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Monotonicity in Condorcet's Jury Theorem with dependent voters

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Abstract Condorcet's Jury Theorem (CJT) provides a theoretical basis of public choice theory and political science. This paper provides an extension of CJT for random subcommittees consisting of dependent heterogeneous experts. Necessary and sufficient conditions for beneficial augmentation (reduction) of the size of a random subcommittee are provided. These results are applied in several dependency models.

1 Introduction

1.1 Historical Background

Issues related to democracy raise actual questions in any society and era. A lot of works concerned with public choice theory, social science and political science explore various aspects of these questions. Mathematics, together with other sciences, contributes its significant part in the investigation of these issues. One of the origins of these studies goes back as far as Condorcet (1785). In his *Essai sur l'application de l'analyse à la probabilité des décisions méthodiques*, he demonstrated what is known as "Condorcet's Jury Theorem", which deals with some properties of majority voting in the dichotomous case. Condorcet considered

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the following situation: a committee consisting of an odd number of members is required to select one of two alternatives, of which exactly one is correct. For example, a committee of jurors has to decide whether a certain person is guilty. The dichotomous choice model appears in a variety of areas, such as medicine, law, management and others. The classical version of CJT states that, if all decision makers are independent and have the same qualifications, expressed by the probability p of each of them to make the correct choice, where $p > 1/2$, then:

(i) The probability of the committee to make the correct choice when utilizing the simple majority rule is larger than the correctness probability p of any of its members.
(ii) The correctness probability of the committee increases monotonically to 1 as n tends to infinity.

Obviously, this implies that, if $p < 1/2$, then the conclusions are the opposite. If $p = 1/2$, then the correctness probability of the committee is always $1/2$. Part (i) is referred to as the *non-asymptotic part* of Condorcet's statement, and (ii) as the *asymptotic part*.

1.2 Previous results

In practice, the classical assumptions of CJT, namely the homogeneity of the committee and independence among the experts, are seldom realistic in most practical situations. Thus, various attempts to generalize the theorem were made in several directions. One may be concerned with heterogeneous correctness probabilities (cf. Grofman et al. (1983); Nitzan and Paroush (1985); Miller (1986); Young (1989); Paroush (1998); Berend and Paroush (1998); Kanazawa (1998)). Similarly, it is interesting to incorporate dependence among the experts in the model. The first work to show the significance of independence in the dichotomous model is due to Nitzan and Paroush (1984). From their work one can conclude that, without the assumption of independence, simple majority is not necessarily the optimal rule even if the decision makers possess identical competence. Other models of dependence among the experts were considered by Boland (1989), Berg (1993), and Ladha (1993, 1995). There have been numerous other directions in which the theorem was extended, such as hierarchical voting systems dealing with collective decisions at several stages (cf. Berg (1997); Berg and Paroush (1998)), strategic versions of CJT removing the assumption of voter truthfulness and showing that the truthfulness is not generally incentive-compatible, even when the voters have the same objective (cf. Austen-Smith and Banks (1996); Feddersen and Pesendorfer (1998)) and others.

For our purposes, one of the most interesting of these is the study of the connection between the quality of the committee's decision and the size of the committee (cf. Feld and Grofman (1984); Paroush and Karotkin (1989); Maranon (2000); Ben-Yashar and Paroush's (2000); Karotkin and Paroush (2003); Berend and Sapir (2005); Sapir (2005)).

1.3 Scope of paper

In this paper we focus on the non-strategic version of the Condorcet jury model, where the voters express their judgments truthfully. However, we omit both classical assumptions in CJT, allowing heterogeneous correctness probabilities and dependence among the experts. We extend further the ideas of Boland (1989), Ladha (1993, 1995), Ben-Yashar and Paroush (2000) and Berend and Sapir (2005). A slight adjustment of Condorcet's statement was considered by Ben-Yashar and Paroush (2000), who proved that, under the assumption of independence, the probability of a group of competence structure $(p_1, p_2, \ldots, p_n)$, with $p_i \geq 1/2$ for each i, to reach the correct decision when utilizing the simple majority rule is always larger than the probability of a random group member to do so. Later, under the same assumptions, Berend and Sapir (2005) generalized the result of Ben-Yashar and Paroush (2000) and proved the monotonicity of the probability of a correct choice as a function of the size of the (randomly selected) subcommittee. This implies that, regardless of the specific competence structure of the group, but under the assumption of independence, the augmentation of a random subcommittee is always beneficial.

This paper provides an extension of CJT for random subcommittees consisting of dependent heterogeneous experts. Omitting the independence assumption, the paper raises several questions, mainly concerned with an optimal size of a random subcommittee. We start with the question whether an augmentation of a random subcommittee is still beneficial for dependent voters. We show that, in general, the answer to this question is negative. Moreover, we illustrate in the sequel that, even if the members are uncorrelated, the augmentation is still not necessary beneficial. Our main result focuses on necessary and sufficient conditions for beneficial augmentation of the size of a random subcommittee for arbitrary dependent experts and on their intuitive meaning. We exemplify these conditions on two models of correlation, considered by Boland (1989), and provide connections between the correlation and monotonicity. For these models we propose some practical recommendations.

The rest of this paper is organized as follows: In Sect. 2 we describe the setup more carefully. Section 3 contains the main results, and Sect. 4 their proofs. In Sect. 5 we summarize and raise some ideas for further research.

2 Model and notations

We start with several notations. We have an initial committee $E_n = \{1, 2, \ldots, n\}$. For an arbitrary fixed member i, $1 \leq i \leq n$, of E_n, define a random variable X_i by

$$X_i = \begin{cases} 1, & \text{the } i\text{ th expert chooses the correct alternative,} \\ 0, & \text{the } i\text{ th expert chooses the incorrect alternative.} \end{cases} \tag{1}$$

Let $p_i = P(X_i = 1)$ be the probability of the i th expert to choose the correct alternative, and $q_i = 1 - p_i$ be his probability of making a mistake. An n-tuple of individual decisions $\vec{X} = (X_1, X_2, \ldots, X_n)$ is a *decision profile*.

