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Distribution-neutral provision of public goods

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Abstract The paper describes a normative approach to budget incidence, given the provision of a public good. The objective is to balance the budget by an income tax such that the budget is distribution-neutral. This property requires that every consumer's net welfare gain, i.e. the benefit from consuming the public good and the tax burden in financing it, does not change the inequality inherent in the income distribution. The properties of the distribution-neutral tax are investigated and completely determined. The marginal willingness to pay for the public good is decisive for the degree of progression of the tax schedule.

1 Introduction

The paper describes how to finance a given public good by means of an income tax such that the distribution of net benefits (the benefits of the public good minus the tax liability) is neutral. The analysis is normative[1] but – in contrast to mainstream public sector theory – it is not based on the concept of social welfare which incorporates the aspects of efficiency and distribution at the same time. Instead an equity principle is imposed in order to define the income tax.

[1] Ebert and Tillmann (2006) present a positive analysis of budget incidence.

My co-author Georg Tillmann passed away in March 2006, much too soon. He was my friend, more than a colleague. I dedicate this paper to his memory.

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As a reference case the benefits-received principle is employed: The tax liability just equals the benefits of the public good. It is a principle of considerable practical interest since it has undoubted popular appeal to the general public. More generally, two criteria of distribution neutrality[2] are proposed and their implications for financing the public good in a distribution-neutral manner are derived.

The benefits of the public good are measured as the market-equivalent of the public good which is equal to the marginal willingness to pay $\times$ the quantity of the good. The income tax has to fulfil two requirements: as usual it has to raise the revenue necessary to finance the public good and it has to be distribution-neutral. Distribution neutrality requires that the absolute [relative] Lorenz curves of the original income distribution and of the new one, which also takes into account the net benefits, are identical. The objective is to examine the existence and the properties of a tax schedule satisfying these requirements.

The analysis of this paper is based on an equal-preference model. Consumers possess the same preference ordering and may differ in income. In this framework we are able to derive the distribution-neutral income tax. We show that the income elasticity of the marginal willingness to pay for the public good (i.e. of the Lindahl price) is the crucial variable for the structure and the properties of the tax schedule. The income tax can be regressive, proportional, or progressive. In each case considered here, the properties of the tax are identified and completely described.

There is an analogy between our approach and sacrifice principles. Though distribution neutrality is introduced differently it is equivalent to an equal net benefit principle. There is only one paper in the literature dealing with the benefit and sacrifice principle simultaneously:[3] Neill (2000) considers a synthesis of both principles. But – as usual in this theory – his analysis is based on a cardinal utility function that depends on the public good and income. In our framework the utility function is (completely) ordinal. We use net benefits since we are interested in distribution neutrality. Because of these differences one cannot compare the two approaches directly.

The organization of the paper is as follows. Section 2 introduces the model, defines (net) benefits, and discusses some properties of the marginal willingness to pay function for the public good. Section 3 at first discusses the benefits-received tax. Then two criteria of distribution neutrality are introduced and their relationship to other normative criteria is examined. The properties of corresponding distribution-neutral tax schedules are derived. Section 4 presents three examples and some discussion, and Sect. 5 offers some conclusions.

[2] See also Kaplow (2004) for a recent discussion of distribution neutrality.

[3] Burgat and Jeanrenaud (1996) also consider benefit taxation. But they compare the benefit and equal absolute sacrifice rule only for one specific form of utility function.

2 Framework

There are n private market goods $Z_1, \ldots, Z_n$ and a pure public good G. We use an equal-preference model: consumers possess the same strictly convex and monotonic preference ordering R over commodity bundles $(\mathbf{Z}, G) = (Z_1, \ldots, Z_n, G)$, but possibly differ with respect to income. It is represented by a three times differentiable direct utility function $U(\mathbf{Z}, G)$ which is strictly quasi-concave and strictly increasing in each of its arguments. G is a normal good and at least one private good has to be essential. $X > 0$ denotes a consumer's income, i.e. labor supply is fixed in order to make the problem under review manageable. The population is described by a continuous density function $f(X)$.

We assume constant returns to scale for private goods and denote their prices by $\mathbf{p} = (p_1, \ldots, p_n)$. A consumer is unable to influence the quantity G and given her exogenous income X she maximizes utility under the budget constraint. Then $Z_i(\mathbf{p}, G, X)$, $i = 1, \ldots, n$, denotes the conditional demand functions and $V(\mathbf{p}, G, X)$ the conditional indirect utility function. The marginal willingness to pay for the public good can be derived by Roy's identity: $w(\mathbf{p}, G, X) = \frac{\partial V(\mathbf{p},G,X)/\partial G}{\partial V(\mathbf{p},G,X)/\partial X}$ and can be interpreted as the Lindahl price. Since prices $\mathbf{p}$ and the quantity G are fixed, they are often suppressed: we use $w(X)$ instead of $w(\mathbf{p}, G, X)$. Denoting the elasticity of x with respect to y by $\eta(x, y)$ we define the income elasticity of the marginal willingness to pay for the public good by $\varepsilon(X) := \eta(w(\mathbf{p}, G, X), X)$ for given $\mathbf{p}$ and G. Normality of the public good implies that $\varepsilon(X) > 0$ (see Ebert 2003, p. 447).

The total benefit enjoyed from consuming G can be measured in various ways, e.g. by a Hicksian welfare measure (cf. Ebert 1993) or by gross consumer's surplus. We use the total amount of money the consumer would be willing to pay *in a market*. Since no market exists for the public good, the market price is replaced by the marginal willingness to pay or the Lindahl price (see e.g. Tresch 2002). Then the benefit enjoyed is $w(\mathbf{p}, G, X)\, G$. It is a market-equivalent measure of the benefit received from the consumption of the public good.

There are a number of arguments supporting our choice of the market-equivalent $w(X)G$.[4] Income X equals the expenditure for and the income necessary to buy the private goods consumed ($X = \Sigma p_i Z_i$). The market-equivalent $w(X)G$ is therefore measured in terms of income and is analogous to the market evaluation of private goods. w corresponds to the Lindahl price of the public good in the same way as the price p_i of a private good corresponds to the marginal willingness to pay for it. The corresponding net consumer surplus is ignored, but 'it is such neglect that necessitates discussion of the distribution of income as distinct from, say, the distribution of utilities' (Brennan 1976, p. 392).

