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Characteristic properties of list proportional representation systems

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Abstract In this paper, three characterizations are given of a rule that models list systems of proportional representation: the plurality ranking rule. It is shown that a social preference rule is the plurality ranking rule if and only if it satisfies three independent conditions: consistency, faithfulness, and first score cancellation. It is also shown that first score cancellation is implied by neutrality, anonymity, and topsonlyness. This means a second characterization is found, containing deeper axioms than the previous one. A third characterization contains the notion of top monotonicity. In order to motivate topsonlyness, we show that a scoring seat allocation rule is proof against party fragmentation if and only if it is topsonly. Various other properties of the plurality ranking rule are related to its characteristic properties.

1 Introduction

Since Arrow's theorem (Arrow, 1963), the characteristic axioms of various social decision rules have been determined. Famous examples are the axiomatization of the simple majority rule by May (1952) and the axiomatization of the Borda choice correspondence by Young (1974a). These axiomatizations make it possible to evaluate and compare these social decision rules on the basis of their properties. May's result, for example, shows that if we want a rule to satisfy the conditions of neutrality, anonymity, and strong monotonicity, the simple majority rule is the only possible one.

Special kinds of social decision rules are used in Western democracies to choose representatives. A great variety of rules are used for this purpose (see Lijphart

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(1994), Farrell (1997) and Swart et al. (2003)) that produce very different outcomes given the same preferences of the individual citizens (see, for example, Mueller 1989, p. 220–222). Very little is known about the properties of these electoral systems (see Deemen (1993)). This means that the basis to evaluate and compare these systems based on their properties is lacking.

The object of this paper is to give three characterizations of a rule that models list systems of proportional representation (PR). In list systems of proportional representation, parties are assigned a number of seats in parliament that is proportional to the number of votes they received. The degree of proportionality that is actually achieved in these various systems is influenced by characteristics like assembly size, district magnitude, electoral thresholds, and electoral formula. These distortions of proportionality are not considered here. The rule we consider is a rule that assigns to each combination of individual preference orderings of the parties a social ordering of these parties, where a party is ranked higher (receives more seats) when it is the first preference of more voters (receives more first votes). This rule is known as the plurality ranking rule.

2 Preliminaries

Let V be a finite set of voters. Since we want our social preference rule to be applicable to variable voter sets, we consider finite, nonempty subsets I of V. The set of alternatives, in this case (candidates of) political parties, will be identified with A. $R \subseteq A \times A$ will be conceived of as a preference relation. This means that the statement $\langle x, y\rangle \in R$ will be interpreted in the sense that $x \in A$ is at least as good as $y \in A$. Instead of $\langle x, y\rangle \in R$ we usually write $x \succeq_R y$. $x \succ_R y$ is defined by: $\langle x, y\rangle \in R$ and $\langle y, x\rangle \notin R$; $x \sim_R y$ is by definition: $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$. A preference relation R is a weak ordering on A if R is complete and transitive. R is a linear ordering on A if R is complete, transitive and antisymmetric. The set of all weak orderings on A will be denoted by $W(A)$, and the set of all linear orderings on A will be denoted by $L(A)$.

We assume that each individual $i \in I$ orders the parties in a strict way, i.e., for all $i \in I$, $R_i \in L(A)$. For each $I \subseteq V$, a specific combination of the linear orderings on A of the individuals in I, a so-called preference profile on I, can thus be formulated as a function $c : I \to L(A)$. The set of all preference profiles on I is equal to $L(A)^I$. For the linear ordering of individual $i \in I$ in preference profile c on I, we will write c_i. $t(c_i) = x$ denotes that x is the top of c_i. For $c \in L(A)^I$, $\pi_c(x) = |\{i \in I \mid t(c_i) = x\}|$.

Given a set of voters $I \subseteq V$, a *social preference rule* is a function $F : L(A)^I \to W(A)$. Thus, for each finite set I of voters, a social preference rule assigns to each preference profile $c \in L(A)^I$ a weak (social) ordering $F(c)$ of the parties. By definition,

$$t(F(c)) = \{x \in A \mid x \text{ is a top element of } F(c)\}.$$

3 Desirable properties

There are a number of properties that are generally considered to be desirable for any social decision function. Examples are neutrality, anonymity, and (weak) Pareto optimality. In addition, Young (1974a) presented some properties of social choice correspondences which we may consider if we allow the number of voters to vary. A social choice correspondence assigns to each profile a set of social choices. Two of Young's conditions, consistency and faithfulness, may be regarded as desirable, since they are closely connected to the Pareto condition: if a choice correspondence is faithful and consistent, then this choice correspondence is Pareto optimal. The consistency condition relates choices made by disjoint subsets of voters to choices made by their union. It says that if two disjoint subsets of voters choose the same alternatives using a choice correspondence, then their union should choose exactly the same alternatives, using this same choice correspondence. Faithfulness demands of a choice correspondence that, if society consists of a single individual, it must choose the most preferred alternative of this individual. A third desirable property introduced by Young is the cancellation property. A choice correspondence has the cancellation property if it declares a tie between all alternatives if for all pairs (x, y) of alternatives the number of voters who prefer x to y equals the number of voters who prefer y to x.

Similar properties can also be considered desirable for social preference rules. Social preference rules differ from social choice correspondences in that they assign a social preference ordering to each profile instead of a set of social choices. Thus, the conditions should be adapted in order to apply them to social preference rules. The consistency condition relates choices made by disjoint subsets of voters to choices made by their union. Given a social preference rule F, we can demand that, if two disjoint sets of voters I and J both socially prefer party x to party y, using F, then their union should also socially prefer party x to party y, using F. Similarly, we can demand that if party x is socially preferred to party y by voter set I, using F, and voter set J is socially indifferent regarding the choice between party x and party y, using F, then party x should also be socially preferred by the union of I and J, using F. Note that Young's consistency condition would be too strong in the case of social preference rules. The fact that some parts of the social ordering coincide for two groups of voters should not imply anything for other parts of the social ordering.

Definition 1 (Consistency) *A social preference rule F is consistent if whenever $c \in L(A)^I$, $c' \in L(A)^J$ are preference profiles for disjoint sets of voters $I \subseteq V$ and $J \subseteq V$ and $c + c'$ is the profile on $I \cup J$ that corresponds with c on I and with c' on J, for all $x, y \in A$:*

if $x \succ_{F(c)} y$ and $x \succeq_{F(c')} y$, then $x \succ_{F(c+c')} y$.

