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Utility function of fuzzy preferences on a countable set under max-$*$-transitivity

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Abstract We determine, by means of max-$*$-transitivity, necessary and sufficient conditions for a fuzzy binary relation R defined on a countable (finite or denumerable) set A to be representable by a utility function. We display one example of its application.

Keywords Countable set · Fuzzy binary relation · Max-$*$-transitivity · Utility function

1 Introduction

A basic assumption made by pioneers of classical microeconomics such as Edgeworth and Pareto was that the ranking of a consumer's preferences could always be measured numerically, by associating to each possible consumption bundle a real number that measures its utility: the greater the utility, the more preferred the bundle was, and conversely. Consequently, there has appeared a

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general mathematical problem to justify this assumption of the representation of preferences.

When individual (consumer) has crisp preferences, many authors (Bridges and Metha 1995; Debreu 1959; Fishburn 1970; Kreps 1990; Varian 2002) solved the problem of the representation of a crisp binary relation (usually called classical representation) which is defined as follows: "Given a crisp binary relation R on a set of alternatives A, find necessary and sufficient conditions of the existence of a real-valued mapping f on A such that for any elements $x, y \in A, xRy$ if and only if $f(x) \geq f(y)$".

Since 1980, Economists recognized that consumer's preferences are not always crisp, they could be vague or ambiguous. Accordingly, as in many works (Barrett et al. 1990; Billot 1992, 1995; Dasgupta and Deb 1996; Fodor and Roubens 1994; Fono 2004; Fono and Andjiga 2005; Fono and Gwét 2003; Jain 1990; Ponsard 1987; Salles 1998; Sengupta 1999), we assume in this paper that individual or consumer has crisp or ambiguous preferences on A, which are modeled by a Fuzzy Weak Preference Relation R on A formally defined as follows:

Definition 1 (Barrett et al. 1990; Billot 1995; Dasgupta and Deb 1996; Fono 2004; Fono and Andjiga 2005; Fono and Gwét 2003; Salles 1998; Sengupta 1999) *A fuzzy binary relation is a function* $R : A \times A \to [0, 1]$.

- *R is crisp if* $\forall x, y \in A, R(x, y) \in \{0, 1\}$.
- *R is reflexive if* $\forall x \in A, R(x, x) = 1$.
- *R is connected if* $\forall x, y \in A, R(x, y) + R(y, x) \geq 1$.
- *R is strongly connected if* $\forall x, y \in A, max(R(x, y), R(y, x)) = 1$.
- *A fuzzy weak preference relation (FWPR) is a reflexive and connected fuzzy binary relation.*
- *R is bin-transitive if* $\forall x, y, z \in A$,

$$[R(x, y) \geq R(y, x) \text{ and } R(y, z) \geq R(z, y)] \quad \text{imply} \quad R(x, z) \geq R(z, x).$$

In this case, for all $x, y \in A$, the real $R(x, y)$ is interpreted as the degree to which "x is at least as good as y" and the inequality $R(x, y) \geq R(y, x)$ means "x is preferred or indifferent to y".

Thus, researchers in Fuzzy Mathematical Economics attempt to formulate and analyze, when preferences are ambiguous, problems of classical Economic Theory which have been already solved with crisp preferences. In particular, some scholars (Billot 1992, 1995; Fono 2004) formulated the mathematical problem of the representation of crisp or fuzzy preferences as follows:

Definition 2 (Billot 1995; Fono 2004; Salles 1998) *Let R be a FWPR.*
R is numerically representable if there exists a real-valued mapping f on A satisfying

$$\forall x, y \in A, \quad R(x, y) \geq R(y, x) \Leftrightarrow f(x) \geq f(y).$$

When such f exists, we say that R is numerically representable by f on A or f is a utility function of R where the real $f(x)$ measures the utility of x for any alternative x of A.

Authors such as Gwét (2001) and Ovchinnikov (2000) among others, tackled the problem of representation of fuzzy relations with formulations, definitions and assumptions distinct from the ones used in this paper. This is due to the fact that many crisp properties can be generalized in different ways in the fuzzy case.

In the crisp case, that is, when consumer's preferences are crisp and modeled by a crisp binary relation, we have the following well-known result:

Proposition 1 (Bridges and Metha 1995; Debreu 1959; Fishburn 1970; Kreps 1990; Varian 2002) *Let R be a crisp binary relation on a countable set A.*

R is numerically representable on A iff R is a crisp ordering (reflexive, connected and transitive crisp binary relation).

The determination of representable FWPR's is given by the following proposition due to Billot's results (Billot 1995).

Proposition 2 *Let R be a FWPR on a countable set A. Then R is numerically representable on A iff R is bin-transitive.*

It is important to emphasize that in the fuzzy set theory, we have standard, basic and traditional definitions of fuzzy transitivity which are defined by means of a t-norm $*$ and called max-$*$-transitivity, that is, $\forall x, y, z \in A, R(x, z) \geq R(x, y) * R(y, z)$. Max-$*$-transitivity is the natural fuzzification of the crisp concept of transitivity. It is also a general notion of transitivity of a FWPR mostly used in the Fuzzy Decision Making (see Barrett et al. 1990; Fodor and Roubens 1994; Fono 2004; Fono and Andjiga 2005; Fono and Gwét 2003; Jain 1990; Salles 1998; Sengupta 1999). Given the importance of the max-$*$-transitivity, it is necessary to seek if bin-transitivity is borne of the former.

The following example shows that this is false.

Example 1 Let $A = \{x, y, z\}$ and R be a bin-transitive FWPR defined by: $\forall a \in A, R(a, a) = 1; R(x, y) = 0.75; R(y, x) = 0.6; R(x, z) = 0.55; R(z, x) = R(z, y) = 0.5$ and $R(y, z) = 1$.

For all t-norm $*$, R violates max-$*$-transitivity (since $R(x, z) < R(x, y) * R(y, z)$).

Thus, we remark that since bin-transitivity is not borne to max-$*$-transitivity, the result of Proposition 2 severely hampers the preference modellers in the determination of all the numerically representable FWPRs in the following three sets: the sets of $*$-fuzzy orderings (that are, max-$*$-transitive FWPRs) denoted FO^*, the set of strong $*$-fuzzy orderings (that are, strongly connected and max-$*$-transitive FWPRs) denoted SFO^* and the set of fuzzy orderings (that are, max-min transitive FWPRs) denoted FO which are the usual subsets of the set of FWPRs.

