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İpek Özkal-Sanver · M. Remzi Sanver

Ensuring pareto optimality by referendum voting

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Abstract We consider a society confronting the decision of accepting or rejecting a list of (at least two) proposals. Assuming separability of preferences, we show the impossibility of guaranteeing Pareto optimal outcomes through anonymous referendum voting, except in the case of an odd number of voters confronting precisely two proposals. In this special case, majority voting is the only anonymous social choice rule which guarantees Pareto optimal referendum outcomes.

1 Introduction

Given a society confronting the decision of accepting or rejecting a list of proposals, a (social choice) outcome indicates whether each proposal is accepted or rejected. To be more concrete, given m proposals, assuming that each proposal will either be accepted or rejected, there are 2^m possible outcomes. It is, of course, possible to model the problem in a standard social choice framework where the basic information is voters' preferences on outcomes. On the other hand, this may cause practical problems as the number of outcomes can explode (e.g., there would

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İ. Özkal-Sanver (✉) · M. R. Sanver
Department of Economics, İstanbul Bilgi University, 80310 İstanbul, Turkey
E-mail: isanver@bilgi.edu.tr

be 131,072 possible outcomes when there are only 17 proposals) and voters would confront difficulties in ranking such a high number of outcomes.

A typical solution to this problem is to decide over proposals separately, an approach called as referendum voting. Under referendum voting, the preference of voters about the acceptance/rejection of every proposal is aggregated into a social decision regarding that particular proposal, usually but not necessarily, through majority voting.

Referendum voting has reduced information requirements. Each voter indicates "Yes" or "No" on each proposal, which is equivalent to specifying a single outcome that can be interpreted as the voter's most preferred outcome. In general, further information from each voter would be needed to describe the social choice problem in a much better manner. However, this additional information can be seen as redundant by assuming separable preferences, which is standard in this context. A voter is said to have separable preferences over outcomes if for every proposal the voter either always prefers that proposal to be accepted or always prefers it to be rejected, independent of what happens to the remaining proposals.[1]

Although referendum voting is not the only way of handling the complications due to the size of the social choice problem,[2] it is a very popular one. That is why we concentrate on referendum voting and ask whether it can ensure Pareto optimality. We consider a social choice problem with at least two proposals and two voters. A voter votes for each proposal to be accepted or rejected. A fixed social choice rule is applied to each proposal separately to decide whether it should be accepted or rejected, hence leading to the social outcome. While the Pareto optimality of this outcome depends on how voters order outcomes, the only information we have on hand is their first best. To be able to analyze the efficiency of the social choice, we assume that voters have separable preferences over outcomes. In other words, once we know the first best outcome of a voter, we allow her to have any separable ordering over outcomes. As a result, any list of individual opinions of voters regarding the acceptance/rejection of proposals leads to a set of admissible preference profiles over outcomes obtained through the separability axiom. We say that a referendum voting rule is Pareto ensuring (for separable preferences) if and only if given any list of individual opinions regarding the proposals, it picks an outcome which is Pareto optimal according to every admissible preference profile over outcomes.

We illustrate these concepts through an example. Consider three voters who have to decide over three proposals. Let (Y, Y, N) be the most preferred outcome of the first voter who wishes the first two proposals to be accepted and the third one to be rejected. Similarly, let the second and third voters vote as (Y, N, Y) and (N, Y, Y), respectively. Assume we use the majority rule on each issue. The first proposal receives two approvals both by the first and the second voters; hence, it is accepted. In the same manner, the second and third proposals also receive two

[1] Kilgour (1997) and Bradley et al. (2006) give a formal treatment of this property. Brams et al. (1997) analyze referendum voting under nonseparable preferences.

