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Population growth and poverty measurement

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Abstract If the absolute number of poor people goes up, but the fraction of people in poverty comes down, has poverty gone up or gone down? The economist's instinct, framed by population replication axioms that undergird standard measures of poverty, is to say that in this case poverty has gone down. But this goes against the instinct of those who work directly with the poor, for whom the absolute numbers notion makes more sense as they cope with more poor on the streets or in the soup kitchens. This paper attempts to put these two conceptions of poverty into a common framework. Specifically, it presents an axiomatic development of a family of poverty measures without a population replication axiom. This family has an intuitive link to standard measures, but it also allows one or other of "the absolute numbers" or the "fraction in poverty" conception to be given greater weight by the choice of relevant parameters. We hope that this family will prove useful in empirical and policy work, where it is important to give both views of poverty—the economist's and the practitioner's—their due.

1 Introduction

The World Bank's calculations show that from 1987 to 1998, the number of people in the world surviving on less than two dollars a day increased from 2.5 billion to 2.8 billion. But the world's population was increasing sufficiently fast so that the incidence of poverty, the percentage of people below the poverty line, fell from 61.0 to 56.1%. (See Tables 1 and 2). Did world poverty fall or stay constant during this turbulent period of globalization? One answer to this question is to say that it is a nonquestion—the answer depends on what is meant by an increase in poverty. But this is precisely the point.

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Table 1 Population living on less than $2 per day, 1987 and 1998

Regions	Number of people living on less than $2 day (millions)	
	1987	1998
East Asia and the Pacific	1,052.3	884.9
(excluding China)	299.9	252.1
Eastern Europe and Central Asia	16.3	98.2
Latin America and the Caribbean	147.6	159.0
Middle East and North Africa	65.1	85.4
South Asia	911.0	1,094.6
Sub-Saharan Africa	356.6	489.3
Total	2,549.0	2,811.5
(excluding China)	1,796.6	2,178.7

(Adapted from World Bank, 2001 Poverty Update)

There appear to be two substantively different views of poverty increase (or decrease). One is associated with absolute numbers of the poor, the other with their number relative to the total population (or the "incidence" of poverty). The economist's instinct is to go with the latter. The instinct of those on the ground, for example, those who have to face the absolute requirements of increased demands on soup kitchens or homeless shelters, is to think that poverty has gone up when the number of mouths to feed or beds to find goes up.

In the axiology of poverty measurement, which is where economists draw their instincts from, variously labeled axioms of population replication assure a neutrality with respect to population scale. These axioms basically argue the following: Take two identical societies and merge them to create a society with twice the population size. The poverty index in the merged society is the same as in the component societies even though the absolute number of the poor is twice as great because the total population is twice as large as well.

Table 2 Percent of people living on less than $2 per day, 1987 and 1998

Regions	Percent of people living on less than $2 day (millions)	
	1987	1998
East Asia and the Pacific	67.0	48.7
(excluding China)	62.9	44.3
Eastern Europe and Central Asia	3.6	20.7
Latin America and the Caribbean	35.5	31.7
Middle East and North Africa	30.0	29.9
South Asia	86.3	83.9
Sub-Saharan Africa	76.5	78.0
Total	61.0	56.1
(excluding China)	58.2	57.9

(Adapted from World Bank, 2001 Poverty Update)

The distinction between the two views of poverty is no mere technicality. As argued in Kanbur (2001), in Ghana between 1987 and 1991, the incidence of poverty came down by about one percentage point per year, while the absolute number of the poor increased because total population was growing by around two percentage points a year. The World Bank and the International Monetary Fund trumpeted the first as a measure of the success of their recommended "structural adjustment" policies, while those in civil society who criticized these policies did so, at least partly, because as they looked around them they could see more poor people in the streets. The global figures on poverty reproduced in Tables 1 and 2 show many comparisons where change in absolute numbers and change in incidence move in opposite directions or, when they move in the same direction, do so at very different rates. For example, in South Asia the number of poor people increased by more than 180 million people, while the incidence of poverty fell by 2.7 percentage points. Even in East Asia excluding China, where both absolute numbers and incidence fell, the rate of fall was very different. Absolute numbers fell by 16%, while the incidence fell by 30%. In sub-Saharan Africa, where both numbers and incidence rose, absolute numbers rose by 38%, while incidence rose by a bare 2%. These contrasts raise questions about the recent United Nations "millennium target" for income poverty reduction, which has been specified in terms of the incidence of poverty rather than in terms of the absolute numbers of the poor.

