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Marc Vorsatz

Approval voting on dichotomous preferences

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Abstract The aim of this paper is to find normative foundations of Approval Voting when individuals have dichotomous preferences. We show that a social choice function is anonymous, neutral, strategy-proof and strictly monotone if and only if it is Approval Voting and interpret this result as an extension of May's theorem (Econometrica 20:680–684, 1952). Then, we show that Approval Voting is the only strictly symmetric, neutral and efficient social choice function. This result is related to a characterization of Baigent and Xu (Math Soc Sci 21:21–29, 1991).

1 Introduction

The main objective of this paper is to study set-valued social choice functions axiomatically when individuals have dichotomous preferences. Unlike standard approaches to such issues, it is not assumed that all individuals necessarily vote, nor that all alternatives are necessarily available. The main results offer two characterizations of Approval Voting (Brams and Fishburn 1978), one of the most prominent procedures in both theory and practice.

More concretely, we are interested in the following kind of problems: consider a job offer for specialized candidates. Often, firms decide in a multi-stage procedure whom to contract (e.g. firms invite a number of candidates for an assessment center or a personal interview before taking the final decision), because the amount of extractable information from the applications may be rather low and purchasing external information can be very expensive. In these circumstances, preferences of the recruiting committee members are likely to have a simple structure at the beginning of the decision process. In an extreme case, every member of the recruiting committee classifies candidates either as "acceptable" or as "non-acceptable";

M. Vorsatz
Department of Economics (AE1), University of Maastricht,
P.O. Box 616, 6200 MD Maastricht, The Netherlands
E-mail: m.vorsatz@algec.unimaas.nl

that is, individuals have dichotomous preferences on the set of candidates. The main purpose of this paper is to study how decision makers should aggregate their opinions in these kind of situations and determine the set of pre-selected candidates.

In Sect. 3, we characterize Approval Voting by means of anonymity, neutrality, strategy-proofness and strict monotonicity (Theorem 1). Further axiomatic representations of Approval Voting are due to Fishburn (1978a) and Sertel (1988), but the result which is most similar to Theorem 1 is presented in Fishburn (1978b). There Fishburn shows that on the dichotomous preference domain, a family of social choice correspondences (the set of alternatives is fixed whereas the set of voters is allowed to vary) is anonymous, neutral, strategy-proof and consistent if and only if it is Approval Voting.[1]

Although the two sets of axioms are nearly the same we can identify at least two reasons why the results are nevertheless fundamentally distinct. First, consider the following version of May's Theorem (1952): For the case of two alternatives a social choice function is anonymous, neutral and strictly monotone if and only if it is the Majority Rule (Condorcet Rule). Since Brams and Fishburn (1978) have shown that Approval Voting is equal to the Condorcet Rule whenever individuals have dichotomous preferences (the set of Condorcet winners on this domain is non-empty according to Inada (1964)), the main interpretation of Theorem 1 is that May's Theorem can be extended to any arbitrary number of alternatives if strategy-proofness is added to the original set of properties. Second, Moulin (1988) has pointed out that for dichotomous preferences, the social welfare function associated with Approval Voting is characterized by neutrality, anonymity, strict monotonicity and independence of irrelevant alternatives (IIA). Thus, Theorem 1 indicates some equivalence between strategy-proofness of a social choice function and IIA of the corresponding social welfare function. Such an equivalence has been formally established for strict preference domains by and Blair and Muller (1983) but is so far unknown for dichotomous preferences.

In Sect. 4, we show that Approval Voting is the only strictly symmetric, neutral and efficient social choice function (Theorem 2).[2] This result is related to the following characterization of Baigent and Xu (1991): a choice aggregation procedure is neutral, strictly monotone and satisfies independence of symmetric substitution (ISS) if and only if it is Approval Voting. Choice aggregation procedures and social choice functions are generally not comparable, because the domain of the former is the set of all subsets of alternatives (the alternatives an individual votes for) and not preferences. But if preferences are restricted to be dichotomous and we interpret the observed ballots as the set of acceptable alternatives, then voting decisions reveal preferences and the two concepts coincide. In this case, strict symmetry implies ISS, but it turns out that this strengthening is necessary in order to apply efficiency instead of strict monotonicity.

The remainder of the paper is organized as follows. In the next section, we introduce notation and definitions. The characterizations are presented in the

[1] Consistency means that if some alternatives are selected for two disjoint electorates, then exactly those alternatives have to be chosen whenever all individuals within the two disjoint electorates and no individual outside these electorates participate in the election.

[2] Strict symmetry means that the effect on the image of an alternative to be acceptable is independent of who considers this alternative to be acceptable and which other alternatives are acceptable from the very same individual.

Sect. 3 and 4. Afterwards, we conclude. Proofs and additional examples can be found in the Appendix.

2 Basic notation and definitions

Consider a group of individuals $N = \{1, \ldots, n\}$ with preferences on the universal set of alternatives $K \cup \{\emptyset\}$, $|K| \equiv k \geq 3$, whose objective is to choose a non-empty subset of alternatives. Since individuals may abstain from voting, the actual electorate $\bar{N}$ is assumed to be a subset of N. Moreover, it may happen that not all alternatives are implementable, and therefore, we restrict the set of feasible alternatives $\bar{K}$ to be equal to $\bar{K} \subseteq K$. The aggregation problem is interesting only if $|\bar{K}| \equiv \bar{k} \geq 2$ and $|\bar{N}| \equiv \bar{n} \geq 2$.