[2] For example one can use "Yes-No voting" proposed by Brams and Fishburn (1993) whereby a voter can indicate multiple packages of proposals she supports. Another example is the "minimax procedure" proposed by Brams et al. (2004) . This procedure finds a compromise on multilateral treaties which minimizes the maximum distance between the agreement and the top preferences of all players. The idea is based on "Fallback Bargaining", introduced by Brams and Kilgour (2001).

approvals each. Thus, the referendum outcome via the majority rule is (Y, Y, Y), i.e., every proposal is accepted. However, every voter may prefer the rejection of all proposals, i.e., the outcome (N, N, N) to (Y, Y, Y). This is perfectly compatible with separability and when this is the case, the referendum voting outcome is not Pareto optimal.[3]

This small example shows that referendum voting via the majority rule is not Pareto ensuring with three voters and three proposals. We question the generality of this result and ask whether it depends on the size of the social choice problem or the social choice rule via which the referendum is made. Interestingly, the class of Pareto ensuring referendum voting rules can be characterized in terms of an escape from the paradox for multiple elections introduced by Brams, Kilgour and Zwicker (1998). This paradox, which we refer to as the BKZ-paradox, is about referendum voting rules that pick an outcome which is voted by no voter. We show that a referendum voting rule is Pareto ensuring if and only if it escapes the BKZ-paradox. Moreover, except for a particular case of two proposals and an odd number of voters, no anonymous voting rule can escape the BKZ-paradox, hence be Pareto ensuring. So, our main result is the impossibility of guaranteeing Pareto optimality by referendum voting.[4]

Our paper proceeds as follows: In Sect. 2 we give the basic notions. Section 3 contains the main results and Sect. 4 makes some concluding remarks.

2 Basic notions

We consider a society N with $\#N = n \geq 2$ confronting a set of proposals M with $\#M = m \geq 2$. Let $A = \{-1, 1\}^M$ stand for the set of all possible outcomes. So an outcome $x \in A$ is an m-tuple where for each $j \in M$, the entry $x^j \in \{-1, 1\}$ reflects the social decision about proposal j in the following manner: If $x^j = 1$ (resp., $x^j = -1$), then proposal j is accepted (resp., rejected). The *vote* of a voter $i \in N$ is an m-tuple $v_i \in A$ where for each $j \in M$, the entry $v_i^j \in \{-1, 1\}$ reflects her opinion over the proposal j with the usual interpretation that $v_i^j = 1$ (resp., $v_i^j = -1$) means voter i votes for proposal j to be accepted (resp., rejected).[5] Ignoring strategic considerations, the vote of a voter is an outcome which we interpret as her most preferred one. We write $v = \{v_i\}_{i \in N} \in V$ for a *vote profile* of the society where $V = A^N$ is the set of vote profiles.

Let $\mathcal{R}$ be the set of all complete and transitive binary relations over A. Every voter $i \in N$ has a preference $R_i \in \mathcal{R}$ on A. For all $x, y \in A$, $x R_i y$ means that

[3] For instance, let the (Y, Y, N) voter order the outcomes as (Y, Y, N), (N, Y, N), (Y, N, N), (N, N, N), (Y, Y, Y), (N, Y, Y), (Y, N, Y), (N, N, Y). Similarly, let (Y, N, Y), (Y, N, N), (N, N, Y), (N, N, N), (Y, Y, Y), (Y, Y, N), (N, Y, Y), (N, Y, N) be the ordering of the (Y, N, Y) voter. Finally, the (N, Y, Y) voter orders the outcomes as (N, Y, Y), (N, Y, N), (N, N, Y), (N, N, N), (Y, Y, Y), (Y, Y, N), (Y, N, Y), (Y, N, N). All three orderings are separable while all voters prefer (N, N, N) to (Y, Y, Y).

[4] A related result is due to Lacy and Niou (2000) who show that nonseparable preferences can lead to the social choice of an outcome which is a Condorcet loser or even Pareto dominated. A similar analysis is made by Benoit and Kornhauser (1994) who examine the efficiency properties of voting systems electing assemblies as a function of the preferences of voters over individual candidates.

[5] Note that we do not allow for indifferences in neither individual nor social preference. This is a matter which we analyze at the end of Sect. 3.

voter i finds outcome x at least as good as outcome y. We write $x P_i y$ whenever the preference relation is strict, i.e., $x R_i y$ but not $y R_i x$. Similarly, $x I_i y$ stands for the indifference counterpart of R_i, i.e., we have $x I_i y$ whenever $x R_i y$ and $y R_i x$ both hold. An n-tuple $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$ of these binary relations reflects a preference profile of the society over the possible outcomes.