The issue of population size in evaluating welfare in general has a venerable tradition in economics and philosophy, going back at least as far as the debate around Parfit's (1984) "repugnant conclusion" (see also, Broome 1996). In the realm of poverty measurement, Kundu and Smith (1982) launched a debate by proving an impossibility theorem on measuring poverty with variable population. Most recently, Subramanian (2002) has tackled directly the issue of normalizing by population size and proved some possibility results. Our object is to build on this literature, but with a focus on an axiomatic derivation of a class of poverty measures that is operational and allows different weights on the normalized and nonnormalized views of poverty.

The analysis in this paper puts the two conceptions on poverty measures—one, that the poverty measure should rise when the number of poor increases, and the other that the poverty measure should fall when, holding the number of poor constant, total population increases—into a common framework. Section 2 sets out the axiomatic framework and derives the basic characterization of a family of poverty measures without a population replication axiom. Section 3 discusses the basic result further, and shows how, with different parameterizations, the two different views can be given different weights within this family of measures. Section 4 concludes.

2 Framework and basic result

For a population of size n, the set of income distributions is given by R_+^n, the nonnegative orthant of the n-dimensional Euclidean space R^n. A typical element of R_+^n is $x = (x_1, x_2, \ldots, x_n)$, where $x_i \geq 0$ is the income of person i. The set of all income distributions is $R_+ = \underset{n \in N}{\cup} R_+^n$, where N is the set of natural numbers.

A person i is said to be poor if $x_i < z$, where $z \in R^1_{++}$ is the exogenously given poverty line, with R^1_{++} being the strictly positive part of the real line. For any $n \in N,\ x \in R^n_+$, the set of poor persons is $S(x) = \{1 \leq i \leq n : x_i < z\}$ and the cardinality of $S(x)$, that is, the absolute number of poor persons is $q(x)$. The censored income distribution associated with x is $x^* = \left(x_1^*, x_2^*, \ldots, x_n^*\right)$, where $x_i^* = z$ if $x_i \geq z$, and $x_i^* = x_i$ if $x_i < z$.

A poverty index is a real valued function of individual incomes, the population size and the poverty line. More precisely, a poverty index is a function $P : R_+ \times R^1_{++} \times N \to R^1$. The restriction of P on $R^n_+ \times R^1_{++} \times \{n\}$ is denoted by P^n, where $n \in N$ is arbitrary. For any $n \in N$, $x \in R^n_+$, $P^n(x;z;n)$ indicates the poverty level associated with the income distribution x distributed over the concerned population of size n and the poverty line z.

The poverty index is assumed to satisfy certain desirable properties. These are:

- Focus (FOC) For all $n \in N, x, y \in R^n_+, z \in R^1_{++}$, if $S(x) = S(y)$ and $x_i = y_i$ for all $i \in S(x)$, then $P^n(x; z; n) = P^n(y; z; n)$.
- Monotonicity (MON) For all $n \in N, x, y \in R^n_+, z \in R^1_{++}$, if $x_j = y_j$ for all $j \neq i$, $i \in S(x)$, $x_i > y_i$, then $P^n(x; z; n) < P^n(y; z; n)$.
- Transfers principle (TRP) For all $n \in N, x, y \in R^n_+, z \in R^1_{++}$, if $x_i = y_i$ for all $i \neq j, k$ and $x_j > y_j \geq y_k > x_k, x_j - y_j = y_k - x_k$, for $k \in S(x)$ and $S(x) = S(y)$, then $P^n(x; z; n) > P^n(y; z; n)$.
- Symmetry (SYM) For all $n \in N, x, y \in R^n_+, z \in R^1_{++}$, if y is a permutation of x, then $P^n(x; z; n) = P^n(y; z; n)$.
- Increasingness in subsistence income (ISI) For all $n \in N, x \in R^n_+, P^n(x; z; n)$ is increasing in z over R^1_{++}.
- Continuity (CON) For all $n \in N, z \in R^1_{++}, P^n(x; z; n)$ is continuous in $x \in R^n_+$, $z \in R_+$.
- Scale invariance (SCI) For all $n \in N, x \in R^n_+,\ z \in R^1_{++}, P^n(x; z; n) = P^n(cx; cz; n)$, where $c > 0$ is any scalar.