Let R_i be the weak preference relation of individual i on $K \cup \{\emptyset\}$. We assume that R_i is reflexive, complete and transitive. The strict and the indifference preference relations associated with R_i are denoted by P_i and I_i, respectively. The set of all weak preferences on $K \cup \{\emptyset\}$ is denoted by $\mathcal{R}$. The preference relation R_i is dichotomous if the set K can be divided into at most two indifference classes, the set of good and the set of bad alternatives. Given $R_i \in \mathcal{R}$, define the set of good alternatives associated with R_i as $G(R_i) = \{x \in K : x R_i y \text{ for all } y \in K \text{ and } x P_i \emptyset\}$. Similarly, let $B(R_i) = \{x \in K : y R_i x \text{ for all } y \in K \text{ and } \emptyset P_i x\}$ be the set of bad alternatives corresponding to R_i. The cardinalities of the two sets are equal to $g(R_i)$ and $b(R_i)$. Then, $R_i \in \mathcal{R}$ is *dichotomous* if and only if $g(R_i) + b(R_i) = k$.[3]

The domain of all dichotomous preferences is given by $\mathcal{D} \subset \mathcal{R}$ and $D_i \in \mathcal{D}$ denotes a particular dichotomous preference relation for individual i. Given the electorate $\bar{N}$, a preference profile $D_{\bar{N}} = (D_i)_{i \in \bar{N}} \in \mathcal{D}^{\bar{N}}$ is a $\bar{n}$-tuple of dichotomous preference relations. The i-variant preference profile $\left(D_i', D_{\bar{N}\backslash\{i\}}\right)$ is obtained by changing the preference relation of individual i in the profile $D_{\bar{N}}$ from D_i to $D_i' \in \mathcal{D}$. Given the preference profile $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, let $N\left(D_{\bar{N}}; x, y\right)$ be the individuals who weakly prefer x to y at $D_{\bar{N}}$; that is, $N\left(D_{\bar{N}}; x, y\right) = \{i \in \bar{N} : x \in G(D_i) \text{ or } y \in B(D_i)\}$. Finally, given the preference profile $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $N_x\left(D_{\bar{N}}\right) = |\{i \in \bar{N} : x \in G(D_i)\}|$ denotes the support of alternative x at $D_{\bar{N}}$.

Consider any set of feasible alternatives $\bar{K}$ and any electorate $\bar{N}$. The *social choice function* $f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}$ selects for all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ (remember that preferences are defined on $K \cup \{\emptyset\}$ and not on $\bar{K}$) a non-empty set of feasible alternatives $f^{\bar{K},\bar{N}}\left(D_{\bar{N}}\right)$. With a slight abuse of notation we write $f^{\bar{K}}\left(D_{\bar{N}}\right)$ instead of $f^{\bar{K},\bar{N}}\left(D_{\bar{N}}\right)$ and suppress indexes throughout whenever no restriction is made on the set of feasible alternatives or the set of individuals. A *family of social choice functions* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ takes then into account the variability of the sets $\bar{K}$ and $\bar{N}$.

[3] We add the empty-set as a dummy to the universal set of alternatives in order to introduce a reference point. By doing so we allow unconcerned individuals to distinguish whether all alternatives are good or bad. This way of defining preferences has no influence on the results but leads to a considerably more compact and intuitive notation in the proof of Lemma 2.

Two consistency conditions keep track on how the set of selected alternatives varies as the set of feasible alternatives or the electorate changes. The family of social choice functions $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}}\backslash\{\emptyset\}\right\}_{\bar{K},\bar{N}}$ is *consistent in alternatives* if for all sets of feasible alternatives $S \subset T \subseteq K$, all electorates $\bar{N}$, and all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $f^S\left(D_{\bar{N}}\right) = f^T\left(D_{\bar{N}}\right) \cap S$ whenever $f^T\left(D_{\bar{N}}\right) \cap S \neq \emptyset$. This condition, introduced by Arrow (1959), states that given the electorate $\bar{N}$, the subfamily $\left\{f^{\bar{K},\bar{N}}\right\}_{\bar{K}}$ satisfies the well-known properties α and β.[4] Hence, this subfamily is rationalizable by a social preference ordering (see Sen 1977).

Given the electorates $A \subset C \subseteq N$ and the preference profile $D_C \in \mathcal{D}^C$, let $D_C|_A \in \mathcal{D}^A$ be the profile obtained by restricting $D_C \in \mathcal{D}^C$ to A. The family of social choice functions $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}}\backslash\{\emptyset\}\right\}_{\bar{K},\bar{N}}$ is *consistent in individuals* if for all pairs of alternatives (x, y), all electorates $A \subset C \subseteq N$, and all preference profiles $D_A \in \mathcal{D}^A$ and $D_C \in \mathcal{D}^C$ which are such that $D_A = D_C|_A$ and xI_iy for all $i \in C\backslash A$, the condition $f^{\{x,y\}}(D_A) = f^{\{x,y\}}(D_C)$ holds. Hence, unconcerned individuals cannot alter the outcome in the two alternatives case.

A *social choice rule* is a family of social choice functions $\left\{f^{\bar{K},\bar{N}}\right\}_{\bar{K},\bar{N}}$ that is consistent in alternatives and individuals. One particular social choice rule is Approval Voting. According to it all feasible alternatives with the highest support from the electorate are selected.

Definition 1 *The social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}}\backslash\{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is Approval Voting if for all sets of feasible alternatives* $\bar{K}$*, all electorates* $\bar{N}$*, and all dichotomous preference profiles* $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $x \in f^{\bar{K}}\left(D_{\bar{N}}\right)$ *if and only if* $N_x\left(D_{\bar{N}}\right) \geq N_y\left(D_{\bar{N}}\right)$ *for all* $y \in \bar{K}$.

We denote the generic social choice function $f^{\bar{K},\bar{N}}$ associated with Approval Voting by $f_A^{\bar{K},\bar{N}}$. Now, we are ready to introduce the axioms of the first characterization. The first property, strategy-proofness, states that truth-telling is a dominant strategy in the preference revelation game. But since our primitives are social choice correspondences, we have to know how individuals compare non-empty subsets of alternatives in order to define strategy-proofness properly. In particular, we assume that the reflexive, complete and transitive preference relation $\succsim_{D_i}$ on $2^K\backslash\{\emptyset\}$ derived from the dichotomous preference relation D_i satisfies the subsequent properties proposed by Brams and Fishburn (1978):

1. *Condition* P: $\{x\} \succ_{D_i} \{x, y\} \succ_{D_i} \{y\}$ if and only if xP_iy.
2. *Condition* R: For all $S, T \subseteq 2^K\backslash\{\emptyset\}$, if $S \subseteq G(D_i)$ or $T \subseteq B(D_i)$ or $[S\backslash T \subseteq G(D_i)$ and $T\backslash S \subseteq B(D_i)]$, then $S \succsim_{D_i} T$.