We assume that the vote v_i and the preference R_i of a voter $i \in N$ are related. This relationship is established through a binary relation $B(v_i)$ over A which is defined for any $x, y \in A$ as follows: $x B(v_i) y$ if and only if for every $j \in M$ we have

$$x^j \geq y^j \quad \text{whenever } v_i^j = 1 \quad \text{and} \quad x^j \leq y^j \text{ whenever } v_i^j = -1.$$

So given any voter $i \in N$ with a vote $v_i \in A$, the outcome x beats the outcome y through $B(v_i)$ if and only if for every separate proposal, voter $i \in N$ finds the decision according to x at least as good as the decision according to y. Note that $B(v_i)$ is transitive but not complete.

Given any $i \in N$, a preference $R_i \in \mathcal{R}$ is said to be *separable* with respect to $v_i \in A$ if and only if for all $x \in A$ and $y \in A \backslash \{x\}$ with $x B(v_i) y$ we have $x P_i y$.

Remark 2.1 *For any $i \in N$ with a vote $v_i \in A$, any $x \in A$ and any $y \in A \backslash \{x\}$, the following are true*:

(i) *If x $B(v_i)$ y holds, then there exists no preference $R_i \in R$ with y R_i x while R_i is separable with respect to v_i.*
(ii) *If x $B(v_i)$ y fails to hold, then there exists a preference $R_i \in R$ with y P_i x while R_i is separable with respect to v_i.*

We say that a preference profile $R \in \mathcal{R}^N$ is separable with respect to a vote profile $v \in V$ whenever R_i is separable with respect to v_i for all $i \in N$. Given any $v \in V$, we write $\Sigma(v) \subset \mathcal{R}^N$ for the set of preference profiles over A which are separable with respect to v.

A *voting rule* is a function $F : V \to A$ which assigns an outcome $F(v) \in A$ to every vote profile $v \in V$.

An outcome $x \in A$ is said to be *Pareto optimal* at $R \in \mathcal{R}^N$ whenever there exists no $y \in A$ such that y R_i x for all $i \in N$ and y P_j x for some $j \in N$. A voting rule is *Pareto ensuring (for separable preferences)* if and only if given any vote profile $v \in V$, the outcome $F(v)$ is Pareto optimal at every $R \in \Sigma(v)$.

3 Results

We quote a version of a paradox introduced by Brams et al. (1998). A voting rule $F : V \to A$ is said to escape the *BKZ-paradox* if and only if for all $v \in V$, there exists $i \in N$ such that $F(v) = v_i$. So escaping the BKZ-paradox means that, at every vote profile, the voting rule picks an outcome that matches exactly the vote of at least one voter. In other words, a voting rule exhibits the BKZ-paradox if and only if at some vote profile it picks an outcome that is not fully supported by any voter.

Interestingly, the class of Pareto ensuring voting rules can be characterized through the escape from the BKZ-paradox, as the following theorem states:

Theorem 3.1 *A voting rule $F : V \to A$ is Pareto ensuring for separable preferences if and only if F escapes the BKZ-paradox.*

Proof To show the "if" part, consider a voting rule $F : V \to A$ which escapes the BKZ-paradox. Take any vote profile $v \in V$. Let $x = F(v)$. As F escapes the BKZ-paradox, there exists some $i \in N$ such that $x = v_i$. Now take any $y \in A \setminus \{x\}$. Clearly $xB(v_i)y$. So, by part (i) of Remark 2.1, at every $R \in \Sigma(v)$, we have $xP_i y$. Thus $x = F(v)$ is Pareto optimal at every $R \in \Sigma(v)$, showing that F is Pareto ensuring.