FOC says that the poverty index is independent of the incomes of nonpoor persons. According to MON, a reduction in the income of poor must increase poverty. TRP demands that a transfer of income from a poor (j) to a richer poor (k) that does not change the set of poor persons increases poverty. SYM means that any characteristic other than income, e.g., the names of the individuals, is irrelevant to the measurement of poverty. Since given the income distribution, an increase in the poverty line makes the poor people more deprived in terms of income shortfalls from the poverty line, the poverty index should increase if the poverty line increases. This is what is demanded by ISI. CON ensures that minor observational errors in incomes will generate minor changes in the poverty index. SCI says that the poverty index is independent of the unit in which incomes and the poverty line are measured. (For further discussions on these properties, see Sen 1976; Donaldson and Weymark 1986; Cowell 1988; Chakravarty 1990; Foster and Shorrocks 1991; Zheng 1997 and Dutta 2002).

We now adopt an additional axiom, which when combined with other axioms, leads to the desired form of the poverty index. This axiom is Structural Specification (STS):

(a) For all m, $n \in N$, $x \in R^m_+, y \in R^n_+$; $P^{m+n}(x, y; z; m+n) = B(P^m(x; z; m), P^n(y; z; n))$, with $B : R^2 \to R^1$ being increasing in its arguments, and

(b) for any $m \in N, x \in R^m_+, P^m(x; z; m) = G\left(\sum_{i=1}^{m} h(x_i); z; m\right)$, where G is increasing in first two arguments.

In STS we impose a mild structure on the poverty index. It consists of two parts. Part (a) is a separability assumption. It is quite standard in the literature (see, for example, Blackorby et al. 1978). It says that overall poverty can be calculated from poverty levels in two or more subgroups, which form a partition of the population. Here, for simplicity, we consider a two-subgroup partitioning of the population. This specific form of poverty index excludes indices that are based on ordinal rank weights (see, for example, Sen 1976; Blackorby and Donaldson 1980, Kakwani 1980; Thon 1983; Shorrocks 1995; and Chakravarty 1997). The second part of the axiom makes a stronger assumption about the form of the poverty index. It says that the poverty index is a function of the sum of some transformation h of individual incomes, the poverty line and the population size. As we will note, under FOC and SCI, we may interpret the transformation h as an individual deprivation function in that it can be viewed in terms of some type of gap between the poverty line and the censored income. We may refer to $\sum_{i=1}^{m} h(x_i)$ as an aggregate deprivation or illfare function. Thus, part (b) of STS says that we view poverty as a function of a composite indicator of deprivation/illfare, defined in an unambiguous way, the poverty line and the population size. In other words, our poverty index is formulated in terms of a summary statistic of incomes, the poverty line and the population size. Several poverty indices can be put into this structure (see, for example, Watts 1968). It may be noted that part (a) of this axiom does not necessarily imply part (b). For example, assuming FOC and SCI, if we define the poverty index as $\Pi_i \left(\frac{x_i^*}{z}\right)^{-r}$, where $r > 1$, then part (b) does not hold, although part (a) holds. A second example that supports our claim is $\left\{1 - \min_i x_i^*/z\right\}$.

We now state a theorem that characterizes the family of poverty indices, which satisfy the above axioms. Note, in particular, that the axiom set does not include any of the variants of the "population replication" axiom.

Theorem 1: A poverty index $P : R_+ \times R^1_{++} \times N \to R^1$ satisfies FOC, MON, TRP, CON, SCI, and STS if and only if for all $m \in N$, $x \in R^m_+$, $z \in R^1_{++}$, $P^m(x; z; m)$ is ordinally equivalent to

$$a\left(\sum_{i=1}^{m} p\left(\frac{x_i^*}{z}\right) - \alpha m\right), \tag{1}$$

where $p : [0, 1] \to R^1$ is continuous, decreasing, strictly convex and $a > 0$, α are constants.

Proof: In Appendix.

The monotonicity principle we have used in Theorem 1 was suggested by Sen (1976). However, the index in Eq. 1 satisfies a stronger monotonicity condition, which requires poverty to decrease if there is an increase in a poor person's income (see Donaldson and Weymark 1986). This latter condition includes the possibility that the beneficiary of the income increase may become rich. Analogously, the index satisfies the Sen (1976) version of the transfer axiom, a stronger requirement than the TRP considered by Donaldson and Weymark (1986). The Sen version of the transfer axiom requires poverty to increase under a transfer of income from poor to anyone richer. Note that in this case, if the two persons involved in the transfer are poor, then the transfer may make the recipient rich so that the set of poor persons changes. Clearly, we can regard the function p in Eq. 1 as the individual deprivation function.