[4] The definition of the two properties are as follows. Property α: For all $S \subset T$, if $x \in f^T\left(D_{\bar{N}}\right) \cap S$, then $x \in f^S\left(D_{\bar{N}}\right)$. Property β: If $x, y \in f^S\left(D_{\bar{N}}\right)$ and $S \subset T$, then $x \in f^T\left(D_{\bar{N}}\right)$ whenever $y \in f^T\left(D_{\bar{N}}\right)$. We do not impose any restriction on $f^S\left(D_{\bar{N}}\right)$ if $f^T\left(D_{\bar{N}}\right) \cap S = \emptyset$.

The reader should observe that *condition* P and R induce for any dichotomous preference relation an incomplete pre-order on the set of all non-empty subsets of alternatives. Individuals are then allowed to hold any transitive and reflexive completion of this pre-order. One example of the large variety of preferences included in this definition is expected utility maximization when individuals believe that all selected alternatives have an equal chance of being declared the unique winner of the election. In this case, individuals care only about the proportion of good alternatives within the set of selected alternatives (see Definition 7 in Sect. 4). The notion of manipulability analyzes consequently whether individuals can improve by lying under at least one of the possible preference extensions. Formally, given the set of feasible alternatives $\bar{K}$ and the electorate $\bar{N}$, the social choice function $f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}$ is *manipulable by i* if for some $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ and $D'_i \in \mathcal{D}$, $f^{\bar{K}}\left(D'_i, D_{\bar{N}\backslash\{i\}}\right) \succ_{D_i} f^{\bar{K}}\left(D_{\bar{N}}\right)$.

Definition 2 *The social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is strategy-proof if for all sets of feasible alternatives* $\bar{K}$ *and all electorates* $\bar{N}$*,* $f^{\bar{K},\bar{N}}$ *is not manipulable by any individual.*

Anonymity (Neutrality) formalizes the democratic idea that there is no a priori bias in favor of some individual (alternative). Given the preference profile $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ and the permutation σ of $\bar{N}$, let $D_{\sigma(\bar{N})} \in \mathcal{D}^{\bar{N}}$ be the preference profile obtained by permuting individuals according to σ; that is, $D_{\sigma(\bar{N})} = \left(D_{\sigma(i)}\right)_{i\in\bar{N}}$.

Definition 3 *The social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is anonymous if for all sets of feasible alternatives* $\bar{K}$*, all electorates* $\bar{N}$*, all preference profiles* $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$*, and all permutations* σ *of* $\bar{N}$*,* $f^{\bar{K}}\left(D_{\sigma(\bar{N})}\right) = f^{\bar{K}}\left(D_{\bar{N}}\right)$.

Given the preference profile $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ and the permutation μ of K, $\mu\left(D_{\bar{N}}\right) \in \mathcal{D}^{\bar{N}}$ is the preference profile obtained by permuting alternatives according to μ; that is, for all $i \in \bar{N}$ and $x \in K$, $x \in \mu\left(G(D_i)\right)$ if and only if $\mu^{-1}(x) \in G(D_i)$.

Definition 4 *The social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is neutral if for all sets of feasible alternatives* $\bar{K}$*, all electorates* $\bar{N}$*, all preference profiles* $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$*, and all permutations* μ *of* K*,* $f^{\mu(\bar{K})}(\mu(D_{\bar{N}})) = \mu\left(f^{\bar{\mu}}\right)\left(D_{\bar{N}}\right)$.

The sets $\mu\left(f^{\bar{K}}\left(D_{\bar{N}}\right)\right)$ and $\mu(\bar{K})$ are obtained by applying μ to $f^{\bar{K}}(D_{\bar{N}})$ and $\bar{K}$, respectively. To explain the last property, strict monotonicity, suppose that x and y are the only feasible alternatives and that both alternatives are selected for some profile. The tie occurring in this situation should then be broken in favor of x whenever x receives additional support everything else unchanged.

Definition 5 *The social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \setminus \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is strictly monotone if for all pairs of alternatives* (x, y), *all electorates* $\bar{N}$, *and all preference profiles* $D_{\bar{N}}, D'_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ *which are such that* $x P_i y$ *and* $x I'_i y$ *for some* $i \in \bar{N}$ *and* $D_{\bar{N}\setminus\{i\}} = D'_{\bar{N}\setminus\{i\}}$, *the condition* $x \in f^{\{x,y\}}\left(D'_{\bar{N}}\right)$ *implies* $f^{\{x,y\}}\left(D_{\bar{N}}\right) = \{x\}$.

3 A Characterization with strategy-proofness

Our main result characterizes Approval Voting by means of strategy-proofness, anonymity, neutrality and strict monotonicity. The proof we provide is organized along an important lemma showing that if a social choice rule is neutral and strategy-proof, then it depends on the individuals who prefer x to y and y to x whenever there are no other feasible alternatives. Formally, the social choice rule $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}}\setminus\{\emptyset\}\right\}_{\bar{K},\bar{N}}$ satisfies IIA if for all pairs of alternatives (x, y), all electorates $\bar{N}$, and all preference profiles $D_{\bar{N}}, D'_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ which are such that $N\left(D_{\bar{N}}; x, y\right) = N\left(D'_{\bar{N}}; x, y\right)$ and $N\left(D_{\bar{N}}; y, x\right) = N\left(D'_{\bar{N}}; y, x\right)$, the condition $f^{\{x,y\}}\left(D_{\bar{N}}\right) = f^{\{x,y\}}\left(D'_{\bar{N}}\right)$ holds.

Lemma 1 *If the social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}}\setminus\{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is neutral and strategy-proof, then it satisfies IIA.*

Proof See the Appendix. □

We are able to characterize Approval Voting for the case of two alternatives with the help of Lemma 1 and the properties of neutrality, anonymity and strict monotonicity. Afterwards, we apply consistency in alternatives to generalize this result to all sets of feasible alternatives.