Taxation is described by a tax schedule $T : \mathbb{R}_{++} \to \mathbb{R}$. It is required that net income $N(X) := X - T(X)$ is strictly positive for $X > 0$. The consumer

[4] The idea of imputing benefits from a nonmarket good in this way goes back at least to Aaron and McGuire (1970). They manage to make the concept operational (see also Maital 1973 for a discussion) in a specific situation by using a utility function which is separable between the private and the public good. In the present paper no a priori restrictions on the ordering R are imposed.

has to take into consideration taxation when she maximizes her utility (so that the budget constraint becomes $\mathbf{p}Z = X - T(X)$). The government's budget constraint is given by $\int_0^\infty T(X)f(X)\,\mathrm{d}X \geq C(G)$, where $C(G)$ denotes the total cost function of the public good G.

Residual progression[5] is defined by $\rho(X) := \eta(N(X), X)$. We introduce two concepts of tax progression for $X > 0$:

$$T(X) \text{ is } \begin{cases} \text{*-progressive} \\ \text{*-proportional} \\ \text{*-regressive} \end{cases} \begin{bmatrix} \text{progressive} \\ \text{proportional} \\ \text{regressive} \end{bmatrix} \text{at } X$$
$$\Leftrightarrow \; T'(X) \gtreqless 0 \; \left[\frac{\mathrm{d}\big(T(X)/X\big)}{\mathrm{d}X} \gtreqless 0 \right].$$

This definition of income tax progression is related to the measurement of absolute [relative] inequality: a tax schedule is *-progressive [progressive] if and only if the distribution of net income absolutely [relatively] Lorenz dominates the distribution of gross income for an arbitrary income distribution (Moyes 1988; Jakobsson 1976).

For an investigation of budget incidence we have to take into account the benefits enjoyed and the tax burden the consumer bears. The tax burden is determined by her tax liability $T(X)$ which depends on gross income X. Since in our case the consumer also pays income tax $T(X)$, her (gross) benefit $w(X - T(X))G$ depends on net income. Therefore, in the remainder of this paper we will consider the net benefit $\mathrm{NB}(X, T, G) := w(X - T(X))G - T(X)$ and full net income $X + \mathrm{NB}(X, T, G)$.

Finally, for later use we present some basic results. The properties of the benefit $w(X)G$ depend exclusively on the properties of the marginal willingness to pay function $w(X)$ since its income elasticity is identical with $\varepsilon(X)$, the income elasticity of $w(X)$. We will also consider some limits for $X = 0$ and $X = \infty$. We denote them by $T(0)$, $w'(0)$, etc. and analogously by $T(\infty)$, $\varepsilon(\infty)$, etc. Whenever we use one of these limits, we suppose that it exists and that the respective function is continuous in X at the boundary.

We establish[6]

Lemma 1 (a) $w'(X) > 0$ *for* $X > 0$. (b) $w(0) = 0$.

$$\text{(c)} \;\; w'(0) = \begin{cases} 0 \\ \bar{w} \in \mathbb{R}_{++} \\ \infty \end{cases} \Rightarrow \varepsilon(0) \begin{cases} \geq 1 \\ = 1 \\ \leq 1 \end{cases} \textit{ and } \varepsilon(0) \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \Rightarrow w'(0) \begin{cases} = 0 \\ \in \mathbb{R}_+ \cup \{\infty\} \\ = \infty \end{cases}.$$

5 See Lambert (2001) for the basic concepts of tax theory.

6 All proofs have been relegated to the Appendix.

$$\text{(d)}\ \ w'(\infty)=\begin{cases}0\\ \bar{w}\in\mathbb{R}_{++}\\ \infty\end{cases}\Rightarrow\varepsilon(\infty)\begin{cases}\leq 1\\ =1\\ \geq 1\end{cases}\ \textit{and}\ \varepsilon(\infty)\begin{cases}<1\\ =1\\ >1\end{cases}\Rightarrow w'(\infty)\begin{cases}=0\\ \in\mathbb{R}_{+}\cup\{\infty\}.\\ =\infty\end{cases}$$

The marginal willingness to pay (function) $w(X)$ is always strictly increasing in income X and in the limit it equals zero if income tends to zero. It turns out that the relationship between the derivative of $w(X)$ and the income elasticity $\varepsilon(X)$ at the boundaries of the income space is well defined, but it is not one-to-one. These results will be helpful below when the properties of tax functions are derived.

3 Distribution neutrality

We assume that the government has taken a decision on the quantity of the public good G and that it knows the preference ordering R and the income distribution. Its objective is to finance the provision by means of a distribution-neutral income tax. Therefore, it has to choose a tax schedule $T(X)$ such that the program $(G, T(X))$ is distribution-neutral. In order to get a reference case we at first postulate that the income distribution is not changed, i.e. that

$$X + \mathrm{NB}(X, T, G) = X \quad \text{for all } X > 0. \tag{1}$$

The income tax satisfying this condition is denoted by $\tilde{T}(X)$. Then the program $(G, \tilde{T}(X))$ is distribution-neutral in a simple, naïve sense. Full net income has to be identical with the original income X. The tax schedule $\tilde{T}(X)$ offsets the benefit and is defined in accordance with the benefits-received principle (cf. Tresch 2002). The consumers neither gain nor lose compared to the status quo. Though the tax liability $T(X) = w(X)G$ depends on the marginal willingness to pay (Lindahl price) of a consumer having income X there is no guarantee that the program $(G, \tilde{T}(X))$ is efficient: even if the situation without provision of G and without taxation is efficient the Samuelson rule need not be satisfied. Distribution neutrality (1) is merely an *equity* principle.

As far as the budget constraint is concerned we have to distinguish three cases. First, the costs of providing G are not covered by the tax revenue – given $\tilde{T}(X)$. Then the program cannot be carried out. Second, the budget constraint is satisfied. Third, there is a budget surplus.

