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The TAL-family of rules for bankruptcy problems

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Abstract This paper analyzes a family of rules for bankruptcy problems that generalizes the Talmud rule (T) and encompasses both the constrained equal-awards rule (A) and the constrained equal-losses rule (L). The family is defined by means of a parameter $\theta \in [0, 1]$ that can be interpreted as a measure of the distributive power of the rule. We provide a systematic study of the structural properties of the rules within the family and its connections with the existing literature.

1 Introduction

A bankruptcy problem describes a situation in which an arbitrator has to allocate a given amount of a perfectly divisible commodity among a group of agents when the available amount is not enough to satisfy all their claims. Most rationing problems can be given this form. Solving bankruptcy problems means finding a procedure or "rule" that exhibits some desirable properties and determines, for each specific problem, a well-defined allocation satisfying two basic restrictions: (1) no agent gets more than she claims nor less than zero; and (2) the entire available amount is distributed. The reader is referred to Young (1994), Herrero and Villar (2001), Moulin (2002) or Thomson (2003) for reviews of this literature.

We present in this paper a family of rules, called the *TAL-family*, that extends the classical Talmud rule (T) and encompasses two other classical rules: the constrained equal awards rule (A) and the constrained equal losses rule (L). The rules in the TAL-family apply the same principle that underlies the Talmud rule. Namely, they generate allocations in which individual rationing is of the same type

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as collective rationing. Indeed, in the Talmud rule no agent gets more than half of her claim if the amount is less than half of the aggregate claim and nobody gets less than half of her claim if the amount exceeds half of the aggregate claim. The rules in the TAL-family extend this principle by considering all possible shares of the amount to divide in the aggregate claim. That is, for any given value of the parameter $\theta \in [0, 1]$, the rule R^{θ} distributes the amount accordingly so that nobody gets more than a fraction θ of her claim if the amount to divide is smaller than θ times the aggregate claim and nobody gets less than a fraction θ of her claim if the amount to divide exceeds θ times the aggregate claim.

The parameter θ that generates the family can be regarded as an index of the *distributive power* of the rule, in the following sense. Higher values of θ imply that R^{θ} gives higher satisfaction to agents with relatively lower claims, whereas lower values of θ imply that R^{θ} favors agents with relatively larger claims.

The rules in the TAL-family exhibit a precise duality relationship: the dual of the rule associated with the parameter θ is that rule associated with the parameter $(1-\theta)$. Furthermore, they satisfy many of the standard properties in the axiomatic theory of bankruptcy. In particular, they belong to the class of "parametric rules", they are order preserving in awards and losses, monotonic with respect to claims and resources, and satisfy homogeneity, and either independence of claims truncation and securement or composition from minimal rights and the dual property of securement.

The paper is organized as follows: Sect. 2 describes the reference model and presents the TAL-family; Sect. 3 analyzes the structural properties of the rules in this family; Sect. 4 concludes with a discussion on the connection between this work and other results in the literature; all the proofs are relegated to the Appendix.

2 The model

2.1 Preliminary definitions

Let $\mathbb{N}$ represent the set of all potential agents (a set with an infinite number of members) and let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents $N = \{1, 2, \ldots, n\}$, where $|N| = n$. A *bankruptcy problem* (O'Neill 1982) is a triple (N, E, c), where N is the set of agents, $E \in \mathbb{R}_+$ represents the *amount to be divided*, and $c \in \mathbb{R}^n_+$ is a *vector of claims* whose ith component is c_i. The very notion of bankruptcy problem requires $\sum_{i \in N} c_i \geq E > 0$. The family of all problems is $\mathbb{B}$. To simplify notation we write, for any given problem $(N, E, c) \in \mathbb{B}$, $C = \sum_{i \in N} c_i$. We denote by c_{-i} the vector c from which the ith coordinate has been removed. Finally, we assume, without loss of generality, that agents are labelled so that $c_1 \leq c_2 \leq \cdots \leq c_n$.

A *rule* is a mapping R that associates with every $(N, E, c) \in \mathbb{B}$ a unique point $R(N, E, c) \in \mathbb{R}^n$ such that:

(i) $0 \leq R(N, E, c) \leq c$; and
(ii) $\sum_{i \in N} R_i(N, E, c) = E$.

The point $R(N, E, c)$ represents a desirable way of dividing E among the agents in N. Requirement (i) is that each agent receives an award that is non-negative and bounded above by her claim. Requirement (ii) is that the entire amount be allocated. These two requirements imply that $R(N, E, c) = c$ whenever $E = \sum_{i \in N} c_i$.

Consider the following classical rules:

The ***constrained equal-awards*** *rule, A, selects for all $(N, E, c) \in \mathbb{B}$, the vector $(\min\{c_i, \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.*

The ***constrained equal-losses*** rule*, L , selects for all $(N, E, c) \in \mathbb{B}$, the vector $(\max\{0, c_i - \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.*

The ***Talmud*** *rule, T, selects for all $(N, E, c) \in \mathbb{B}$, the vector $(\min\{(1/2)c_i, \lambda\})_{i \in N}$ if $E \leq (1/2)C$ and the vector $(\max\{(1/2)c_i, c_i - \mu\})_{i \in N}$ if $E \geq (1/2)C$, where λ and μ are chosen so that $\sum_{i \in N} T_i(N, E, c) = E$.*

The constrained equal awards rule distributes the amount equally among all agents, subject to no agent receiving more than she claims. The constrained equal losses rule imposes that losses are as equal as possible subject to no one receiving a negative amount. Finally, the Talmud rule behaves like the constrained equal awards rule (respectively the constrained equal losses rule) using half-claims instead of claims, provided the amount to divide falls short (respectively exceeds) one half of the aggregate claim. Indeed, the Talmud rule can also be given the following representation:

$$T\,(N, E, c) = \begin{cases} A(N, E, \frac{1}{2}c) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + L(N, E - \frac{1}{2}C, \frac{1}{2}c) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

That is, for “small” values of E the Talmud rule behaves as the constrained equal awards rule and for “large” values of E as the constrained equal losses rule.

2.2 The TAL-family

We now present a family of rules that generalizes the Talmud rule (T) and encompasses the constrained equal awards rule (A) and the constrained equal losses rule (L). The analysis of this family will provide us with further insights into the relationship between them.

The Talmud rule is an allocation method that depends on the size of the amount to divide relative to the aggregate claim. It can be regarded as implementing a criterion according to which nobody gets more than half of her claim if the amount to be distributed is less than half of the aggregate claim, and nobody gets less than half of her claim if the amount exceeds half of the total demand. The TAL-family generalizes this idea by applying the same principle to E and θC, and using θc as the switch point, for all values θ in the interval [0, 1]. Formally, the following holds.