Theorem 1 *The social choice rule* $\left\{f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}}\setminus\{\emptyset\}\right\}_{\bar{K},\bar{N}}$ *is strategy-proof, neutral, anonymous and strictly monotone if and only if it is Approval Voting.*

Proof See the Appendix. □

We show in the Appendix that Theorem 1 is tight.[5] Fishburn (1978b) proposes a very similar result, because he characterizes Approval Voting as the only strategy-proof, neutral, anonymous and consistent family of social choice functions (it is allowed for a variable electorate whereas the set of feasible alternatives is assumed to be fixed). Using current notation consistency is defined as follows: Let $\bar{\mathcal{D}}$ be the domain of dichotomous preferences without the two preference relations

[5] One referee pointed out that it would be desirable to present a similar characterization for a fixed set of alternatives and a fixed electorate. Such a result could in principle be obtained, because we have not succeeded to establish the independence of the consistency conditions with respect to the other axioms. Since we have no hint how a possible proof could be constructed, we have to regard this question as an open problem.

indicating that an individual is indifferent between all alternatives. The family of social choice functions $\left\{f^{\bar{N}} : \bar{\mathcal{D}}^{\bar{N}} \to 2^K\backslash\{\emptyset\}\right\}_{\bar{N}}$ is *consistent* if for all disjoint electorates $\hat{N}$ and $\tilde{N}$ and all preference profiles $\bar{D}_{\hat{N}} \in \bar{\mathcal{D}}^{\hat{N}}$ and $\bar{\mathcal{D}}_{\tilde{N}} \in \bar{\mathcal{D}}^{\tilde{N}}$, $f\left(\bar{D}_{\hat{N}\cup\tilde{N}}\right) = f\left(\bar{D}_{\hat{N}}\right) \cap f\left(\bar{D}_{\tilde{N}}\right)$ whenever $f\left(\bar{D}_{\hat{N}}\right) \cap f\left(\bar{D}_{\tilde{N}}\right) \neq \emptyset$. In the former definition, the preference profile $\bar{D}_{\hat{N}\cup\tilde{N}} \in \mathcal{D}^{\hat{N}\cup\tilde{N}}$ is obtained by unifying the other two preference profiles.

At this point it is crucial to analyze whether the properties of consistency and strict monotonicity are independent from each other, because otherwise one characterization would base on a stronger set of axioms and follow as a corollary. The corresponding definitions provide evidence why this is not the case: While consistency deals with situations where two disjoint electorates are put together, strict monotonicity analyzes what happens if an alternative gets additional support. The proper objective of both properties is very different, thereby establishing the independence of the two sets of axioms.

The main difference between the results is that only Theorem 1 can be interpreted as an extension of the following version of May's Theorem (1952): Suppose that $K = \{x, y\}$. The social choice function $f : \mathcal{R}^N \to 2^K\backslash\{\emptyset\}$ is anonymous, neutral and strictly monotone if and only if it is the Majority (Condorcet) Rule; that is, for all $R \in \mathcal{R}^N$, $w \in K$ and $z \neq w$, $w \in f(R)$ if and only if $|\{i \in N : wR_iz\}| \geq |\{i \in N : zR_iw\}|$. Since Approval Voting is equivalent to the Condorcet Rule on the domain of dichotomous preferences (see Brams and Fishburn 1978), Theorem 1 establishes that May's Theorem can be extended to any number of alternatives if we restrict our attention to social choice rules and add strategy-proofness to the original properties.[6]

4 A characterization with efficiency

From now on we restrict our attention to the special case when all individuals reveal their preferences and all alternatives are feasible. Specially, it is our aim to focus in the second characterization on the notion of efficiency meaning that it is not possible to make some individual better off without hurting others.

Definition 6 *The social choice function* $f : \mathcal{D}^N \to 2^K\backslash\{\emptyset\}$ *is said to be efficient if there does not exist a preference profile* $D \in \mathcal{D}^N$ *and a set* $S \subseteq 2^K\backslash\{\emptyset\}$ *such that* $S \succsim_{D_i} f(D)$ *for all* $i \in N$ *and* $S \succ_{D_j} f(D)$ *for some* $j \in N$.

Approval Voting is not efficient for all reflexive, complete and transitive preference relations $\succsim_{D_i}$ on $2^K\backslash\{\emptyset\}$ that satisfy *condition* P and R, because if both good

[6] Approval Voting is the only strategy-proof voting system for dichotomous preferences according to Brams and Fishburn (1978, 2002). Formally, they define a *voting system* π to be a non-empty subset of the integers $\{1, 2, .., k\}$ with the understanding that if $\pi = \{s_1, s_2, \ldots, s_l\}$, then individuals have to vote either for s_1, or for s_2,..., or for s_l alternatives. Given all voting decisions, all alternatives with the highest amount of votes are selected. We obtain Approval Voting if $\pi = \{1, 2, \ldots, k\}$. If $\pi \subset \{1, 2, \ldots, k\}$, then voting decisions do not reveal dichotomous preferences completely, and therefore, these voting systems are not representable by means of Definition 1. Approval Voting is consequently the only aggregation rule that is both social choice function and voting system. Hence, Theorem 1 is not a corollary of the result presented in Brams and Fishburn (1978, 2002).

and bad alternatives are selected, then taking away a bad alternative can leave an individual indifferent.

Example 1 Let $K = \{x, y, z\}$ and $N = \{1, 2, 3\}$. If the profile $D \in \mathcal{D}^N$ is equal to $G(D_1) = \{x, y\}$, $G(D_2) = \{x, z\}$ and $G(D_3) = \{y, z\}$, then $f_A(D) = K$. If the preference relations $\succsim_{D_1}$, $\succsim_{D_2}$ and $\succsim_{D_3}$ are such that $\{x, y\} \succ_{D_1} K$, $\{x, y\} \sim_{D_2} K$ and $\{x, y\} \sim_{D_3} K$, then neither *condition* P nor *condition* R is contradicted. Thus, Approval Voting is not efficient.

This problem disappears if we put more structure on the preference extension. In particular, we are going to assume from now on that every individual evaluates the set $S \subseteq 2^K \backslash \{\emptyset\}$ according to the proportion of good alternatives contained in S.

Definition 7 *The preference relation $\succsim_{D_i}$ on $2^K \backslash \{\emptyset\}$ is cohesive with respect to D_i whenever for all $S, T \in 2^K \backslash \{\emptyset\}$, $S \succsim_{D_i} T$ if and only if $(|G(D_i) \cap S|)/|S| \geq (|G(D_i) \cap T|)/|T|$ ($\succsim_{D_i}$ is strict whenever the inequality is strict).*

Proposition 1 of Bogomolnaia et al. (2005) states that Approval Voting is efficient whenever preferences are cohesive.[7] The following interpretation makes this preference extension particularly appealing: If we think of $f(D)$ as the set of preselected alternatives from which a unique winning alternative has to be determined via a lottery and individuals are expected utility maximizers, then individuals care only about the probability that a good alternative is chosen. If, in addition, individuals assign to all alternatives belonging to $f(D)$ the same winning probability, then the lottery with support on S is weakly preferred to the lottery with support on T if and only if $S \succsim_{D_i} T$.