Let $E = \{e_1, e_2, \ldots, e_m\}$ be a subcommittee of E_n consisting of m members, where $1 \le e_1 < e_2 < \cdots < e_m \le n$. Denote by $P_{m,j}(E) = P_{m,j}(e_1, e_2, \ldots, e_m)$, $0 \le j \le m$, the probability for exactly j of the members of E to choose the correct alternative:

$$P_{m,j}(E) = P\left(\sum_{i \in E} X_i = j\right), \qquad 0 \le j \le m.$$

For example, for $E = \{1, 2, 3\}$, we have:

$$P_{3,1}(E) = P(X_1 = 1, X_2 = 0, X_3 = 0) + P(X_1 = 0, X_2 = 1, X_3 = 0)$$

$$+P(X_1 = 0, X_2 = 0, X_3 = 1).$$

Let $P_{m,\ge j}(E) = P_{m,\ge j}(e_1, e_2, \ldots, e_m)$, $\ 0 \le j \le m$, be the probability for at least j of the members of E to choose the correct alternative, namely:

$$P_{m,\ge j}(E) = \sum_{i=j}^{m} P_{m,i}(E).$$

For example, for $E = \{1, 2, 3\}$:

$$P_{3,\ge 2}(E) = P(X_1 = 1, X_2 = 1, X_3 = 0) + P(X_1 = 1, X_2 = 0, X_3 = 1)$$

$$+ P(X_1 = 0, X_2 = 1, X_3 = 1) + P(X_1 = 1, X_2 = 1, X_3 = 1).$$

For E of odd size $m \ge 1$, let $M(E) = M(e_1, e_2, \ldots, e_m)$ be the probability of E to make the correct choice when utilizing the simple majority rule:

$$M(E) = P_{m,\ge (m+1)/2}(E).$$

For example, if $E = \{1, 2, 3\}$, then $M(E) = P_{3,\ge 2}(E)$.

$S_{n,m,j}$ will denote the sum of the $P_{m,j}(E)$'s as E ranges over all $\binom{n}{m}$ subsets of E_n of size m. Namely, denoting by $\bar{n}_m$ this collection of subsets:

$$S_{n,m,j} = \sum_{E \in \bar{n}_m} P_{m,j}(E).$$

For example,

$$S_{4,3,1} = P_{3,1}(1, 2, 3) + P_{3,1}(1, 2, 4) + P_{3,1}(1, 3, 4) + P_{3,1}(2, 3, 4).$$

Denote by $\bar{P}_{m,j}$ the probability that exactly j of the members of a randomly selected subcommittee of size m of E_n will choose the correct alternative, namely:

$$\bar{P}_{m,j} = \frac{S_{n,m,j}}{\binom{n}{m}}.$$

Similarly, let $\bar{P}_{m,\geq j},\ \ 0 \leq j \leq m$, be the probability for at least j of the members of a randomly selected subcommittee of size m of E_n to choose the correct alternative, namely:

$$\bar{P}_{m,\geq j}(E) = \sum_{i=j}^{m} \bar{P}_{m,i} = \frac{1}{\binom{n}{m}} \sum_{E \in \bar{n}_m} P_{m,\geq j}(E).$$

Denote by $\bar{M}_m$ the probability of a randomly selected subcommittee of E_n of size m to decide correctly when utilizing the simple majority rule. That is, for odd $m \geq 1$

$$\bar{M}_m = \frac{1}{\binom{n}{m}} \sum_{E \in \bar{n}_m} M(E) = \bar{P}_{m,\geq (m+1)/2}. \qquad (2)$$

For example, if $n = 4$ and $m = 3$, then

$$\bar{M}_3 = \frac{1}{4}(M(1,2,3) + M(1,2,4) + M(1,3,4) + M(2,3,4)),$$

or, equivalently,

$$\bar{M}_3 = \bar{P}_{3,\geq 2}.$$

In particular, the probability of a random committee member to decide correctly is $\bar{M}_1 = \bar{p} = (1/n)\sum_{i=1}^{n} p_i$. If n is odd, then $\bar{M}_n$ is the probability of a committee of size n, utilizing the simple majority rule, to make the correct choice.

3 The main results

The motivation for this paper derives from the results of Ben-Yashar and Paroush (2000). They considered a slight adjustment of Condorcet's statement, which is valid regardless of the specific competence structure of the group. Since the simple majority rule is well-defined only for odd-sized committees we assume, by default, that n is odd (otherwise we mention it specifically). Under the assumption of independence, they proved that the probability of a group with competence structure $(p_1, p_2, \ldots, p_n)$, where $p_i \geq 1/2$ for each i, to reach the correct decision when utilizing the simple majority rule is larger than the probability $\bar{p} = (1/n)\sum_{i=1}^{n} p_i$ of a random group member to do so. Later, under the same assumptions, Berend and Sapir (2005) generalized this result and proved the monotonicity of the probability of a correct choice as a function of the size of the selected subcommittee. Namely, they proved

Theorem 1 [Berend and Sapir (2005), Theorem 1] *Let E_n be a committee of size n with competence structure $(p_1, p_2, \ldots, p_n)$. If the members are independent and $p_i \geq 1/2$ for each i, then*

$$\begin{cases} \bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_n, & n \equiv 1 \pmod 2, \\ \bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_{n-1}, & n \equiv 0 \pmod 2. \end{cases}$$

This result raises

Question 1 Are the above conclusions still valid if the voters are dependent?

It is easy to see that the answer is negative in general. Ladha (1992) gave an example to that effect with $n = 3$. The following example generalizes his example.

Example 1 Let $n = 3$ and $0 \leq c \leq 3/4$. Suppose that the voters are exchangeable in the sense of Ladha (1993), and $P_{3,0}(E_3) = 0$, $P_{3,1}(E_3) = c$, $P_{3,2}(E_3) = \sqrt{3c} - 2c$ and $P_{3,3}(E_3) = 1 + c - \sqrt{3c}$. It is easy to see that $p_i = \bar{M}_1 = \sum_{j=1}^{3}[jP_{3,j}(E_3)]/3 = 1 - \sqrt{c/3} \geq 1/2$, $i = 1, 2, 3$, and $\bar{M}_3 = P_{3,2}(E_3) + P_{3,3}(E_3) = 1 - c$. Since $\bar{M}_3 - \bar{M}_1 = \sqrt{c/3} - c$, we easily find that

(i) $\bar{M}_3 > \bar{M}_1$ for $0 < c < 1/3$,
(ii) $\bar{M}_3 = \bar{M}_1$ for $c = 0$ and $c = 1/3$,
(iii) $\bar{M}_1 > \bar{M}_3$ for $1/3 < c \leq 3/4$.

Note that for $c \neq 0$ the voters are not independent, since

$$P_{3,3}(E_3) = 1 + c - \sqrt{3c} \neq \prod_{i=1}^{3} p_i = \left(1 - \sqrt{\frac{c}{3}}\right)^3 = 1 + c - \sqrt{3c} - \left(\sqrt{\frac{c}{3}}\right)^3.$$

However, they are uncorrelated, since for $i \neq j$:

$$\begin{aligned} \operatorname{cov}(X_i, X_j) &= E(X_i \cdot X_j) - E(X_i)E(X_j) \\ &= P(X_i = 1, X_j = 1) - P^2(X_i = 1) \\ &= \tfrac{1}{3}P_{3,2}(E_3) + P_{3,3}(E_3) - p_1^2 \\ &= \tfrac{1}{3}\left(\sqrt{3c} - 2c\right) + 1 + c - \sqrt{3c} - \left(1 - \sqrt{\tfrac{c}{3}}\right)^2 = 0. \end{aligned}$$

Note that for $c = 0.6075$ we obtain Example 1 of Ladha (1992).