A surplus can be distributed among all consumers by lowering their tax liability and thus increasing full net income. There is an infinite variety of tax 'reforms' which could be performed. Since then the distribution of full net income in general differs from the distribution of original income we have to provide a more general definition of distribution neutrality. In Sect. 3.1 two criteria are presented. Sects. 3.2 and 3.3 discuss their implications for the tax schedule that finances the public good and guarantees distribution neutrality.

3.1 Normative criteria

Now we suppose that there is a budget surplus for the reference tax schedule $\tilde{T}(X)$. A change in an income distribution is called distribution-neutral if the inequality inherent in the original and the new distribution is the same. We will consider the concept of absolute and relative inequality and use the criterion of Lorenz dominance. The absolute [relative] concept requires that equal absolute [relative] changes in all incomes do not change the level of inequality. For distribution neutrality we postulate that the original income distribution and the new distribution of full net income are *equivalent* according to the criterion of Lorenz-dominance. Therefore, we introduce

Definition $(G, T(X))$ *is called absolutely* $[relatively]$ *distribution-neutral if there is* $\alpha \geq 0 \quad [\lambda \geq 0]$[7] *such that*

$$X + \text{NB}(X, T, G) = X + \alpha \quad \textit{for all} \;\; X > 0 \tag{2A}$$

$$\left[X + \text{NB}(X, T, G) = X + \lambda X \quad \textit{for all} \;\; X > 0\right]. \tag{2R}$$

The benefits-received principle with the tax function $\tilde{T}(X)$ is the limiting criterion for $\alpha = \lambda = 0$. For $\alpha > 0 \quad [\lambda > 0]$ the provision of G and its financing by $T(X)$ lead to an equal absolute [relative] increase in full net income. The net gain (measured in monetary terms) is strictly positive for every consumer. There is no reranking of incomes. The realization of the program $(G, T(X))$ is worthwhile. It should be stressed that it is not necessary to impose any measurability or comparability conditions on the utility function(s). Since net benefits are measured in monetary units the approach used is entirely ordinal.

Below it will turn out that $T(X)$ is decreasing in α [respectively λ] for any fixed X, which implies that tax revenue will also decrease. Therefore there is an upper limit for α [and λ] if the costs of production $C(G)$ are to be covered. In other words, since we assume that there is a budget surplus for $\alpha = \lambda = 0$, we can find maximal α^* and λ^* such that the government's budget constraint is satisfied. The results we will derive hold for every $\alpha[\lambda]$ such that $0 \leqslant \alpha \leqslant \alpha^*\left[0 \leqslant \lambda \leqslant \lambda^*\right]$. The government will choose $\alpha^*\left[\lambda^*\right]$ in order to maximize the welfare gain. As discussed above the program $(G, T(X))$ is in general not efficient.

The criterion of distribution neutrality can be justified equivalently by some other normative concepts. First, it is well known (cf. Moyes 1988; Foster 1985) that Lorenz dominance can also be characterized by means of inequality measures. One distribution absolutely [relatively] Lorenz dominates another one if and only if the first one is unanimously preferred to the second one by all anonymous absolute [relative] inequality measures which satisfy the principle

[7] In (1) we have dealt with the limiting situation $\alpha = \lambda = 0$.

of progressive transfers and the principle of population. Thus distribution neutrality requires that the level of inequality is not changed for every inequality measure belonging to the corresponding class. Second, we can consider the net benefit of the program $(G, T(X))$ for every individual and postulate that there is $\alpha \geqslant 0 \quad [\lambda \geqslant 0]$ such that

$$\mathrm{NB}(X, T, G) = \alpha \quad \left[\mathrm{NB}(X, T, G)/X = \lambda\right] \quad \text{for all } X > 0. \tag{3}$$

The criterion can be called the principle of equal absolute [relative] net benefits.[8] This formulation of distribution neutrality demonstrates that our approach has some similarity to Neill's (2000). He combines the benefit and sacrifice principles, but his synthesis is still based on a cardinal utility function which depends on the quantity of the public good and income and is therefore more general than the utility function usually employed in the definition of equal sacrifice.

Third, we can also examine the incidence of net benefits and employ the tools of tax theory. One can distribute (potential) welfare gains in many ways, e.g. progressively, proportionally or regressively, thereby favoring consumers with low or high incomes. For neutrality a suggestive postulate is that the distribution of net benefits be *-proportional [proportional]. This requirement is equivalent to:

$$\frac{\mathrm{d}}{\mathrm{d}X}\mathrm{NB}(X, T, G) = 0 \quad \left[\frac{\mathrm{d}}{\mathrm{d}X}\left(\mathrm{NB}(X, T, G)/X\right) = 0\right] \quad \text{for all } X > 0. \tag{4}$$

Given the budget surplus, then there is $\alpha \geqslant 0 \quad [\lambda \geqslant 0]$ such that (2A) [2R] is fulfilled.

To sum up, distribution neutrality is a well-founded criterion which can be motivated and supported in various ways.

3.2 Implications: the absolute view

We now assume that the quantity of the public good G is given and want to derive the properties of a tax schedule $T(X)$ satisfying (2A). We establish

Proposition 2 *Assume that G is given, $\alpha \geqslant 0$ and $L := w(\infty)G > \alpha$.*

(a) *There exists a unique twice differentiable tax schedule $T(X)$ satisfying* (2A).
(b) *If $(G, T(X))$ is absolutely distribution-neutral, then*

  (i) *the structure of $T(X)$ is described by Table 1,*
  (ii) $\operatorname{sign} T''(X) = \operatorname{sign} w''(X - T(X))$,
  (iii) *$T(X)$ is *-progressive.*
  (iv) $\mathrm{d}T(X)/\mathrm{d}\alpha < 0$ *and* $\operatorname{sign} \mathrm{d}T'(X)/\mathrm{d}\alpha = \operatorname{sign} w''(X - T(X))$.

[8] The values α^* and λ^* maximize the individual absolute and relative net benefit given the government's budget constraint, i.e. we have a maximal equal net-benefit principle, analogous to the principle of minimal equal sacrifice (see e.g. Moyes 2003).