Neutrality and strict symmetry are the other properties applied in the second characterization. The intuition of strict symmetry is simple: suppose that there are two different preference profiles which differ from each other just because some alternative that is good for the first individual and bad for the second individual according to the first preference profile is good for the second individual and bad for the first individual according to the second preference profile. This variation in preferences should not provoke any change in the chosen set of alternatives. Formally, the preference profiles $D, D' \in \mathcal{D}^N$ are *x-symmetric* if for some pair of individuals (i, j), $G(D'_i) \cup \{x\} = G(D_i)$, $G\left(D'_j\right) = G(D_j) \cup \{x\}$, where $x \notin G\left(D'_i\right) \cup G(D_j)$, and $D'_l = D_l$ for all $l \neq i, j$.

Definition 8 *The social choice function $f : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ is strictly symmetric if for all $x \in K$ and all x-symmetric preference profiles $D, D' \in \mathcal{D}^N$, $f(D) = f\left(D'\right)$.*

Lemma 2 shows that a social choice function depends on the k-dimensional vector $(N_x(D))_{x \in K}$ if and only if f is strictly symmetric. Formally, the social choice function $f\colon \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ *depends on the support of the alternatives* if

[7] Apart from this result the two papers follow different objectives, because Bogomolnaia et al. (2005) establish various impossibility results when expected utility maximizers have dichotomous preferences whereas we look for positive results with respect to Approval Voting.

for all preference profiles $D, D' \in \mathcal{D}^N$ which are such that $N_x(D) = N_x\left(D'\right)$ for all $x \in K$, the condition $f(D) = f\left(D'\right)$ holds.

Lemma 2 *The social choice function* $f\colon \mathcal{D}^N \to 2^K \setminus \{\emptyset\}$ *is strictly symmetric if and only if it depends on the support of the alternatives.*

Proof See the Appendix. □

Now, we are ready to state our second characterization.

Theorem 2 *The social choice function* $f\colon \mathcal{D}^N \to 2^K \setminus \{\emptyset\}$ *is strictly symmetric, neutral and efficient for the cohesive preference extension if and only if it is Approval Voting.*

Proof See the Appendix. □

We show in the Appendix that Theorem 2 is tight. Baigent and Xu (1991) characterize the *choice aggregation procedure* corresponding to Approval Voting. Formally, let $M(R_i) \in 2^K$ be the alternatives individual i votes for when her/his preference relation is R_i. A choice aggregation procedure $c : \left(2^K\right)^N \to 2^K \setminus \{\emptyset\}$ aggregates the collective voting decisions $(M(R_1), \ldots, M(R_n))$ by selecting a nonempty set of alternatives. To relate choice aggregation procedures and social choice functions for dichotomous preferences we just have to interpret the alternatives an individual votes for as the set of her/his good alternatives (incentive-compatibility legitimates this approach); that is, for all $i \in N$ and all $D_i \in \mathcal{D}$, $M(D_i) = G(D_i)$. In this way, we are able to recover preferences from the observed voting decision, or, to say it differently, choice aggregation procedures and social choice functions coincide.

In Baigent and Xu (1991) Approval Voting is characterized by means of Independence of ISS, strict monotonicity and neutrality. Using our notation ISS is defined as follows: The social choice function $f : \mathcal{D}^N \to 2^K \setminus \{\emptyset\}$ satisfies ISS if for all profiles $D, D' \in \mathcal{D}^N$ which are such that for some pair (i, j) and some pair (x, y), $G\left(D'_i\right) \cup \{y\} = G(D_i) \cup \{x\}$ and $G\left(D'_j\right) \cup \{x\} = G(D_j) \cup \{y\}$ where $y \notin G\left(D'_i\right) \cup G(D_j)$ and $x \notin G(D_i) \cup G\left(D'_j\right)$, and $D'_l = D_l$ for all $l \neq i, j$, the condition $f\left(D'\right) = f(D)$ holds. Since strict symmetry is stronger than ISS, it is important to analyze whether we can relax our set of axioms and characterize Approval Voting using ISS, neutrality and efficiency. In the Appendix, we present a social choice function different from Approval Voting that satisfies these properties. In this sense, the strengthening from ISS to strict symmetry is not only sufficient but also necessary for exploring the efficiency of Approval Voting.

5 Conclusion

Approval Voting is without any doubt of great practical importance. Hence, it is a logical consequence to ask for theoretical support of this rule. Our goal has been to look for new normative foundations of Approval Voting under the assumption of dichotomous preferences.

Working with dichotomous preferences is surely not innocuous, but if one aims to compare Approval Voting axiomatically with well known social choice functions such as Scoring Rules or Voting by Committees, then we necessarily have to restrict ourselves to this preference domain. To see this in an easy example suppose that there are three alternatives x, y and z and let the preference relation for individual i be such that $x P_i y P_i z$. In this case, we expect individual i to vote either for alternative x or for the set $\{x, y\}$ (see e.g. Luo et al. 1996). But if $M_i(R_i) = \{x, y\}$, then we cannot deduce that $x P_i y$. Similarly, if $M_i(R_i) = \{x\}$, then we do not know that $y P_i z$. Things become only different if preferences are dichotomous, because then individuals want to vote exactly for their set of good alternatives and the observed voting decisions are as if individuals had fully revealed their preferences. Thus, the assumption of dichotomous preference is necessary for defining Approval Voting as a social choice function.

Finally, it should be noted that the literature on social choice has neglected to a large extend the importance of indifferences in collective choice problems. The dichotomous preference domain constitutes without any doubt an important benchmark case in the analysis of those situations.