The ***TAL-family*** *consists of all rules with the following form: for some $\theta \in$ [0, 1], for all $(N, E, c) \in \mathbb{B}$, and all $i \in N$,*

$$R_i^{\theta}\,(N, E, c) = \begin{cases} \min\{\theta c_i, \lambda\} & \text{if } E \leq \theta C \\ \max\{\theta c_i, c_i - \mu\} & \text{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} R_i^{\theta}\,(N, E, c) = E$.

The rule R^θ in the TAL-family applies the following principle: nobody gets more than a fraction θ of her claim if the amount to be distributed is less than θ times the aggregate claim and nobody gets less than a fraction θ of her claim if the amount exceeds θ times the aggregate claim. It is straightforward to show that all rules within the family are well defined. Note that the constrained equal-losses rule corresponds to the case $\theta = 0$ ($R^0 = L$), whereas the constrained equal-awards rule corresponds to the other extreme value, $\theta = 1$ ($R^1 = A$). Obviously, the Talmud rule is obtained for $\theta = 1/2$ ($R^{1/2} = T$).

The TAL-family can also be given the following equivalent expression:

$$R^\theta (N, E, c) = \begin{cases} A(N, E, \theta c) & \text{if } E \leq \theta C \\ \theta c + L(N, E - \theta C, (1-\theta)c) & \text{if } E \geq \theta C \end{cases},$$

or, equivalently,

$$R^\theta (N, E, c) = A(N, \min\{E, \theta C\}, \theta c) + L(N, \max\{E - \theta C, 0\}, (1-\theta)c).$$

Therefore, one can visualize the rule R^θ as follows. First, it applies equal division until the claimant with the smallest claim has obtained a fraction θ of her claim. Then, that agent stops receiving additional units and the remaining amount is divided equally among the other agents until the claimant with the second smallest claim gets the fraction θ of her claim. The process continues until every agent has received a fraction θ of her claim, or the available amount is distributed. If there is still something left after this process, agents are invited back to receive additional shares. Now, agents receive additional amounts sequentially starting with those with larger claims and applying equal division of their losses (see Fig. 1 for further details).

Aumann and Maschler (1985) define the *dual* of a rule R, denoted R^*, as follows: for all $(N, E, c) \in \mathbb{B}$, $R^*(N, E, c) = c - R(N, C - E, c)$. The rule R^* shares awards as R shares losses. When a rule and its dual produce the same outcomes it is called *self-dual*. There exists a precise duality relationship between the members of the TAL-family which is as follows:

Proposition 1 *Let R^θ be a rule in the TAL-family $\{R^\theta\}_{\theta \in [0,1]}$. Then, the dual rule of R^θ is $R^{1-\theta}$, that is, $(R^\theta)^* = R^{1-\theta}$.*

Two well-known results are immediately derived from this duality relationship. One is that L and A are dual rules. The other that T is self-dual (in fact, there is no other self-dual rule in the TAL-family).

The notion of duality can also be applied to the properties a rule satisfies. That is, $\mathcal{P}^*$ is the *dual property of* $\mathcal{P}$ if for every rule R it is true that R satisfies $\mathcal{P}$ if and only if its dual rule R^* satisfies $\mathcal{P}^*$. This will be important in the rest of the paper.

3 Properties of the TAL-family

The rules in the TAL-family satisfy many properties that are common in the bankruptcy literature. We shall consider some of them here. In the first part, we analyze the most basic properties which are satisfied by all rules in the family (symmetry, continuity, consistency, order preservation, monotonicity and homogeneity). Next,

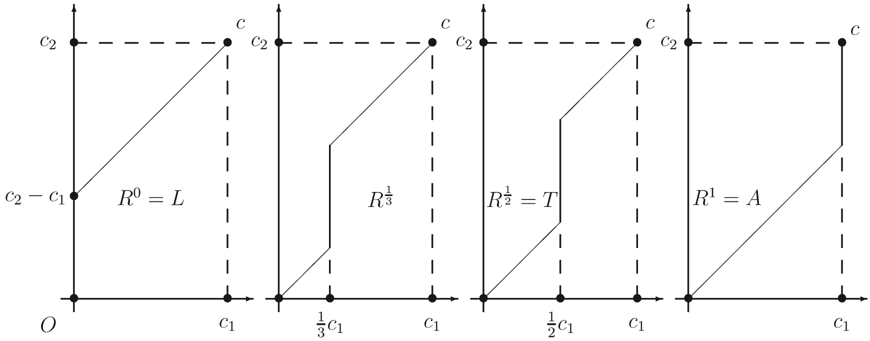

Fig. 1 Rules in the two-claimant case. This figure illustrates the "path of awards" of some rules within the TAL-family for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. The path of awards for c (the locus of the awards vector chosen by a rule as the amount to divide E varies from 0 to $c_1 + c_2$) of $R^0 = L$ follows the vertical axis until the average loss coincides with the lowest claim, i.e., until $E = c_2 - c_1$. After that, it follows the line of slope 1 until it reaches the vector of claims. The path of awards of $R^{1/3}$ follows the 45° line until claimant 1 obtains one-third of her claim. Then, it is a vertical line until $E = c_2 - (1/3)c_1$, from where it follows the line of slope 1 until it reaches the vector of claims. The path of awards of $R^{1/2} = T$ follows the 45° line until claimant 1 obtains half of her claim. Then, it is a vertical line until $E = c_2$, from where it follows the line of slope 1 until it reaches the vector of claims. Finally, the path of awards of $R^1 = A$ follows the 45° line until it gives the whole claim to the lowest claimant, i.e., until $E = 2c_1$, from where it is vertical until it reaches the vector of claims

we discuss more specific properties that are satisfied by the members of the family depending on the value of the parameter θ (independence of claims truncation, composition from minimal rights and securement). Then, we consider some properties that are only satisfied by the extreme members of the family (composition up, composition down, sustainability, independence of residual claims, exemption and exclusion). Finally, we refer to the effect of a change in the parameter θ over the spread of the allocations generated by the corresponding rule.

The most important of these results is that in which we show that $\theta = 1/2$ is the threshold that separates those rules in the family that satisfy independence of claims truncation from those that satisfy composition from minimal rights.

3.1 Basic properties

Let us start by showing that each rule in the TAL-family is parametric. According to Young (1987), a rule is parametric if the ith agent's award is a function that only depends on c_i and a parameter λ, which is related to the size of the amount to divide. More precisely:
A rule R is ***parametric*** *if there exists a function $f\colon [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $[a, b] \subset \mathbb{R} \cup \{\pm\infty\}$, continuous and weakly monotonic in its first argument, such that:*

(i) $R_i(N, E, c) = f(\lambda, c_i)$ *for all* $(N, E, c) \in \mathbb{B}$ *and for some* $\lambda \in [a, b]$;
(ii) $f(a, x) = 0$, *for all* $x \in \mathbb{R}_+$; *and*
(iii) $f(b, x) = x$, *for all* $x \in \mathbb{R}_+$.