Thus, the answer to Question 1 is negative even if the members are assumed to be uncorrelated.

The next example illustrates a more extreme situation for any odd n, where the probability of a random subcommittee to choose correctly decreases monotonically as the subcommittee's size increases, namely $\bar{M}_1 > \bar{M}_3 > \cdots > \bar{M}_n$.

Example 2 Consider a committee of size n. Suppose that $P_{n,n}(E_n) = c$, and that each decision profile, in which exactly $(n-1)/2$ of the committee members are correct, has the same probability $(1-c)/\binom{n}{(n-1)/2}$, where $0 < c < 1$. (We mention in passing that these probabilities make the votes exchangeable in the sense of Ladha (1993).) Thus, $P_{n,(n-1)/2}(E_n) = 1-c$, and $P_{n,j}(E_n) = 0$, for $j \neq n, (n-1)/2$. It is easy to see that $p_i = \bar{M}_1 = c + (n-1)/2n(1-c)$ for $1 \leq i \leq n$, and $\bar{M}_n = c$. Assume that $c \geq 1/(n+1)$, which condition guarantees that $p_i \geq 1/2$ for $1 \leq i \leq n$. Thus $\bar{M}_1 > \bar{M}_n$, which is in contradiction with Condorcet's belief. Moreover, in this situation:

$$\bar{M}_1 > \bar{M}_3 > \cdots > \bar{M}_n.$$

Indeed, for arbitrary fixed odd m, $1 \leq m \leq n$,

$$\bar{M}_m = c + \frac{1-c}{\binom{n}{(n-1)/2}} \sum_{j=(m+1)/2}^{m} \binom{m}{j} \binom{n-m}{[(n-1)/2]-j}$$

(where a binomial coefficient $\binom{a}{b}$ with $b < 0$ is to be taken as 0). Thus for $3 \leq m \leq n$ we have:

$$\bar{M}_{m-2} - \bar{M}_m = \frac{1-c}{\binom{n}{(n-1)/2}} \left(\sum_{j=(m-1)/2}^{m-2} \binom{m-2}{j} \binom{n-m+2}{(n-1)/2-j} - \sum_{j=(m+1)/2}^{m} \binom{m}{j} \binom{n-m}{(n-1)/2-j} \right).$$

To prove that $\bar{M}_{m-2} > \bar{M}_m$ we will use the identity

$$\binom{a+2}{b} = \binom{a}{b} + 2\binom{a}{b-1} + \binom{a}{b-2}$$

for $a = n-m$ and $b = [(n-1)/2] - j$ to obtain:

$$\sum_{j=(m-1)/2}^{m-2} \binom{m-2}{j} \binom{n-m+2}{[(n-1)/2]-j}$$
$$= \sum_{j=(m-1)/2}^{m-2} \binom{m-2}{j} \binom{n-m}{[(n-1)/2]-j}$$

$$+2\sum_{j=(m-1)/2}^{m-2}\binom{m-2}{j}\binom{n-m}{[(n-1)/2]-j-1}$$
$$+\sum_{j=(m-1)/2}^{m-2}\binom{m-2}{j}\binom{n-m}{[(n-1)/2]-j-2}. \quad (3)$$

Similarly, for $a=m-2$ and $b=j$ we have:

$$\begin{aligned}\sum_{j=(m+1)/2}^{m}\binom{m}{j}\binom{n-m}{[(n-1)/2]-j} &= \sum_{j=(m+1)/2}^{m}\binom{m-2}{j}\binom{n-m}{[(n-1)/2]-j}\\ &\quad +2\sum_{j=(m+1)/2}^{m}\binom{m-2}{j-1}\binom{n-m}{[(n-1)/2]-j}\\ &\quad +\sum_{j=(m+1)/2}^{m}\binom{m-2}{j-2}\binom{n-m}{[(n-1)/2]-j}\\ &= \sum_{j=(m+1)/2}^{m}\binom{m-2}{j}\binom{n-m}{[(n-1)/2]-j}\\ &\quad +2\sum_{j=(m-1)/2}^{m-1}\binom{m-2}{j}\binom{n-m}{[(n-1)/2]-j-1}\\ &\quad +\sum_{j=(m-3)/2}^{m-2}\binom{m-2}{j}\binom{n-m}{[(n-1)/2]-j-2}.\end{aligned} \quad (4)$$

By (3) and (4):

$$\begin{aligned}&\sum_{j=(m-1)/2}^{m-2}\binom{m-2}{j}\binom{n-m+2}{[(n-1)/2]-j}-\sum_{j=(m+1)/2}^{m}\binom{m}{j}\binom{n-m}{[(n-1)/2]-j}\\ &\quad =\binom{m-2}{(m-1)/2}\binom{n-m}{(n-m)/2}-\binom{m-2}{(m-3)/2}\binom{n-m}{(n-m-2)/2}>0.\end{aligned}$$

Hence the probability of a random subcommittee to choose correctly decreases monotonically as the subcommittee's size increases.

Suppose we have a committee of size n, and want to choose a subcommittee which will make its decisions by majority vote. The polar options are to take all the n members and to take but one of them. Naturally, anyone who believes in democratic traditions would hope that the first option is preferable. As mentioned earlier, Ben-Yashar and Paroush's (2000) result confirms this belief for independent voters (with $p_i \geq 1/2$ for each i). As we have seen in Examples 1 and 2, the assumption of independence is critical. This raises

Question 2 Under what conditions is the non-asymptotic part of Condorcet's statement, namely $\bar{M}_n \geq \bar{M}_1$, valid?

Partial answers were provided by several authors for special models of dependence (cf. Boland (1989), Berg (1993)). The following result, which contains a complete answer to this question, is implicit in Ladha (1993). (See the proof of Lemma 2 in his paper.)

Proposition 1 [Ladha 1993] *For a group E_n of an odd size n with any competence structure $(p_1, p_2, \ldots, p_n)$, we have*

$$\bar{M}_n \geq \bar{M}_1$$

if and only if

$$\sum_{j=1}^{(n-1)/2} j(P_{n,n-j}(E_n) - P_{n,j}(E_n)) \geq 0. \tag{5}$$

Note that the values $P_{n,n}(E_n)$ and $P_{n,0}(E_n)$ are immaterial in (5). The reason is that the first of these contributes all its value to both $\bar{M}_n$ and $\bar{M}_1$ (as well to any $\bar{M}_j$), whereas the latter contributes to none.

