Soc Choice Welfare (2005) 24: 475–496
DOI: 10.1007/s00355-003-0311-1

Social Choice and Welfare

Factorization of fuzzy preferences

Bonifacio Llamazares

Departamento de Economía Aplicada (Matemáticas), Universidad de Valladolid, Avda. Valle de Esgueva 6, 47011 Valladolid, Spain (e-mail: boni@eco.uva.es)

Received: 14 August 2001/Accepted: 24 November 2003

Abstract. In the ordinary framework, the factorization of a weak preference relation into a strict preference relation and an indifference relation is unique. However, in fuzzy set theory, the intersection and the union of fuzzy sets can be represented different ways. Furthermore, some equivalent properties in the ordinary case have generalizations in the fuzzy framework that may be not equivalent. For these reasons there exist in the literature several factorizations of a fuzzy weak preference relation. In this paper we obtain and characterize different factorizations of fuzzy weak preference relations by means of two courses of action which are equivalent in the ordinary framework: axioms and definitions of strict preference and indifference.

1 Introduction

In social choice theory agents' preferences on a set of alternatives are usually represented through complete binary relations. These weak preference relations show whether an alternative is at least as good as another or vice versa. Starting from a weak preference relation R, it is possible to obtain a strict preference relation P and an indifference relation I. This factorization is unique, so that $P = R \cap (R^{-1})^c$, which is equivalent to $P = (R^{-1})^c$, and $I = R \cap R^{-1}$. Moreover, these binary relations satisfy the following properties: P is asymmetric, I is symmetric, $P \cap I = \emptyset$ and $R = P \cup I$. These properties can

This work is partially financed by the Junta de Castilla y León (Consejería de Educación y Cultura, Proyecto VA057/02), Ministerio de Ciencia y Tecnología, Plan Nacional de Investigación, Desarrollo e Innovación Tecnológica (I + D + I) (Proyecto BEC2001-2253) and ERDF. I am indebted to José Luis García-Lapresta and an anonymous referee for his helpful comments.

be considered as the starting point, and, in this case, we will refer to them as axioms.

However, human preferences are often vague, and the previous model does not allow for this fact. Vagueness can be taken into account by means of ordinal concepts (see, for instance, Basu et al. [6]) or through fuzzy logic. The use of fuzzy binary relations in social choice theory for representing individuals' preferences has been justified by several authors, such as Blin [8], Basu [5], Barrett et al. [3], Dutta [12] and Barrett and Pattanaik [4], among others.

In fuzzy set theory, several triangular norms and conorms are used for defining the intersection and the union of fuzzy sets, respectively. Furthermore, some ordinary properties can be generalized in different ways to the fuzzy framework. Consequently, various factorizations of fuzzy weak preference relations have been given and studied in the literature. See, for example, Orlovsky [19], Ovchinnikov [20] and [21], Dutta [12], Banerjee [2], Dasgupta and Deb [9] and [10] and Richardson [24], among others.

It is important to emphasize that in the theory of preference modelling some authors such as Roubens and Vincke [25], Barrett and Pattanaik [4], Ovchinnikov and Roubens [23], Fodor [13], Van de Walle et al. [29] and Llamazares [16], among others, also suppose an incomparability relation. A survey of this development can be found in De Baets and Fodor [11].

In this paper we try to clarify the generalization of some properties from the ordinary to the fuzzy framework. This is very important because the definitions of asymmetry and completeness determine the characterizations of P and I. In the same way, the choice of P, I and the transitivity property plays a crucial role in the generalization of Arrow's Impossibility Theorem to the fuzzy framework. For instance, Dutta [12, p. 225] and Banerjee [2, p. 126] obtain opposite results because they utilize different definitions of P.

Moreover, we generalize and characterize some factorizations from the literature by means of two courses of action: axioms and definitions of P and I. When we consider the axioms we conclude that the only interesting t-norms are ϕ–transforms of Lukasiewicz's t-norm. Under the assumption of reciprocal order automorphisms, if we require additional conditions such as $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$ (or $P = (R^{-1})^c$), then we obtain generalizations of the factorizations given by Orlovsky [19] and Barrett and Pattanaik [4], respectively. On the other hand, when we establish the definitions of P and I as the starting point, we prove that P and I satisfy the axioms only for generalizations of the factorization given by Barrett and Pattanaik [4].

The paper is organized as follows: In Sect. 2 we introduce notation and basic definitions. In Sect. 3 we focus our analysis on factorizations obtained from axioms, while in Sect. 4 we develop our research based on definitions of P and I. We conclude in Sect. 5.

2 Preliminaries

Let A be a not empty set of alternatives with $|A| \geq 2$. An *ordinary binary relation Q on A* is an ordinary subset of $A \times A$. We will use aQb to denote

$(a,b) \in Q$. Given an ordinary binary relation Q on A, the ordinary binary relations Q^{-1} and Q^c are defined by

$$Q^{-1} = \{(a,b) \in A \times A | (b,a) \in Q\},$$
$$Q^c = \{(a,b) \in A \times A | (a,b) \notin Q\}.$$

An ordinary binary relation Q on A is:

1. *Reflexive* if aQa for all $a \in A$.
2. *Irreflexive* if not aQa for all $a \in A$.
3. *Symmetric* if $Q \subseteq Q^{-1}$ $\quad (aQb \Rightarrow bQa,$ for all $a,b \in A)$.
4. *Asymmetric* if $Q \cap Q^{-1} = \emptyset$ $\quad (aQb \Rightarrow$ not $bQa,$ for all $a,b \in A)$.
5. *Complete* if $Q \cup Q^{-1} = A \times A$ $\quad (aQb$ or $bQa,$ for all $a,b \in A)$.