The aim of this paper is to determine by means of max-$*$-transitivity, necessary and sufficient conditions on a given FWPR R such that it is numerically representable on a countable set A.

The paper is organized as follows: Sect. 2 completes some basic concepts on FWPRs and fuzzy operators. In Sect. 3, we introduce and analyze four conditions on FWPRs which are used to establish by means of max-$*$-transitivity, necessary and sufficient conditions of the existence of a utility function of a given FWPR R on A. We determine all the numerically representable FWPRs in the traditional and usual subsets of FWPRs: FO^*, SFO^* and FO. We give an example of the application of the obtained results on the Theory of revealed preference in Economics. Section 4 contains some concluding remarks.

2 Other preliminaries

Throughout this paper, for each $a, b \in [0,1], a \vee b = \max(a,b)$.

Definition 3 (Fodor and Roubens 1994; Fono 2004; Fono and Andjiga 2005; Fono and Gwét 2003; Klement et al. 2000; Ovchinnikov 2000; Salles 1998)

1. *A t-norm is a continuous function* $* : [0,1] \times [0,1] \to [0,1]$ *satisfying for all* $a,b,c,d \in [0,1]$, *(i)* $a * 1 = a$; *(ii)* $a * b \leq d * c$ *if* $a \leq d$ *and* $b \leq c$; *(iii)* $a * b = b * a$ *and (iv)* $(a * b) * c = a * (b * c)$.
2. *Let* $*$ *be a t-norm. An implication (or quasi-inverse) of* $*$ *is the internal composition law denoted by* $|$ *and defined over* $[0,1]$ *by* $\forall a,b \in [0,1], a|b = max\{t \in [0,1]/a * t \leq b\}$.

Remark 1 Let R be a FWPR, $*$ be a t-norm and $(x,y) \in A \times A$.

1. If $*$ is the Zadeh's min t-norm denoted $\wedge$, then the max-$*$-transitivity is called max-min transitivity.
2. If R is crisp, then bin-transitivity becomes classical transitivity.
3. Strong connectedness implies reflexivity and connectedness.

We shall need the following properties of a t-norm. (For proof, one may consult Fodor and Roubens 1994; Fono and Gwét 2003; Gwét 1997): let $*$ be a t-norm and $|$ its implication. For all $a,b,c \in [0,1]$,

$$\text{(i)}\ \ a * b \leq a \wedge b, \quad \text{(ii)}\ \ a \leq b \Rightarrow \begin{cases} b|c \leq a|c \\ c|a \leq c|b \end{cases} \quad \text{and} \quad \text{(iii)}\ b \leq a|b. \qquad (1)$$

We recall the following four reals which we need in this paper (see Fono and Andjiga 2005):
$\alpha_1^*(x,y,z) = R(z,y) * R(y,x)$,
$\alpha_2^*(x,y,z) = R(x,y) * R(y,z)$,
$\alpha_3^*(x,y,z) = (R(y,z)|R(y,x)) \wedge (R(x,y)|R(z,y))$,
$\alpha_4^*(x,y,z) = (R(y,x)|R(y,z)) \wedge (R(z,y)|R(x,y))$
where $*$ is a t-norm, $|$ its implication, R is a FWPR and $x,y,z \in A$.

We establish in the following lemma, the comparisons of these reals.

Lemma 1 *Let R be a FWPR and $*$ be a t-norm. For all $x, y, z \in A$, if*

$$\begin{pmatrix} (R(x,y) > R(y,x) \ \text{and} \ R(y,z) \geq R(z,y)) \\ \text{or} \\ (R(x,y) = R(y,x) \ \text{and} \ R(y,z) > R(z,y)) \end{pmatrix} \tag{2}$$

then

$$\begin{cases} (i) \quad \alpha_1^*(x,y,z) \leq \alpha_3^*(x,y,z) < 1 \\ (ii) \quad \alpha_1^*(x,y,z) \leq \alpha_2^*(x,y,z) \\ (iii) \quad \alpha_3^*(x,y,z) \leq \alpha_4^*(x,y,z) \end{cases} \tag{3}$$

Proof Let $x, y, z \in A$. Suppose (2) and show (3).

1. Let us show (i) of (3).
   (1-1) Let us show that $\alpha_1^*(x,y,z) \leq \alpha_3^*(x,y,z)$.
   (iii) of (1) implies

$$R(z,y) \leq R(x,y)|R(z,y) \quad \text{and} \quad R(y,x) \leq R(y,z)|R(y,x) \tag{4}$$

   Since (i) of (1) implies $\alpha_1^*(x,y,z) \leq R(z,y) \wedge R(y,x)$, (4) implies the result.
   (1-2) Let us show that $\alpha_3^*(x,y,z) < 1$. Assume the contrary, then $\alpha_3^*(x,y,z) = 1$. Therefore $R(x,y)|R(z,y) = R(y,z)|R(y,x) = 1$. Thus, the definition of implication gives $R(x,y) \leq R(z,y)$ and $R(y,z) \leq R(y,x)$. This contradicts (2).
2. Let us show (ii) of (3).
   (2) implies $R(y,x) \leq R(x,y)$ and $R(z,y) \leq R(y,z)$. The definition of $*$ gives $R(y,x) * R(z,y) \leq R(x,y) * R(y,z)$. Thus $\alpha_1^*(x,y,z) \leq \alpha_2^*(x,y,z)$.
3. Let us show that $\alpha_3^*(x,y,z) \leq \alpha_4^*(x,y,z)$. We distinguish two cases:
   If $R(y,z) \leq R(y,x)$, then (2) implies $R(z,y) \leq R(x,y)$. Thus $R(y,z)|R(y,x) = R(z,y)|R(x,y) = 1$. And we obtain $\alpha_3^*(x,y,z) = R(x,y)|R(z,y)$ and $\alpha_4^*(x,y,z) = R(y,x)|R(y,z)$. Since (2) implies $R(y,x) \leq R(x,y)$ and $R(z,y) \leq R(y,z)$, then (ii) of (1) implies $\alpha_3^*(x,y,z) \leq R(y,x)|R(z,y) \leq R(y,x)|R(y,z) = \alpha_4^*(x,y,z)$.
   If $R(y,z) > R(y,x)$, then $R(y,x)|R(y,z) = 1$. And $\alpha_4^*(x,y,z) = R(z,y)|R(x,y)$. Since (2) implies $R(y,x) \leq R(x,y)$ and $R(z,y) \leq R(y,z)$, then (ii) of (1) implies $R(y,z)|R(y,x) \leq R(z,y)|R(y,x) \leq R(z,y)|R(x,y) = \alpha_4^*(x,y,z)$. And as $\alpha_3^*(x,y,z) \leq R(y,z)|R(y,x)$, then $\alpha_3^*(x,y,z) \leq \alpha_4^*(x,y,z)$.