Intuitively, condition (5) means that there is a bias towards having more members voting correctly rather than incorrectly. In particular, Proposition 1 immediately implies

Corollary 1 *If for each "small" $j \in \{1, 2, \ldots, (n-1)/2\}$ the probability of the number of correct opinions to be j does not exceed that of the number of correct opinions to be $n-j$, namely*

$$P_{n,n-j}(E_n) \geq P_{n,j}(E_n), \qquad 1 \leq j \leq \frac{n-1}{2}, \tag{6}$$

then

$$\bar{M}_n \geq \bar{M}_1.$$

As mentioned above, for independent juries with $p_i \geq 1/2, \quad 1 \leq i \leq n$, the probability of a random subcommittee to choose correctly increases monotonically as a function of its size. Using continuity, it is clear, if the voters are "almost" independent, then we still have monotonicity. Another simple corollary of Theorem 1 is

Corollary 2 *Let $j \in \{3, 5, \ldots, n\}$ be an arbitrary fixed number. If in E_n every j experts are independent, then*

$$\bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_{j-2} \leq \bar{M}_j. \tag{7}$$

Thus, having only partial information about the dependence among the experts, (knowing only that every j experts in the committee are independent), Corollary 7 provides some "recommendations" about the size of the random subcommittee. However, this information is insufficient to determine the optimal size of the (random) subcommittee. In particular, both of the two polar situations

$$\bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_{j-2} \leq \bar{M}_j \leq \bar{M}_{j+2} \leq \cdots \leq \bar{M}_{n-2} \leq \bar{M}_n$$

and

$$\bar{M}_n \leq \bar{M}_{n-2} \leq \cdots \leq \bar{M}_{j+2} \leq \bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_{j-2} \leq \bar{M}_j$$

are compatible with this assumption.

The following example illustrates such a phenomenon.

Example 3 We illustrate the two polar situations for $n = 5$. In both of the two cases below, every three experts in the committee are independent, but all the five members are dependent.

Case 1 Suppose that $P_{5,5}(E_5)=0.024$, $P_{5,4}(E_5)=0.48$, $P_{5,3}(E_5)=0, P_{5,2}(E_5)=0.48$, $P_{5,1}(E_5) = 0$, $P_{5,0}(E_5) = 0.016$. Also suppose that the voters are exchangeable in the sense of Ladha (1993). Clearly, $p_i = P(X_i = 1) = 0.6$ for $1 \leq i \leq 5$, so that $\bar{M}_1 = 0.6$, and

$$P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1, X_5 = 1) = P_{5,5}(E_5) = 0.024 \neq \prod_{i=1}^{n} p_i.$$

Thus, the members of the committee are dependent. Since the votes are exchangeable, to prove the independence of any three committee members, it suffices to verify that $P(X_1 = 1, X_2 = 1) = {p_1}^2$ and $P(X_1 = 1, X_2 = 1, X_3 = 1) = {p_1}^3$. Indeed, we have:

$$\begin{aligned} P(X_1 = 1, X_2 = 1) &= P_{5,5}(E_5) + \frac{3}{5} P_{5,4}(E_5) + \frac{3}{10} P_{5,3}(E_5) + \frac{1}{10} P_{5,2}(E_5) \\ &= 0.36 = 0.6^2 = p_1^2, \end{aligned}$$

and

$$\begin{aligned} P(X_1 = 1, X_2 = 1, X_3 = 1) &= P_{5,5}(E_5) + \frac{2}{5} P_{5,4}(E_5) + \frac{1}{10} P_{5,3}(E_5) \\ &= 0.216 = 0.6^3 = {p_1}^3. \end{aligned}$$

By symmetry:

$$\bar{M}_3 = 3P(X_1 = 1, X_2 = 1, X_3 = 0) + P(X_1 = 1, X_2 = 1, X_3 = 1)$$
$$= 3\left(\frac{1}{5}P_{5,4}(E_5) + \frac{2}{10}P_{5,3}(E_5) + \frac{1}{10}P_{5,2}(E_2)\right) + {p_1}^3 = 0.648,$$

and

$$\bar{M}_5 = P_{5,3}(E_5) + P_{5,4}(E_5) + P_{5,5}(E_5) = 0.504.$$

Thus:

$$\bar{M}_5 = 0.504 < \bar{M}_1 = 0.6 < \bar{M}_3 = 0.648.$$

Case 2 Suppose that $P_{5,5}(E_5)=0.1$, $P_{5,4}(E_5)=0.16$, $P_{5,3}(E_5)=0.52, P_{5,2}(E_5)=0.08$, $P_{5,1}(E_5)=0.14$, $P_{5,0}(E_5)=0$. Also suppose that the voters exchangeable in the sense of Ladha (1993). As in Case 1, a routine calculation shows that the members are dependent, although any three of them are independent, but this time:

$$\bar{M}_1 = 0.6 < \bar{M}_3 = 0.648 < \bar{M}_5 = 0.78.$$

Now let us focus on a minimal augmentation (namely, by two members) of a random subcommittee's size. The following theorem provides necessary and sufficient conditions for such an augmentation to be beneficial.

Theorem 2 *For a group E_n of size n with any competence structure $(p_1, p_2, \ldots, p_n)$ and arbitrarily fixed odd m, $3 \le m \le n$, the following three conditions are equivalent:*

(i) $\bar{M}_m \ge \bar{M}_{m-2}$.

(ii) $\bar{P}_{m,(m+1)/2} \ge \bar{P}_{m,(m-1)/2}$.

(iii) $$\sum_{j=(m-1)/2}^{(n-1)/2} c_{j,m}(P_{n,n-j}(E_n) - P_{n,j}(E_n)) \ge 0,$$

where $c_{j,m} = \frac{n-2j}{n-j-[(m-1)/2]} \cdot \binom{j}{(m-1)/2}\binom{n-j}{(m+1)/2}$
for $j = (m-1)/2, \ (m+1)/2, \ldots, (n-1)/2$.