Table 1 Absolute inequality, $\alpha \geqslant 0$

	$X = 0$	$0 < X < \infty$	$X = \infty$	
			$L < \infty$	$L = \infty$
$T(X)$	< 0 if $\alpha > 0$ 0 if $\alpha = 0$	> 0 if $\alpha = 0$	$L - \alpha$	∞
$T'(X)$	$0 < T'(0) < 1$ if $\alpha > 0$ or if $\alpha = 0$ and $0 < w'(0) < \infty$	$0 < T'(X) < 1$	0	$= 0$ if $w'(\infty) = 0$ and $\alpha > 0$ $-0 < T'(\infty) < 1$ otherwise $= 1$ if $w'(\infty) = \infty$

The existence of a tax function is guaranteed. (If L is finite we require the condition $L > \alpha$ in order to get $T(X) > 0$ for some X.) $T(X)$ is unique and is an admissible tax schedule, i.e. $N(X) > 0$ for $X > 0$.

As the Proof of Proposition 2 demonstrates, for $\alpha > 0$ consumers with low incomes always have to be subsidized: their gross benefit from consuming the public good is (too) small. Therefore, they have to receive a transfer in order to get the (monetary) welfare gain α. Since by assumption the gross benefit exceeds α for high incomes there exists a unique income $\tilde{X}$ such that $T(\tilde{X}) = 0$. Consumers with income $X > \tilde{X}$ have to pay taxes. The tax liability is bounded if L is finite. Otherwise $T(X)$ goes to infinity if income tends to infinity.[9] The marginal tax rate on high incomes depends, of course, on $w'(\infty)$ and thus on the income elasticity of the marginal willingness to pay for G (cf. Lemma 1d). If $\varepsilon(X)$ strictly exceeds (is strictly less than) unity for $X \to \infty$, $T'(X)$ tends to unity (zero). The shape of the tax schedule is determined by the shape of the marginal willingness to pay function $w(N(X))$.

As $T'(X) > 0$ for $X > 0$, the tax schedule is always *-progressive. The reason is that gross benefits are strictly increasing with income. Furthermore, an increase in α always lowers the tax liability $T(X)$, but nevertheless the marginal tax rates can increase (uniformly).

For completeness, we also evaluate $T(X)$ by means of the usual concept of progression. We get

Proposition 3 *Assume that* $(G, T(X))$ *satisfies* (2A) *and that* $T\left(\tilde{X}\right) = 0$.

(a) *For* $\alpha > 0$ *there is* $\delta > 0$ *such that* $T(X)$ *is progressive for* $X \in \left[0, \tilde{X} + \delta\right]$.

(b) *For* $\alpha \geqslant 0$:

$$T(X) \text{ is } \begin{cases} \text{progressive} \\ \text{proportional} \\ \text{regressive} \end{cases} \Leftrightarrow \varepsilon(N(X)) \left(1 + \frac{\alpha}{T(X)}\right) \gtreqless 1.$$

As consumers with low incomes receive a transfer and this transfer decreases absolutely, the tax is always progressive for the lowest incomes. The income

[9] Consideration of $X = \infty$ allows us to determine the structure of $T(X)$ for high incomes.

elasticity $\varepsilon(N(X))$ is the crucial variable[10] (but we have to observe that $\varepsilon(N(X))$ and $T(X)$ are not independent!). If L is bounded (and therefore $\varepsilon(\infty) = 0$) the tax is regressive for high incomes. If ε is always greater than unity the income tax is progressive, but even if ε is strictly less than unity the tax schedule may be progressive. If L is infinite, nothing general can be said. But it is clear that taxation is determined by gross benefits: $T(X) = w(N(X))G - \alpha$, i.e., if benefits increase more (less) than proportionally, the same is true for the tax burden. (For high incomes α can be neglected.) Then taxation has to be progressive (regressive).

3.3 Implications: the relative view

Now we adopt the concept of relative inequality and investigate the implications of imposing the criterion introduced by (2R). We assume again that the quantity of the public good G is given and want to derive the properties of the tax schedule $T(X)$ satisfying (2R). We obtain

Proposition 4 *Assume that G is given and $\lambda > 0$.*[11]

(a) *There is a unique twice differentiable tax schedule $T(X)$ satisfying (2R).*
(b) *If $(G, T(X))$ is relatively distribution-neutral, then*

(i) *the structure of $T(X)$ is described by Table* 2.
(ii) $\operatorname{sign} T''(X) = \operatorname{sign} w''(X - T(X))$.
(iii) $T(X)$ *is* $\begin{cases} \textit{progressive} \\ \textit{proportional at } X \Leftrightarrow \quad \varepsilon(N(X)) \gtreqless 1. \\ \textit{regressive} \end{cases}$
(iv) $\mathrm{d}T(X)/\mathrm{d}\lambda < 0$ *and* $[w''(X - T(X)) \leqslant 0$ *implies that* $\mathrm{d}T'(X)/\mathrm{d}\lambda < 0\,]$. *If* $\eta\big(w'(X), X\big)\big|_{X=X-T(X)} \leqslant 1$, *then* $\mathrm{d}T'(X)/\mathrm{d}\lambda < 0$.

A solution according to this equity criterion exists, is unique, and is twice continuously differentiable. The function derived possesses the properties of a tax function, i.e. $X - T(X) > 0$ for $X > 0$.

There are some general properties of $T(X)$. For $X = 0$ we always obtain $T(0) = 0$. The marginal tax rate is strictly less than unity for strictly positive incomes. But it may be negative. Leaving the government's budget constraint aside, $T(X)$ could be a transfer for all $X > 0$ (but this does not really make sense). For $X = \infty$ more or less everything is possible.

Employing Lemma 1 we consider two interesting cases. First, if the marginal willingness to pay increases more than proportionally ($\varepsilon(X) > 1$ for all $X > 0$), then $T(X)$ is a transfer for low and a tax for high incomes. The reason is that for the poor the benefit is not sufficient to guarantee the welfare gain λX, whereas

[10] A similar result is derived by Kovenock and Sadka (1981) and Snow and Warren (1983) for a Lindahl tax if $\alpha = 0$.

[11] The case $\lambda = 0$ is identical with the case $\alpha = 0$ already considered above.