□

We end this section by a recall of an useful characterization of a max-$*$-transitivity on $\{x, y, z\}$.

Lemma 2 (Fono and Andjiga 2005, Lemma 1, p. 375) *Let R be a FWPR and $*$ be a t-norm. For all $\{x, y, z\} \subseteq A$,*

$$R \ \text{is max-}*\text{-transitive on} \ \{x,y,z\} \ \Leftrightarrow \begin{cases} \text{(i) } R(x,z) \in [\alpha_2^*(x,y,z), \alpha_4^*(x,y,z)] \\ \text{(ii) } R(z,x) \in [\alpha_1^*(x,y,z), \alpha_3^*(x,y,z)]. \end{cases} \tag{5}$$

In the next section, to solve by means of a t-norm the problem of the existence of a utility function of a given FWPR when A is countable, we will proceed as follows:

- In Sect. 3.1, we recall two conditions and introduce two new conditions on FWPRs.
- In Sect. 3.2, we establish by means of a t-norm, necessary and sufficient conditions of the existence of a utility function of a given FWPR.
- In Sect. 3.3, we deduce all the numerically representable FWPRs in the usual sets FO^*, SFO^* and FO.
- In Sect. 3.4, we show that numerically representable FWPRs could generate consistent choices.

3 New result and application

3.1 Conditions on FWPR

Let us recall the two first conditions introduced by Fono and Andjiga (2005).

Definition 4 (Fono and Andjiga 2005, Definition 6, p. 379) *Let R be a FWPR and $*$ be a t-norm.*

1. *R satisfies condition Q_1^* if $\forall x, y, z \in A$,*

$$(R(x,y) > R(y,x) \text{ and } R(y,z) > R(z,y)) \text{ imply}$$
$$\left(\begin{pmatrix} R(x,z) \in [\alpha_2^*(x,y,z), \alpha_3^*(x,y,z)] \\ \text{and} \\ R(z,x) \in [\alpha_2^*(x,y,z), \alpha_3^*(x,y,z)] \end{pmatrix} \Rightarrow R(x,z) > R(z,x)\right)$$

2. *R satisfies condition Q_2^* if $\forall x, y, z \in A$,*

$$\begin{pmatrix} R(x,y) > R(y,x) \text{ and } R(y,z) = R(z,y) \\ \text{or} \\ R(x,y) = R(y,x) \text{ and } R(y,z) > R(z,y) \end{pmatrix}$$

imply

$$\left(\begin{pmatrix} R(x,z) \in [\alpha_2^*(x,y,z), \alpha_3^*(x,y,z)] \\ \text{and} \\ R(z,x) \in [\alpha_2^*(x,y,z), \alpha_3^*(x,y,z)] \end{pmatrix} \Rightarrow R(x,z) > R(z,x)\right)$$

We also recall the following result which gives properties of Q_1^* and Q_2^* in two particular cases: R is strongly connected and $*$ is the Zadeh's min t-norm. We need the following classical condition T (see Fono and Andjiga 2005; Fono and Gwét 2003; Sengupta 1999):

R satisfies condition T if for all $x, y, z \in A$,

$$R(x,y) = R(y,x) = R(y,z) = R(z,y) \text{ implies } R(x,z) = R(z,x).$$

Proposition 3 (Fono and Andjiga 2005, Proposition 5, p. 380) *Let R be a FWPR and* $*$ *be a t-norm.*

1. *If R is strongly connected, then R satisfies conditions* Q_1^* *and* Q_2^*.
2. (i) *If* $*$ *is the Zadeh's min t-norm, then R satisfies condition* Q_1^*.
   (ii) *If* $*$ *is the Zadeh's min t-norm and R is max-min transitive, then conditions* Q_2^* *and T are equivalent.*

We introduce the two new conditions using the five following intervals of $[0, 1]$:

$$\begin{aligned}
I_1^*(x,y,z) &= [0, \alpha_1^*(x,y,z)],\\
I_2^*(x,y,z) &=]\alpha_1^*(x,y,z), \alpha_2^*(x,y,z) \wedge \alpha_3^*(x,y,z)],\\
I_3^*(x,y,z) &=]\alpha_3^*(x,y,z), \alpha_2^*(x,y,z) \vee \alpha_3^*(x,y,z)],\\
I_4^*(x,y,z) &=]\alpha_2^*(x,y,z) \vee \alpha_3^*(x,y,z), \alpha_4^*(x,y,z)],\\
I_5^*(x,y,z) &=]\alpha_4^*(x,y,z), 1],
\end{aligned}$$

where $*$ is a t-norm, $|$ its implication, R is a FWPR and $x, y, z \in A$ satisfying (2).

It is noticeable that I_2^*, I_3^*, I_4^* and I_5^* can be empty sets.

Definition 5 *Let R be a FWPR and* $*$ *be a t-norm.*

1. *R satisfies condition* H_1^* *if* $\forall x, y, z \in A$,

$$\begin{pmatrix} (R(x,y) > R(y,x) \text{ and } R(y,z) \geq R(z,y)) \\ \text{or} \\ (R(x,y) = R(y,x) \text{ and } R(y,z) > R(z,y)) \end{pmatrix} \text{ imply}$$
$$\forall i \in \{1,2,3,4\}, \left(\alpha_i^*(x,y,z) < R(z,x) \Rightarrow \alpha_i^*(x,y,z) < R(x,z)\right).$$

2. *R satisfies condition* H_2^* *if* $\forall x, y, z \in A$,

$$\begin{pmatrix} (R(x,y) > R(y,x) \text{ and } R(y,z) \geq R(z,y)) \\ \text{or} \\ (R(x,y) = R(y,x) \text{ and } R(y,z) > R(z,y)) \end{pmatrix} \text{ imply}$$
$$\forall i \in \{1,2,3,4,5\}, \left(R(x,z) \text{ and } R(z,x) \in I_i^*(x,y,z) \Rightarrow R(x,z) > R(z,x)\right).$$

We can notice that Fono and Andjiga (2005, p. 382–384) gave some explanations of conditions Q_1^* and Q_2^*.