We shall exemplify, on two versions of a model suggested by Boland (1989), how these conditions can be verified, and understand better their intuitive significance. In both models, the committee consists of regular members and a "leader". In Boland's first model the leader is part of the decision making team, while in the second model the leader has external influence on the voters, but does not participate in the voting itself. For example, in the first model the leader may be thought as the chairman of the committee, and in the second it may be advertising or promotional campaign, which effects the voters without being a voter itself. Let X_i be a random variable as in (1), corresponding to the i th regular expert, and let Y be the analogous random variable for the leader. In the first model, the outcome of the voting is determined by the variables Y and X_i, $1 \le i \le n-1$, and in the second – by the variables X_i, $1 \le i \le n$. Assume that Y takes values 1 and 0 with probabilities p and q, respectively, and that the X_i's are independent given Y, with

$$P(X_i = 1|Y = 1) = p + rq,$$

and

$$P(X_i = 1|Y = 0) = p - rp,$$

for a certain parameter r. Note that each X_i is distributed as Y, namely $B(1,p)$, and that r is related to the correlations between the variables: $\rho(X_i, Y) = r$, and $\rho(X_i, X_j) = r^2$, $i \neq j$. To have all probabilities in the range [0, 1], we need to require that $-q/p \le r \le 1$.

Boland (1989) proved an extended version of CJT for both models. He showed that, in each of them, for positive correlation r between the regular experts and the leader, the probability of the majority rule to yield the correct answer decreases as a function of r. However, in each model, for $p > 1/2$ and $0 \le r < 1$, the group competence is still greater than the individual ones, namely, $\bar{M}_1 < \bar{M}_n$, and thus the non-asymptotic part of CJT is still valid.

Let us start with Boland's first model. For $n = 3$ we generalize his result, allowing any value of r. Indeed, to check (ii) or (iii) of Theorem 2, for $p \ge 1/2$ and $-q/p \le r < 1$, we compute:

$$\begin{aligned} P_{3,2}(E_3) - P_{3,1}(E_3) &= 2p(p+rq)(q-rq) + q(p-rp)^2 \\ &\quad -2q(q+rp)(p-rp) - p(q-rq)^2 \\ &= 3pq(1-r)^2(p-q) > 0. \end{aligned}$$

Thus, condition (iii) in Theorem 2 holds, and therefore $\bar{M}_1 < \bar{M}_3$ for any r.

Let us turn to $n = 5$. In principle, there may be six possible rankings of the $\bar{M}_i$'s:

(a) $\bar{M}_1 \le \bar{M}_3 \le \bar{M}_5$,
(b) $\bar{M}_1 \le \bar{M}_5 \le \bar{M}_3$,
(c) $\bar{M}_5 \le \bar{M}_1 \le \bar{M}_3$,
(d) $\bar{M}_5 \le \bar{M}_3 \le \bar{M}_1$,
(e) $\bar{M}_3 \le \bar{M}_1 \le \bar{M}_5$,
(f) $\bar{M}_3 \le \bar{M}_5 \le \bar{M}_1$.

We claim that, in fact, we necessarily have $\bar{M}_1 \le \bar{M}_3$, so that the chains in (d), (e), and (f) are impossible (unless at least two of the $\bar{M}_i$'s coincide). As the computations required to verify these claims are routine, we shall present a single case as an example. Checking Theorem 2(iii), we obtain that $\bar{M}_3 \le \bar{M}_5$

Table 1 All monotonicity chains in Boland's first model for $n = 5$

p	r						
	$[-1, -0.78)$	$[-0.78, -0.77)$	$[-0.77, -0.65)$	$[-0.65, -0.6)$	$[-0.6, 0.4)$	$[0.4, 0.52)$	$[0.52, 1)$
(ps*)	(c)	(b)	(b)	(a)	(a)	(a)	(b)
(pl*)		(c)		(b)		(b)	

if and only if $p(1-p) \geq r(r-0.4)/(1-r)^2$. Thus, for $0 \leq r \leq 0.4$ we have $\bar{M}_3 \leq \bar{M}_5$, which, combined with the inequality $\bar{M}_1 \leq \bar{M}_3$, immediately implies chain (a). Similarly, for all other values of r we can show that the ordering for this model is as in Table 1. More precisely, put:

$$r_1 = -\frac{7}{9} \approx -0.78, \qquad r_2 = -\frac{1+\sqrt{13}}{6} \approx -0.77,$$

$$r_3 = -\frac{1+\sqrt{76}}{15} \approx -0.65, \quad r_4 = \frac{-1+\sqrt{76}}{15} \approx 0.52,$$

and

$$p^*(r) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{\frac{7+9r}{3(1-r)}}, & r_1 \leq r < r_2, \\ \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4r(r-0.4)}{(1-r)^2}}, & r_3 \leq r < -0.6 \bigvee 0.4 \leq r < r_4, \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

Clearly, for $p = 0.5$ and $p = 1$ we have $\bar{M}_1 = \bar{M}_3 = \bar{M}_5$. For arbitrary fixed $-1 \leq r < 1$, the condition $r \geq -q/p$ implies $p \leq 1/(1-r)$. We shall distinguish between "small" and "large" values of p:

(ps*) $0.5 < p < p^*(r)$,
(pl*) $p^*(r) \leq p < \min(1/(1-r), 1)$.

(In fact, in case (pl*) we allow p to assume the value $1/(1-r)$ when $r < 0$, but not the value 1, which was selected above, when $r \geq 0$.) With these notations, Table 1 provides all possible monotonicity chains for Boland's first model for $n = 5$. (The empty cells in the table correspond to empty sets.)

The table confirms, in a quantitative way, a few natural beliefs. The case $r = 0$ (for Boland's model) is the independent case, in which we have monotonicity [chain (a)], and in particular the non-asymptotic part of CJT holds ($\bar{M}_1 \leq \bar{M}_5$ – chains (a) and (b) in this model). Intuitively, it is clear that, if the correlation r is "sufficiently small," (i.e. $-0.6 \leq r < 0.4$), then voters are "almost" independent

Table 2 All monotonicity chains in Boland's second model for $n = 5$

p	r					
	$[-1, -0.5)$	$[-0.5, -0.45)$	$[-0.45, -0.39)$	$[-0.39, 0.5)$	$[0.5, 0.64)$	$[0.64, 1)$
(ps**)		(c)				
(pm**)	(d)	(b)	(b)	(a)	(a)	(b)
(pl**)		(a)	(a)		(b)	

and the augmentation of the random subcommittee will be always beneficial [chain (a)]. Since the condition $\bar{M}_1 \leq \bar{M}_5$ is weaker than that of monotonicity, it seems clear that the non-asymptotic part of CJT is valid for a much wider range of r, which contains both positive and negative values of the correlation (as confirmed by the table).