Assuming that $p\left(\frac{x_i^*}{z}\right) = 0$ for $x_i > z$ and q is fixed, α can be interpreted as the amount by which the poverty index reduces when the number of rich persons in the society increases by 1. This clearly shows that under these ceteris paribus assumptions, nonnegativity of α is a reasonable requirement. However, the situation is not so straightforward if we allow both the numbers of poor and the rich to increase simultaneously. In such a case, there is a trade-off between the increase in poverty due to the higher number of poor and the reduction in poverty resulting from the higher number of rich. Different indices may well evaluate the trade-off in different directions. The value of α becomes helpful in determining the direction of trade-off. As shown in the next section, if the absolute number of poor and the number of rich reduce so as to keep the fraction of population in poverty constant, then to ensure that the trade-off balances out in favor of poverty reduction, a negative value of α may be used. Thus, in this case we take the view that the reduction in poverty due to a lower number of poor should outweigh the increase in poverty from a lower rich population size. Since α can be arbitrary, its choice becomes an issue of value judgement. Note that for $\alpha = 0$ under FOC, the general poverty index in Eq. 1 is independent of both nonpoor population sizes and their income distribution. For a given income distribution over a given population size, an increase in the value of α reduces the general index. The constant a is a scale parameter: given other things, an increase in the value of a increases the poverty index. Therefore,without loss of generality, for all future discussions we can set $a = 1$.

To relate Theorem 1 with existing results, let us denote the first term of Eq. 1 by T^m. Foster and Shorrocks (1991) showed that all population replication invariant subgroup consistent relative poverty indices must be of the form $F(T^m/m)$, where F is continuous and increasing. Subgroup consistency demands that for any partitioning of the population into subgroups, aggregate poverty will fall if one subgroup's poverty is reduced, ceteris paribus. Clearly, there are some important differences between the class isolated in Theorem 1 and the Foster–Shorrocks family. While the latter is population replication invariant, the former is not. Another source of difference is the appearance of the term αm in Eq. 1, which enables us to consider different views on poverty change under population growth. Specifically, notice that the first term of Eq. 1 is an "aggregate" (not normalized by total population) version of standard poverty measures that emerge from settings where population replication axioms are imposed. The second term depends purely on total population and its impact depends on the choice of the parameter α and the population size m. These two terms allow us to see the different implications of population growth for the measure of poverty. The next section discusses these implications.

3 Discussion

In the rest of the paper, we will assume, for simplicity, that p satisfies the normalization condition $p(1) = 0$. We will now show how aggregate counterparts to different population replication invariant subgroup consistent indices can be derived as particular cases of Eq. 1. For this we make the assumption that $\alpha = 0$. In this case a k-fold replication of the population will multiply the poverty index by k, as in the replication scaling principle of Subramanian (2002).

As a first example, let $p(t) = (1 - t)^{\delta}$, where for MON and TRP to hold we need $\delta > 1$. The underlying index becomes

$$P_{\delta}^{m}(x; z; m) = \sum_{i \in S(x)} \left(1 - \frac{x_i}{z}\right)^{\delta}. \tag{2}$$

P_{δ} is the aggregate version of the Foster-Greer-Thorbecke (1984) index. For $0 < \delta \leq 1$ the index satisfies MON but not TRP. As $\delta \to 0$, $P_{\delta}^{m} \to q(x)$ the absolute number of poor. For $\delta = 1$, P_{δ}^{m} becomes the aggregate income gap ratio of the poor, which can be rewritten as

$$P_{\delta}^{m} = q(x) I(x), \tag{3}$$

where $I(x) = \sum_{i \in S(x)} (1 - x_i/z)/q(x)$ is the income gap ratio of the poor. On the other hand, if $\delta = 2$, the index becomes

$$P_{\delta}^{m} = q(x) \left\lfloor (I(x))^2 + (1 - I(x))^2 C^2(x) \right\rfloor, \tag{4}$$

where $C(x)$ is the coefficient of variation of the income distribution of the poor. Thus, over the income distributions with the same number of poor and the same mean income of the poor, the ranking of distributions generated by P_{δ} (for $\delta = 2$) is same as that produced by C. Note that the number of nonpoor incomes and their distribution are immaterial for this ranking. It is easy to check that an increase in the value of $\delta > 2$ makes the index more sensitive to transfers lower down the scale.