The following result shows that when R is strongly connected, conditions H_1^* and H_2^* become merely:

(i) Condition $H_{1,fc}^* : \forall x, y, z \in A$,

$$\begin{pmatrix} (R(x,y) > R(y,x) \text{ and } R(y,z) \geq R(z,y)) \\ or \\ (R(x,y) = R(y,x) \text{ and } R(y,z) > R(z,y)) \end{pmatrix} \text{ imply}$$
$$\forall i \in \{1,3\}, \quad \left(\alpha_i^*(x,y,z) < R(z,x) \Rightarrow \alpha_i^*(x,y,z) < R(x,z)\right)$$

and

(ii) Condition $H_{2,fc}^* : \forall x, y, z \in A$,

$$\begin{pmatrix} (R(x,y) > R(y,x) \text{ and } R(y,z) \geq R(z,y)) \\ or \\ (R(x,y) = R(y,x) \text{ and } R(y,z) > R(z,y)) \end{pmatrix} \text{ imply}$$
$$\left(R(x,z) \quad \text{and} \quad R(z,x) \in I_3^*(x,y,z) =]\alpha_3^*(x,y,z), 1] \Rightarrow R(x,z) > R(z,x)\right)$$

respectively.

Proposition 4 *Let R be a FWPR and $*$ be a t-norm. If R is strongly connected, then conditions H_1^* and H_2^* are equivalent to conditions $H_{1,fc}^*$ and $H_{2,fc}^*$ respectively.*

Proof Suppose that R is a strongly connected FWPR. Thus, for all $x, y, z \in A$ such that (2) is satisfied, we have $R(x,y) = R(y,z) = 1$ and $\alpha_1^*(x,y,z) \leq \alpha_3^*(x,y,z) < \alpha_2^*(x,y,z) = \alpha_4^*(x,y,z) = 1$. And, the five intervals become:

$$I_1^*(x,y,z) = [0, \alpha_1^*(x,y,z)], I_2^*(x,y,z) =]\alpha_1^*(x,y,z), \alpha_3^*(x,y,z)],$$
$$I_3^*(x,y,z) =]\alpha_3^*(x,y,z), 1] \quad \text{and} \quad I_4^*(x,y,z) = I_5^*(x,y,z) = \emptyset.$$

The equalities $\alpha_2^*(x,y,z) = \alpha_4^*(x,y,z) = 1$ imply that for $i \in \{2,4\}$, the assertion $\alpha_i^*(x,y,z) < R(z,x) \Rightarrow \alpha_i^*(x,y,z) < R(x,z)$ is satisfied. Hence conditions H_1^* and $H_{1,fc}^*$ are equivalent.

Let us consider for $i \in \{1,2,4,5\}$, the assertion:

$$\left(R(x,z) \text{ and } R(z,x) \in I_i^*(x,y,z)\right) \Rightarrow R(x,z) > R(z,x) \tag{6}$$

Since $I_4^*(x,y,z) = I_5^*(x,y,z) = \emptyset$, then for $i \in \{4,5\}$, (6) is satisfied.

As R is strongly connected, we have $R(x,z) = 1$ or $R(z,x) = 1$. Moreover, since the upper bounds $\alpha_1^*(x,y,z)$ and $\alpha_3^*(x,y,z)$ of the intervals $I_1^*(x,y,z)$ and $I_2^*(x,y,z)$ are lower than 1, then these two intervals can no contain the two reals $R(x,z)$ and $R(z,x)$. Thus, for $i \in \{1,2\}$, (6) is satisfied. Hence conditions H_2^* and $H_{2,fc}^*$ are equivalent. □

The next result shows that the four conditions are satisfied when R is numerically representable.

Proposition 5 *Let R be a FWPR and $*$ be a t-norm. If R is numerically representable, then R satisfies conditions Q_1^*, Q_2^*, H_1^* and H_2^*.*

Proof Let $x, y, z \in A$ such that (2) is satisfied. It is sufficient to show that $R(x,z) > R(z,x)$. As R is numerically representable, there exists a utility function f of R on A. Thus (2) implies ($f(x) > f(y)$ and $f(y) \geq f(z)$) or ($f(x) = f(y)$ and $f(y) > f(z)$). This gives $f(x) > f(z)$, i.e., $R(x,z) > R(z,x)$. And we obtain the result. □

Let us now give by means of max-$*$-transitivity, necessary and sufficient conditions of the existence of a utility function for a FWPR which is the main result of our paper.

3.2 New result with standard transitivity

Theorem 1 *Let R be a FWPR and $*$ be a t-norm. The two following statements are equivalent:*

1. *R is numerically representable.*
2. *$\forall\{x,y,z\} \subseteq A$, one of the two following conditions is satisfied:*
   (a) *R is max-$*$-transitive and satisfies conditions Q_1^* and Q_2^* on $\{x,y,z\}$.*
   (b) *R violates max-$*$-transitivity and satisfies conditions H_1^* and H_2^* on $\{x,y,z\}$.*

The proof of this theorem is due to the two following lemmas.

Lemma 3 *Let R be a FWPR and $*$ be a t-norm. Given $\{x,y,z\} \subseteq A$, if R is max-$*$-transitive and satisfies conditions Q_1^* and Q_2^* on $\{x,y,z\}$, then R is bin-transitive on $\{x,y,z\}$.*

Lemma 4 *Let R be a FWPR and $*$ be a t-norm. Given $\{x,y,z\} \subseteq A$, if R violates max-$*$-transitivity and satisfies conditions H_1^* and H_2^* on $\{x,y,z\}$, then R is bin-transitive on $\{x,y,z\}$.*

Proof of Lemma 3 Suppose that R is max-$*$-transitive and satisfies conditions Q_1^* and Q_2^* on $\{x,y,z\}$ and show that R is bin-transitive on $\{x,y,z\}$.