Table 2 Relative inequality, $\lambda > 0$

	$X = 0$	$0 < X < \infty$	$X = \infty$	
			$L < \infty$	$L = \infty$
$T(X)$	0		$-\infty$	$-\infty$ if $w'(\infty) = 0$ $-\infty < T(\infty) < \infty$ otherwise $+\infty$ if $w'(\infty) = \infty$
$T'(X)$	$-\lambda$ if $w'(0) = 0$ $-\lambda < T'(0) < 1$ otherwise 1 if $w'(0) = \infty$	< 1 sign $T'(X)$ $=$ sign $(w'(X)G - \lambda)$	$-\lambda$	$-\infty < T'(\infty) < 1$ if $w'(\infty) < \infty$ $= 1$ if $w'(\infty) = \infty$

for the rich the benefit grows faster than λX. Second, it may be that $\varepsilon(X) < 1$ for all $X > 0$. (This case is also similar to the situation in which $L < \infty$.) Then we get the converse result. Since benefits increase less than proportionally, high incomes have to be subsidized and low incomes have to be taxed in order to satisfy criterion (2R). Whether $T(X)$ is progressive, proportional or regressive at X depends only on the magnitude[12] of $\varepsilon(N(X))$.

If benefits are proportional to income for all X, ($\varepsilon(X) \equiv 1$), the tax is also proportional. If $\varepsilon(X) = 1$ only for $X = 0$ or $X = \infty$, things are more complicated. If $\varepsilon(X) > 1$ for all X (poor individuals are subsidized, rich individuals taxed), the tax schedule is progressive, if $\varepsilon(X) < 1$ for all X (poor individuals are taxed, rich ones subsidized) the tax schedule is regressive.

Furthermore the tax liability has to decrease in λ, since the welfare gain – measured with respect to the status quo – is equal to λX and has to increase with λ. The shape of the tax schedule is again determined by the marginal willingness to pay function $w(N(X))$. In this case an increase in λ can decrease the marginal tax rate even for $w''(N(X)) > 0$.

If the concept of *-progression is employed, then anything can occur since the marginal tax rate might be positive, zero, or negative depending on the circumstances.

Corollary 5 *Assume that* $(G, T(X))$ *satisfies* (2R).
Then $T(X)$ *can be *-progressive,*-proportional, or*-regressive.*

4 Examples and discussion

In order to illustrate our findings we consider a CES-function $U(\mathbf{Z}, G)$ with an elasticity of substitution σ. Then we obtain $w(\mathbf{p}, G, X) = a(\mathbf{p})(X/G)^{1/\sigma}$. The income elasticity of the marginal willingness to pay for G is constant: $\varepsilon(X) = 1/\sigma$. Therefore, $w(X)$ is convex in income for $\sigma < 1$, proportional for $\sigma = 1$, and concave for $\sigma > 1$.

Confining ourselves to $\sigma = 1/2$, 1, and 2 we are able to calculate the distribution-neutral tax schedules for the absolute and relative inequality views.

[12] If benefits are measured by a Hicksian measure or consumer's surplus the analysis is much more complicated.

Figure 1 illustrates these cases. The shapes of the tax schedules, of course, meet our expectation. Inspection of Fig. 1 also demonstrates that for a given preference ordering the tax schedule $T(X,\alpha)$ (for the absolute inequality concept) and the corresponding schedule $T(X,\lambda)$ (for the relative concept) cross once,

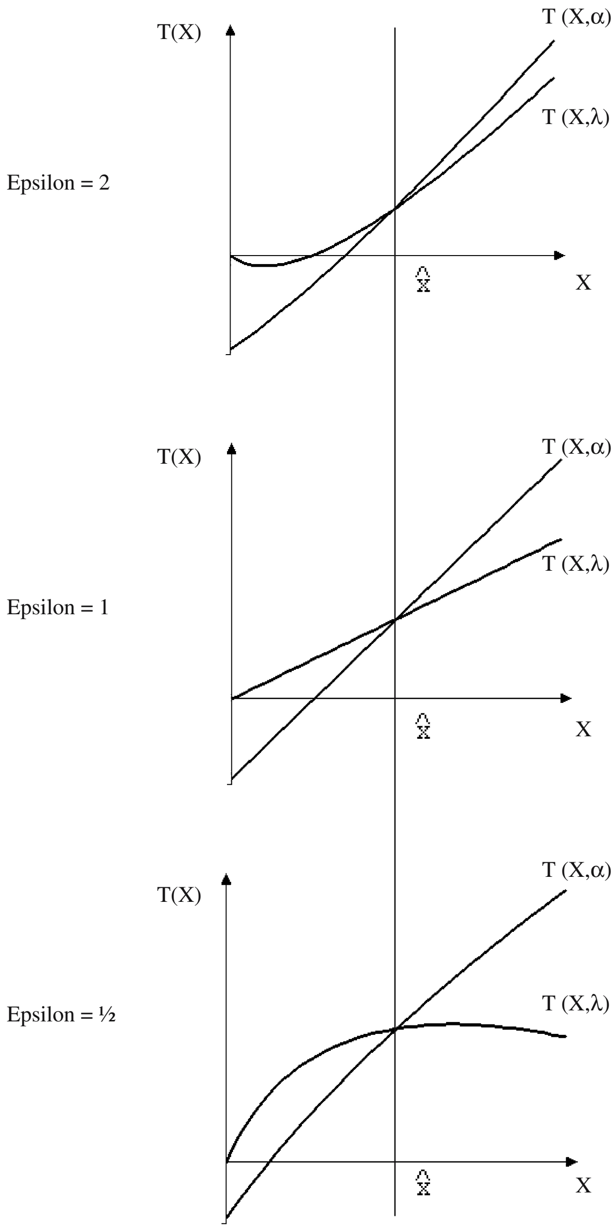

Fig. 1 Tax schedules for alpha = 0.5, lambda = 0.5, a=1

independently of the preference ordering. Furthermore, it seems that they cross at the same $X = \hat{X}$. This result is no coincidence. We obtain

Proposition 6 *Assume that* $0 < \alpha \leqslant \alpha^*$ *and* $0 < \lambda \leqslant \lambda^*$ *and define* $\hat{X} := \alpha/\lambda$.