Table 2 illustrates the results for Boland's second model. The picture is "quite similar" to the previous one. However, for a "sufficiently negative" correlation r, in this model we may obtain a situation where enlargement of the subcommittee is always non-beneficial, namely $\bar{M}_1 \geq \bar{M}_3 \geq \cdots \geq \bar{M}_n$. In particular, for $n = 3$ the inequality $\bar{M}_1 < \bar{M}_3$ holds if and only if $r > -0.5$. Similarly, for $n = 5$ denote

$$r_5 = \frac{-21 + \sqrt{105}}{24} \approx -0.45, \;\; r_6 = \frac{1 - \sqrt{17}}{8} \approx -0.39, \;\; r_7 = \frac{1 + \sqrt{17}}{8} \approx 0.64,$$

$$p_{**}(r) = \begin{cases} \dfrac{1}{2} + \sqrt{\dfrac{12r^2 + 21r + 7}{12(1-r)(1+4r)}}, & -0.5 \leq r < r_5, \\[2ex] \dfrac{1}{2}, & \text{otherwise}, \end{cases} \tag{9}$$

and

$$p^{**}(r) = \begin{cases} \dfrac{1}{2} + \sqrt{\dfrac{(1+r)(-4r^2 + r + 1)}{4(1-r)^2(1+4r)}}, & -0.5 \leq r < r_6 \bigvee 0.5 \leq r < r_7, \\[2ex] 1, & \text{otherwise}. \end{cases} \tag{10}$$

Clearly, for $p = 0.5$ and $p = 1$ we have $\bar{M}_1 = \bar{M}_3 = \bar{M}_5$. For arbitrary fixed $-1 \leq r < 1$, distinguish between small, medium and large $p \in \left[0.5, \min\left(1/(1-r), 1\right)\right]$:

(ps^{**}) $0.5 < p \leq p_{**}(r)$,
(pm^{**}) $p_{**}(r) \leq p < p^{**}(r)$,
(pl^{**}) $p^{**}(r) \leq p < \min\left(1/(1-r), 1\right)$.

(In fact, in case (pl**) we allow p to assume the value $1/(1-r)$ when $r < 0$, but not the value 1, which was selected above, when $r \geq 0$.) With these notations Table 2 provides all possible monotonicity chains for Boland's second model for $n = 5$. (The empty cells in the table correspond to empty sets).

Finally, note that in both models there exists a wide range of values of the correlation r around 0, for which an augmentation of the subcommittee is always beneficial, namely $\bar{M}_1 \leq \bar{M}_3 \leq \bar{M}_5$.

We return to the discussion of the general case. The equivalence of Theorem 2(i) and (iii) immediately implies

Corollary 3

1. *If* $P_{n,n-j}(E_n) \geq P_{n,j}(E_n)$ *for* $1 \leq j \leq (n-1)/2$, *then an augmentation of the (random) subcommittee is always beneficial, namely* $\bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_n$.
2. *If* $P_{n,n-j}(E_n) \leq P_{n,j}(E_n)$ *for* $1 \leq j \leq (n-1)/2$, *then an augmentation of the (random) subcommittee is always non-beneficial, namely* $\bar{M}_1 \geq \bar{M}_3 \geq \cdots \geq \bar{M}_n$.

In the following two propositions, we apply Corollary 3 to obtain, for both of Boland's models, monotonicity in certain regions.

Proposition 2 *Under the assumptions of Boland's first model, if*

1. $-\frac{1}{2} \leq r \leq 0, \qquad 1 - \frac{1}{2(1-r)} \leq p \leq \frac{1}{1-r},$
   or
2. $0 < r \leq \frac{1}{1+e} \approx 0.27, \qquad \frac{e}{(1+e)(1-r)} \leq p \leq 1,$
   then
   $\bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_n$.

Proposition 3 *Under the assumptions of Boland's second model, if*

1. $-\frac{1}{2} \leq r \leq 0, \qquad 1 - \frac{1}{2(1-r)} \leq p \leq \frac{1}{1-r},$
   or
2. $0 < r \leq \frac{1}{2}, \qquad \frac{1}{2(1-r)} \leq p \leq 1,$
   then
   $\bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_n$.

This means that there exists a range of values of the correlation r around 0, for which an augmentation of the subcommittee is always beneficial. If the correlation r belongs to this range, then it is best to collect all the experts opinions and follow the majority. In particular, in both cases the monotonicity implies the validity of the non-asymptotic part of CJT, namely that $\bar{M}_1 \leq \bar{M}_n$, as long as $r \geq -0.5$. Recall that Boland (1989) proved that $\bar{M}_1 \leq \bar{M}_n$ for any $r \geq 0$. Note that, already in our analysis of Boland's model for $n = 5$, we have seen that the inequality $\bar{M}_1 \leq \bar{M}_n$ does not necessarily implies monotonicity, i.e., it is possible to have, say, $\bar{M}_1 \leq \bar{M}_n \leq \bar{M}_i$ for some odd i, $3 \leq i \leq n-2$.

The equivalence of Theorem 2(i) and (ii) immediately implies

Corollary 4 *For each* $1 \le i \le n$, *denote by* $\bar{n}_{i,m}$ *the set of all subcommittees of given odd size* $m \le n$, *containing the* ith *expert of the given committee* E_n. *Let* $E_{i,j,m}$, $1 \le j \le \binom{n-1}{m-1}$, *be the subcommittees belonging to* $\bar{n}_{i,m}$. *For an arbitrary fixed* j, *let* $A_j(i)$ *be the event whereby the opinions in the subcommittee* $E_{i,j,m} \backslash \{i\}$, *are split evenly. If for all* $1 \le i \le n$ *we have*

$$P(X_i = 1 | A_j(i)) \ge \frac{1}{2}, \qquad 1 \le j \le \binom{n-1}{m-1}, \tag{11}$$

then

$$\bar{M}_m \ge \bar{M}_{m-2}.$$

The intuitive meaning of condition (11) is as follows. Any single committee member is usually "unimportant". In fact, his vote makes a difference only when all other members are evenly split. Condition (11) ensures that each member has a positive contribution in any situation in which his vote is important. That is, if he is a member of any subcommittee, at which the other members are evenly split, there is a probability of at least 1/2 that his vote will bring about the correct decision. Thus, Corollary 4 implies the following far-reaching generalization of the monotonicity result of Berend and Sapir (2005).

Corollary 5 *If condition* (11) *takes place for all odd* m, $3 \le m \le n$, *then*

$$\bar{M}_1 \le \bar{M}_3 \le \cdots \le \bar{M}_n.$$

Note that, for the condition in Corollary 5 to hold, we only need each member to have a probability at least 1/2 of being correct, given that in any arbitrary fixed subset of the committee the opinions are evenly split. Thus, members are only required to have good decisional ability (i.e., probability at least 1/2) when they are not pressed in any direction by the opinions of others. If their vote is negatively effected only when a majority of the other members are of some opinion, the performance of a random subcommittee is still better is still better as the size of the subcommittee grows.

4 Proofs

In this section we prove all the results of the paper, except for those which were proved in other papers, and those which follow easily from the other results.