Fono and Andjiga (2005, Proposition 7, p. 381) show that:
Let R be a max-$*$-transitive FWPR.
R satisfies conditions Q_1^* and Q_2^* iff (for all $x,y,z \in A$, ($R(x,y) \geq R(y,x)$ and $R(y,z) \geq R(z,y)$) imply $R(x,z) \geq R(z,x)$).

This result and the previous hypotheses imply that R is bin-transitive on $\{x,y,z\}$. □

Proof of Lemma 4 Suppose that R violates max-$*$-transitivity and satisfies conditions H_1^* and H_2^* on $\{x,y,z\}$. Let us show that R is bin-transitive on $\{x,y,z\}$.

Suppose that $R(x,y) \geq R(y,x)$ and $R(y,z) \geq R(z,y)$. Let us show that $R(x,z) \geq R(z,x)$.

With Lemma 2, R violates max-$*$-transitivity on $\{x,y,z\}$, i.e.,

$$R(x,z) \notin [\alpha_2^*(x,y,z), \alpha_4^*(x,y,z)] \text{ or } R(z,x) \notin [\alpha_1^*(x,y,z), \alpha_3^*(x,y,z)] \quad (7)$$

We distinguish two cases.

Case 1 Suppose (2) and show that $R(x,z) > R(z,x)$. Here, we distinguish six cases noted from (1-1) to (1-6).

(1-1) Suppose that $R(z,x) \in I_6^*(x,y,z) =]\alpha_2^*(x,y,z) \wedge \alpha_3^*(x,y,z), \alpha_3^*(x,y,z)]$. As $R(z,x) \in I_6^*(x,y,z)$, then $\alpha_2^*(x,y,z) \wedge \alpha_3^*(x,y,z) = \alpha_2^*(x,y,z) < R(z,x) \leq \alpha_3^*(x,y,z)$. Therefore by condition H_1^*, we have $R(x,z) > \alpha_2^*(x,y,z)$.
The inequality $\alpha_2^*(x,y,z) < \alpha_3^*(x,y,z)$ and (ii) and (iii) of (3) give $\alpha_1^*(x,y,z) \leq \alpha_2^*(x,y,z) < \alpha_3^*(x,y,z) \leq \alpha_4^*(x,y,z)$. Thus, as $R(z,x) \in]\alpha_2^*(x,y,z), \alpha_3^*(x,y,z)]$, then $R(z,x) \in [\alpha_1^*(x,y,z), \alpha_3^*(x,y,z)]$. By (7), we deduce that $R(x,z) \notin [\alpha_2^*(x,y,z), \alpha_4^*(x,y,z)]$. The inequality $R(x,z) > \alpha_2^*(x,y,z)$ implies $R(x,z) > \alpha_4^*(x,y,z)$. We have $R(z,x) \leq \alpha_3^*(x,y,z) \leq \alpha_4^*(x,y,z) < R(x,z)$. Hence $R(x,z) > R(z,x)$.

(1-2) Suppose that $R(z,x) \in I_1^*(x,y,z)$. Here, we distinguish two cases:
- (1-2-i) If $R(x,z) > \alpha_1^*(x,y,z)$, then $R(z,x) < R(x,z)$.
- (1-2-ii) If $R(x,z) \in I_1^*(x,y,z)$, then (2) and condition H_2^* imply $R(x,z) > R(z,x)$.

(1-3) Suppose that $R(z,x) \in I_2^*(x,y,z)$.
If $R(x,z) > \alpha_2^*(x,y,z) \wedge \alpha_3^*(x,y,z)$, then $R(z,x) < R(x,z)$.
If $R(x,z) \leq \alpha_2^*(x,y,z) \wedge \alpha_3^*(x,y,z)$.
As $R(z,x) \in I_2^*(x,y,z)$, then $R(z,x) > \alpha_1^*(x,y,z)$. Therefore (2) and condition H_1^* imply $R(x,z) > \alpha_1^*(x,y,z)$. Thus $R(x,z) \in I_2^*(x,y,z)$. Hence (2) and condition H_2^* imply $R(x,z) > R(z,x)$.

(1-4) Suppose that $R(z,x) \in I_3^*(x,y,z) =]\alpha_3^*(x,y,z), \alpha_2^*(x,y,z) \vee \alpha_3^*(x,y,z)]$.
If $R(x,z) > \alpha_2^*(x,y,z) \vee \alpha_3^*(x,y,z)$, then $R(x,z) > R(z,x)$.
If $R(x,z) \leq \alpha_2^*(x,y,z) \vee \alpha_3^*(x,y,z)$.
As $R(z,x) \in I_3^*(x,y,z)$, then $\alpha_3^*(x,y,z) < R(z,x) < \alpha_2^*(x,y,z) \vee \alpha_3^*(x,y,z) = \alpha_2^*(x,y,z)$. Therefore, by condition H_1^*, we have $R(x,z) > \alpha_3^*(x,y,z)$. Thus $R(x,z) \in I_3^*(x,y,z)$. (2) and condition H_2^* imply $R(x,z) > R(z,x)$.

(1-5) Suppose that $R(z,x) \in I_4^*(x,y,z) =]\alpha_3^*(x,y,z) \vee \alpha_2^*(x,y,z), \alpha_4^*(x,y,z)]$.
If $R(x,z) > \alpha_4^*(x,y,z)$, we have $R(x,z) > R(z,x)$.
If $R(x,z) \leq \alpha_4^*(x,y,z)$, as $R(z,x) \in I_4^*(x,y,z)$, then $\alpha_3^*(x,y,z) \vee \alpha_2^*(x,y,z) < R(z,x)$. Therefore, condition H_1^* implies $R(x,z) > \alpha_3^*(x,y,z)$ and $R(x,z) > \alpha_2^*(x,y,z)$. Thus $R(x,z) \in I_4^*(x,y,z)$. Hence (2) and condition H_2^* imply $R(x,z) > R(z,x)$.