To show the "only if" part, suppose F exhibits the BKZ-paradox. So, there exists some $v \in V$ such that $F(v) \neq v_i$ for all $i \in N$. Write $F(v) = x$ and $-F(v) = y$. Take any $i \in N$. There exists some $j \in M$ such that $x^j \neq v_i^j$. Note that $v_i^j = y^j$. Hence $xB(v_i)y$ does not hold. In other words, $F(v)B(v_i) - F(v)$ fails to hold for every $i \in N$. So by part (ii) of Remark 2.1, there exists $R \in \Sigma(v)$ such that $-F(v)P_i F(v)$ for all $i \in N$, showing that F is not Pareto ensuring, thus completing the proof. □

Remark 3.1 *By definition of Pareto ensurance, the equivalence established by Theorem 3.1 can be stated for any set of admissible preferences over outcomes containing the set $\Sigma(v)$ of separable preferences. In particular, Theorem 3.1 would hold without assuming separability and allowing voters to have any ordering over outcomes, independent of the vote they cast.*

Remark 3.2 *The assertions of Remark 2.1 hold for additively separable preferences as well.*[6] *Hence, it is possible to strengthen the "only if" part of Theorem 3.1 by replacing separability with additive separability.*

We now show a basic impossibility of escaping the BKZ-paradox, hence ensuring Pareto optimal outcomes, through anonymous referendum voting when there are an even number of voters or at least three proposals. First, we give the necessary definitions. A voting rule $F : V \to A$ is said to be *simple* whenever the number of proposals $m = 1$. Now for every $j \in M$, let $v^j = (v_1^j, \ldots, v_n^j)$ the list of opinions of the voters for proposal j. We call a voting rule F *referendum voting* if and only if there exists a simple voting rule f such that $F(v) = (f(v^1), \ldots, f(v^m))$. We refer to f as the corresponding simple rule of the referendum voting rule F. So a referendum voting rule is one where a given simple voting rule is applied to all proposals separately. A voting rule $F : V \to A$ is said to be *anonymous* if and only if given any vote profile $v = (v_1, \ldots, v_n)$ and any permutation $\tau : N \longleftrightarrow N$ of voters, we have $F(v_1, \ldots, v_n) = F(v_{\tau(1)}, \ldots, v_{\tau(n)})$. Note that anonymity of a referendum voting rule F implies the anonymity of its corresponding simple voting rule f.

Theorem 3.2 *Let $m \geq 3$ or n be even. There exists no anonymous referendum voting rule $F : V \to A$ which escapes the BKZ-paradox.*

[6] We say that R_i is *additively separable* with respect to v_i if and only if there exists an m-tuple of (strictly) positive real numbers $u = (u^1, \ldots, u^m)$ such that given any $x, y \in A$ we have $xR_i y$ if and only if $\sum_{k \in M} x^k v^k u^k \geq \sum_{k \in M} y^k v^k u^k$. Additive separability is stronger than separability. Bradley et al. (2006) clarify the distinction between additive and separable preferences in the context of referendum voting. We thank Christopher Chambers and two anonymous referees who pointed to an error in an earlier version of the paper.

Proof Let $F : V \to A$ be any anonymous referendum voting rule. So given any vote profile $v \in V$, writing $v^j = (v_1^j, \ldots, v_n^j)$ for the list of opinions of the voters for proposal $j \in M$, we have $F(v) = (f(v^1), \ldots, f(v^m))$ for some simple voting rule f.

We define n^* as the least integer not less than $n/2$. First, let n be even and consider the following vote profile $v \in V$ where for every $j \in M \backslash \{m\}$ we have $v_i^j = 1$ for all $i \in \{1, \ldots, n^*\}$ and $v_i^j = -1$ for all $i \in \{n^*+1, \ldots, n\}$. On the other hand, $v_i^m = -1$ for all $i \in \{1, \ldots, n^*\}$ and $v_i^m = 1$ for all $i \in \{n^* + 1, \ldots, n\}$. Note that by the anonymity of f, we have $f(v^i) = f(v^j)$ for all $i, j \in M$. Hence $F(v) \in \{(-1, -1, \ldots, -1), (1, 1, \ldots, 1)\}$. However, there exists no $i \in N$ for whom $v_i \in \{(-1, -1, \ldots, -1), (1, 1, \ldots, 1)\}$. Thus, F exhibits the BKZ-paradox.