Faithfulness requires of a decision rule that it chooses according to the individual preference ordering in a one-person situation. A social preference rule will be called faithful if, in case society consists of a single individual whose most preferred party is party x, it orders this party x first.

Definition 2 (Faithfulness) *Let $\{i\} \subseteq V$ be a set of voters consisting of a single individual. A social preference rule F is faithful if, for all $i \in N$, for all $c_i \in L(P)^{\{i\}}$, and for all $x \in A$, if $t(c_i) = x$, then $t(F(c_i)) = x$.*

Faithfulness and consistency are related to a special kind of Pareto property, named first score (FS) Pareto optimality. A social preference rule will be called FS Pareto optimal if, in case all individuals prefer a party x to all other parties, it orders this party x first.

Definition 3 (FS Pareto optimality) *A social preference rule F is FS Pareto optimal if, for all $x \in A$, for all $I \subseteq V$, and for all $c \in L(A)^I$,*
if for all $i \in I$, $t(c_i) = x$, then $t(F(c)) = x$.

Lemma 1 *If F is faithful and consistent, then F is FS Pareto optimal.*

Proof Let F be faithful and consistent. Let $x \in A$, let $I \subseteq V$, and let $c \in L(A)^I$ be such that for all $i \in I$, $t(c_i) = x$. By faithfulness, for all c_i, $t(F(c_i)) = x$. By repeated application of consistency, $t(F(c)) = x$. □

Similar to Young's cancellation requirement, we could demand that any voter's statement 'x is preferred to all other parties' is cancelled or balanced by any other voter's statement 'y is preferred to all other parties'. A social preference rule will be said to have the FS cancellation property if it declares a tie between party x and party y in case the number of individuals who prefer party x most (order x first) equals the number of individuals who prefer party y most (order y first).

Definition 4 (FS cancellation) *A social preference rule F has the FS cancellation property if, for all $I \subseteq V$, for every $c \in L(A)^I$, and for all $x,\ y \in A$,*

$$\text{if } \pi_c(x) = \pi_c(y),\ \text{then } x \sim_{F(c)} y.$$

First score cancellation is implied by three deeper axioms. These are anonymity, neutrality, and topsonlyness. Anonymity and neutrality are defined in the usual way. Topsonlyness requires that whenever the tops of the individual preference orderings correspond for two profiles, then the social preference rule should choose the same outcome for both profiles.

Definition 5 (Anonymity) *A social preference rule F is anonymous if, for all $I \subseteq V$, for every permutation σ of I, and for all preference profiles $c \in L(A)^I$, $F(c \circ \sigma) = F(c)$.*

Definition 6 (Neutrality) *A social preference rule F is neutral if, for every permutation λ of A, for all $I \subseteq V$, and for every preference profile $c \in L(A)^I$, $F(\lambda c) = \lambda F(c)$.*

Definition 7 (Topsonlyness) *A social preference rule F is topsonly if $F(c) = F(c')$ whenever $c, c' \in L(A)^I$ are such that for all $i \in I$ and for all $x \in A$, $t(c_i) = x$ iff $t(c'_i) = x$.*

Lemma 2 *If F is anonymous, neutral, and topsonly, then F has the FS cancellation property.*

Proof Let F be neutral, anonymous, and topsonly. Let $c \in L(A)^I$ be such that $\pi_c(x) = \pi_c(y)$. We should prove that $x \sim_{F(c)} y$.

Let $1, \ldots, k$ be the voters that voted for x and let $k+1, \ldots, 2k$ be the voters that voted for y. Let σ be the permutation of I in which $\sigma(i) = 2k - (i-1)$. Since F is anonymous, $F(c \circ \sigma) = F(c)$ (i). Let λ be the permutation of c with $\lambda(x) = y$ and $\lambda(y) = x$. Since F is neutral, $F(\lambda c) = \lambda F(c)$ (ii). Since $c \circ \sigma$ and λc are such that, for all $i \in I$ and for all $x \in A$, $t(c_i \circ \sigma) = x$ iff $t(\lambda c_i) = x$, by topsonlyness, $F(c \circ \sigma) = F(\lambda c)$ (iii). By (i), (ii), and (iii), $F(c) = \lambda F(c)$. Hence, $x \sim_{F(c)} y$. For suppose $x \succ_{F(c)} y$. Then, $y \succ_{\lambda F(c)} x$ and, thus, $\lambda F(c) \neq F(c)$. Similarly, if $y \succ_{F(c)} x$, then, $x \succ_{\lambda F(c)} y$ and, thus, $\lambda F(c) \neq F(c)$. □

The axioms of consistency and faithfulness in our characterizations can be replaced by an axiom called top monotonicity. This axiom requires that whenever some party x is socially weakly preferred to party y and one voter changes his top preference to party x, all other things being equal, party x must now be ordered before party y.

Definition 8 (Top monotonicity) *Let $c, c' \in L(A)^I$ and $x \in A$ be such that for all $i \in I$, with $t(c_i) = x$, $t(c'_i) = x$, for some $j \in I$ with $t(c_j) \neq x$, $t(c'_j) = x$ and for all $i \in I$ with $t(c'_i) \neq x$, $t(c_i) = t(c'_i)$. A social preference rule F is top monotonous if, for all $I \subseteq V$, for all $x \in A$ and for all $y \in A$, $y \neq x$,*

$$x \succeq_{F(c)} y \text{ implies } x \succ_{F(c')} y.$$

4 The plurality ranking rule

In list systems of proportional representation, every voter is allowed to cast a single vote. This implies that a voter can only vote for the party that is ordered first in his individual preference ordering. Of course, a voter is not obliged to vote for the party he or she prefers most. A voter may, for example, try to influence the coalition formation by voting for another party. Whether individual preference orders represent true or manipulated preferences is not relevant here, however. In this section, we only describe how the individual linear orderings, manipulated or not, are transformed to a (weak) social ordering over the parties.

We define the following score function, which makes explicit the score of a party $x \in A$ given an individual $i \in I$ with linear ordering c_i in a preference profile $c \in L(A)^I$.

$$\pi_{c_i}(x) = \begin{cases} 1 & \text{iff } t(c_i) = x \\ 0 & \text{otherwise.} \end{cases}$$

Thus, given an individual voter, the score of a party is 1 if it is ordered first by this individual voter. Otherwise, its score is 0. The score of a party $x \in A$ at preference profile $c \in L(A)^I$ is equal to the sum of the scores of this party over all the individuals in profile c.

$$\pi_c(x) = \sum_{i \in I} \pi_{c_i}(x) = |\{i \in I \mid t(c_i) = x\}|$$

The plurality ranking rule is then the function $D : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{D(c)} y$ iff $\pi_c(x) \geq \pi_c(y)$.