(1-6) Suppose that $R(z,x) \in I_5^*(x,y,z) =]\alpha_4^*(x,y,z), 1]$. As $R(z,x) \in I_5^*(x,y,z)$, then $R(z,x) > \alpha_4^*(x,y,z)$. Therefore condition H_1^* implies $R(x,z) > \alpha_4^*(x,y,z)$. Thus $R(x,z) \in I_5^*(x,y,z)$. Hence (2) and condition H_2^* imply $R(x,z) > R(z,x)$.

Case 2 Let us show that $R(x,y) = R(y,x)$ and $R(y,z) = R(z,y)$ imply $R(x,z) = R(z,x)$.

Assume to the contrary that $R(x,y) = R(y,x), R(y,z) = R(z,y)$ and $R(x,z) \neq R(z,x)$.

If $R(x,z) > R(z,x)$, then the equality $R(y,z) = R(z,y)$ and the result of the first case imply that $R(x,y) > R(y,x)$. This contradicts $R(x,y) = R(y,x)$.
If $R(z,x) > R(x,z)$, then the equality $R(x,y) = R(y,x)$ and the result of the first case imply that $R(z,y) > R(y,z)$. This contradicts $R(z,y) = R(y,z)$. □

Proof of Theorem 1 (1) ⇒ (2) Obvious by Proposition 5.
(2) ⇒ (1) By Lemmas 3 and 4, R is bin-transitive on $\{x,y,z\}$ for all $\{x,y,z\} \subseteq A$. Therefore, R is bin-transitive and, by Proposition 2, R is numerically representable. □

In the literature (see Fono and Andjiga 2005; Fono and Gwét 2003; Salles 1998), we have $CO \subset SFO^* \subset FO^*$ for all t-norm $*$, where CO is the set of crisp orderings on A. Since elements of FO^* (and therefore those of SFO^*) are standard and traditional fuzzy extensions of crisp orderings, the previous theorem brings us into determining all the $*$-fuzzy orderings and all the strong $*$-fuzzy orderings which are numerically representable.

3.3 Subsets of numerically representable FWPRs

Proposition 6 *Let R be a FWPR.*

1. *If R is a $*$-fuzzy ordering, then*

$$(R \text{ is numerically representable}) \Leftrightarrow (R \text{ satisfies conditions } Q_1^* \text{ and } Q_2^*).$$

2. *If R is a strong $*$-fuzzy ordering, then R is numerically representable.*

Proof 1. Suppose that R is a $*$-fuzzy ordering, then R is max-$*$-transitive on A. Thus Theorem 1 gives the first result.
2. Suppose that R is a strong $*$-fuzzy ordering, then the first result of Proposition 3 and the first result of this proposition imply the second result. □

In the literature (see Fono and Andjiga 2005; Fono and Gwét 2003; Salles 1998), if $*$ is the Zadeh's min t-norm, then a $*$-fuzzy ordering and a strong $*$-fuzzy ordering are simply called a fuzzy ordering and a strong fuzzy ordering respectively, and their sets are therefore denoted respectively FO and SFO. We have $CO \subset SFO \subset FO$. Let us deduce in the sets FO and SFO, all the numerically representable elements.

Corollary 1 *Let R be a FWPR.*

1. *If R is a fuzzy ordering, then*

$$(R \text{ is numerically representable}) \Leftrightarrow (R \text{ satisfies condition } T).$$

2. *If R is a strong fuzzy ordering, then R is numerically representable.*

Proof 1. Suppose that R is a fuzzy ordering, then the two last results of Proposition 3 and the first result of Proposition 6 give the first result of this corollary. 2. Suppose that R is a strong fuzzy ordering, then the second result of Proposition 6 gives the second result. □

Remark 2 Let $*$ be a t-norm. We denote by FO^*NR the set of all the numerically representable $*$-fuzzy orderings.

The previous results give the following interesting sets relations:

– $FO^*NR = \{R \in FO^*/R$ satisfies conditions Q_1^* and $Q_2^*\}$.
– $CO \subset SFO^* \subset FO^*NR \subset FO^*$.

In other words, the set of all the numerically representable $*$-fuzzy orderings which is the set of all $*$-fuzzy orderings satisfying conditions Q_1^* and Q_2^* contains the two usual sets: CO and SFO^*. Furthermore, the two sets SFO^* and FO^*NR contain the set of crisp orderings CO which is also the set of numerically representable crisp binary relations.

In the last subsection, we give an example of the application of Theorem 1 on the "Theory of revealed preference" in Economics.

3.4 Example of application

In real life, the individual, even though his preference is fuzzy, will have to make a choice, necessarily exact. This raises, on the "Theory of revealed preference", the question as how unambiguous or exact choice are generated by fuzzy preference and whether the exact choices induced by fuzzy preference satisfy certain plausible rationality conditions.

In this subsection, we introduce one traditional alternative rule for generating exact choices from FWPRs and determine by means of Theorem 1, $*$-fuzzy orderings under which that rule satisfies eight classical and fairly weak rationality conditions.

We assume that: A is a finite set of alternatives and $|A| \geq 3$; $\mathcal{A}$ is the set of nonempty crisp subsets of A and G is the set of FWPRs on A.

Definition 6 (Barrett et al. 1990, p. 198) *A preference based choice function (PCF) is a mapping* $\mathcal{C} : \mathcal{A} \times G' \to \mathcal{A}$ *satisfying*

$$\emptyset \neq G' \subset G \text{ and } \forall S \in \mathcal{A}, \forall R \in G', \emptyset \neq \mathcal{C}(S, R) \subseteq S.$$

Remark 3 (Barrett et al. 1990, p. 198, Remark 2.5) Intuitively, G' figuring in Definition 6 constitutes the set of "admissible" FWPRs. Given an admissible FWPR R, and given a crisp set S of available alternatives, $\mathcal{C}(S, R)$ constitutes the exact set of alternatives chosen from S, on the basis of R.

In this paper, we consider the following usual PCF (see Barrett et al. 1990; Jain 1990; Salles 1998; Sengupta 1999):

$$\mathcal{C} : \begin{array}{l} \mathcal{A} \times G' \to \mathcal{A} \\ (S, R) \mapsto \mathcal{C}(S, R) = \{x \in S / \ \forall y \in S, R(x, y) \geq R(y, x)\} \end{array} \tag{8}$$

The previous PCF generates exact choices from fuzzy preference. Intuitively, given an admissible FWPR R, and given a crisp set S of available alternatives, the previous PCF stipulates: an alternative x of S is chosen iff it is preferred or indifferent to every other alternative of S.