Now let $m \geq 3$ and n be odd. Consider the following vote profile $v \in V$ where $v_i^1 = 1$ for all $i \in \{1, \ldots, n^*\}$ and $v_i^1 = -1$ for all $i \in \{n^* + 1, \ldots, n\}$. On the other hand $v_i^2 = -1$ for all $i \in \{1, \ldots, n^* - 1\}$ and $v_i^2 = 1$ for all $i \in \{n^*, \ldots, n\}$. Finally, let $v_i^3 = 1$ for all $i \in \{1, \ldots, n^* - 1, n^* + 1\}$ and $v_i^3 = -1$ for all $i \in \{n^*, n^* + 2, \ldots, n\}$. In case $m > 3$, let $v^j = v^1$ for all $j \in \{4, \ldots, m\}$. Note that by the anonymity of f, we have $f(v^i) = f(v^j)$ for all $i, j \in M$. Hence $F(v) \in \{(-1, -1, \ldots, -1), (1, 1, \ldots, 1)\}$. However, there exists no $i \in N$ for whom $v_i \in \{(-1, -1, \ldots, -1), (1, 1, \ldots, 1)\}$. Thus, F exhibits the BKZ-paradox, completing the proof. □

Theorems 3.1 and 3.2 lead to the following theorem as a corollary:

Theorem 3.3 *Let $m \geq 3$ or n be even. There exists no anonymous referendum voting rule $F : V \to A$ which is Pareto ensuring for separable preferences.*

Our impossibility results do not cover the case where there are two proposals and an odd number of voters. In fact, for this particular case, we do have a unique Pareto ensuring and anonymous referendum voting rule which uses the well-known majority rule as its corresponding simple voting rule. To be sure, a simple voting rule $f : V \to A$ is the majority rule if and only if $f(v) = \text{sgn}(\sum_{i \in N} v_i)$.[7]

Theorem 3.4 *Let $m = 2$ and n be odd. An anonymous referendum voting rule is Pareto ensuring for separable preferences if and only if its corresponding simple voting rule is the majority rule.*

Proof Let $m = 2$ and n be odd. To show the "if" part, consider the referendum voting rule $F : V \to A$ with its corresponding simple voting rule f being the majority rule. For every vote profile $v \in V$, write $F(v) = (f(v^1), f(v^2))$. By definition of f, we have $\#\{i \in N : v_i^1 = f(v^1)\} > n/2$ and $\#\{i \in N : v_i^2 = f(v^2)\} > n/2$. By the Pigeonhole Principle, there must exist at least one agent $i \in N$ with $v_i = \left(v_i^1, v_i^2\right) = \left(f(v^1),\ f(v^2)\right) = F(v)$. Thus, F escapes the BKZ-paradox. Hence, by Theorem 3.1, F is Pareto ensuring.

To show the "only if" part, consider any anonymous referendum voting rule $F : V \to A$ having a corresponding simple voting rule f which is not the majority rule. Again, for every vote profile $v \in V$, write $F(v) = \left(f(v^1), f(v^2)\right)$. As

[7] Given any real number r, sgn(r) respectively equals 1, 0 and -1 when r >0, r= 0, r<0. Note that when n is odd, $\sum_{i \in N} v_i \neq 0$

f is not the majority rule, there exists some $v \in V$ and $j \in \{1, 2\}$ such that $\#\{i \in N : v_i^j = f(v^j)\} = r < n - r$. Let $j = 1$ without loss of generality. Now define a vote profile $u \in V$ as follows: $u_i^1 = f(v^1)$ for all $i \in \{1, \ldots, r\}$, $u_i^1 = -f(v^1)$ for all $i \in \{r + 1, \ldots, n\}$, $u_i^2 = -f(v^1)$ for all $i \in \{1, \ldots, n - r\}$ and $u_i^2 = f(v^1)$ for all $i \in \{n - r + 1, \ldots, n\}$. By anonymity of f, we have $f\left(u^1\right) = f\left(u^2\right) = f(v^1)$, i.e., $F(u) = \left(f(v^1),\ f(v^1)\right)$. However, there exists no $i \in N$ for whom $u_i = \left(f(v^1),\ f(v^1)\right)$, showing that F exhibits the BKZ-paradox, which, again by Theorem 3.1, proves that F is not Pareto ensuring. □