5 Characterizations

Theorem 1 *Let $F : L(A)^I \to W(A)$, for $I \subseteq V$, be a social preference rule. Then, F is the plurality ranking rule D if and only if F is consistent, faithful, and has the FS cancellation property.*

Proof It is left to the reader to verify that the plurality ranking rule is consistent, faithful, and has the FS cancellation property. Conversely, let F be a social preference rule that satisfies these conditions. Let $I \subseteq V$ and let $c \in L(A)^I$. We should prove that for all $x,\ y \in A$, $x \succeq_{F(c)} y$ iff $x \succeq_{D(c)} y$, i.e., $x \succeq_{F(c)} y$ iff $\pi_c(x) \geq \pi_c(y)$. It is sufficient to prove that, for all x, $y \in A$,

(1) if $\pi_c(x) = \pi_c(y)$, then $x \sim_{F(c)} y$.
(2) if $\pi_c(x) > \pi_c(y)$, then $x \succ_{F(c)} y$.

For, suppose $x \succeq_{F(c)} y$ and $\pi_c(x) < \pi_c(y)$. Then, by (2) $x \prec_{F(c)} y$. Contradiction. So, if $x \succeq_{F(c)} y$, then $\pi_c(x) \geq \pi_c(y)$.

Case 1: Let $\pi_c(x) = \pi_c(y)$. By FS cancellation, $x \sim_{F(c)} y$.
Case 2: Let $\pi_c(x) > \pi_c(y)$. Then, either for all $i \in I$, $t(c_i) = x$; or for some $i \in I$, $t(c_i) \neq x$. If for all $i \in I$, $t(c_i) = x$, then, by Lemma 1, $t(F(c)) = x$. Hence, in particular, $x \succ_{F(c)} y$.

Suppose, on the other hand, that for some $i \in I$, $t(c_i) \neq x$. Then, there are $I', I'' \subseteq I$ such that $I \neq \emptyset$, $I = I' \cup I''$, $I' \cap I'' = \emptyset$ and there are $c' \in L(A)^{I'}$ and $c'' \in L(A)^{I''}$ such that $c = c' + c''$, $t(c'_i) = x$ for all $i \in I'$ and $\pi_{c''}(x) = \pi_{c''}(y)$.

By Lemma 1, $t(F(c')) = x$ and, in particular, $x \succ_{F(c')} y$. By FS cancellation, $x \sim_{F(c'')} y$. By consistency, $x \succ_{F(c)} y$. □

It is left to the reader to verify that the plurality ranking rule satisfies anonymity, neutrality, and topsonlyness. The proof of Theorem 1, together with the proof of Lemma 2 gives then the proof of a second theorem:

Theorem 2 *Let $F : L(A)^I \to W(A)$, for $I \subseteq V$, be a social preference rule. Then, F is the plurality ranking rule D if and only if F is consistent, faithful, anonymous, neutral, and topsonly.*

We are also able to prove the following theorem, in which consistency and faithfulness are replaced by monotonicity.

Theorem 3 *Let $F : L(A)^I \to W(A)$, for $I \subseteq V$ be a social preference rule. Then, F is the plurality ranking rule D if and only if F is top monotonous, anonymous, neutral, and topsonly.*

Proof It is left to the reader to verify that the plurality ranking rule is top monotonous. Conversely, let $F : L(A)^I \to W(A)$, for $I \subseteq V$ be a social preference rule that satisfies top monotonicity, anonymity, neutrality, and topsonlyness. Let $I \subseteq V$ and let $c \in L(A)^I$. We should prove that for all $x,\ y \in A$, $x \succeq_{F(c)} y$ iff $x \succeq_{D(c)} y$, i.e., $x \succeq_{F(c)} y$ iff $\pi_c(x) \geq \pi_c(y)$. We can prove this part by using top monotonicity instead of consistency and faithfulness in the proof of Theorem 1. By Lemma 2, F has the FS cancellation property; so, case 1 in the proof of Theorem 1 remains the same. Consider case 2, where $c \in L(A)^I$ is such that $\pi_c(x) > \pi_c(y)$. We can obtain this profile c from a profile $c' \in L(A)^I$ with $\pi_{c'}(x) = \pi_{c'}(y)$ by assuming that for a sufficient number of individuals $t(c_i) = x$, but $t(c'_i) \neq x$, all other things being equal. By FS cancellation $x \sim_{F(c')} y$. By top monotonicity $x \succeq_{F(c')} y$ implies $x \succ_{F(c)} y$. □

6 Why topsonlyness

Our second characterization shows that if we want a topsonly social preference rule to satisfy the conditions of neutrality, anonymity, consistency and faithfulness, we should pick the plurality ranking rule. Similarly, our third characterization tells us that we should choose the plurality ranking rule if we want such a rule to satisfy the conditions of top monotonicity, anonymity and neutrality.

Whether we would want to choose a topsonly social preference rule may be a matter of discussion. Usually, topsonlyness is not regarded as a very attractive property, since it means that information about the second and third preferences of the individuals is not used. However, in this section we shall show that there are strong arguments for requiring topsonlyness for seat allocation rules. Since in this paper social preference rules model electoral systems and since seat allocation rules provide for an even better model of electoral systems, this means that topsonlyness is also a desirable property for our social preference rules.

First, when allocating seats in a legislature, it is possible in principle for virtually everybody to be represented by the party he or she prefers most. In fact, under pure list PR, everybody does get a representative of the party he or she chooses, and thus everybody actually gets his first choice. We can argue that the fact that John's most preferred alternative is Mark's least preferred one is irrelevant since the representative's job is to present John, not Mark.