We now recall eight classical and fairly rationality conditions of a PCF. Each of them lay down sufficient conditions of choosing alternative, given an admissible FWPR R, and given a feasible set of alternatives. The two first are defined in Barrett et al. (1990) and the six others in Sen (1986).

Definition 7 (Barrett et al. 1990; Sen 1986) *Let $\mathcal{C}$ be a PCF.*

1. (i) *$\mathcal{C}$ satisfies RPWD (Reward for Pairwise Weak Dominance) if $\forall S \in \mathcal{A}, \forall R \in G', \forall x \in S$,*

$$(\forall y \in S - \{x\}, R(x, y) \geq R(y, x)) \Rightarrow x \in \mathcal{C}(S, R).$$

   (ii) *$\mathcal{C}$ satisfies RPSD (Reward for Pairwise Strict Dominance) if $\forall S \in \mathcal{A}, \forall R \in G', \forall x \in S$,*

$$(\forall y \in S - \{x\}, R(x, y) > R(y, x)) \Rightarrow x \in \mathcal{C}(S, R).$$

2. (i) *$\mathcal{C}$ satisfies condition α (Standard contraction consistency) if*

$$\forall S, K \in \mathcal{A}, \forall R \in G', \forall x \in A, [x \in \mathcal{C}(S, R) \ \textit{and} \ x \in K \subseteq S] \Rightarrow x \in \mathcal{C}(K, R).$$

   (ii) *$\mathcal{C}$ satisfies condition weak α if $\forall S, K \in \mathcal{A}, \forall R \in G', \forall x \in A$,*

$$[x \in K \subseteq S \ \textit{and} \ \forall Y \subseteq S \ \textit{such that} \ Y \neq K \ \textit{and} \ x \in \mathcal{C}(Y, R)] \Rightarrow x \in \mathcal{C}(K, R).$$

   (iii) *$\mathcal{C}$ satisfies condition β^+ if $\forall S, K \in \mathcal{A}, \forall R \in G', \forall x, y \in A$,*

$$[x \in \mathcal{C}(S, R) \ \textit{and} \ y \in S \subseteq K] \Rightarrow [y \in \mathcal{C}(K, R) \Rightarrow x \in \mathcal{C}(K, R)].$$

   (iv) *$\mathcal{C}$ satisfies condition β if $\forall S, K \in \mathcal{A}, \forall R \in G', \forall x, y \in A$,*

$$[x, y \in \mathcal{C}(S, R) \ \textit{and} \ S \subseteq K] \Rightarrow [y \in \mathcal{C}(K, R) \Rightarrow x \in \mathcal{C}(K, R)].$$

   (v) *$\mathcal{C}$ satisfies condition γ (Standard expansion consistency) if for all $(S_j)_{j \in J}$ a class of elements of $\mathcal{A}, \forall R \in G', \forall x \in A$,*

$$[\forall j \in J, x \in \mathcal{C}(S_j, R)] \Rightarrow x \in \mathcal{C}(\cup_{j \in J} S_j, R).$$

(vi) *$\mathcal{C}$ satisfies condition δ if* $\forall S, K \in \mathcal{A}, \forall R \in G', \forall x, y \in A$,

$$[x, y \in \mathcal{C}(S, R) \;\; and \;\; S \subseteq K] \Rightarrow [\{x\} \neq \mathcal{C}(K, R) \;\; and \;\; \{y\} \neq \mathcal{C}(K, R)].$$

Remark 4 (see Barrett et al. 1990; Sen 1986)

1. RPWD implies RPSD.
2. Condition α implies condition weak α.
3. Condition β^+ implies conditions β, γ and δ.

Given one of the eight rationality conditions, the main question is now to determine a set of FWPRs under which the PCF defined by (8) satisfies the given condition.

The following proposition shows that when admissible FWPRs are $*$-fuzzy orderings, then the classical PCF defined by (8) satisfies the five first conditions and violates conditions β^+ and β. Furthermore, it shows that the PCF violates condition δ if $*$ is the product t-norm (defined by $\forall a, b \in [0,1], a * b = ab$) and admissible FWPRs are $*$-fuzzy orderings.

Proposition 7 *Let $\mathcal{C}$ defined by (8) and $*$ be a t-norm.*

1. *If $G' \subset FO^*$, then $\mathcal{C}$ satisfies conditions RPWR, RPSD, α, weak α and γ.*
2. *If $G' = FO^*$, then $\mathcal{C}$ violates conditions β^+ and β.*
3. *If $*$ is the product t-norm and $G' = FO^*$, then $\mathcal{C}$ violates condition δ.*

Proof 1. Suppose that $G' \subset FO^*$. Let us show that $\mathcal{C}$ satisfies conditions RPWD and RPSD. As RPWD implies RPSD, it suffices to show that $\mathcal{C}$ satisfies condition RPWD.
Let $S \in \mathcal{A}, R \in G'$ and $x \in S$ such that $(\forall y \in S - \{x\}, R(x, y) \geq R(y, x))$. Show that $x \in \mathcal{C}(S, R)$, that is, $\forall s \in S, R(x, s) \geq R(s, x)$.
Let $s \in S$. If $s = x$, then $R(x, x) = R(x, x)$. If $s \in S - \{x\}$, then $R(x, s) \geq R(s, x)$. Hence the result.
Let us show that $\mathcal{C}$ satisfies conditions α and weak α.
As condition α implies condition weak α, it suffices to show that $\mathcal{C}$ satisfies condition α.
Let $S, K \in \mathcal{A}, R \in G'$ and $x \in A$ such that $x \in \mathcal{C}(S, R)$ and $x \in K \subseteq S$. Show that $x \in \mathcal{C}(K, R)$, that is, $\forall t \in K, R(x, t) \geq R(t, x)$.
Let $t \in K$. As $K \subseteq S$, we have $t \in S$. And as $x \in \mathcal{C}(S, R)$, we have $R(x, t) \geq R(t, x)$. Hence the result.
Let us show that $\mathcal{C}$ satisfies condition γ.
Let $(S_j)_{j \in J}$ be a class of elements of $\mathcal{A}$, $R \in G'$ and $x \in A$ such that $x \in \mathcal{C}(S_j, R)$ for all $j \in J$. Show that $x \in \mathcal{C}(\cup_{j \in J} S_j, R)$, that is, $\forall s \in \cup_{j \in J} S_j, R(x, s) \geq R(s, x)$.
Let $s \in \cup_{j \in J} S_j$. There exists $j_0 \in J$ such that $s \in S_{j_0}$. As $x \in \mathcal{C}(S_{j_0}, R)$ and $s \in S_{j_0}$, then $R(x, s) \geq R(s, x)$. Hence the result.