Proof of Theorem 2 (i) $\Leftrightarrow$ *(ii)* Viewing a random subcommittee of size $m-2$ of E_n as a random subcommittee of size m from which two random members i and k are dropped, we derive from (2):

$$\bar{M}_{m-2} = \frac{1}{\binom{n}{m-2}} \sum_{E \in \bar{n}_m} \frac{1}{\binom{n-(m-2)}{2}} \sum_{\{i,k\} \subseteq E} \sum_{j=(m-1)/2}^{m-2} P_{m-2,j}(E \backslash \{i,k\}).$$

Since $\binom{n}{m-2}\binom{n-(m-2)}{2} = \binom{n}{m}\binom{m}{2}$, we have

$$\bar{M}_{m-2} = \frac{1}{\binom{n}{m}\binom{m}{2}} \sum_{E\in\bar{n}_m} \sum_{\{i,k\}\subseteq E} \sum_{j=(m-1)/2}^{m-2} P_{m-2,j}(E\backslash\{i,k\}).$$

Writing each $P_{m-2,j}(E\backslash\{i,k\})$ as a sum

$$\begin{aligned} &P_{m-2,j}(E\backslash\{i,k\}) \\ &\quad = P\left(\sum_{s\in E\backslash\{i,k\}} X_s = j, X_i = 0, X_k = 0\right) + P\left(\sum_{s\in E\backslash\{i,k\}} X_s = j, X_i = 1, X_k = 0\right) \\ &\qquad + P\left(\sum_{s\in E\backslash\{i,k\}} X_s = j, X_i = 0, X_k = 1\right) + P\left(\sum_{s\in E\backslash\{i,k\}} X_s = j, X_i = 1, X_k = 1\right), \end{aligned}$$

we arrive at

$$\begin{aligned} \bar{M}_{m-2} = \frac{1}{\binom{n}{m}\binom{m}{2}} \sum_{E\in\bar{n}_m} \Bigg[&\binom{\frac{m+1}{2}}{2} P_{m,(m-1)/2}(E) \\ &+ \left(\binom{(m-1)/2}{2} + \binom{(m-1)/2}{1}\binom{(m+1)/2}{1}\right) P_{m,(m+1)/2}(E) \\ &+ \sum_{\text{odd } t=3}^{n} \left(\binom{(m-t)/2}{2} + \binom{(m-t)/2}{1}\binom{(m+t)/2}{1}\right. \\ &\qquad \left. + \binom{(m+t)/2}{2}\right) P_{m,(m+t)/2}(E) \Bigg]. \end{aligned}$$

One can easily check that for odd t, $1 \le t \le m$, we have:

$$\binom{(m-t)/2}{2} + \binom{(m-t)/2}{1}\binom{(m+t)/2}{1} + \binom{(m+t)/2}{2} = \binom{m}{2}.$$

Thus

$$\begin{aligned} \bar{M}_{m-2} = \frac{1}{\binom{n}{m}\binom{m}{2}} \sum_{E\in\bar{n}_m} \Bigg[&\binom{(m+1)/2}{2} P_{m,(m-1)/2}(E) \\ &+ \left(\binom{m}{2} - \binom{(m+1)/2}{2}\right) P_{m,(m+1)/2}(E) + \binom{m}{2} \sum_{j=(m+3)/2}^{m} P_{m,j}(E) \Bigg]. \end{aligned}$$

Using (2) we obtain:

$$\begin{aligned}\bar{M}_m - \bar{M}_{m-2} &= \frac{\binom{(m+1)/2}{2}}{\binom{n}{m}\binom{m}{2}} \sum_{E \in \bar{n}_m} \left(P_{m,(m+1)/2}(E) - P_{m,(m-1)/2}(E)\right) \\ &= \frac{\binom{(m+1)/2}{2}}{\binom{m}{2}} \left(\bar{P}_{m,(m+1)/2} - \bar{P}_{m,(m-1)/2}\right),\end{aligned}$$

which proves the equivalence of (i) and (ii).

Now we will show that (ii) $\Leftrightarrow$ (iii). Recall that $\bar{P}_{m,(m+1)/2}$ is the probability that exactly $(m+1)/2$ experts in a random subcommittee of size m make the right choice. Each decision profile, which contains exactly $(m+1)/2$ correct individual decision of an arbitrary fixed subcommittee $E \in \bar{n}_m$ of size m, we complete by the all possible 2^{n-m} decisions of the complementary $n-m$ members of $E_n \setminus E$. We do it for each subcommittee $E \in \bar{n}_m$. This yields:

$$\bar{P}_{m,(m+1)/2} = \frac{1}{\binom{n}{m}} \sum_{j=(m+1)/2}^{n-(m-1)/2} \binom{j}{(m+1)/2}\binom{n-j}{(m-1)/2} P_{n,j}(E_n).$$

In the same way:

$$\bar{P}_{m,(m-1)/2} = \frac{1}{\binom{n}{m}} \sum_{j=(m-1)/2}^{n-(m+1)/2} \binom{j}{(m-1)/2}\binom{n-j}{(m+1)/2} P_{n,j}(E_n).$$

By the last two equalities:

$$\begin{aligned}\bar{P}_{m,(m+1)/2} - \bar{P}_{m,(m-1)/2} = \frac{1}{\binom{n}{m}} \sum_{j=(m-1)/2}^{n-(m-1)/2} &\left(\binom{j}{(m+1)/2}\binom{n-j}{(m-1)/2}\right. \\ &\left. - \binom{j}{(m-1)/2}\binom{n-j}{(m+1)/2}\right) P_{n,j}.\end{aligned}$$

Since for $(m-1)/2 \le j \le (n-1)/2$ all coefficients of $P_{n,j}(E_n)$ in the last sum are negative and all the coefficients of $P_{n,n-j}(E_n)$ are positive, and for each $(m-1)/2 \le j \le (n-1)/2$ the terms $P_{n,n-j}(E_n)$ and $-P_{n,j}(E_n)$ have the same coefficient $c_{j,m}$, we obtain

$$\bar{P}_{m,(m+1)/2} - \bar{P}_{m,(m-1)/2} = \frac{1}{\binom{n}{m}} \sum_{j=(m-1)/2}^{(n-1)/2} c_{j,m}(P_{n,n-j}(E_n) - P_{n,j}(E_n)),$$

which completes the proof.