All rules within the TAL-family are parametric. To check this, let $\theta \in [0, 1]$ be given and define $f^{\theta}\colon \mathbb{R} \cup \{\pm\infty\} \times \mathbb{R}_{+} \to \mathbb{R}_{+}$ as follows:

$$f^{\theta}(\lambda, c_i) = \begin{cases} \min\{-\frac{1}{\lambda}, \theta c_i\} & \text{if } \lambda < 0 \\ \max\{c_i - \frac{1}{\lambda}, \theta c_i\} & \text{if } \lambda \geq 0 \end{cases}.$$

Clearly, f^{θ} is continuous and weakly monotonic in its first argument, with $\lim_{\lambda \to -\infty} f^{\theta}(\lambda, c_i) = 0$ and $\lim_{\lambda \to +\infty} f^{\theta}(\lambda, c_i) = c_i$ for all $c_i \in \mathbb{R}_{+}$. As a result, the Darboux property shows that, for all $(N, E, c) \in \mathbb{B}$, there exists $\lambda_0 \in \mathbb{R}_{+}$ such that $E = \sum_{i \in N} f^{\theta}(\lambda_0, c_i)$. Thus, $f^{\theta}(\lambda_0, c_i) = R_i^{\theta}(N, E, c)$, which shows that f^{θ} is a parametric representation of R^{θ}.

Remark 1 Note that these functions f^{θ} are S-shaped in claims. That is, f^{θ} is concave on its second argument when $\lambda < 0$ (which corresponds to a "small" amount to divide) whereas it is convex when $\lambda > 0$ (which corresponds to a "large" amount to divide). Therefore, the rules in the TAL-family evaluate awards and losses differently, depending on the size of the amount to divide. This feature appears in the psychology literature, where it is well known that losses loom larger than gains (e.g., Kahneman and Tversky 1979).

Young (1987) shows that a rule is parametric if and only if it satisfies symmetry, continuity and consistency. *Symmetry* is a property of impartiality that requires allotting equal amounts to those agents with equal claims. *Continuity* is an axiom of non-arbitrariness that says that small changes in the parameters of the problem should not induce large changes in the corresponding solution. *Consistency* is a principle that relates the solution of a given problem to the solutions of the subproblems that appear when we consider a subgroup of agents as a new population and the amounts gathered in the original problem as the available amount to be distributed. Consistency requires that the application of the rule to each subproblem produces precisely the allocation that the subgroup obtained in the original problem.[1] More formally, a rule R is consistent if, for all $(N, E, c) \in \mathbb{B}$, all $Q \subset N$ and all $i \in Q$, we have $R_i(N, E, c) = R_i(Q, E_Q, c_Q)$, where $E_Q = \sum_{i \in Q} R_i(N, E, c)$ and $c_Q = (c_i)_{i \in Q}$.

Therefore, the preceding discussion shows the following:

Proposition 2 *All rules in the TAL-family are symmetric, continuous and consistent.*

The following related properties are also pertinent to this analysis: order preservation and homogeneity. *Order preservation*, which implies symmetry, says that agents with larger claims receive larger awards and face larger losses. That is, $c_i \geq c_j$ implies that $R_i(N, E, c) \geq R_j(N, E, c)$ and $c_i - R_i(N, E, c) \geq c_j - R_j(N, E, c)$, for all $(N, E, c) \in \mathbb{B}$, all $i, j \in N$. The rules in the family satisfy this property. *Homogeneity* says that if claims and amount available are multiplied by the same positive number, then so should all awards. It implies that the units in which claims and amount available are measured have no influence on the outcome. The rules in the family also satisfy this property.

Proposition 3 *All rules in the TAL-family are order preserving and homogeneous.*

[1] Thomson (1996) provides us with a survey of the many applications that have been made on the idea of consistency.

3.2 Solidarity properties

In this section, we consider monotonicity properties, also referred to as axioms of solidarity in similar contexts (see, for instance, Moreno-Ternero and Roemer 2004, and the literature cited therein).

Resource monotonicity says that when there is more to be divided, other things being equal, nobody should lose, i.e., a larger cake should not harm anybody. Formally, a rule R is resource monotonic if, for each $(N, E, c) \in \mathbb{B}$ and $(N, E', c) \in \mathbb{B}$ such that $E \leq E'$, then $R(N, E, c) \leq R(N, E', c)$. *Claims monotonicity* says that if an agent's claim increases, ceteris paribus, she should receive at least as much as she did initially, i.e., a larger claim should not harm the claimant. Its dual property says that if an agent's claim and the amount to divide increase by the same amount, the agent's award should increase by at most that amount. Formally, a rule R is claims monotonic if, for all $(N, E, c) \in \mathbb{B}$ and all $i \in N$, $c_i \leq c_i'$ implies $R_i(N, E, (c_i', c_{-i})) \geq R_i(E, (c_i, c_i))$. A rule satisfies the dual of claims monotonicity if, for all $(N, E, c) \in \mathbb{B}$ and $i \in N$, $R_i(E+\varepsilon, (c_i+\varepsilon, c_{-i})) \leq R_i(E, c)+\varepsilon$.

Similar properties turned out to be very strong in other domains of problems, sometimes even being incompatible with very elementary requirements of efficiency and fairness (e.g., Thomson 1987). In the context of bankruptcy, they are quite weak however. As it stands out, all rules within the TAL-family satisfy these properties.

Proposition 4 *All rules in the TAL-family satisfy resource monotonicity, claims monotonicity and its dual.*

3.3 Independence and related properties

We start this section considering the axioms of independence of claims truncation and composition from minimal rights. *Independence of claims truncation* postulates that the part of a claim that is above the amount to divide should be ignored. That is, $R(N, E, c) = R(N, E, t(N, E, c))$, where $t_i(N, E, c) = \min\{E, c_i\}$ for all $i \in N$. *Composition from minimal rights* ensures each agent a minimal amount $m_i(N, E, c) = \max\{0, E - \sum_{j\in N-\{i\}} c_j\}$, which is the portion of the amount to divide that is left to her when the claims of all other agents are fully honored, provided this amount is non-negative. That is,

$$R(N, E, c) = m(N, E, c) + R(N, E - M(N, E, c), c - m(N, E, c)) ,$$

where $m(N, E, c) = [m_i(N, E, c)]_{i\in N}$ and $M(N, E, c) = \sum_{i\in N} m_i(N, E, c)$.

These two axioms were first studied by Curiel et al. (1897). The Talmud rule is the only rule that satisfies symmetry, independence of claims truncation, composition from minimal rights, and consistency (Aumann and Maschler 1985; Dagan 1996; Herrero and Villar 2001). A characterization of all rules that satisfy homogeneity, independence of claims truncation and composition from minimal rights is also available (Hokari and Thomson 2003).