Second, topsonlyness turns out to be an important requirement for a seat allocation rule when we bear in mind that a legislature will on its turn take decisions using the majority rule. A rule that satisfies all the characteristic properties of list PR systems except topsonlyness is the Borda rule. The Borda rule satisfies consistency, faithfulness, anonymity and neutrality. The Borda rule is not topsonly, however - the outcome is not fully dependent on the tops of the individual preference orderings. The following example, due to Anthony McGann (see E. van der Hout and A.J. McGann, submitted) shows that the Borda rule is not fit as a seat share allocation rule. This is not surprising, as it was originally proposed as a rule for ranking candidates, not for distributing representatives. Suppose we have two parties x and y, each of which has two voters who favor it:

Voter	1	2	3	4
	x	x	y	y
	y	y	x	x

Parties x and y both receive a Borda count of 2, and thus receive an equal allocation of seats. However, now let us assume that party x splits up into a party x' and a party x'', giving us the following preference distribution:

Voter	1	2	3	4
	x'	x''	y	y
	x''	x'	x''	x'
	y	y	x'	x''

Party x', party x'' and party y now all get a Borda count of 4, so all get equal representation. However, this means that the combined representation of parties x' and x'' is now double that of y. By dividing into two, the original party x has increased its representation at the expense of y. This property of the Borda procedure makes sense when we are ranking candidates—if a new candidate enters the race who is almost identical to x, that candidate should score almost identically to x. However, it is not a desirable quality when distributing seats, because it does not take into account the similarity of candidates or parties. This leads to some potentially undesirable consequences, such as encouraging party fragmentation and possibly excluding minority representation. Apart from these consequences, the results will be arbitrary, as they depend as much on the number of candidates of each type running as on the preferences of the voters.

We are able to prove that a scoring seat allocation rule is party fragmentation proof if and only if it is topsonly. A seat allocation rule is a function that assigns to each profile for each party a seat share of this party.

In the remainder of this section we assume that given m alternatives (m a natural number), a scoring vector v is an m-tuple $\langle v_m, \ldots, v_1 \rangle$ such that $v_m \geq v_{m-1} \geq \cdots \geq v_1 \geq 0$ and $v_m > 0$. The idea is that the score of an alternative $x \in A$ given an individual $i \in I$ with linear ordering c_i in preference profile $c \in L(A)^I$, denoted as $\tau_{v,c_i}(x)$, is $v_{m-(k-1)}$ iff x is the kth preference of this voter. Given a scoring vector v and a preference profile c, the total score of a party $x \in A$, $\tau_{v,c}(x)$, is equal to the sum of the scores for party $x \in A$ over all the individuals in profile c, i.e., $\tau_{v,c}(x) = \Sigma_{i \in I} \tau_{v,c_i}(x)$. Given a scoring vector v, the total score of a profile c, summed over all parties, is denoted as $\tau_v(c)$, i.e., $\tau_v(c) = \Sigma_{x \in A} \tau_{v,c}(x)$.

Definition 9 (scoring seat allocation rule) *Let m be the number of alternatives in A and $v = \langle v_m, \ldots, v_1 \rangle$ be a scoring vector.*

A score seat allocation rule is a function $F_v : L(A)^I \to [0, 1]^A$, defined as follows: $F_v(c) : A \to [0, 1]$ is the seat share function that assigns to any party $x \in A$ its seat share $\frac{\tau_{v,c}(x)}{\tau_v(c)}$. So, $F_v(c)(x)$ is by definition equal to $\frac{\tau_{v,c}(x)}{\tau_v(c)}$.

Note that the range of F_v is the unit simplex of dimension $|A|$, i.e., $\{f \in [0, 1]^A; \Sigma_{x \in A} f(x) = 1\}$.

With party fragmentation proof we mean that a party x cannot obtain a larger seat share by splitting up into a party x_1 and x_2 with similar policy positions. Let $A' = (A - \{x\}) \cup \{x_1, x_2\}$ and let $c' \in L(A')^I$ be the profile that corresponds with profile $c \in L(A)^I$, except that party x_1 and party x_2 take the position of party x in the preference orderings of the voters. So, if for example, x is ordered second by some individual at c, then x_1 is ordered second and x_2 is ordered third at c' or x_2 is ordered second and x_1 is ordered third at c'.

Definition 10 (party fragmentation proof) *A seat allocation rule $F_v : L(A)^I \to [0, 1]^A$ is party fragmentation proof if there exists no party x and profile c such that $F_{v'}(c')(x_1) + F_{v'}(c')(x_2) > F_v(c)(x)$, where x_1 and x_2 result from splitting up party x in to two parties with similar policy positions, c' results from c as described above and v' is some scoring vector $\langle v_m, v_{m-1}, \ldots, v_1, v_0 \rangle$ if $v = \langle v_m, v_{m-1}, \ldots, v_1 \rangle$.*

Definition 11 (topsonly) *Let v be the scoring vector $\langle v_m, \ldots, v_1 \rangle$.*
$F_v : L(A)^I \to [0, 1]^A$ is topsonly iff for all $i \in \{1, \ldots, m - 1\}$, $v_i = 0$.

Theorem 4 *A scoring seat allocation rule is party fragmentation proof if, and only if, it is topsonly.*

Proof Let $|A| = m$ and let $v = \langle v_m, v_{m-1}, \ldots, v_1 \rangle$ be a scoring vector. Let $F_v : L(A)^I \to [0, 1]^A$ be a scoring seat allocation rule. It will be clear that if F_v is topsonly, then F_v is party fragmentation proof. In order to show the converse, suppose that F_v is not topsonly, and hence $v_{m-1} > 0$. Let c be such that x is the first preference of at least one of the voters, and such that y is the first preference of at least one of the other voters for some $y \in A$, $y \neq x$. Notice that $\tau_v(c) > \tau_{v,c}(x)(*)$. □

The fact that party x splits up into a party x_1 and a party x_2 means that at c' the first and the second place are occupied by the parties x_1 and x_2 in the preference ordering of the voters that ordered x first at c. Let $v' = \langle v_m, v_{m-1}, \ldots, v_1, v_0 \rangle$. Then, because $v_{m-1} > 0$,

$$\tau_{v',c'}(x_1) + \tau_{v',c'}(x_2) > \tau_{v,c}(x). \tag{1}$$

Since, for all $y \in A$, $y \neq x$, y occupies at c' either the same position in the preference orderings of the voters or a lower position, we have:

$$\tau_{v',c'}(y) \leq \tau_{v,c}(y) \quad \text{for all } y \in A,\ y \neq x. \tag{2}$$

From Eq. 1 it follows that $\tau_{v',c'}(x_1) + \tau_{v',c'}(x_2) = \tau_{v,c}(x) + k$ for some k. Now, in order to show that F_v is not party fragmentation proof, we shall prove that the seat share $\frac{\tau_{v,c}(x)+k}{\tau_{v'}(c')} > \frac{\tau_{v,c}(x)}{\tau_v(c)}$.

