}\} \subset F(R_2', R_{-2})$. By the definition of $\succsim_2^{\text{top}}$, it can be seen that $F(R_2', R_{-2}) \succ_2^{\text{top}} F(R_N)$, which is a contradiction with E^{PP} strategy-proofness. Thus, $x_2 \in F(R_N)$.

Therefore, $\{x_1, x_2, x_{|X|}\} \subset F(R_N)$. Now, let R_1' be a linear order such that $\max R_1' = x_1$. By Lemma 5.5, F satisfies unanimity, and hence $F(R_1', R_{-1}) = \{x_1\}$. Therefore,

$$F(R_1', R_{-1}) \succ_1^{\text{bot}} F(R_N),$$

which is a contradiction with E^{PP} strategy-proofness. □

Proof of Theorem 3.2 In the proof of Theorem 3.1, replace $\succsim_2^{\text{top}}$ with $\succsim_2^{\max}$ and replace $\succsim_1^{\text{bot}}$ with $\succsim_1^{\min}$. □

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