Appendix

Proof of Lemma 1

Suppose otherwise. Then, there is an electorate $\bar{N}$ and two preference profiles $D_{\bar{N}}, D'_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ which are such that for some pair of alternatives (x, y), $N(D_{\bar{N}}; x, y) = N\left(D'_{\bar{N}}; x, y\right)$ and $N\left(D_{\bar{N}}; y, x\right) = N\left(D'_{\bar{N}}; y, x\right)$, whereas the social choice function $f^{\{x,y\},\bar{N}}$ satisfies $f^{\{x,y\}}\left(D_{\bar{N}}\right) = \{x\}$ and $f^{\{x,y\}}\left(D'_{\bar{N}}\right) \in \{\{y\}, \{x, y\}\}$. Let $i \in C \subseteq \bar{N}$ if and only if $x P_i y$ or $y P_i x$. If $C = \emptyset$, then $x I_i y$ for all $i \in \bar{N}$. In this case, it has to be that $f^{\{x,y\}}\left(D_{\bar{N}}\right) = \{x, y\}$, because the function $f^{\{x,y\},\bar{N}}$ is neutral by assumption and the empty set cannot be selected. This is a contradiction to $f^{\{x,y\}}\left(D_{\bar{N}}\right) = \{x\}$, and therefore, $C \neq \emptyset$. Now, apply consistency in individuals to obtain that $f^{\{x,y\}}(D_{\bar{N}}\big|_C) = \{x\}$. For simplicity let the preference profile $D_C \in \mathcal{D}^C$ be such that $D_C = D_{\bar{N}}\big|_C$.

We prove that for all $j \in C$, $f^{\{x,y\}}\left(D'_j, D_{C\setminus\{j\}}\right) = \{x\}$. Suppose otherwise; that is, $f^{\{x,y\}}\left(D'_j, D_{C\setminus\{j\}}\right) \in \{\{y\}, \{x, y\}\}$. If $x P_j y$, then it follows from *condition* P that j can manipulate $f^{\{x,y\},C}$ at $\left(D'_j, D_{C\setminus\{j\}}\right)$ via D_j. On the other hand, if $y P_j x$, then j can manipulate $f^{\{x,y\},C}$ at D_C via D'_j. This contradicts strategy-proofness, and therefore, we can conclude that $f^{\{x,y\}}\left(D'_j, D_{C\setminus\{j\}}\right) = \{x\}$. If we change preferences for any $i \in C\setminus\{j\}$ one by one from D_i to D'_i, then it follows from the repeated application of strategy-proofness that the image of $f^{\{x,y\},C}$ remains unaffected. Hence, we arrive at $f^{\{x,y\}}\left(D'_C\right) = \{x\}$ once all preferences have been changed. Finally, $f^{\{x,y\}}\left(D'_C, D'_{\bar{N}\setminus C}\right) = f^{\{x,y\}}\left(D'_{\bar{N}}\right) = \{x\}$, because all individuals $\bar{N} \backslash C$ are indifferent between x and y at $D'_{\bar{N}}$ and the subfamily $\left\{ f^{\{x,y\},\bar{N}} \right\}_{\bar{N}}$ is consistent in alternatives. This contradicts $y \in f^{\{x,y\}} \left(D'_{\bar{N}} \right)$. □

Proof of Theorem 1

Approval Voting is neutral, anonymous, strictly monotone and strategy-proof. Therefore, suppose that the social choice rule $\left\{ f^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\} \right\}_{\bar{K},\bar{N}}$ satisfies the four properties. At first we show that given $\bar{K} = \{x, y\}$, the subfamily $\left\{ f^{\{x,y\},\bar{N}} : \mathcal{D}^{\bar{N}} \to \{\{x\}, \{x, y\}, \{y\}\} \right\}_{\bar{N}}$ orders x and y according to Approval Voting; that is, for all electorates $\bar{N}$ and all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, (a) if $N_x \left(D_{\bar{N}} \right) = N_y \left(D_{\bar{N}} \right)$, then $f^{\{x,y\}} \left(D_{\bar{N}} \right) = \{x, y\}$ and (b) if $N_x (D_{\bar{N}}) > N_y \left(D_{\bar{N}} \right)$, then $f^{\{x,y\}} \left(D_{\bar{N}} \right) = \{x\}$. Afterwards, we generalize this result.

(a) Since the family $\left\{ f^{\{x,y\},\bar{N}} \right\}_{\bar{N}}$ satisfies IIA by Lemma 1 and all social choice rules are anonymous by assumption, we deduce that for all electorates $\bar{N}$, $f^{\{x,y\},\bar{N}}$ depends on the numbers $|N \left(D_{\bar{N}}; x, y \right)| = N_x \left(D_{\bar{N}} \right) + |\{i \in \bar{N} : x, y \in B(D_i)\}|$ and $|N \left(D_{\bar{N}}; y, x \right)| = N_y \left(D_{\bar{N}} \right) + |\{i \in \bar{N} : x, y \in B(D_i)\}|$. It follows from neutrality that $f^{\{x,y\}} \left(D_{\bar{N}} \right) = \{x, y\}$ whenever $N_x \left(D_{\bar{N}} \right) = N_y \left(D_{\bar{N}} \right)$.

(b) Suppose that given the electorate $\bar{N}$ and the preference profile $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $N_x \left(D_{\bar{N}} \right) - N_y \left(D_{\bar{N}} \right) = 1$. Construct the preference profile $D'_{\bar{N}} \in \mathcal{D}^{\bar{N}}$ in the following way: For some individual i whose preference relation D_i is such that $x P_i y$, D'_i satisfies $x I'_i y$. Moreover, let $D'_j = D_j$ for all $j \neq i$. Since $N_x \left(D'_{\bar{N}} \right) = N_y \left(D'_{\bar{N}} \right)$, it follows from part (a) that $f^{\{x,y\}} \left(D'_{\bar{N}} \right) = \{x, y\}$. Apply strict monotonicity to see that $f^{\{x,y\}} \left(D_{\bar{N}} \right) = \{x\}$. A simple induction argument establishes finally that $f^{\{x,y\}} \left(D_{\bar{N}} \right) = \{x\}$ whenever $N_x \left(D_{\bar{N}} \right) > N_y \left(D_{\bar{N}} \right)$.