From Eqs. 1 and 2 it follows that

$$\tau_{v'}(c') \leq \tau_v(c) + k. \tag{3}$$

So, $\frac{1}{\tau_{v'}(c')} \geq \frac{1}{\tau_v(c)+k}$ and, thus, $\frac{\tau_{v,c}(x)+k}{\tau_{v'}(c')} \geq \frac{\tau_{v,c}(x)+k}{\tau_v(c)+k}$. Hence, in order to show that F_v is not party fragmentation proof, it suffices to show that $\frac{\tau_{v,c}(x)+k}{\tau_v(c)+k} > \frac{\tau_{v,c}(x)}{\tau_v(c)}$. In other words, $[\tau_{v,c}(x)+k]\tau_v(c) > [\tau_v(c)+k]\tau_{v,c}(x)$. Or equivalently, $k\tau_v(c) > k\tau_{v,c}(x)$, i.e., $\tau_v(c) > \tau_{v,c}(x)$, which, by (*), is indeed the case.

So, we have shown that if F_v is not topsonly, then it is possible to construct a profile c such that the sum of the seat shares of x_1 and x_2 at c' is larger than the seat share of x at c. □

Remark 1 Notice that the proof of Theorem 4 remains valid if we take $v' = \langle v_m, v_{m-1}, \ldots, v_k, v_k, \ldots, v_1 \rangle$, where v_k is the least score party x receives from an individual in profile c.

7 Independence

Consistency, faithfulness, and FS cancellation are independent. A function that violates FS cancellation, while satisfying consistency and faithfulness, is the Borda ranking rule. This is a function $F1 : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{F1(c)} y$ if and only if Borda score $x(c) \geq$ Borda score y (c), for any $c \in L(A)^I$. Given m alternatives, Borda score $x(c)$ is equal to a score of $m - a$ each time alternative x is the ath preference of some voter, summed over all voters.

The function $F2 : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{F2(c)} y$ if and only if $\pi_c(x) \leq \pi_c(y)$, for any $c \in L(A)^I$, does not satisfy faithfulness, while satisfying FS cancellation and consistency. A function that does not satisfy consistency, while satisfying faithfulness and FS cancellation, is the function that ranks the parties that received the most votes first, and ranks all parties that did not receive the most votes second. In order to give a formal description of this rule, for $c\in L(A)^I$, let $V := \{x\in A \mid \forall y \neq x[\pi_c(x) \geq \pi_c(y)]\}$ and let $W := \{x\in A \mid \exists y \neq x[\pi_c(x) < \pi_c(y)]\}$. The function that does not satisfy consistency, while satisfying faithfulness and FS cancellation, is, then, the function $F3 : L(A)^I \to W(A)$, for $I \subseteq V$, defined by (i) for all $x,\ y\in V$, $x \sim_{F3(c)} y$, (ii) for all $r,\ s\in W$, $r \sim_{F3(c)} s$, (iii) for all $x\in V$ and $r\in W$, $x \succ_{F3(c)} r$. This function is not consistent: let $c\in L(A)^I$ and $c'\in L(A)^J$ be preference profiles for disjoint sets of voters $I \subseteq V$ and $J \subseteq V$ such that $\pi_c(x) = 5$, $\pi_c(y) = 4$, $\pi_c(z) = 1$, $\pi_{c'}(x) = 1$, $\pi_{c'}(y) = 4$, and $\pi_{c'}(z) = 5$. Then, $x \succ_{F3(c)} y$ and $x \sim_{F3(c')} y$. Hence, consistency requires that, for the profile $c + c'$, which corresponds with c on I and with c' on J, $x \succ_{F3(c+c')} y$. However, when we apply $F3$ to the profile $c + c'$, we find that $y \succ_{F3(c)} x$.

Anonymity, neutrality, topsonlyness, faithfulness and consistency are also independent. The function $F1$ is not topsonly, but does satisfy all the other properties. The function $F2$ satisfies all properties except faithfulness, and the function $F3$ satisfies all properties but consistency. A function that is not anonymous but does satisfy all the other properties is obtained as follows. Let j be the least element in V and define (i) $\pi'_{c_i}(x) = 1$ if $i \neq j$ and $t(c_i) = x$, (ii) $\pi'_{c_i}(x) = 100$ if $i = j$ and $t(c_i) = x$, and (iii) $\pi'_{c_i}(x) = 0$ in all other cases. The function we look for is the function $F4 : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{F4(c)} y$ iff $\pi'_c(x) \geq \pi'_c(y)$, for any profile c on any set $I \subseteq V$.

The function that satisfies all the properties except neutrality is the function that differs from the plurality ranking rule in that, for a certain party r, party r is ordered before the other parties if an absolute majority of the voters orders party r first, and else this party r is ordered last. Formally, it is the function $F5 : L(A)^I \to W(A)$, for $I \subseteq V$, defined by, for $r\in A$, if $\pi_c(r) > 1/2\ |I|$, then $r \succ_{F5(c)} y$ for all $y\in A$ and if $\pi_c(r) \leq 1/2\ |I|$, $y \succ_{F5(c)} r$ for all $y\in A$ and for $x,\ y\in A$, $x \neq r$, $y \neq r$, $x \succeq_{F5(c)} y$ iff $\pi_c(x) \geq \pi_c(y)$, for any profile c on any set $I \subseteq V$.

The independence of the axioms in the third characterization can also be shown. The function $F1$ satisfies neutrality, anonymity and top monotonicity, but is not topsonly. The function $F4$ satisfies all four axioms except anonymity, and the function $F5$ satisfies all four axioms except neutrality. A function that satisfies all four axioms except top monotonicity is the function $F6 : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{F6(c)} y$ iff $\pi_c(x) \leq \pi_c(y)$, for any profile c on any set $I \subseteq V$.

8 Related axioms

It is well known that the plurality ranking rule fails to satisfy strong monotonicity and Pareto optimality (see Van Deemen (1997)). It does not satisfy independence of irrelevant alternatives (IIA), either. In this section, it is shown how these deficiencies are related to the properties that characterize this rule. On the other hand, the plurality ranking rule does satisfy weak Pareto optimality and weak monotonicity. It is also shown in which way these properties are related to the characteristic ones.