An alternative of interest arises from the specification $p(t) = 1 - t^{c}$, where $0 < c < 1$ ensures that MON and TRP are fulfilled. The corresponding index is given by

$$P_{c}^{m}(x; z; m) = \sum_{i \in S(x)} \left(1 - \left(\frac{x_i}{z}\right)^{c}\right), \tag{5}$$

which is the aggregate version of the Chakravarty (1983) index. For any $0 < c < 1$, a transfer of income from poor to rich increases P_{c}^{m} by a larger amount, the poorer the donor is. If we assume $c > 1$, then TRP is violated but MON is satisfied and as $c \to \infty, P_c^m \to q(x)$. For $c = 1$, P_c^m coincides with $q(x)I(x)$.

As a last example, assuming that all incomes are positive, let us suppose that $p(t) = -\log t$. This generates the aggregate form of the Watts (1968) poverty index

$$P_w^m(x; z; m) = \sum_{i \in S(x)} \log\left(\frac{z}{x_i}\right). \tag{6}$$

P_w is more sensitive to transfers at the lower end of the distribution.

Setting $\alpha = 0$ in Eq. 1, an increase in the number of poor increases poverty. Next, assuming positivity of α, we note that the poverty index in Eq. 1 decreases unambiguously as the total population increases, keeping the number of poor and their income distribution constant. This shows how the two views concerning poverty change as a consequence of change in the population size have been incorporated in a general structure. We refer to these two views as V1 and V2, respectively.

Kundu and Smith (1982) demonstrated that there does not exist any poverty index that meets the Sen (1976) version of the transfer principle and the two above conceptions on poverty change because of population growth. The main difference between the Kundu–Smith formulation and ours is that we do not impose the two population growth criteria at the outset; rather, we derive the two views separately as implications of our general formula (Eq. 1). Subramanian (2002) proved a similar impossibility result under a population growth principle, which states that if all the poor persons have the same income and a person having this common income joins the society and if there is at least one nonpoor person in the society, then poverty must go up. It is easy to check that this form of population growth principle is verified by our index in Eq. 1 under the assumption that $\alpha = 0$.

Let us now consider a fourth view concerning poverty change resulting from population growth and call it R1. It is the requirement that for a given head count ratio q/m, if the absolute number of poor q reduces, then poverty declines. We note the distinction between V1 and R1. While both require poverty reduction under a decrease in the size of the poor population, the latter demands additionally that the head count ratio is a constant. To combine R1 with V2, we rewrite V2 as the condition that for a given absolute number of poor and the income distribution of the poor, poverty declines if the head count ratio declines and call it R2.

We now want to examine whether R1 and R2 can be incorporated in our general framework simultaneously. For this, let us suppose for simplicity that T^m in Eq. 1 is the absolute number of poor q so that the poverty indicator becomes

$$P^m(x; z; m) = q - \alpha m. \tag{1a}$$

If $q_1 > q_2$ and $q_1/m_1 = q_2/m_2 = \lambda$, say (so that $m_1 > m_2$), then R1 would demand that $P_2^m < P_1^m$, which in turn gives $q_2 - \alpha m_2 < q_1 - \alpha m_1$, that is, $\alpha < (q_2 - q_1)/(m_2 - m_1) = \lambda$. Thus, this restriction is met if α is smaller than the given head count ratio.

Next, given $q_1 = q_2 = q$ (say) and $q_1/m_1 > q_2/m_2$ (so that $m_1 < m_2$), R2 will demand $P_2^m < P_1^m$, that is, $(q - \alpha m_2) - (q - \alpha m_1) < 0$, which gives $\alpha(m_1 - m_2) < 0$. Hence, α must be positive.

We summarize the above observations in the following:

Proposition 1 A poverty index of the form Eq. 1a will satisfy the following conditions simultaneously for a small positive α:

- (R1) For a given head count ratio, if the absolute number of poor declines, then poverty should decline.
- (R2) For a given absolute number of poor and the income distribution of the poor subpopulation, if the head count ratio declines then poverty should decline.