It remains to apply consistency in alternatives to prove that for all sets of feasible alternatives $\bar{K}$, all electorates $\bar{N}$, and all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $x \in f^{\bar{K}} \left(D_{\bar{N}} \right)$ if and only if $N_x \left(D_{\bar{N}} \right) \geq N_y \left(D_{\bar{N}} \right)$ for all $y \in \bar{K}$. Suppose that $x \in f^{\bar{K}} \left(D_{\bar{N}} \right)$. Then, it follows from consistency in alternatives that $x \in f^{\{x,y\}} \left(D_{\bar{N}} \right)$ for all $y \in \bar{K} \backslash \{x\}$. This together with the fact that the subfamily $\left\{ f^{\{x,y\},\bar{N}} \right\}_{\bar{N}}$ orders x and y according to Approval Voting implies that $N_x \left(D_{\bar{N}} \right) \geq N_y \left(D_{\bar{N}} \right)$ for all $y \in \bar{K}$. To show the other inclusion suppose that $N_x \left(D_{\bar{N}} \right) \geq N_y \left(D_{\bar{N}} \right)$ for all $y \in \bar{K}$. Then, $x \in f^{\{x,y\}} \left(D_{\bar{N}} \right)$ for all $y \in \bar{K} \backslash \{x\}$ because the subfamily $\left\{ f^{\{x,y\},\bar{N}} \right\}_{\bar{N}}$ orders alternatives according to Approval Voting. If there is an alternative $z \neq x$ such that $z \in f^{\bar{K}} \left(D_{\bar{N}} \right)$, then $f^{\bar{K}} \left(D_{\bar{N}} \right) \cap \{x, z\} \neq \emptyset$ and it follows from consistency in alternatives that $f^{\{x,z\}}\left(D_{\bar{N}}\right) = f^{\bar{K}}\left(D_{\bar{N}}\right) \cap \{x, z\}$. Now, since $x \in f^{\{x,z\}}\left(D_{\bar{N}}\right)$ it also has to be the case that $x \in f^{\bar{K}}\left(D_{\bar{N}}\right)$. If there does not exist any alternative $z \neq x$ such that $z \in f^{\bar{K}}\left(D_{\bar{N}}\right)$, then $x \in f^{\bar{K}}\left(D_{\bar{N}}\right)$ because $f^{\bar{K}}\left(D_{\bar{N}}\right) \neq \emptyset$ by assumption. □

Tightness of Theorem 1

Strategy-proofness

Any scoring rule can be represented by a vector of non-negative numbers $s = (s_0, s_1, \ldots, s_{k-1}) \in \mathbb{R}^k$ satisfying the conditions $s_{j-1} \leq s_j$ for all $j = 1, \ldots, k-1$ and $s_0 < s_{k-1}$. For dichotomous preferences points are assigned to alternatives in the following way: Given the preference relation $D_i \in \mathcal{D}$, assign $1/g(D_i) \sum_{i=b(D_i)}^{k-1} s_i$ points to all $x \in G(D_i)$ and $1/b(D_i) \sum_{i=0}^{b(D_i)-1} s_i$ to all $y \in B(D_i)$. Then, we select for all preference profiles all feasible alternatives with the highest sum of points. All scoring rules are anonymous, neutral and strictly monotone. It is shown in Vorsatz (2004) that on the dichotomous preference domain, the Borda Count (e.g. $s_j = j$ for all $j = 1, \ldots, k-1$) is equivalent to Approval Voting and all other scoring rules are manipulable.

Neutrality

The social choice rule $\left\{f_1^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \setminus \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ is such that for all sets of feasible alternatives $\bar{K}$, all electorates $\bar{N}$ and all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $f_1^{\bar{K}}\left(D_{\bar{N}}\right) = \{x\}$ whenever $x \in f_A^{\bar{K}}\left(D_{\bar{N}}\right) \cap \bar{K}$. Otherwise, $f_1^{\bar{K}}\left(D_{\bar{N}}\right) = f_A^{\bar{K}}\left(D_{\bar{N}}\right)$. This rule is anonymous, strictly monotone and strategy-proof. The following example illustrates that it is not neutral: Let $K = \{x, y\}$ and $N = \{1, 2\}$. If the preference profile $D \in \mathcal{D}^N$ is equal to $G(D_1) = G(D_2) = \{x, y\}$, then $f_1(D) = \{x\}$. Define the permutation μ of K as $\mu(x) = y$ and $\mu(y) = x$. Since $\mu(D) = D$, it has to be that $f_1(\mu(D)) = f_1(D) = \{x\}$. On the other hand, we observe that $\mu(f_1(D)) = \{y\}$. This contradicts neutrality.

Anonymity

Consider a n-tuple $(q_i)_{i \in N}$ of strictly positive numbers which satisfies $q_i \neq q_j$ for some pair of alternatives $(i, j) \in N^2$. The social choice rule $\left\{f_2^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \setminus \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ is as follows: For all sets of feasible alternatives $\bar{K}$, all electorates $\bar{N}$, and all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $x \in f_2^{\bar{K}}\left(D_{\bar{N}}\right)$ if and only if $\sum_{i \in \bar{N}: x \in G(D_i)} q_i \geq \sum_{i \in \bar{N}: y \in G(D_i)} q_i$ for all $y \in \bar{K}$. This rule is neutral, strictly monotone and strategy-proof. The following example illustrates that it is not anonymous: Let $K = \{x, y\}$ and $N = \{1, 2\}$. Moreover suppose that $q_1 > q_2$. If the preference profile $D \in \mathcal{D}^N$ is such that $G(D_1) = \{x\}$ and $G(D_2) = \{y\}$, then

$f_2(D) = \{x\}$. Define the permutation σ of N as $\sigma(1) = 2$ and $\sigma(2) = 1$. We observe that $f_2(\sigma(D)) = \{y\}$. This contradicts anonymity.

Strict monotonicity

The social choice rule $\left\{f_3^{\bar{K},\bar{N}} : \mathcal{D}^{\bar{N}} \to 2^{\bar{K}} \backslash \{\emptyset\}\right\}_{\bar{K},\bar{N}}$ is such that for all sets of feasible alternatives $\bar{K}$, all electorates $\bar{N}$, and all preference profiles $D_{\bar{N}} \in \mathcal{D}^{\bar{N}}$, $f_3^{\bar{K}}(D_{\bar{N}}) = \bar{K}$. This rule is strategy-proof, neutral and anonymous. The following example illustrates that it is not strictly monotone: Suppose that $K = \{x, y\}$ and $N = \{1, 2\}$. Let the preference profiles $D, D' \in \mathcal{D}^N$ be such that $G(D_1) = \{x, y\}$, $G\left(D'_1\right) = \{x\}$ and $G(D_2) = G\left(D'_2\right) = \{x, y\}$. Then, $f_3(D) = f_3\left(D'\right) = \{x, y\}$ contradicting strict monotonicity.