Definition 12 (Pareto optimality) *A social preference rule F is Pareto optimal if for all $x,\ y \in A$ with $x \neq y$, for all $I \subseteq V$, and for all $c \in L(A)^I$, if $x \succ_{c_i} y$ for all $i \in I$, then $x \succ_{F(c)} y$.*

Proposition 1 *If F has the FS cancellation property, then F is not Pareto optimal.*

Proof Let F have the FS cancellation property. Let $x \succ_{c_i} y$ for all $i \in I$ and let $t(c_i) \neq x$ for all $i \in I$. By FS cancellation, $x \sim_{F(c)} y$. Hence, F is not Pareto optimal. □

Definition 13 (Strong monotonicity) *A social preference rule F is strongly monotonic if the following holds: Let $c, c' \in L(A)^I$ and $x \in A$ be such that*

(i) for all $i \in I$ and for all $x^* \neq x$ and $y^* \neq x$, $x^* \succ_{c'_i} y^*$ iff $x^* \succ_{c_i} y^*$.
(ii) for some $j \in I$ and $y \in A$, $y \succ_{c_j} x$ but $x \succ_{c'_j} y$ and for all $i \in I$, $i \neq j$, and for all y^*, $x \succ_{c_i} y^*$ implies $x \succ_{c'_i} y^*$,.

Then, if $x \succeq_{F(c)} y$, then $x \succ_{F(c')} y$.

Proposition 2 *If F has the FS cancellation property, then F is not strongly monotonic.*

Proof Let $c, c' \in L(A)^I$ be like in the definition of strong monotonicity. Furthermore, let $c, c' \in L(A)^I$ be such that, for all $i \in I$, $t(c_i) \neq x$, $t(c_i) \neq y$, $t(c'_i) \neq x$, $t(c'_i) \neq y$. By FS cancellation, $x \sim_{F(c)} y$. By FS cancellation also, $x \sim_{F(c')} y$. Hence, F is not strongly monotonic. □

Definition 14 (Independence of irrelevant alternatives -IIA) *A social preference rule F is IIA if, for all $x,\ y \in A$, for all $I \subseteq V$, and for all $c,\ c' \in L(A)^I$, if $c\ |_{\{x,\ y\}} = c'\ |_{\{x,\ y\}}$, then $F(c)\ |_{\{x,\ y\}} = F(c')\ |_{\{x,\ y\}}$, where $c\ |_{\{x,\ y\}}$ denotes the preference profile c restricted to the individual preferences over x and y, and $F(c)\ |_{\{x,\ y\}}$ denotes the social preference ordering under F given c restricted to the social preference over x and y.*

The proofs of the following propositions in this section do not use the fact that a faithful, consistent, social preference rule that satisfies the FS cancellation property, is the plurality ranking rule (Theorem 1).

The following result has a similar flavor as Theorem 2 in Young and Levenglick (1978), although the setting is different.

Proposition 3 *If F is faithful, consistent, and has the FS cancellation property, then F is not IIA.*

Proof Let $c,\ c' \in L(A)^I$ be such that $x \succ_{c_i} y$ and $x \succ_{c'_i} y$ for all $i \in I$ and such that $t(c_i) = x$ and $t(c'_i) \neq x$ for all $i \in I$. Then $c\ |_{\{x,\ y\}} = c'\ |_{\{x,\ y\}}$. By lemma 1, $x \succ_{F(c)} y$. By FS cancellation, $x \sim_{F(c')} y$. Hence, F is not IIA.

□

Definition 15 (Weak Pareto optimality) *A social preference rule F is weakly Pareto optimal if, for all parties $x,\ y \in A$ with $x \neq y$, for all $I \subseteq V$, and for all preference profiles $c \in L(A)^I$, if $x \succ_{c_i} y$ for all $i \in I$, then $x \succeq_{F(c)} y$.*

Proposition 4 *If F is faithful, consistent, and has the FS cancellation property, then F is weakly Pareto optimal.*

Proof Let F be faithful, consistent, and have the FS cancellation property. Let $x \succ_{c_i} y$ for all $i \in I$. We should prove that either $x \succ_{F(c)} y$, or $x \sim_{F(c)} y$.

There are two cases: (i) $t(c_i) \neq x$ for all $i \in I$, (ii) $t(c_i) = x$ for some $i \in I$. Case (i): By FS cancellation, $x \sim_{F(c)} y$. Case (ii): Either for all $i \in I$, $t(c_i) = x$; or for some $i \in I$, $t(c_i) \neq x$. If for all $i \in I$, $t(c_i) = x$, then by Lemma 1, $t(F(c)) = x$. Hence, in particular, $x \succ_{F(c)} y$. Suppose, on the other hand, for some $i \in I$, $t(c_i) \neq x$. Then, there are $I', I'' \subseteq I$ such that $I = I' \cup I''$, $I' \cap I'' = \emptyset$ and there are $c' \in L(A)^{I'}$ and $c'' \in L(A)^{I''}$ such that $c = c' + c''$, $t(c'_i) = x$ for all $i \in I'$ and $\pi_{c''}(x) = \pi_{c''}(y)$. By Lemma 1, $t(F(c')) = x$ and, in particular, $x \succ_{F(c')} y$. By FS cancellation, $x \sim_{F(c'')} y$. By consistency, $x \succ_{F(c)} y$. □

Definition 16 (Weak monotonicity) *A social preference rule F is weakly monotonic if the following holds. Let $c, c' \in L(A)^I$ and $x \in A$ be such that for all $i \in I$*

(i) for all $x^* \neq x$ and $y^* \neq x$, $x^* \succ_{c'_i} y^*$ if and only if $x^* \succ_{c_i} y^*$,
(ii) for all y^*, $x \succ_{c_i} y^*$ implies $x \succ_{c'_i} y^*$.

Then, if $x \succ_{F(c)} y$, then $x \succ_{F(c')} y$.

Proposition 5 *If F is faithful, consistent, and has the FS cancellation property, then F is weakly monotonic.*

Proof Suppose (i) and (ii) in the definition of weak monotonicity hold, F is faithful, consistent and satisfies the FS cancellation property, and $x \succ_{F(c)} y$. Then by case 1 in the proof of Theorem 1, $\pi_c(x) \neq \pi_c(y)$, and by case 2 in the proof of Theorem 1, not $\pi_c(y) > \pi_c(x)$. Hence, $\pi_c(x) > \pi_c(y)$. According to condition (ii), all voters with $t(c_i) = x$ have $t(c'_i) = x$. According to conditions (i) and (ii), all voters with $t(c_i) = y$ have $t(c'_i) = y$ or $t(c'_i) = x$. Thus, $\pi_{c'}(x) \geq \pi_c(x)$ and $\pi_{c'}(y) \leq \pi_c(y)$. Using $\pi_c(x) > \pi_c(y)$, it follows that $\pi_{c'}(x) > \pi_{c'}(y)$. Again by case 2 in the proof of Theorem 1 it follows that $x \succ_{F(c')} y$. □

9 Discussion and comparison to other scoring rules

Since our characterizations contain the axiom of topsonlyness, the characteristic properties we found can easily be compared to the characteristic properties of other topsonly social preference rules. On the other hand, since the rule we characterize uses as input the complete linear preference orderings of the individuals, its characteristic properties can also easily be compared to those of social preference rules that do use the other preferences of the individuals.