As the following result shows, $\theta = 1/2$ is the precise value of the parameter that separates the rules in the family that satisfy independence of claims truncation from those that satisfy composition from minimal rights.

Theorem 1 *Let R^θ be a rule in the TAL-family $\{R^\theta\}_{\theta\in[0,1]}$. The following statements hold:*

(i) *R^θ satisfies independence of claims truncation if and only if $\theta \in [1/2, 1]$.*
(ii) *R^θ satisfies composition from minimal rights if and only if $\theta \in [0, 1/2]$.*

A direct consequence of this result is that there is only one rule in the family that satisfies these two properties simultaneously: the Talmud rule, $T = R^{1/2}$.

A related axiom is that of securement. *Securement* ensures each agent a minimal share of her individual claim, no matter what the other claims are. In particular, for a problem involving n agents, securement establishes that any agent holding a feasible claim (a claim not larger than the amount to divide) will get at least one nth of her claim. And also that, those agents whose individual claims are unfeasible will get at least one nth of the amount to divide. Formally, a rule R satisfies securement if, for all $(N, E, c) \in \mathbb{B}$, $R_i(N, E, c) \geq (1/n)\min\{c_i, E\}$. Its dual property is also an interesting one. This property provides an upper bound to each claimant involved in the problem. Formally, a rule R satisfies the dual of securement if, for all $(N, E, c) \in \mathbb{B}$, $R_i(N, E, c) \leq c_i - (1/n)\min\{c_i, C - E\}$.

These two axioms are introduced in Moreno-Ternero and Villar (2004). Recently, Dominguez and Thomson (2006) have presented the unique rule satisfying the requirement that for each problem the awards vector can be obtained either directly or in two steps: by first assigning to each claimant the lower bound that the property of securement recommends and then reapplying the rule to the revised problem.

In Moreno-Ternero and Villar (2004), we show that the Talmud rule is the unique consistent rule that satisfies securement and its dual property. Thus, by Proposition 2, no other rule within the TAL-family can satisfy both properties simultaneously. Furthermore, the Talmud rule is also the unique consistent rule that satisfies securement and composition from minimal rights (Moreno-Ternero and Villar 2006; Yeh 2006). Thus, by Proposition 2 and Theorem 1, no rule R^θ in the TAL-family such that $\theta \in [0, 1/2)$ satisfies securement. We show in the next result that all the remaining rules within the family satisfy securement.

Theorem 2 *Let R^θ be a rule in the TAL-family $\{R^\theta\}_{\theta\in[0,1]}$. The following statements hold:*

(i) *R^θ satisfies securement if and only if $\theta \in [1/2, 1]$.*
(ii) *R^θ satisfies the dual of securement if and only if $\theta \in [0, 1/2]$.*

From Theorems 1 and 2, one might think that there exists a logical relation between independence of claims truncation and securement. That is not the case, as shown in Moreno-Ternero and Villar (2004), where we provide examples of rules satisfying one property but not the other.

3.4 Composition properties

We now consider two additional properties dealing with the solvability of a bankruptcy problem in stages. To motivate these properties think of the following situation: after having divided the allocation of the available amount among its creditors, it turns out that the actual value of the amount is larger than was initially assumed. Then, two options are open: either the tentative division is cancelled altogether and the actual problem is solved, or we add to the initial distribution the result of applying the rule to the remaining amount. The requirement formulated next is that both ways of proceeding should result in the same award vectors. Formally, a rule R satisfies *composition up* (Young 1988) if, for all $(N, E, c) \in \mathbb{B}$, and all $E_1, E_2 \in \mathbb{R}_{++}$ such that $E_1+E_2 = E$, $R(N, E, c) = R(N, E_1, c)+R\left[N, E_2, c - R(N, E_1, c)\right]$.

Think now of the dual case. Namely, after having divided the available amount among its creditors one finds that the actual value of the amount to divide falls short of what was assumed. Here again, we can ignore the initial division and apply the rule to the revised problem, or we can apply the rule to the problem in which the initial claims are substituted by the (unfeasible) allocation initially proposed. The next requirement is that both ways of proceeding should result in the same award vectors. Formally, a rule R satisfies *composition down* (Moulin 1987) if, for all $(N, E, c) \in \mathbb{B}$, and all $E' > E$, we have $R(N, E, c) = R\left[N, E, R(N, E', c)\right]$.

In stark contrast with the properties mentioned above, these properties are only satisfied by the constrained equal awards rule and the constrained equal losses rule within the TAL-family. More precisely:

Theorem 3 *The following statements hold:*

(i) *The only rules in the TAL-family that satisfy composition up are* $R^0 = L$ *and* $R^1 = A$.
(ii) *The only rules in the TAL-family that satisfy composition down are* $R^0 = L$ *and* $R^1 = A$.

There are only three symmetric rules that satisfy homogeneity, consistency, composition up and composition down. They are the proportional rule, the constrained equal awards rule and the constrained equal losses rule (Moulin 2000).[2] Thus, Propositions 2 and 3 imply that $R^0 = L$ and $R^1 = A$ are the only rules in the TAL-family that satisfy both composition up and composition down. Theorem 3 complements this corollary showing that there is no other rule within the family satisfying only one of the properties.

Similarly, Dagan (1996) shows that the constrained equal awards rule is the only symmetric rule that satisfies independence of claims truncation and composition up. By duality, the constrained equal awards rule is the only symmetric rule that satisfies composition from minimal rights and composition down. As before, these two results and Theorem 1 imply that $R^0 = L$ and $R^1 = A$ are the only rules in the TAL-family that satisfy both composition up and composition down. Furthermore, they imply that the only rule in the family $\left\{R^\theta\colon \theta \in [1/2, 1]\right\}$ that satisfies composition up is $R^1 = A$; and also that the only rule in the family $\left\{R^\theta\colon \theta \in [0, 1/2]\right\}$ that satisfies composition down is $R^0 = L$.[3]

[2] The proportional rule is the rule that selects, for all $(N, E, c) \in \mathbb{B}$, the vector $[(E/C) \cdot c_i]_{i \in N}$. Observe that, for a given problem $(N, E, c) \in \mathbb{B}$, the rule R^θ within the TAL-family, where $\theta = E/C$, yields a solution $R^{E/C}(N, E, c)$ that coincides with the allocation provided by the proportional rule to this problem. Yet, there is no θ for which R^θ is the proportional rule (i.e., the proportional rule is *not* a member of the TAL-family).

[3] We thank an anonymous referee for raising these implications between existing results in the literature and those of our paper.