Note that the majority rule is the unique simple voting rule which can induce a Pareto ensuring referendum voting rule at least for some social choice problems with two proposals and an odd number of voters. We state this in the following corollary as a new characterization of the majority rule:

Corollary 3.1 *A simple voting rule is the majority rule if and only if it is anonymous and it can induce a Pareto ensuring referendum voting rule at some size of the social choice problem.*

In spite of the restricted positive result announced by Theorem 3.4, what we establish is a basic impossibility in ensuring Pareto optimal outcomes through referendum voting. We now ask whether a possibility result could be obtained for a larger class of voting rules which allow for social indifference in outcomes. In this more general world, an outcome is an m-tuple $x \in \{-1, 0, 1\}^m$ where $x^j = 0$ means social indifference for proposal $j \in M$. We write $\overline{A} = \{-1, 0, 1\}^m$ for the set of all possible outcomes. So, a voting rule is a mapping $F : V \to \overline{A}$. A voting rule $F : V \to \overline{A}$ is said to be *decisive* if and only if given any $v \in V$, writing $x = F(v)$, we have $x_j = 0$ for no $j \in M$. So, our results upto now are for decisive voting rules. On the other hand, it is not possible to escape the established impossibilities by allowing social indifference in outcomes, as decisiveness of a referendum voting rule is a necessary condition for its being Pareto ensuring. We state this in the following theorem:

Theorem 3.5 *A referendum voting rule $F : V \to \overline{A}$ is Pareto ensuring for separable preferences only if F is decisive.*

Proof Take any anonymous voting rule $F : V \to \overline{A}$ which is not decisive. So there exists some $v \in V$ where, writing $x = F(v)$, we have $x^k = 0$ for some $k \in M$. Fix that particular $k \in M$ as well as the list v^k of individual opinions over k. Let f be the corresponding simple voting rule of F. Note that $f(v^k) = 0$.

First, consider the case where $v_i^k = v_j^k$ for all $i, j \in N$. Let $y \in \overline{A}$ be defined as $y^r = x^r$ for all $r \in M \backslash \{k\}$ and $y^k = v_i^k$ for some $i \in N$. Clearly $yB(v_i)x$ for all $i \in N$. So given any $R \in \Sigma(v)$, we have yP_ix for all $i \in N$. Thus, F is not Pareto ensuring.

Next, consider the case where there exist $i, j \in N$ such that $v_i^k = 1$ and $v_j^k = -1$. Take a vote profile $u \in V$ where $u^r = v^k$ for all $r \in M$. So we have $F(u) = [f(u^1), \ldots, f(u^m)] = (0, \ldots, 0)$. Let $y \in \overline{A}$ be defined as $y^1 = 1$, $y^2 = -1$ and $y^r = 0$ for all $r \in M \backslash \{1, 2\}$. It is straightforward to check that $F(u)B(u_i)y$ holds for no $i \in N$. Hence, there exists some $R \in \Sigma(u)$ such that $yP_iF(u)$ for all $i \in N$, i.e., there exists some $R \in \Sigma(u)$ according to which $F(u)$ is not Pareto optimal, showing that F is not Pareto ensuring, thus completing the proof. □

4 Final remarks

We state an impossibility result about the existence of anonymous Pareto ensuring referendum voting rules except for a particular case of two proposals and an odd number of voters. This result is established through the equivalence of Pareto ensurance and the escape from the paradox of multiple elections introduced by Brams et al. (1998). In the particular case of two proposals with an odd number of voters, Pareto optimality of the resulting referendum outcome can be guaranteed if and only if we use majority voting to decide on separate proposals. This restricted but positive result can be interpreted as another characterization of the majority rule in the context of referendum voting.[8]