Since the plurality ranking rule is a kind of scoring rule, our result is related to the works of Smith (1973), and Young (1974; 1975), who characterized the class of scoring rules. Scoring rules are rules that assign for each individual a certain number of points to each of the various alternatives. Choice correspondences then choose the alternative(s) that received most points, while social preference rules give a rank ordering of the alternatives, in which an alternative that received more points is ranked higher.

Scoring rules differ as regards the scores they assign. Under the plurality (ranking) rule, for each individual, a score of 1 is assigned to the voter' most preferred alternative, and a score of 0 to all the other alternatives. For each voter, the Borda rule assigns to each alternative a score that is equal to the number of alternatives to which it is weakly preferred. Under approval voting, a score of 1 is assigned to all the alternatives the voter approves of and a score of 0 to all the alternatives he or she does not approve of.

Young (1974b) defined scoring functions as follows (p. 1129): "...there is a finite sequence $s^1, s^2, \ldots, s^k$ from vectors from R^m [m is the number of alternatives] (scoring vectors) such that every voter gives score s_i to his ith most preferred alternative, and if alternative a gets a higher total score than b, then a is socially preferred to b. Ties relative to s^1 are solved using s^2, and so forth." Smith (1973) was the first to give a characterization of these scoring functions. He showed that the only social preference rules that are anonymous, neutral and consistent are scoring functions. In Young (1974), it is shown that the only subpreference functions that are anonymous, neutral and consistent are scoring functions. A subpreference function is a function that assigns to each profile a partial, asymmetric transitive relation.

In Young (1975), Young characterizes scoring functions that are choice correspondences. He shows that (simple or composite) scoring functions are characterized by anonymity (A), neutrality (N), and consistency (CS). Young also shows that simple scoring functions are characterized by anonymity, neutrality, consistency and continuity (CT). A simple scoring function is defined as a function f that, given m alternatives and a profile, assigns a score of s_i (s_i a real number) to each voter' ith most preferred alternative and then chooses the alternative(s) with the highest total score. Examples of simple scoring functions are the plurality choice correspondence and the Borda choice correspondence. A composite scoring function is a composition of scoring functions such that ties are resolved. Consistency is defined as in Young (1974a). Continuity is defined as follows: Given some profile c, whenever $f(c) = \{x\}$, then, for any profile c', there is a sufficiently large integer n such that $f(c' + n'c) = \{x\}$ for all $n' \geq n$. Young calls this a kind of domination of large number principle. It means that if a certain committee chooses a certain alternative x and if we replicate this committee a sufficient number of times, then, given any second committee disjoint from the first, it will overwhelm this second committee in a combined vote and the same alternative x will be chosen.

Saari (1994) uses his geometric tools to prove an extended form of Young's (1975) theorem. He proves that a non-constant procedure that satisfies anonymity, neutrality, and consistency is equivalent to scoring rules, where ties can be broken using a second or third scoring rule in runoff elections. Saari also gives a characterization of positional voting methods (PVM). Positional voting methods are scoring methods with $s^1 \geq s^2 \geq s^3$. They are characterized by anonymity, neutrality, consistency, and two new properties called 'eventual responsiveness' (E) and 'balancing' (BA). Eventual responsiveness requires that, if a sufficiently large subset of a given set of voters ranks alternative x first, then this alternative x will be chosen. A choice procedure is balancing if, whenever society consists of two voters who each have the same two alternatives top-ranked, then at least one of these alternatives is chosen. Saari also finds that a number of conditions, relating the outcome of pairwise majority voting to the alternative that is actually chosen, separate the Borda choice correspondence from other positional methods. The Borda choice, or the Borda choice with a runoff, is, for example, the only positional method that never selects the Condorcet loser. The characterizations of the Borda choice that Saari finds in this way do not even need the axioms of balancing and eventual responsiveness. The only axioms he needs are anonymity, neutrality, consistency and some pairwise ranking property (P).

Myerson (1995) gives a characterization of scoring rules defined in a more general way. He remarks that the assumption in Smith (1973) and Young (1974b; 1975) that the expressed preferences of the individual voters are rankorders is very restrictive, since it would exclude approval voting. In Myerson's characterization, he imposes no assumption on the structure of the set of permissible votes, except that it is a nonempty finite set. Scoring rules are characterized by reinforcement (= Young's consistency), overwhelming majorities (= Young's continuity), and neutrality. Anonymity is implied by the way profiles are defined.

Besides characterizations of the general class of scoring rules, various characterizations of the plurality choice correspondence exist. Richelson (1987) gives a characterization of the choice correspondence that relies on Young's (1975) characterization of the class of scoring rules. He shows that the plurality rule is the only rule that satisfies anonymity, neutrality, continuity, consistency and reduction. Richelson thus obtains a characterization of the plurality rule by adding a condition named reduction to the conditions that were used by Young. Reduction (R) states that, if there exists a candidate y such that all individuals prefer y over x, then the outcome when x is included in the set of candidates is the same as when x is excluded from the set of candidates. In other words, it says that removing Pareto-dominated alternatives does not alter the social choice.

Ching (1996) gives a characterization of the choice correspondence that strengthens the result of Richelson (1987). He shows that the plurality rule is the only choice correspondence satisfying neutrality, anonymity, consistency and reduction. Hence, he does not need Young's (1975) and Richelson's (1987) continuity.

An unpublished paper of Merlin and Naeve (1999) contains another characterization of the plurality rule. They proved that a social choice function is self implementable in demanding equilibrium (s.i.d.e.) if, and only if, it satisfies bottom invariance (B), (weak) monotonicity (M), upper conditional independence, and top equivalence. They also proved that the only non-constant scoring rule which is s.i.d.e. for any population size is the plurality rule. Merlin and Naeve proved this second theorem by showing that the only non-constant scoring function that is bottom invariant and (weakly) monotonic is the plurality rule. From Young (1975), we know that anonymity, neutrality, consistency and continuity characterize simple scoring functions. Hence, Merlin and Naeve, in fact, proved that a rule is the plurality rule if, and only if, it is non-constant and satisfies anonymity, neutrality, consistency, continuity, bottom invariance and (weak) monotonicity. Bottom invariance requires of a choice correspondence f that, for every profile c and for every $k \in \mathbb{N}$, if $f(c)$ is the kth choice of individual i with preference ordering c_i, and c_i' equals c_i on the first k alternatives, then $f(c/c_i') = f(c)$.