3.5 Protective properties

We now turn our attention to a group of properties that appear in the literature referring to the application of some protective criteria for agents with "very large" or "very small" claims: sustainability, independence of residual claims, exemption and exclusion (Herrero and Villar 2001, 2002; Yeh 2005).[4] They establish restrictions on the behavior of a rule when claims are very unequal. Both exemption and sustainability require that agents with relatively small claims be fully reimbursed. In the former property, smallness is defined as having a claim below equal division of the available amount. In the latter property, a claim is considered small when substituting it for the claim of any other agent whose claim is higher, there would be enough to compensate everyone. Formally, a rule R satisfies *sustainability* if, for all $(N, E, c) \in \mathbb{B}$, and each $i \in N$, if $\sum_{j \in N} \min\{c_i, c_j\} \leq E$ then $R_i(N, E, c) = c_i$. Similarly, a rule R satisfies *exemption* if, for all $(N, E, c) \in \mathbb{B}$, $c_i \leq E/n$ implies $R_i(N, E, c) = c_i$. Although they are equivalent in the two-claimant case, sustainability implies exemption in the general case of n claimants (Herrero and Villar 2002). Conversely, if a rule satisfies exemption and consistency then it also satisfies sustainability (Yeh 2005).

Dually, one could adopt the viewpoint that agents with larger claims are given priority so that agents with very small claims should not receive anything. That kind of value judgement makes sense when the claims represent needs or when we want to compensate more to those who have risked larger amounts. We say that a claim is "residual" when the aggregate excess claim relative to the agent holding this claim exceeds the worth of the amount available. That is, $E \leq \sum_{j \in N} \max\{0, c_j - c_i\}$. Independence of residual claims requires that if an agent's claim is residual, she should get nothing. Formally, a rule R satisfies *independence of residual claims* if, for all $(N, E, c) \in \mathbb{B}$, $E \leq \sum_{j \in N} \max\{0, c_j - c_i\}$ implies $R_i(N, E, c) = 0$. Exclusion requires that if an agent's claim does not reach the average loss, she gets nothing. Formally, a rule R satisfies *exclusion* if, for all $(N, E, c) \in \mathbb{B}$, $c_i \leq (C - E)/n$ implies $R_i(N, E, c) = 0$. It is straightforward to show that independence of residual claims and exclusion are the dual properties of sustainability and exemption, respectively. Therefore, independence of residual claims implies exclusion, whereas exclusion, in conjunction with consistency, implies independence of residual claims.

As the next result shows, the constrained equal awards rule is the only rule in the TAL-family that satisfies exemption. It is also the only rule that satisfies sustainability. On the other hand, the constrained equal losses rule is the only rule in the TAL-family that satisfies exclusion. It is also the only one that satisfies independence of residual claims.

Theorem 4 *The following statements hold:*

(i) *The only rule in the TAL-family that satisfies sustainability is* $R^1 = A$.
(ii) *The only rule in the TAL-family that satisfies exemption is* $R^1 = A$.
(iii) *The only rule in the TAL-family that satisfies independence of residual claims is* $R^0 = L$.
(iv) *The only rule in the TAL-family that satisfies exclusion is* $R^0 = L$.

[4] The notions of sustainability and independence of residual claims are referred as conditional full compensation and conditional null compensation, respectively, by Thomson (2003).

Theorem 4 relates with some characterization results of the constrained equal awards rule and the constrained equal losses rule that appear in the literature. For instance, Herrero and Villar (2002) show that the constrained equal awards rule is the only rule that satisfies sustainability and composition down. By duality, the constrained equal losses rule is the only rule that satisfies independence of residual claims and composition down. Yeh (2005) replaces the axioms of composition in the previous results by claims monotonicity and its dual property, respectively. Yeh (2005) also shows that the constrained equal awards rule is the only rule satisfying exemption, order preservation and consistency. By duality, the constrained equal losses rule is the only rule satisfying exclusion, order preservation and consistency.[5]

Since all rules within the TAL-family are consistent (Proposition 2), order preserving (Proposition 3) and satisfy claims monotonicity and its dual (Proposition 4) the content of Theorem 4 follows from Yeh's results and duality.

3.6 Distributional effects

Finally, we consider the effect of changes in the parameter θ that generates the TAL-family on the resulting distribution corresponding to a given problem. In order to do that, we compare the allocations generated by different rules in the family, for a given problem, by means of the classical Lorenz ordering. Given $x, y \in \mathbb{R}^n$ satisfying $x_1 \leq x_2 \leq \cdots \leq x_n$, $y_1 \leq y_2 \leq \cdots \leq y_n$, and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, we say that x *is greater than* y *in the Lorenz ordering* if $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$, for all $k = 1, \ldots, n-1$, with at least one strict inequality. This criterion induces a partial ordering on allocations which reflects their relative spread. When x is greater than y in the Lorenz ordering, the distribution x is unambiguously "more egalitarian" than the distribution y. It is well known that this property is equivalent to saying that y can be obtained from x by means of a finite sequence of transfers "from the richer to the poorer" , and that $I(y) > I(x)$ for any sensible inequality index $I(\cdot)$, i.e., for any symmetric relative index satisfying the Pigou-Dalton transfer principle and the Dalton population principle [see Atkinson (1970), Dasgupta et al. (1973), Rostchild and Stiglitz (1973) or Chakravarty (1999), among others].

We now propose what appears the obvious move: we say that a rule R *Lorenz dominates* a rule R', which we write as $R \succsim_L R'$, when for all $(N, E, c) \in \mathbb{B}$, $R(N, E, c)$ is greater than $R'(N, E, c)$ in the Lorenz ordering. The following result is obtained:

Theorem 5 (Moreno-Ternero and Villar 2005) *For all* $\theta_1, \theta_2 \in [0, 1]$ *with* $\theta_1 \geq \theta_2$*,* $R^{\theta_1} \succsim_L R^{\theta_2}$.

Theorem 5 says that all rules within the TAL-family are fully ranked in terms of the Lorenz dominance criterion. In particular, Theorem 5 says that higher values of θ imply higher protection for those agents with lower claims (more redistribution), whereas lower values of θ entail higher shares for those with larger claims (less redistribution). Therefore, the parameter θ that generates the family can be regarded as an index of the *distributive power* of the rule. Choosing θ can be interpreted as giving a degree of priority θ in the distribution to those agents with lower

[5] Note that order preservation and consistency are self-dual properties.

claims and a degree of priority $(1-\theta)$ to those with higher demands. From this perspective, the Talmud rule, $R^{1/2}$, is a balanced compromise between the different types of claimants.

4 Final remarks

We have presented in this paper a family of bankruptcy rules, the TAL-family, that generalizes the Talmud rule and encompasses the constrained equal awards rule and the constrained equal losses rule. This family depends on a parameter $\theta \in [0, 1]$ that refers to the relative magnitude of the amount to divide with respect to the aggregate claim and can be interpreted in terms of an index of the distributive power of the rule. We have explored the behavior of the rules in the TAL-family with respect to the standard properties in the literature. Our findings are summarized in Table 1.