While R2 holds unambiguously for all $\alpha > 0$, the choice of α for R1 to be satisfied would be dictated by specific situations. To see this explicitly, let us set $q_1 = 2, q_2 = 1$, and (since $q_1/m_1 = q_2/m_2$), $m_1 = 2m_2$ in the R1-based inequality $\alpha < (q_2 - q_1)/(m_2 - m_1)$. This inequality then becomes $\alpha < 1/m_2$. Since m_2 can be made as large as possible, effectively, α will have to be some small positive number quite close to zero. Thus, in any pairwise poverty comparison of two income distributions, the major determinants of the ranking become the aggregate head counts in the two distributions, with the head count ratios playing a negligible part.

Another interesting observation that can be made here is that the R1-based inequality $\alpha < (q_2 - q_1)/(m_2 - m_1)$ can hold for negative values of α as well. Since we require positivity of α for R2 to hold, in such a case Proposition 1 turns out to be an impossibility result, which says that simultaneous satisfaction of R1 and R2 is not possible.

A natural question that arises at this stage is: Do there exist other poverty indices that can incorporate these two views simultaneously? The answer is yes, and an example of the desired type of index is $P_A^m(x; z; m) = q^2/m$. P_A^m was introduced by Arriaga (1970) as an urbanization index. In Arriaga's framework, the numerator of P_A^m is the square of total resident population in the urban community (if there are k urban communities, then it will be the sum of squares of such population sizes).

Next, suppose that q increases from q_1 to q_2 and m increases from m_1 to m_2. Then, for the overall poverty index in Eq. 1a to increase, but allowing for a discounting for population growth, the index should fulfill the inequality $(q_2 - \alpha m_2) - (q_1 - \alpha m_1) < q_2 - q_1$, which we can rewrite as $(q_2 - q_1) - \alpha(m_2 - m_1) < q_2 - q_1$. Evidently, this inequality is satisfied whenever $\alpha > 0$.

It may now be worthwhile to illustrate the index P^m in Eq. 1a using different values of α. From Tables 1 and 2, we note that for South Asia and for the World as a whole, the head count ratio has gone down between 1987 and 1998. But the absolute number of poor has gone up in both places during the period. For $\alpha = 0.3$, the directional change in $q - \alpha m$ over the concerned period has been found to be the same as that in q for both regions. For $\alpha = 0.5$, while a similar picture was observed for South Asia, a reverse situation occurred for the World. For $\alpha = 0.75$, the trend observed was completely opposite to that found for $\alpha = 0.3$. As stated earlier, this is natural. For low positive values of α, q becomes the dominant factor, particularly, if it is high. Therefore, it is likely that the directional change in $q - \alpha m$ will be controlled by the directional change in q. However, for large positive values of α, a different directional change may occur.

Proposition 1 and the numerical illustration provided above are based purely on head counting, which is a well-known violator of MON and TRP. The same remark applies to the Arriaga index as well. An example of an index that satisfies R1 and R2 along with MON and TRP is

$$P_{\beta}^{m}(x; z; m) = \frac{1}{m^{\beta}} \sum_{i=1}^{n} p\left(\frac{x_i^*}{z}\right), \tag{1b}$$

where p is same as in Eq. 1 and $0 < \beta < 1$. For a given $(x; z)$, an increase in β decreases the value of P_{β}^{m}. To see satisfaction of R1 by P_{β}, note that for a given head count ratio q/m, for R1 to hold under a reduction of the absolute number of poor, we require proportionate contraction of the nonpoor population size. Under this contraction, poverty value, as measured by a population replication invariant poverty index remains unaltered. Now, we can rewrite P_{β}^{m} in Eq. 1b as $m^{1-\beta}\left(\sum_{i=1}^{m} p\left(x_i^*/z\right) \Big/ m\right) = m^{1-\beta}(T^m/m)$. Clearly, the component T^m/m of P_{β}^{m} is population replication invariant. If the population is replicated $k > 1$ times, then the resulting index becomes $P_{\beta}^{km}(y; z; km) = k^{1-\beta} m^{1-\beta}\left(T^{km}/km\right)$, where y is the k-fold replication of x. By population replication invariance, $T^m/m = T^{km}/km$. Since $0 < \beta < 1$, we must have $P_{\beta}^{km}(y; z; km) > P_{\beta}^{m}(x; z; m)$. Thus, R1 is satisfied.

Next, with a given absolute number of the poor q and an income distribution of the poor subpopulation, a decline in the head count ratio will result from an increase in the number of rich.Clearly, the value of the poverty index in Eq. 1b will reduce in such a case. Thus, R2 is fulfilled by the index P_{β}.