(a) *Then* $T(\hat{X},\alpha) = T(\hat{X},\lambda)$ *and* $T(X,\alpha) < T(X,\lambda)$ *for* $X \in (0,\hat{X})$ *and* $T(X,\alpha) > T(X,\lambda)$ *for* $X \in (\hat{X},\infty)$.
(b) *If* $T(\tilde{X},\alpha) = T(\tilde{X},\lambda) = 0$ *then* $\tilde{X} = \alpha/\lambda = \hat{X}$.

Thus $T(X,\alpha)$ and $T(X,\lambda)$ cross exactly once at $\hat{X}$. The absolute view favors the incomes lower than $\hat{X}$, and the relative one those exceeding $\hat{X}$. These implications are not really surprising since the individual welfare gain, which has to be attained, is larger for the absolute criterion than for the relative criterion if $X < \hat{X}$ and smaller if $X > \hat{X}$. Moreover, if both $T(X,\alpha)$ and $T(X,\lambda)$ become zero at $\tilde{X}$, then $\hat{X} = \tilde{X}$.

5 Conclusion

The topic of this paper is the distribution-neutral provision of a public good. This problem has been investigated and solved completely under the assumption that X is exogenous: if there is no tax defined by the benefits-received principle that covers the costs of the public good for the given quantity G, then the public good should not be provided in the given quantity, since the social (and individual) net benefit would be negative. If the benefit-offsetting tax raises some revenue sufficient to cover costs, then it is always possible to find a distribution-neutral tax which guarantees a maximal individual welfare gain and also allows the public good to be financed. This statement holds true for both concepts of distribution neutrality considered above.

There are four determinants of the result: the quantity of the public good, the income distribution, the cost function, and, of course, the preference ordering of the typical consumer. For the analysis performed it is not necessary to know the entire preference ordering. Knowledge of the marginal willingness to pay function (for the public good) is sufficient since the properties of the distribution-neutral tax schedule depend on the income elasticity of this function, $\varepsilon(X)$. This elasticity can also be determined by means of the price and income elasticity of the pseudo-demand for the public good (cf. Snow and Warren 1983; Ebert 2003).

The analysis of this paper has been based on a number of simplifying assumptions: preferences are identical, benefits are defined by the market-equivalent of the public good, and the consumers' income X is fixed. It might be worthwhile to weaken these restrictions in future work.

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Appendix

Proof of Lemma 1 (a) There is a relationship between the conditional demand system and the ordinary Marshallian demand functions which are denoted by $\hat{Z}_i\left(\mathbf{p}, p_G, \hat{X}\right)$ for $i = 1, \ldots, n$ and $\hat{G}\left(\mathbf{p}, p_G, \hat{X}\right)$ where p_G is the price of G and $\hat{X}$ is virtual income. It is defined by $\hat{X} := X + w(\mathbf{p}, G, X)G$ and represents the income the consumer requires in order to buy the bundle $(\mathbf{Z}, G)$ voluntarily. We obtain the identities: $Z_i(\mathbf{p}, G, X) = \hat{Z}_i\left(\mathbf{p}, w(\mathbf{p}, G, X), \hat{X}\right)$
for $i = 1, \ldots, n$ and $G = \hat{G}\left(\mathbf{p}, w(\mathbf{p}, G, X), \hat{X}\right)$.

Ebert (2003, p. 447) proves that $\operatorname{sign} \varepsilon(X) = \operatorname{sign} \eta\left(\hat{G}, \hat{X}\right)$. Since G is a normal good we get $\varepsilon(X) > 0$ implying that $w'(X) > 0$.

(b) Consider $X = 0$. Then $Z_i(p, G, 0) = 0$ for $i = 1, \ldots, n$. Now assume that $w(0) = \bar{w} > 0$. Since $w'(X) > 0$, the constant $\bar{w}$ is finite. Therefore, $\hat{X} = X + wG = 0 + \bar{w}G > 0$ for $X = 0$. Since there is at least one private good essential, there is i_0, $1 \leqslant i_0 \leqslant n$ such that $\hat{Z}_{i_0}(p, w(p, 0, G), \bar{w}G) > 0$ contradicting $Z_i(p, G, 0) = 0$ for $i = 1, \ldots, n$.

(c) We choose $X > 0$ (small enough) and use Taylor's expansion: $w(Y) = w(X) + w'(X)(Y - X) + \frac{w''(\xi)}{2}(Y - X)^2$ for $0 \leqslant Y < X$ and $0 < \xi < X$. For $Y = 0$ we obtain

$$1 = \frac{w'(X)X}{w(X)} - \frac{w''(\xi)}{w(X)} \frac{X^2}{2}. \tag{A1}$$

(i) $w'(0) = 0$ requires that $w''(X) \geqslant 0$ in a neighborhood of 0. Therefore, $1 \leqslant w'(X)X/w(X) = \varepsilon(X)$ and $\varepsilon(0) \geqslant 1$.

(ii) $w'(0) = \infty$ requires that $w''(X) \leqslant 0$ in a neighborhood of 0. Therefore, $1 \geqslant w'(X)X/w(X) = \varepsilon(X)$ and $\varepsilon(0) \leqslant 1$.

(iii) Suppose that $w'(0) = \bar{w}' \in \mathbb{R}_{++}$. Equation (A1) implies that $\frac{w(X)}{X} = w'(X) - \frac{w''(\xi)}{2}X$.
We get $\lim_{X \to 0} w(X)/X = w'(0) = \bar{w}'$. Therefore,

$$\varepsilon(0) = \lim_{X \to 0} \frac{w'(X)X}{w(X)} = \frac{\lim_{X \to 0} w'(X)}{\lim_{X \to 0} w(X)/X} = \frac{\bar{w}'}{\bar{w}'} = 1.$$

We obtain the second part by using the first one and by contradiction.

(d) We have

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{w(X)}{X}\right) = \frac{1}{X}\left(w'(X) - \frac{w(X)}{X}\right) \tag{A2}$$

and $\lim_{X\to\infty} w(X)/X = \lim_{X\to\infty} w'(X) = w'(\infty)$ for $w(\infty) = \infty$ by L'Hôpital's rule.