Other families of rules extending the Talmud rule appear in the literature. Hokari and Thomson (2003) introduce a family of consistent rules meeting two characteristic properties of the Talmud rule: independence of claims truncation and composition from minimal rights. The rules in this family, however, do not necessarily satisfy the axiom of symmetry.

Thomson (2000, 2003) presents another family of rules (the *ICI-family*) exhibiting another feature of the Talmud rule: the evolution of each claimant's award, as a function of the amount to divide, is increasing first, constant next and finally increasing again. In order to provide the formal definition of the ICI-family, let $\mathcal{G}^N$ be the family of lists $G \equiv \{E_k, F_k\}_{k=1}^{n-1}$, where $n = |N|$, of real-valued functions of the claims vector, satisfying for each $c \in \mathbb{R}_+^N$, the following relations:

$$
\begin{aligned}
\frac{E_1(c)}{n} + \frac{C - F_1(c)}{n} &= c_1 \\
c_1 + \frac{E_2(c) - E_1(c)}{n-1} + \frac{F_1(c) - F_2(c)}{n-1} &= c_2 \\
&\vdots \\
c_{k-1} + \frac{E_k(c) - E_{k-1}(c)}{n-k+1} + \frac{F_{k-1}(c) - F_k(c)}{n-k+1} &= c_k \\
&\vdots \\
c_{n-1} + \frac{-E_{n-1}(c)}{1} + \frac{F_{n-1}(c)}{1} &= c_n
\end{aligned}
$$

The ICI rule relative to $G \equiv \{E_k, F_k\}_{k=1}^{n-1} \in \mathcal{G}^N$, is defined as follows. For each $c \in \mathbb{R}_+^N$, the awards vector is given as the following function of the amount available E, as it varies from 0 to C. As E increases from 0 to $E_1(c)$, equal division prevails; as it increases from $E_1(c)$ to $E_2(c)$, claimant 1's award remains constant, and equal division of each new unit prevails among the other claimants. As E increases from $E_2(c)$ to $E_3(c)$, awards of claimants 1 and 2 remain constant, and equal division of each new unit prevails among the other claimants, and so on. This process goes on until E reaches $E_{n-1}(c)$. The next units go to claimant n until E

Table 1 Structural properties of the TAL-family

Properties	Rules that satisfy the properties
Symmetry	R^θ for all $\theta \in [0, 1]$
Continuity	R^θ for all $\theta \in [0, 1]$
Consistency	R^θ for all $\theta \in [0, 1]$
Order preservation	R^θ for all $\theta \in [0, 1]$
Homogeneity	R^θ for all $\theta \in [0, 1]$
Resource monotonicity	R^θ for all $\theta \in [0, 1]$
Claims monotonicity	R^θ for all $\theta \in [0, 1]$
The dual of claims monotonicity	R^θ for all $\theta \in [0, 1]$
Population-and-resource monotonicity	R^θ for all $\theta \in [0, 1]$
Self-duality	$R^{1/2} = T$
Independence of claims truncation	R^θ for all $\theta \in [1/2, 1]$
Composition from minimal rights	R^θ for all $\theta \in [0, 1/2]$
Securement	R^θ for all $\theta \in [1/2, 1]$
The dual of securement	R^θ for all $\theta \in [0, 1/2]$
Composition up	$R^0 = L,\ R^1 = A$
Composition down	$R^0 = L,\ R^1 = A$
Exclusion	$R^0 = L$
Exemption	$R^1 = A$
Independence of residual claims	$R^0 = L$
Sustainability	$R^1 = A$

reaches $F_{n-1}(c)$, at which point equal division of each new unit prevails among claimants n and $n-1$. This goes on until E reaches $F_{n-2}(c)$, at which point equal division of each new unit prevails among claimants n through $n-2$. The process continues until E reaches $F_1(c)$, at which point claimant 1 re-enters the scene and equal division of each new unit prevails among all claimants.

The TAL-family is a strict subset of the ICI-family. Indeed, for each $\theta \in [0, 1]$, the rule R^θ in the TAL-family corresponds to the ICI rule relative to $G(\theta) \equiv \{E_k(\theta), F_k(\theta)\}_{k=1}^{n-1} \in \mathcal{G}^N$, where

$$E_k(\theta)(c) = \theta \cdot \left(\left(\sum_{l=1}^{k-1} c_l \right) + (n-k+1) \cdot c_k \right),$$

and

$$F_k(\theta)(c) = \theta C + (1-\theta) \left(\left(\sum_{l=k+1}^{n} c_l \right) - (n-k) c_k \right).$$

Furthermore, the TAL-family is made precisely of all consistent and homogeneous rules in the ICI-family.

To conclude, we note that the rules within the TAL-family can be characterized by applying an extension of Moulin's (2002) notions of lower and upper bounds, relative to the parameter θ. It is said that a rule satisfies lower bound if each agent receives at least the minimum of her claim and equal division. Dually, a rule satisfies upper bound if each agent faces at least an equal share of the aggregate loss, unless, of course, this share is higher than her claim (in which case she gets nothing). Formally, a rule R satisfies *lower bound* if, for all $(N, E, c) \in \mathbb{B}$, then $R_i(N, E, c) \geq \min\{c_i, E/n\}$. It satisfies *upper bound* if, for all $(N, E, c) \in \mathbb{B}$, then $R_i(N, E, c) \leq \max\{0, c_i - (C - E)/n\}$. These two axioms are very restrictive. Indeed, by themselves, they characterize the constrained equal awards rule and the constrained equal losses rule, respectively, in the two-claimant case (Moulin 2002). If we add consistency, both rules are characterized in the general case of n claimants.

Lower bound provides a protection for those agents with relatively small claims whereas upper bound provides a protection for those agents with relatively large claims. Upon modulating the protection offered to small and large claimants, by means of the parameter θ, we generate the TAL-family. More precisely, we could say that a rule satisfies *lower bound of degree* θ if each agent receives at least either a fraction θ of her claim or an equal share of the amount to divide. Similarly, a rule would satisfy *upper bound of degree* θ if each agent gets at most a fraction θ of her claim or an equal share of the losses. Formally, let $\theta \in [0, 1]$. A rule satisfies *lower bound of degree* θ, if for all $(N, E, c) \in \mathbb{B}$, $R_i(N, E, c) \geq \min\{\theta c_i, E/n\}$. A rule satisfies *upper bound of degree* θ, if for all $(N, E, c) \in \mathbb{B}$, $R_i(N, E, c) \leq \max\{\theta c_i, c_i - (C - E)/n\}$. It is easy to see that R^θ is the only consistent rule that satisfies these two properties simultaneously, thus providing a characterization of the rules in the TAL-family. One might argue that if the axiomatic characterizations of the constrained equal awards rule and the constrained equal losses rule mentioned above are meaningful results, so is the one concerning the TAL-family. This characterization, however, has not the same appeal of those in Moulin (2002). This is so because the "degree θ axioms" are much closer to the very definition of the rules in the family and have much less intuitive appeal. Therefore, we consider that the proper characterization of the TAL-family is still an open problem. Our conjecture is that the rules in the TAL-family are the only parametric rules satisfying either independence of claims truncation and securement or composition from minimal rights and the dual property of securement. More precisely:

Conjecture 1 *A parametric rule R satisfies independence of claims truncation and securement if and only if* $R \in \left\{R^\theta \colon \theta \in [1/2, 1]\right\}$

Conjecture 2 *A parametric rule R satisfies composition from minimal rights and the dual property of securement if and only if* $R \in \left\{R^\theta \colon \theta \in [0, 1/2]\right\}$.