2. Suppose that $G' = FO^*$ and let us show that $\mathcal{C}$ violates conditions β^+ and β. As β^+ implies β, it suffices to show that $\mathcal{C}$ violates condition β.

Let $A = \{x, y, z\}, S = \{x, y\}, K = \{x, y, z\}$ and $R_1 \in FO^*$ defined on A by: $\forall a \in A, R_1(a, a) = 1, R_1(x, y) = R_1(y, x) = R_1(y, z) = R_1(z, y) = 0.5, R_1(z, x) = 0.8$ and $R_1(x, z) = 0.6$.
We have $S \subseteq K$. As $R_1(x, y) = R_1(y, x)$, we have $\mathcal{C}(S, R_1) = \{x, y\}$. Also, as $R_1(y, z) = R_1(z, y)$ and $R_1(x, y) = R_1(y, x)$, then $y \in \mathcal{C}(K, R_1)$. But we have $x \notin \mathcal{C}(K, R_1)$ since there exists $z \in K$ such that $R_1(z, x) > R_1(x, z)$. Hence $\mathcal{C}$ violates condition β.

3. Suppose that $*$ is the product t-norm and $G' = FO^*$. Let us show that $\mathcal{C}$ violates condition δ.
Let $A = \{x, y, z, v\}, S = \{x, y\}, K = \{x, y, z, v\}$, $*$ be the product-norm and R_2 be the FWPR defined on A by: $\forall a \in A, R_2(a, a) = 1, R_2(y, x) = R_2(x, y) = R_2(y, z) = R_2(z, y) = R_2(z, x) = R_2(v, x) = R_2(x, v) = R_2(y, v) = 0.5, R_2(x, z) = 0.6, R_2(v, z) = R_2(v, y) = 0.7$ and $R_2(z, v) = 0.71$.
Since $\forall g, h, t \in A, R_2(g, t) \geq R_2(g, h)R_2(h, t)$, R is max-$*$-transitive. Thus $R \in FO^*$.
We have $S \subseteq K$. As $R_2(x, y) = R_2(y, x)$, we have $\mathcal{C}(S, R_2) = \{x, y\}$.
Moreover, we have $v \notin \mathcal{C}(K, R_2)$, $z \notin \mathcal{C}(K, R_2)$ and $y \notin \mathcal{C}(K, R_2)$ since $R_2(v, z) < R_2(z, v)$, $R_2(z, x) < R_2(x, z)$ and $R_2(y, v) < R_2(v, y)$ respectively. Furthermore, we have $\forall k \in K, R_2(x, k) \geq R_2(k, x)$. Thus $\mathcal{C}(K, R_2) = \{x\}$. Hence $\mathcal{C}$ violates condition δ. □

Remark 5 If $*$ is the Zadeh's min t-norm, the first result of the previous proposition becomes proposition 4.5 established by Barrett et al. (1990, p. 203).

The two last results of the previous proposition lead us to determine, in the following proposition, a subset of $*$-fuzzy orderings under which $\mathcal{C}$ satisfies conditions β^+, β and δ.

Proposition 8 *Let $\mathcal{C}$ defined by (8) and $*$ be a t-norm.*
If $G' \subset FO^*NR$, then $\mathcal{C}$ satisfies properties β^+, β and δ.

Proof Suppose that $G' \subset FO^*NR$. As β^+ implies β and δ, it is sufficient to show that $\mathcal{C}$ satisfies condition β^+.

Let $S, K \in \mathcal{A}, R \in G'$ and $x, y \in A$ such that $x \in \mathcal{C}(S, R), y \in S \subseteq K$ and $y \in \mathcal{C}(K, R)$. Show that $x \in \mathcal{C}(K, R)$, that is, $\forall t \in K, R(x, t) \geq R(t, x)$.

Let $t \in K$. As $y \in \mathcal{C}(K, R)$ and $t \in K$, then $R(y, t) \geq R(t, y)$. As $x \in \mathcal{C}(S, R)$ and $y \in S$, then $R(x, y) \geq R(y, x)$. Since $R \in G' \subseteq FO^*NR$, the two previous inequalities imply $R(x, t) \geq R(t, x)$. □

Remark 6 All the numerically representable FWPRs in FO^* satisfy all the above conditions. This justify that FWPRs of FO^*NR generate through $\mathcal{C}$ [defined by (8)] rational choices.

4 Concluding remarks

In this paper, we show that numerically representable FWPRs on a countable set A are the $*$-fuzzy orderings satisfying conditions Q_1^* and Q_2^*, or the FWPRs violating max-$*$-transitivity and satisfying conditions H_1^* and H_2^* (Theorem 1).

Although this result does not seem simple, it provides the following interesting results: (a) only the $*$-fuzzy orderings satisfying conditions Q_1^* and Q_2^* are numerically representable (first result of Proposition 6), (b) all the strong *-fuzzy orderings are numerically representable (second result of Proposition 6) and (c) only fuzzy orderings satisfying condition T are numerically representable (Corollary 1). Each of these results is a fuzzy extension of that of classical representation when A is countable.

It is well-known that the classical utility theory has permit the development of traditional Economic Theory, we think that these results will allow the development of the fuzzy Social Choice Theory (and we start to justify it in Subsect. 3.4 of Sect. 3).

This work is the first stage of research on numerical representation of FWPRs. Our next concern is to seek what happens when A is uncountable.

As we raised it in Fono and Andjiga (2005) and Fono and Gwét (2003), contrary to classical and crisp transitivity on $\{0, 1\}$, max-$*$-transitivity on $[0, 1]$ does not handle certain problems relating to rationality on triplets. It is therefore in view of covering this shortcoming that we impose conditions Q_1^*, Q_2^*, H_1^* and H_2^*.

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