(i) $w'(\infty) = 0$. Then $\lim_{X\to\infty} w(X)/X = 0$. Suppose that $\varepsilon(\infty) > 1$. Then there is an X_0 such that $\varepsilon(X) = w'(X)/(w(X)/X) > \bar{\varepsilon} > 1$ for $X > X_0$ and thus $w'(X) > \bar{\varepsilon} w(X)/X$ for $X > X_0$.
Equation (A2) then implies

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{w(X)}{X}\right) > \frac{1}{X}\left(\bar{\varepsilon}\frac{w(X)}{X} - \frac{w(X)}{X}\right) = \frac{w(X)}{X^2}(\bar{\varepsilon}-1) > 0 \quad \text{for } X > X_0.$$

Therefore, $\lim_{X\to\infty} w(X)/X > 0$ since $w(X_0)/X_0 > 0$. We obtain a contradiction.

(ii) $w'(\infty) = \bar{w}'$. Then $w(\infty) = \infty$ and

$$\varepsilon(\infty) = \lim_{X\to\infty} \frac{w'(X)X}{w(X)} = \frac{\lim_{X\to\infty} w'(X)}{\lim_{X\to\infty} w(X)/X} = \frac{\bar{w}'}{\bar{w}'} = 1.$$

(iii) $w'(\infty) = \infty$. Then $w(\infty) = \infty$ and $\lim_{X\to\infty} w(X)/X = \infty$.
We suppose that $\varepsilon(\infty) < 1$. Then there is an X_0 such that $\varepsilon(X) = w'(X)/(w(X)/X) < \bar{\varepsilon} < 1$ for $X > X_0$ and thus $w'(X) < \bar{\varepsilon} w(X)/X$ for $X > X_0$. Equation (6) implies

$$\frac{\mathrm{d}}{\mathrm{dX}}\left(\frac{w(X)}{X}\right) < \frac{1}{X}\left(\bar{\varepsilon}\frac{w(X)}{X} - \frac{w(X)}{X}\right) = \frac{w(X)}{X^2}(\bar{\varepsilon}-1) < 0 \text{ for } X > X_0.$$

Since $w(X_0)/X_0 > 0$ we obtain $\lim_{X\to\infty} w(X)/X < \infty$, a contradiction.
We obtain the second part again by using the first one and then by contradiction. □

Lemma 2 *Suppose that* $R : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *is twice continuously differentiable and that* $R'(X) \geqslant 1$. *Then there exists a unique twice continuously differentiable* $T(X)$ *such that*

$$X - T(X) + w(X - T(X))G = R(X) \quad \textit{for all} \quad X > 0.$$

Proof Using $N(X) = X - T(X)$ we obtain $N(X) + w(N(X))G = R(X)$ and $\big(1 + w'(N(X))G\big) \cdot N'(X) = R'(X)$. Since $w'(N(X)) > 0$ and $R'(X) > 1$ we obtain $0 < N'(X)$. Furthermore, as $R(\infty) = \infty$ the limit of the LHS for $X \to \infty$ has to be equal to infinity which implies that $N(X) \to \infty$ for $X \to \infty$. Furthermore, $w(X)$ is twice continuously differentiable by assumption. This proves the claim. □

Proof of Proposition 2 a, b(i), b(iv): Applying Lemma 2, proved above, and the Implicit Function Theorem to (2A) we obtain

$$\frac{\mathrm{d}T(X)}{\mathrm{d}X}=\frac{w'(X-T(X))G}{w'(X-T(X))G+1}>0 \quad \text{for} \quad X>0 \tag{A3}$$

and

$$\mathrm{d}T(X)\big/\mathrm{d}\alpha=-1\big/\big(w'(X-T(X))G+1\big)<0.$$

Since $w'(X)>0$ by Lemma 1 there exists a function $T(X)$ satisfying (2A). Now let $X=0$. Then there are three cases.

(1) $T(0)=0$. Then (2A) requires that $w(0-0)G-0=\alpha$. This condition can be satisfied only if $\alpha=0$.
(2) $T(0)>0$. This case is impossible since T has to be continuous in $X=0$ and $T(0)>0$ violates the condition $T(X)<X$ for small $X>0$.
(3) $T(0)<0$. In this case (2A) can be satisfied, i.e. $w(-T(0))G-T(0)=\alpha$. Because of (7), $T(0)$ has to be unique.

Condition (7) implies that $0<\mathrm{d}T(X)\big/\mathrm{d}X<1$ for $X>0$. Therefore, we obtain $T(X)<X$ as $T(0)<0$. In other words, there is a unique tax schedule $T(X)$ satisfying (2A). The properties of $T'(0)$ and $T'(\infty)$ are directly implied by (7). If $w(\infty)<\infty$, $T(\infty)=w(\infty)G-\alpha<\infty$, otherwise $T(\infty)=\infty$.
b(ii): We now apply the Implicit Function Theorem to (A3) and get $\frac{\mathrm{d}T'(X)}{\mathrm{d}X}=T''(X)=\frac{w''(N(X))(1-T'(X))G}{(w'(N(X))G+1)^2}$ which proves the claim.
b(iv): We differentiate (2A) with respect to X: $w'(X-T(X))G\big(1-T'(X)\big)-T'(X)=0$. Then we differentiate this equation with respect to α and obtain after rearrangement

$$\mathrm{d}T'(X)\big/\mathrm{d}\alpha=-\frac{w''(X-T(X))\big(1-T'(X)\big)G\mathrm{d}T(X)\big/\mathrm{d}\alpha}{w'(X-T(X))G+1}.$$

Now observe that $\mathrm{d}T(X)\big/\mathrm{d}\alpha<0$ and $w'>0$. □

Proof of Proposition 3 (a) Since in this case $T(0)<0$ and $T'(X)>0$ there is $\tilde{X}>0$ such that $T\big(\tilde{X}\big)=0$. By Table 1 in Ebert and Tillmann (2006) we obtain $\rho(X)<1$ for $X\in\big[0,\tilde{X}\big]$ and also for $X\in\big[0,\tilde{X}+\delta\big]$ for some $\delta>0$ because of continuity.
(b) $T'(X)$ is derived in (A3). Therefore, $1-T'(X)=1\big/\big(w'(N(X))G+1\big)$. Using the liability progression $\eta(T(X),X)$ we consider

$$\begin{aligned}
\frac{\eta(T(X),X)}{\rho(X)} &= \frac{T'(X)}{T(X)}\,\frac{X-T(X)}{1-T'(X)}\\
&= \frac{w'(N(X))G}{1+w'(N(X))G}\,\frac{1+w'(N(X))G}{1}\,\frac{X-T(X)}{T(X)}\\
&= \frac{w'(N(X))}{w(N(X))}N(X)\,\frac{w(N(X))G}{T(X)} = \varepsilon(N(X))\frac{w(N(X))G}{T(X)}\\
&= \varepsilon(N(X))\left(1+\frac{\alpha}{T(X)}\right)
\end{aligned}$$