Note that, as a consequence of Proposition 1, one would just need to prove one conjecture and the other would follow by duality.

5 Appendix: Proofs of the results

5.1 Proof of Theorem 1

(ia) *Independence of claims truncation implies* $\theta \in [1/2, 1]$.

For $\theta = 0$ we have $R^0 = L$, a rule that violates this property (Dagan, 1996). Let $\theta \in (0, 1/2)$ and consider the two-claimant problem $B = (\{1, 2\}, E, (E, E/\theta))$. It is straightforward to check that $R^\theta(B) = (\theta E, (1 - \theta) E)$. The associated problem in which claims are truncated is $(\{1, 2\}, E, (E, E))$, whose solution is $(E/2, E/2)$. Therefore, $R^\theta(\{1, 2\}, E, c) \neq R^\theta(\{1, 2\}, E, t(N, E, c))$

(ib) $\theta \in [1/2, 1]$ *implies independence of claims truncation.*

Let $\theta \in [1/2, 1]$ and $(N, E, c) \in \mathbb{B}$ be given. We prove this part by induction in the cardinality of N.

Suppose first $|N| = 2$. Then the rule R^θ is expressed as follows:

$$R^\theta(N, E, c) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq 2\theta c_1 \\ (\theta c_1, E - \theta c_1) & \text{if } 2\theta c_1 \leq E \leq c_2 + (2\theta - 1)\, c_1 \\ \left(c_1 - \frac{C-E}{2}, c_2 - \frac{C-E}{2}\right) & \text{if } c_2 + (2\theta - 1)\, c_1 \leq E \end{cases}. \quad (1)$$

There are several cases.

Case 1 $E \leq c_1$.

Since $\theta \geq 1/2$, it follows that $E \leq 2\theta c_1$. Thus, $R^\theta\,(N, E, c) = (E/2, E/2)$. Moreover, in this case, $t(N, E, c) = (E, E)$. Given that every rule in the TAL-family is symmetric (Proposition 2), $R^\theta\,(N, E, t(N, E, c)) = (E/2, E/2)$.

Case 2 $c_1 < E < c_2$.

In this case, $t(N, E, c) = (c_1, E)$. We distinguish two subcases:

Subcase 2.1 $E \leq 2\theta c_1$.

Then, $R^\theta\,(N, E, c) = (E/2, E/2) = R^\theta\,(N, E, t(N, E, c))$.

Subcase 2.2 $E > 2\theta c_1$.

Since $\theta \geq 1/2$, $E \leq c_2 + (2\theta - 1)\, c_1$, which implies $R^\theta\,(N, E, c) = (\theta c_1, E - \theta c_1)$. Similarly, since $\theta \geq 1/2$, it follows that $E \leq E + (2\theta - 1)\, c_1 = t_2(N, E, c) + (2\theta - 1)\, t_1(N, E, c)$, and, therefore, $R^\theta\,(N, E, t(N, E, c)) = (\theta c_1, E - \theta c_1)$.

Case 3 $E \geq c_2$.

In this case, $t(N, E, c) = c$ and, therefore, independence of claims truncation holds trivially.

As a consequence, for every two-claimant problem, R^θ satisfies independence of claims truncation, when $\theta \in [1/2, 1]$. We now assume that it is also true when $|N| = k \geq 2$, and we prove it for the case $|N| = k + 1$. Without loss of generality, assume that $N = \{1, 2, \ldots, k + 1\}$ and $c_1 \leq c_2 \leq \cdots \leq c_{k+1}$.

Let us show first that $R_1^\theta\,(N, E, c) = R_1^\theta\,(N, E, t(N, E, c))$. It is straightforward to see that

$$R_1^\theta\,(N, E, c) = \begin{cases} \frac{E}{k+1} & \text{if } E \leq (k+1)\theta c_1 \\ \theta c_1 & \text{if } (k+1)\theta c_1 \leq E \leq C - (k+1)\,(1-\theta)\, c_1 \\ c_1 - \frac{C-E}{k+1} & \text{if } C - (k+1)\,(1-\theta)\, c_1 \leq E \end{cases}.$$

Suppose that $c_1 = t_1(N, E, c)$. Otherwise, $t(N, E, c) = (E, E, \ldots, E)$. In such a case, since $c_1 \leq (k+1)\theta c_1$ and R^θ satisfies symmetry we would have $R^\theta\,(N, E, t(N, E, c)) = (E/(k+1), E/(k+1), \ldots, E/(k+1)) = R^\theta\,(N, E, c)$. As a result, $c_1 \leq t_j(N, E, c)$, for all $j = 2, \ldots, k+1 \ldots$ We can also assume that $t_{k+1}(N, E, c) = E$. Otherwise, we would have $c = t(N, E, c)$, and, therefore, independence of claims truncation would hold trivially. We distinguish two cases.

Case 1 $E \leq (k+1)\theta c_1$.

Then $R_1^\theta\,(N, E, c) = E/(k+1) = R_1^\theta\,(N, E, t(N, E, c))$.

Case 2 $E > (k+1)\theta c_1$.

Let $C^T = \sum_{i \in N} t_i(N, E, c)$. Then:

$$\begin{aligned} C - (k+1)(1-\theta)c_1 &\geq C^T - (k+1)(1-\theta)c_1 \\ &= \sum_{i \in N} t_i(N, E, c) - (k+1)(1-\theta)c_1 \\ &= E + \sum_{i=2}^{k} t_i(N, E, c) + [1 - (k+1)(1-\theta)]c_1 \\ &\geq E + \sum_{i=2}^{k} t_i(N, E, c) - (k-1)\frac{c_1}{2} \\ &\geq E + \sum_{i=2}^{k} t_i(N, E, c) - \frac{1}{2}\sum_{i=2}^{k} t_i(N, E, c) \\ &= E + \frac{1}{2}\sum_{i=2}^{k} t_i(N, E, c) > E. \end{aligned}$$

This implies $R_1^\theta(N, E, t(N, E, c)) = \theta c_1 = R_1^\theta(N, E, c)$.