We mainly consider a world where indifferences are ruled out both in individual preferences and in the final outcome reflecting the social preference. Our main result being negative, expanding the domains of voting rules through allowing indifferences in individual preferences can do no better. Nevertheless, it makes sense to ask whether it is possible to escape the impossibility in ensuring Pareto optimal outcomes by extending the range of voting rules by allowing social indifference in outcomes. Theorem 3.4 is a strong negative answer to this question: Being decisive, i.e., not allowing for social indifference in outcomes, is a necessary condition for a voting rule to be Pareto ensuring. Hence, the positive but restricted result given by Theorem 3.3 is the best one can achieve regarding Pareto optimality in the context of anonymous referendum voting.

To sum up, referendum voting has the important merit of being simple but at the cost of a possible loss of efficiency in case all separable orderings over outcomes are allowed. The heaviness of this cost depends on how often Pareto optimality is violated, a question which is subject to further research.[9]

Another possible direction of research is about the effect of the separability axiom on our results. It is clear that enlarging the set of admissible orderings by allowing for additional non-separable preferences will strengthen the impossibility. Moreover, our impossibility prevails for certain subsets of separable orderings such as additively separable preferences.[10] On the other hand, further restrictions may allow for positive results. For example, assume additively separable preferences where all voters weigh all proposals equally. So every best outcome induces a unique separable ordering where outcomes are ranked according to their number

[8] The majority rule is first characterized by May (1952). For more recent characterizations one can refer to Aşan and Sanver (2002) as well as Woeginger (2003).

[9] An answer to this question may be based on the relationship between the number of possible outcomes (2^m) and the number of voters (n). For example, when n exceeds 2^m by a large margin, one can expect to be confronted with preference profiles where every outcome is the best for at least one voter. In such a case, the BKZ-paradox will be trivially escaped hence ensuring Pareto optimality. On the other hand, efficiency will be a more critical issue when n is less than 2^m.

[10] We know by Remark 3.2 that the "only if" part of Theorem 3.1 holds under additively separable preferences as well while Theorem 3.2 does not use the separability axiom. Hence our main impossibility result expressed by Theorem 3.3 can be strengthened by replacing separability with additive separability.

of disagreements from the best outcome.[11] Brams et al. (2004) show that referendum voting via the majority rule leads to an outcome which minimizes the sum of disagreements from the top preferences of all players. Hence, it is Pareto ensuring when the set of admissible orderings over outcomes is (severely) restricted by assuming additively separable preferences where all voters weigh all proposals equally. We pose the generalization of these observations as an open question.

We close by noting that the impossibility result established by the paper is perhaps not surprising. After all, the Pareto optimality of an outcome depends on the whole preference profile while referendum voting operates under a low informational requirement such as the list of opinions of voters for each proposal. In fact, the impossibility of ensuring Pareto optimality with this much of information is a corollary to Theorem 3.1. On the other hand, there exist Pareto ensuring voting rules which operate with a slightly higher information which is the top preference of each voter. For example, consider the plurality rule which, at each vote profile, picks the outcome voted by the highest number of voters -ties being broken arbitrarily. It is clear that the plurality rule escapes the BKZ-paradox, hence, by Theorem 3.1, it is Pareto ensuring. As referendum voting is typically implemented by asking voters to reveal their top preference, we conclude that the impossibility established by our paper is not only due to the low informational requirements of voting rules but also related to the nature of referendum voting. Thus, an exploration of voting rules which are Pareto ensuring can be an interesting direction of research, as this may end up in viable alternative voting rules which overcome the efficiency problem of referendum voting while preserving the simplicity of its information requirement.

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[11] For example, consider a voter whose best outcome is $(1, -1)$. This voter will be indifferent between $(1, 1)$ and $(-1, -1)$ which both disagree in one component from the best outcome. The worst outcome of this voter is $(-1, 1)$ where there is disagreement in both components.