Roberts (1991) defines a consensus function as a function $f : \bigcup_{n=1}^{\infty} A^n \to 2^A$, where A is the set of alternatives. Such a function assigns to the (first) choices of the voters a subset of A. Assuming that $f(\underline{x}) \neq \emptyset$ for any $\underline{x}$ in $\bigcup_{n=1}^{\infty} A^n$ and $f(x) \neq A$ for any $x \in A$, he shows that f is the plurality function (giving the set of alternatives that receive the largest number of first choice votes) if, and only if, f is anonymous, neutral, consistent and faithful. This characterization resembles our Theorem 2. However, we work with social preference rules assigning to each n-tuple of linear orderings over A a weak ordering of A. This means that the characteristic properties we found can easily be compared to the characteristic properties of other social preference rules. Another advantage is that our proof is much less complex.

Saari (2002) gives a decomposition of profiles specific to the plurality vote.

Besides the plurality rule, other examples of scoring rules are the Borda choice correspondence and approval voting. An axiomatization of the Borda choice correspondence was given in Young (1974a). Young proved that Borda's rule is the only choice correspondence that is neutral, consistent, faithful (F), and has the cancellation property (CN). The definitions of these properties were given in Sect. 3.

An axiomatization of approval voting that is closely related to the work of Smith (1973) and Young (1974a,b, 1975) was given by Fishburn (1978). Fishburn proved that a social choice function is the approval voting function if and only if it satisfies neutrality, consistency and disjoint equality (D). A social choice function is defined here as a function f that adds to each ballot response profile a nonempty subset of the set of candidates. A ballot response profile is defined as a function π from the set of all subsets of the set of candidates into the nonnegative integers. Disjoint equality requires that, whenever there are only two ballots A and B that are disjoint, then the outcome should be the union of A and B. It is this axiom of disjoint equality that distinguishes approval voting from all other neutral and consistent choice functions as they are defined here.

A quite different characterization of approval voting was given by Sertel (1988). The procedure that was characterized by Sertel differs from the approval voting for which Fishburn (1978) offered a characterization, in the case in which every voter rejects every available alternative. In this case, none of the alternatives is selected, whereas, in Fishburn's procedure, all alternatives are selected in these circumstances. This is why Sertel says his procedure always respects unanimity. Sertel proved that approval voting in this definition is characterized by weak unanimity, weak consistency, and strong equality.

For a survey of more than 40 characterizations of scoring methods for preference aggregation see Chebotarev and Shamis (1998).

When we compare our result to Smith's (1973) characterization of the class of scoring functions, we may conclude that faithfulness (F), and topsonlyness (T) distinguish the plurality ranking rule from other scoring functions (see Table 1). This means that the defensibility of the plurality ranking rule against other scoring social preference rules depends on the defensibility of these two axioms. How defensible they should be to survive the comparison depends also on the additional axioms that are needed in the characterizations of other scoring social preference rules. Unfortunately, no characterizations of other scoring social preference rules are known.

The various characterizations that were found for the plurality choice correspondence and other scoring correspondences are compared in Table 2. According to the characterizations of Richelson (1987) and Ching (1996), it is reduction (R) that distinguishes the plurality choice correspondence from simple scoring rules as they were characterized by Young (1975). We remark that reduction implies faith-

Table 1 Social preference rules and their properties

	A	N	CS	F	T	TM
Smith: Scoring rules	1	1	1			
Plurality ranking rule	1	1	1	1	1	
Plurality ranking rule	1	1			1	1

A anonymity, *N* neutrality, *CS* consistency, *F* faithfulness, *T* topsonlyness, *TM* top monotonicity

Table 2 Social choice correspondences and their properties

	A	N	CS	CT	R	D	B	M	F	CN	E	BA	P
Young: SR	1	1	1										
Young: SSR	1	1	1	1									
Saari: PVM	1	1	1								1	1	
Richelson: PR	1	1	1	1	1								
Ching: PR	1	1	1		1								
Merlin: PR	1	1	1	1			1	1					
Fishburn: AV		1	1			1							
Young: BR		1	1						1	1			
Saari: BR	1	1	1										1

SR scoring rules, *SSR* simple scoring rules, *PVM* positional voting methods, *PR* plurality rule, *AV* approval voting, *BR* Borda's rule. *A* anonymity, *N* neutrality, *CS* consistency, *CT* continuity, *R* reduction, *D* disjoint equality, *B* bottom invariance, *M* monotonicity, *F* faithfulness, *CN* cancellation property, *E* eventual responsiveness, *BA* balancing, *P* a pairwise ranking property

fulness as it was defined by Young for choice correspondences in Young (1974a). The distinguishing axioms in the characterization of Merlin and Naeve (1999) are (weak) monotonicity (M) and bottom invariance (B). Our topsonlyness corresponds to a special case of bottom invariance.

10 Conclusion

We have shown that the plurality ranking rule is characterized by consistency, faithfulness, and FS cancellation. A second characterization was found by showing that FS cancellation is implied by anonymity, neutrality, and topsonlyness. We also showed that the plurality ranking rule is characterized by top monotonicity, anonymity, neutrality and topsonlyness. The properties mentioned were found to be independent of each other.

Topsonlyness turns out to be an important requirement for a seat allocation rule. We showed that a scoring seat allocation rule is party fragmentation proof if and only if it is topsonly. This means that topsonlyness is also an important property for social preference rules, since social preference rules model electoral systems and seat allocation rules provide for an even better model of electoral systems.

We also showed in which ways various other properties of the rule are related to the characteristic properties.

The plurality ranking rule, as we characterized it, is distinguished from the class of scoring functions in the axioms of topsonlyness and faithfulness. The axioms we found differ from the axioms that were found by Richelson (1987), Ching (1996), and Merlin and Naeve (1999) in their characterizations of the plurality choice correspondence. We note that faithfulness is implied by the reduction axiom that was used by both Richelson (1987) and Ching (1996). Topsonlyness corresponds to bottom invariance, which was used by Merlin and Naeve (1999).

Since the rule we characterized models list systems of proportional representation, our characterization provides a basis for evaluating these systems and for comparing them with other electoral systems, like FPTP (First Past The Post) systems.

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