Proof of Corollary 4 Clearly, (11) is equivalent to the condition, whereby for any expert $e_i \in E_n,\ \ 1 \le i \le n$, we have:

$$P(X_i = 1 \cap A_j(i)) \ge P(X_i = 0 \cap A_j(i)), \quad 1 \le j \le \binom{n-1}{m-1}.$$

Summing over all i and j, we obtain

$$\frac{1}{\binom{n}{m}} \cdot \sum_{i=1}^{n} \sum_{j=1}^{\binom{n-1}{m-1}} P(X_i = 1 \cap A_j(i)) \ge \frac{1}{\binom{n}{m}} \cdot \sum_{i=1}^{n} \sum_{j=1}^{\binom{n-1}{m-1}} P(X_i = 0 \cap A_j(i)),$$

that is

$$\frac{m+1}{2} \bar{P}_{m,(m+1)/2} = \frac{m+1}{2} \bar{P}_{m,(m-1)/2}.$$

It follows that condition (ii) of Theorem 2 holds, which completes the proof.

Proof of Proposition 2 Denote $q = 1 - p, \quad \alpha = p + rq$ and $\beta = (1 - r)p$. One can easily verify that

$$p(1-\alpha) = q\beta, \quad \alpha \le p + q = 1,$$

$$r = \alpha - \beta, \quad \beta \le (1 + q/p)p = 1.$$

With these notations, for each $1 \le j \le (n-1)/2$ we have

$$\begin{aligned}
&P_{n,n-j}(E_n) - P_{n,j}(E_n) \\
&\quad = p\alpha^{j-1}(1-\alpha)^j \cdot \left(\binom{n-1}{j} \alpha^{n-2j} - \binom{n-1}{j-1} (1-\alpha)^{n-2j} \right) \\
&\qquad - q(1-\beta)^{j-1}\beta^j \cdot \left(\binom{n-1}{j} (1-\beta)^{n-2j} - \binom{n-1}{j-1} \beta^{n-2j} \right) \\
&\quad = q\beta \cdot ((1-\beta)\beta)^{j-1} \\
&\qquad \times \left[\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{j-1} \cdot \left(\binom{n-1}{j} \alpha^{n-2j} - \binom{n-1}{j-1} (1-\alpha)^{n-2j} \right) \right. \\
&\qquad\quad \left. - \left(\binom{n-1}{j} (1-\beta)^{n-2j} - \binom{n-1}{j-1} \beta^{n-2j} \right) \right]. \qquad (12)
\end{aligned}$$

Now we show that the left-hand side of (12) is non-negative under the conditions given in the proposition. Indeed, we have:

$$\frac{\alpha(1-\alpha)}{\beta(1-\beta)} = 1 - \frac{r(p-q)}{p(q+rp)}.$$

If condition (1) is satisfied, then $-q/p \le r \le 0$ and $\alpha \ge 1/2$. Thus $\alpha(1-\alpha) \ge \beta(1-\beta)$, and $\binom{n-1}{j}\alpha^{n-2j} - \binom{n-1}{j-1}(1-\alpha)^{n-2j} > 0$ for $1 \le j \le (n-1)/2$. Since $p \ge q$, we have

$$\alpha + \beta = p + r(1-p) + p(1-r) = 2p(1-r) + r \ge 2(1/2)(1-r) + r = 1,$$

which implies $\alpha \ge 1-\beta$ and $1-\alpha \le \beta$. Using these inequalities in the right-hand side of (12), we obtain that, under (1), we have $P_{n,n-j}(E_n) - P_{n,j}(E_n) \ge 0$. By Corollary 3.1, this completes the first part of the proposition.

Let us turn to the second part. We will show that, if (2) is satisfied, then

$$\binom{n-1}{j}(1-\beta)^{n-2j} - \binom{n-1}{j-1}\beta^{n-2j} < 0, \quad 1 \le j \le \frac{n-1}{2},$$

which is equivalent to

$$\frac{\beta}{1-\beta} > \left(\frac{n-j}{j}\right)^{1/(n-2j)}, \quad 1 \le j \le \frac{n-1}{2}. \tag{13}$$

Indeed, if (2) is satisfied, then we have $e/(1+e) \le p(1-r) \le p + r(1-p) \le 1$, which is equivalent to $e/(1+e) \le \beta \le \alpha \le 1$. Obviously, the function $h(\beta) = \beta/(1-\beta)$ is decreasing and $h\left(e/(1+e)\right) = e$. Hence, for $1 \le j \le (n-1)/2$,

$$\frac{\beta}{1-\beta} \ge e \ge e^{1/j} > \left(\left(1 + \frac{1}{j/(n-2j)}\right)^{j/(n-2j)}\right)^{1/j} = \left(\frac{n-j}{j}\right)^{1/(n-2j)},$$

which proves (13). Thus $P_{n,n-j}(E_n) - P_{n,j}(E_n) \ge 0$ for $1 \le j \le (n-1)/2$, which again by Corollary 3.1 completes the proof.

Proof of Proposition 3 In the second model of Boland we have for $1 \le j \le (n-1)/2$,

$$\begin{aligned} P_{n,n-j}(E_n) - P_{n,j}(E_n) &= \binom{n}{j} p\alpha^j (1-\alpha)^j \left(\alpha^{n-2j} - (1-\alpha)^{n-2j}\right) \\ &\quad + \binom{n}{j} q\beta^j (1-\beta)^j \left(\beta^{n-2j} - (1-\beta)^{n-2j}\right). \end{aligned}$$

Under condition (1) we have $1/2 \le p + (1-p)r \le p(1-r) \le 1$, which is equivalent to $1/2 \le \alpha \le \beta \le 1$. This implies

$$P_{n,n-j}(E_n) - P_{n,j}(E_n) \ge 0, \quad 1 \le j \le \frac{n-1}{2},$$

which proves the first part.

If condition (2) is satisfied, then $1/2 \leq p(1-r) \leq p + (1-p)r \leq 1$, which is equivalent to $1/2 \leq \beta \leq \alpha \leq 1$. This implies

$$P_{n,n-j}(E_n) - P_{n,j}(E_n) \geq 0, \quad 1 \leq j \leq \frac{n-1}{2},$$

and thereby completes the proof.

5 Summary and conclusions

Finally, let us raise the following principal question:

Question 3 Is independence among decision makers critical for democracy?

In democracy all members of the decision body must have the same decisional power. Thus, the strongest form of democracy in our terms means augmenting the decision making body as much as possible. That is, we are concerned with the question:

When is the augmentation always beneficial, namely $\bar{p} = \bar{M}_1 \leq \bar{M}_3 \leq \cdots \leq \bar{M}_n$?

Theorem 1 of Berend and Sapir (2005) proved this monotonicity for independent experts. Theorem 2 implies necessary and sufficient conditions for such monotonicity for arbitrary-dependent experts. We exemplify these conditions on two models of correlation, considered by Boland (1989), and provide connections between the correlation and monotonicity. It seems interesting to study applications of our results in other models of dependence (cf. Berg (1993); Ladha (1995); Dietrich and List (2004)).

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