Proof of Lemma 2

We see that f is strictly symmetric whenever it depends on the support of the alternatives. To show the other inclusion suppose that $f : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ is strictly symmetric and consider two preference profiles $D, D' \in \mathcal{D}^N$ which are such that $N_x(D) = N_x\left(D'\right)$ for all $x \in K$. We have to show that $f(D) = f\left(D'\right)$.

Let $h_x : \left(D, D'\right) \to 2^N$ be the function that identifies for alternative $x \in K$ those individuals who consider x to be good under the first and bad under the second preference profile. Observe that $|h_x\left(D, D'\right)| = |h_x\left(D', D\right)| \equiv h(x)$, because $N_x(D) = N_x\left(D'\right)$ by assumption. Moreover, we introduce the one-to-one mapping $\eta : K \to \{1, 2, \ldots, k\}$ that orders alternatives by assigning to each alternative $x \in K$ a positive integer $1 \leq \eta(x) \leq k$.

Suppose that $\eta(y) = 1$, $i = \min[h_y\left(D, D'\right)]$ and $j = \min[h_y\left(D', D\right)]$. Define the preference profile $D^{1.1} \in \mathcal{D}^N$ as follows: The preference relations $D_i^{1.1}$ and $D_j^{1.1}$ are such that $G\left(D_i^{1.1}\right) = G(D_i)\backslash\{y\}$ and $G\left(D_j^{1.1}\right) = G(D_j) \cup \{y\}$ whereas $G\left(D_l^{1.1}\right) = G(D_l)$ for all $l \neq i, j$. Strict symmetry implies that $f\left(D^{1.1}\right) = f(D)$. If we perform similar changes using the individuals with the second lowest integer, the third lowest integer, …, and apply strict symmetry at every step, then we get $f\left(D^{1.h(y)}\right) = f(D)$. Up to now every individual orders y identically at $D^{1.h(y)}$ and D'. If we apply the same argument to alternative $\eta(z) = 2$, then we obtain $f\left(D^{2.h(z)}\right) = f(D)$. We arrive at $f\left(D'\right) = f(D)$ by repeating the process for all remaining alternatives. □

Proof of Theorem 2

Approval Voting is strictly symmetric, neutral and efficient. In order to prove the other inclusion suppose that the social choice function $f : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ satisfies the three properties. It remains to show that for all preference profiles $D \in \mathcal{D}^N$, $f(D) = \{x \in K : N_x(D) \geq N_y(D) \text{ for all } y \in K\}$.

Given the preference profile $D \in \mathcal{D}^N$, the largest support is given by $q \equiv \max_{y \in K} N_y(D)$. Consider the preference profile $D' \in \mathcal{D}^N$ which is such that for all $i \in N$ and $x \in K$, $x \in G(D'_i)$ if and only if $i \leq N_x(D)$. Since $N_y\left(D'\right) = N_y(D)$ for all $y \in K$ and f depends on the support of the alternatives by Lemma 2, $f\left(D'\right) = f(D)$. To finish the proof we have to establish that $f\left(D'\right) = G\left(D'_q\right)$, because this set contains all alternatives with the largest support under D.

Suppose that $f\left(D'\right) \nsubseteq G\left(D'_q\right)$. Then we have that for some alternative $y \in K$, $y \in f\left(D'\right)$ and $y \notin G\left(D'_q\right)$. Let $x \in G\left(D'_q\right)$ and observe that for all $i \in N$, $\{x\} \succsim_{D'_i} f\left(D'\right)$. The reason is that any subset of good alternatives is as least as good as any other set of alternatives according to *condition* R. For individual q we yield $\{x\} \succ_{D'_q} f\left(D'\right)$, because cohesive preferences imply that any subset of good alternatives is strictly preferred to any set which contains at least one bad alternative. This contradicts efficiency, and therefore, $f\left(D'\right) \subseteq G\left(D'_q\right)$. Finally, apply neutrality to obtain $f\left(D'\right) = G\left(D'_q\right)$. □

Tightness of Theorem 2

Neutrality

The non-neutral social choice function f_1 is strictly symmetric and efficient.

Efficiency

The social choice function $f_4 : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ is as follows: For all preference profiles $D \in \mathcal{D}^N$, $f_4(D) = \{x \in K : N_x(D) \geq 1\}$ whenever this set is non-empty. Otherwise, $f_4(D) = K$. This function is strictly symmetric and neutral. The following example illustrates that it is not efficient: Let $K = \{x, y\}$ and $N = \{1, 2\}$. If the preference profile $D \in \mathcal{D}^N$ is such that $G(D_1) = \{x\}$ and $G(D_2) = \{x, y\}$, then $f_4(D) = \{x, y\}$. Since $\{x\} \succ_{D_1} \{x, y\}$ and $\{x\} \sim_{D_2} \{x, y\}$, we conclude that f_4 is not efficient.

Strict symmetry

The social choice function $f_5 : \mathcal{D}^N \to 2^K \backslash \{\emptyset\}$ is as follows: If the preference profile $D \in \mathcal{D}^N$ is such that for some $i \in N$ and some $y \in K$, $G(D_i) = \{y\}$ and for all $j \neq i$, $G(D_j) = \{x\}$ for some $x \in K \backslash \{y\}$, then $f_5(D) = \{x, y\}$. Otherwise, $f_5(D) = f_A(D)$. This function is efficient, neutral and satisfies ISS. The following example illustrates that it is not strictly symmetric: Let $K = \{x, y\}$ and $N = \{1, 2, 3\}$. If the preference profiles $D, D' \in \mathcal{D}^N$ are such that $G(D_1) = G\left(D'_1\right) = \{x\}$, $G(D_2) = \{x\}$, $G\left(D'_2\right) = \{x, y\}$, $G(D_3) = \{y\}$, and $G\left(D'_3\right) = \emptyset$, then $f_5(D) = \{x, y\}$ and $f_5\left(D'\right) = \{x\}$. Since $f_5(D) \neq f_5\left(D'\right)$, we conclude that f_5 is not strictly symmetric.

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