The above findings can now be summarized in the following:

Proposition 2 A poverty index of the form in Eq. 1b will satisfy R1 and R2 simultaneously.

Since P_{β} does not satisfy STS, it is not a member of the class given by Eq. 1. However, it satisfies a more general condition than part (a) of STS, which requires that $P^{m+n}(x, y; z; m+n) = B(P^m(x; z; m), P^n(y; z; n); m, n)$. But one limitation of P_{β} is that, given other things, the direction of change in poverty, as measured by this index, due to an increase in the number of poor may not be unambiguous.

4 Conclusion

Population replication axioms are now so much a part of the axiology of poverty measurement that economists take them on board without much thought. They have a certain appeal, they are certainly convenient, and help to generate families of poverty measures that we have all become familiar with. But, as we have argued in the introduction, they impose a structure on poverty measures that does not necessarily conform to the intuitions and instincts of those who deal with the daily realities of poor people's lives. We have shown, however, that appealing poverty measures can indeed be derived without population replication axioms. These measures relate intuitively to standard measures and are tractable and applicable in empirical and policy work. They also allow, through choice of parametrizations, for different weights to be given to the "absolute numbers" vs the "fraction in poverty" views. Given these properties, we hope that this family of measures will prove their worth in empirical and policy work.

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Appendix

Proof of Theorem 1

By SCI,

$$P^{m+n}(x,y;z;m+n) = P^{m+n}\left(\frac{x}{z},\frac{y}{z};1;m+n\right) \tag{7}$$

We rewrite the right hand side of Eq. 7 as $E^{m+n}\left(\frac{x}{z},\frac{y}{z};m+n\right)$. Therefore, Eq. 7 becomes

$$P^{m+n}(x,y;z;m+n) = E^{m+n}\left(\frac{x}{z},\frac{y}{z};m+n\right). \tag{8}$$

FOC implies that $P^{m+n}(x,y;z;m+n) = P^{m+n}(x^*,y^*;z;m+n)$. Hence, Eq. 8, under FOC, becomes

$$P^{m+n}(x,y;z;m+n) = E^{m+n}\left(\frac{x^*}{z},\frac{y^*}{z};m+n\right). \tag{9}$$

Given the representation of P^{m+n} in Eq. 9, we apply STS to Eq. 9 and get

$$B\left(F\left(\sum_{i=1}^{m}p\left(\frac{x_i^*}{z}\right);m\right),F\left(\sum_{i=1}^{n}p\left(\frac{y_i^*}{z}\right);n\right)\right) = F\left(\sum_{i=1}^{m}p\left(\frac{x_i^*}{z}\right)+\sum_{i=1}^{n}p\left(\frac{y_i^*}{z}\right);m+n\right). \tag{10}$$

where $G\left(\sum_{i=1}^{m}h\left(\frac{x_i^*}{z}\right),1,m\right) = F\left(\sum_{i=1}^{m}p\left(\frac{x_i^*}{z}\right),m\right)$ and hence F is increasing in the first argument and B is increasing in its arguments. Given continuity of P^m, the general solution to the functional equation (Eq. 10) is given by

$$B(u,v) = f^{-1}(f(u)+f(v)), \tag{11}$$

$$F(l,m) = f^{-1}(al-bm), \tag{12}$$

where f is an arbitrary continuous, strictly increasing function, and a and b are constants (Aczel 1966, Theorem 3 and Corollary 4, pp. 314–315).

Note that l, the first argument of F in Eq. 12, is the value of the function $\sum_i p()$ at x^*/z for some $x \in R^m_+,\ m \in N$ *and* $z \in R^1_{++}$ and F is a representation of the poverty index. Since f is increasing, f^{-1} is so. Increasingness of f^{-1} in l requires that a is positive. In the presence of STS, SCI and FOC, in view of Eq. 12, we now have

$$P^m(x; z; m) = f^{-1}\left(a\left(\sum_{i=1}^{m} p\left(\frac{x_i^*}{z}\right) - \alpha m\right)\right). \tag{13}$$

where $\alpha = b/a$.

Clearly, the domain of p in Eq. 13 is [0, 1]. By increasingness of f^{-1}, P^m in (13) satisfies MON only if p is decreasing. A similar argument shows that for TRP to hold we need strict convexity of p. This gives us the desired form of P^m. The sufficiency is easy to check.

Note that in proving the theorem, we did not assume SYM. However, the poverty index in Eq. 1 satisfies SYM because F in Eq. 10 satisfies it.

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