Now observe that $\rho \gtreqless 1 \Leftrightarrow \eta/\rho \gtreqless 1$. □

Proof of Proposition 4 a, b(i), b(iv): Condition (2R) can be rewritten:

$$N(X) + w(N(X))G - (1+\lambda)X = 0 \quad \text{for all } X > 0. \tag{A4}$$

The Implicit Function Theorem yields $\frac{\mathrm{d}N(X)}{\mathrm{d}X} = \frac{1+\lambda}{w'(N(X))G+1} > 0$ for $X > 0$. Because of Lemma 2 there exists a unique function $N(X)$ satisfying (A4) and therefore $T(X) = X - N(X)$. Using (2R) directly we obtain

$$\frac{\mathrm{d}T(X)}{\mathrm{d}\lambda} = \frac{-X}{w'(N(X))G+1} < 0.$$

Now let X tend to zero. Then $N(0) + w(N(0))G = 0$. We require $N(X) \geqslant 0$ and thus $N(0) = 0 = T(0)$. Further properties of $T(X)$ can be determined from

$$\frac{\mathrm{d}T(X)}{\mathrm{d}X} = \frac{w'(N(X))G - \lambda}{w'(N(X))G+1}.$$

If $T'(\infty) = 1$, then $T(\infty) = \infty$. It is possible that $T(\infty) = -\infty$ since $T'(\infty) < 0$ is admissible.

b(ii): Applying the Implicit Function Theorem again we get

$$\frac{\mathrm{d}T'(X)}{\mathrm{d}X} = T''(X) = \frac{w''(N(X))\big(1-T'(X)\big)G(1+\lambda)}{(w'(N(X))G+1)^2}.$$

b(iii): According to our criterion we postulate that the net benefit $wG - T$ is proportional. By Proposition 2a in Ebert and Tillmann (2006) this property is guaranteed if and only if $\rho(X)(\delta(X) + (1-\delta(X))\varepsilon(X - T(X))) = 1$ for $X > 0$ where $\delta(X) = N(X)/(N(X) + w(N(X))G)$. Insertion of $\delta(X)$ and solving for $\rho(X)$ yields

$$\rho(X) = \frac{N(X) + w(N(X))G}{N(X) + w(N(X))G \cdot \varepsilon(N(X))}.$$

Therefore, $\rho(X) \gtreqless 1 \Leftrightarrow \varepsilon(N(X)) \lesseqgtr 1$.

b(iv): Differentiation of (2R) yields $w'(X - T(X))G\big(1 - T'(X)\big) - T'(X) - \lambda = 0$.

Now differentiate this equation with respect to λ and rearrange. Then

$$\begin{aligned} &w''(X - T(X))G\big(1 - T'(X)\big)\big(-\mathrm{d}T(X)/\mathrm{d}\lambda\big) - 1 \\ &\quad = dT'(X)/\mathrm{d}\lambda\big(w'(X - T(X))G + 1\big). \end{aligned}$$

Since $\mathrm{d}T(X)/\mathrm{d}\lambda < 0$ we obtain

$$\operatorname{sign} \mathrm{d}T'(X)/\mathrm{d}\lambda = \operatorname{sign}\big(w''(X - T(X))G\big(1 - T'(X)\big)\big(-\mathrm{d}T(X)/\mathrm{d}\lambda\big) - 1\big).$$

If $w'' < 0$ the sign is negative. Now suppose that $w'' > 0$. Then we observe that $\mathrm{d}T(X)/\mathrm{d}\lambda = -X/\big(w'(X - T(X))G + 1\big)$ and rewrite the expression on the right-hand side:

$$\begin{aligned} &w'' \cdot G\big(1 - T'\big) \cdot \mathrm{d}T/\mathrm{d}\lambda - 1 \\ &\quad = \frac{w''(X - T(X))(X - T(X))}{w'(X - T(X))}\left(\frac{\big(1 - T'(X)\big)X}{X - T(X)}\right)\frac{w'(X - T(X))G}{w'(X - T(X))G + 1} - 1 \\ &\quad = \eta\big(w'(X), X\big)\big|_{X = X - T(X)} \cdot \rho(X) \cdot \frac{w'(X - T(X))G}{w'(X - T(X))G + 1} - 1. \end{aligned}$$

$w''(X) > 0$ implies that $\varepsilon(X) > 1$. Then $T(X)$ is progressive. Therefore, $\rho(X) < 1$.

In this case $\eta\big(w'(X),X\big)\big|_{X = X - T(X)} \leqslant 1$ implies that this expression is negative. □

Proof of Proposition 6 (a) For $\hat{X} = \alpha/\lambda$ we obtain $\alpha = \lambda\hat{X}$ and therefore

$$w\Big(\hat{X} - T\Big(\hat{X}, \alpha\Big)\Big)G - w\Big(\hat{X} - T\Big(\hat{X}, \lambda\Big)\Big)G = T\Big(\hat{X}, \alpha\Big) - T\Big(\hat{X}, \lambda\Big).$$

If $T\Big(\hat{X}, \alpha\Big) - T\Big(\hat{X}, \lambda\Big) > 0$ then $\hat{X} - T\Big(\hat{X}, \lambda\Big) > \hat{X} - T\Big(\hat{X}, \alpha\Big)$ and the left-hand side has to be negative because of the monotonicity of w. We get a contradiction. We can argue analogously for $T\Big(\hat{X}, \alpha\Big) < T\Big(\hat{X}, \lambda\Big)$.

(b) We use the monotonicity of w. □

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