Once it is shown that $R_1^\theta(N, E, c) = R_1^\theta(N, E, t(N, E, c))$, we conclude by appealing to consistency. Let $S = N \backslash \{1\}$. Denote $E_S = \sum_{i \in S} R_i^\theta(N, E, c)$, $c_s = (c_i)_{i \in S}$, $E_S^T = \sum_{i \in S} R_i^\theta(N, E, t(N, E, c))$, and $c_s^T = (t_i(N, E, c))_{i \in S}$. Now, since all rules within the TAL-family are consistent, $R_i^\theta(N, E, c) = R_i^\theta(S, E_S, c_s)$, for all $i \in S$. By induction hypothesis, $R_i^\theta(S, E_S, c_s) = R_i^\theta\left(S, E_S, c_s^T\right)$. Note that $E_S = E - R_1^\theta(N, E, c) = E - R_1^\theta(N, E, t(N, E, c)) = E_S^T$. Thus, $R_i^\theta\left(S, E_S, c_s^T\right) = R_i^\theta\left(S, E_S^T, c_s^T\right) = R_i^\theta(N, E, t(N, E, c))$, where the last equality holds, again thanks to consistency. In other words, for all $i \in S$, or what is equivalent, for all $i = 2, \ldots, k+1$, $R_i^\theta(N, E, c) = R_i^\theta(N, E, t(N, E, c))$, which concludes the proof.

(ii) Independence of claims truncation and composition from minimal rights are dual properties (Herrero and Villar 2001). Thus, as R^θ satisfies independence of claims truncation if and only if $\theta \in [1/2, 1]$, and $R^{1-\theta}$ is the dual rule of R^θ, R^θ satisfies composition from minimal rights if and only if $\theta \in [0, 1/2]$. □

5.2 Proof of Theorem 2

(i) As mentioned in the text, if $\theta \in [0, 1/2)$, R^θ does not satisfy securement. We fix now $\theta \in [1/2, 1]$ and show that R^θ satisfies securement. Let $(N, E, c) \in \mathbb{B}$ be given. Suppose first that $|N| = 2$. Without loss of generality, assume $N = \{1, 2\}$ and $c_1 \leq c_2$. We distinguish three cases.

Case 1 $E \leq 2\theta c_1$.

In this case, $R_i^\theta(N, E, c) = E/2 \geq (1/2)\min\{c_i, E\}$, for all $i = 1, 2$.

Case 2 $2\theta c_1 < E < c_2 + (2\theta - 1)\, c_1$.

In this case, $R^\theta(N, E, c) = (\theta c_1, E - \theta c_1)$. Since $\theta \geq 1/2$, it follows that $R_1^\theta(N, E, c) = \theta c_1 \geq (1/2)\min\{c_1, E\}$. Similarly, since $2\theta c_1 < E$, it follows that $R_2^\theta(N, E, c) = E - \theta c_1 > E/2 \geq (1/2)\min\{c_2, E\}$.

Case 3 $E \geq c_2 + (2\theta - 1)\, c_1$.

In this case, $R^\theta(N, E, c) = (c_1 - (C - E)/2, c_2 - (C - E)/2)$. Since $\theta \geq 1/2$, it follows that $E \geq c_2 + (2\theta - 1)\, c_1 \geq c_2$. Thus, $R_1^\theta(N, E, c) = c_1 - (C - E)/2 \geq c_1/2 \geq (1/2)\min\{c_1, E\}$. Furthermore, $R_2^\theta(N, E, c) = c_2 - (C - E)/2 \geq E/2 \geq (1/2)\min\{c_2, E\}$.

Thus, we have shown that for every two-claimant problem, R^θ satisfies securement, when $\theta \in [1/2, 1]$. Suppose now, by contradiction, that this does not hold for problems with more than two claims. In other words, suppose that there exists (N, E, c) with $|N| = n > 2$ and $i \in N$ such that

$$R_i^\theta(N, E, c) < \frac{1}{n}\min\{c_i, E\} \leq \frac{c_i}{2}. \tag{2}$$

Since $R_i^\theta(N, E, c) < E/n$, there exists $j \in N$ such that $R_i^\theta(N, E, c) > E/n$ and, therefore,

$$R_i^\theta(N, E, c) < \frac{R_i^\theta(N, E, c) + R_j^\theta(N, E, c)}{2}. \tag{3}$$

Consider now the problem $(\{i, j\}, R_i^\theta(N, E, c) + R_j^\theta(N, E, c), (c_i, c_j))$. Since R^θ is consistent and satisfies securement in the two-agent case, it follows that $R_i^\theta(N, E, c) \geq (1/2)\min\{c_i, R_i^\theta(N, E, c) + R_j^\theta(N, E, c)\}$, which represents a contradiction with (2) and (3).

(ii) Since R^θ satisfies securement if and only if $\theta \in [1/2, 1]$ and $R^{1-\theta}$ is the dual rule of R^θ, R^θ satisfies the dual of securement if and only if $\theta \in [0, 1/2]$. □

5.3 Proof of Theorem 3

(i) The rules $R^0 = L$ and $R^1 = A$ satisfy composition up (Moulin 2000). Let us see that there is no other rule within the TAL-family for which this happens. Let $\theta \in (0, 1)$ be given. Consider the two-claimant problem

$$(N, E, c) = \left(\{1, 2\}, 1, \left(\frac{1}{3\theta}, \frac{1}{\theta}\right)\right),$$

and let $E_1 = 1/2 = E_2$. Then, $E_1 < E = 1 < 1/3 + 1 = \theta \cdot (c_1 + c_2)$. Thus, $R^\theta(N, E, c) = A(N, E, \theta c) = (1/3, 2/3)$ and $R^\theta(N, E_1, c) = A(N, E_1, \theta c) = (1/4, 1/4)$.

Let $c' = c - R^\theta(N, E_1, c) = (1/3\theta - 1/4, 1/\theta - 1/4)$. Then, $E - \theta \cdot (c_1' + c_2') = 1 - \theta \cdot (1/3\theta - 1/4 + 1/\theta - 1/4) = \theta/2 - 5/6 < 0$, so that

$$R^\theta\left(N, E_2, c'\right) = A(N, E_2, \theta c') = \begin{cases} (1/4, 1/4) & \text{if } \theta \leq 1/3 \\ (1/3 - \theta/4, 1/6 + \theta/4) & \text{if } \theta > 1/3 \end{cases}.$$

Thus, $R^{\theta}(N, E, c) \neq R^{\theta}(N, E_1, c) + R^{\theta}\left(N, E_2, c'\right)$ for all $\theta \in (0, 1)$.

(ii) Both A and L satisfy composition down (Moulin 2000). Now, suppose that there exists some $\theta \in (0, 1)$ such that R^{θ} satisfies composition down. Then $R^{1-\theta}$, the dual rule of R^{θ}, should satisfy composition up, which contradicts part (i) of this theorem. □

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