An ordinary binary relation R on A is an *ordinary weak preference relation* if it is complete. Any ordinary weak preference relation can be factorized into an *ordinary strict preference relation*, $P = R \cap (R^{-1})^c$, and an *ordinary indifference relation*, $I = R \cap R^{-1}$. Since R is complete, we have that $P = R \cap (R^{-1})^c$ is equivalent to $P = (R^{-1})^c$. The ordinary binary relations P and I are characterized through the following properties:

1. P is asymmetric.
2. I is symmetric.
3. $P \cap I = \emptyset$.
4. $R = P \cup I$.

Completeness and asymmetry imply reflexivity and irreflexivity, respectively. Therefore, R is reflexive and P is irreflexive. Moreover, I is also reflexive as a consequence of $R = P \cup I$, R reflexive and P irreflexive.

A *fuzzy subset B of A* is defined through its *membership function*, $\mu_B : A \longrightarrow [0,1]$, where $\mu_B(a)$ is the grade of membership of a in B. Given two fuzzy subsets B and C of A, $B \subseteq C$ if $\mu_B(a) \leq \mu_C(a)$ for all $a \in A$. The complement of a fuzzy set B of A, B^c, is defined by $\mu_{B^c}(a) = 1 - \mu_B(a)$ for all $a \in A$. The intersection and the union of fuzzy sets are defined by means of triangular norms and conorms, respectively. These functions satisfy the following properties: commutativity, monotonicity, associativity and a boundary condition. Triangular norms and conorms were widely studied by Schweizer and Sklar [26] in the context of probabilistic metric spaces.

A function $T : [0,1]^2 \longrightarrow [0,1]$ is a *triangular norm (t*-norm) if it satisfies the following conditions:

1. $T(1,x) = x$ for all $x \in [0,1]$.
2. $T(x,y) = T(y,x)$ for all $x,y \in [0,1]$.
3. $T(x,y) \leq T(u,v)$ for all $x,y,u,v \in [0,1]$ such that $x \leq u, y \leq v$.
4. $T(x,T(y,z)) = T(T(x,y),z)$ for all $x,y,z \in [0,1]$.

A function $S : [0,1]^2 \longrightarrow [0,1]$ is a *triangular conorm (t-conorm)* if it satisfies the following conditions:

1. $S(0,x) = x$ for all $x \in [0,1]$.
2. $S(x,y) = S(y,x)$ for all $x,y \in [0,1]$.
3. $S(x,y) \leq S(u,v)$ for all $x,y,u,v \in [0,1]$ such that $x \leq u, y \leq v$.
4. $S(x,S(y,z)) = S(S(x,y),z)$ for all $x,y,z \in [0,1]$.

It is easy to check that $T(x,0) = 0$ and $S(x,1) = 1$, for all $x \in [0,1]$.

Given a *t*-norm T, the function $S(x,y) = 1 - T(1-x, 1-y)$ is a t-conorm and it is called the *dual t-conorm of T*. If T is a *t*-norm and S is its dual t-conorm, the intersection and the union of two fuzzy subsets B and C of A are defined as follows:

1. $\mu_{(B\cap C)}(a) = T(\mu_B(a), \mu_C(a))$ for all $a \in A$.
2. $\mu_{(B\cup C)}(a) = S(\mu_B(a), \mu_C(a))$ for all $a \in A$.

It is important to emphasize that the utilization of dual t-conorms allows De Morgan laws, $(B \cup C)^c = B^c \cap C^c$ and $(B \cap C)^c = B^c \cup C^c$, to be satisfied in fuzzy set theory.

Next we show some *t*-norms and their dual t-conorms which are very usual in the literature.

1. *Minimum*: $\min(x,y)$.
   Maximum: $\max(x,y)$.
2. *Algebraic product*: $\Pi(x,y) = xy$.
   Algebraic sum: $\Pi'(x,y) = x + y - xy$.
3. *Lukasiewicz's t*-norm: $W(x,y) = \max(x+y-1, 0)$.
   Lukasiewicz's t-conorm: $W'(x,y) = \min(x+y, 1)$.

A function $\phi : [0,1] \longrightarrow [0,1]$ is an *order automorphism* if it is bijective and increasing. Any order automorphism ϕ is strictly increasing, continuous and satisfies $\phi(0) = 0$, $\phi(1) = 1$. Furthermore, the function ϕ^{-1} is also an order automorphism. An order automorphism ϕ is *reciprocal* if $\phi(1-x) = 1 - \phi(x)$ for all $x \in [0,1]$. It is easy to check that ϕ is reciprocal if and only if ϕ^{-1} is reciprocal. On this, see García-Lapresta and Llamazares [15].

Given a *t*-norm T and an order automorphism ϕ, the *ϕ–transform of T* is the *t*-norm T_ϕ defined by, $T_\phi(x,y) = \phi^{-1}(T(\phi(x), \phi(y)))$. For instance,

$$W_\phi(x,y) = \phi^{-1}(\max(\phi(x) + \phi(y) - 1, 0)),$$

and its dual t-conorm is

$$W'_\phi(x,y) = 1 - \phi^{-1}(\max(\phi(1-x) + \phi(1-y) - 1, 0)).$$

Moreover, when ϕ is reciprocal we obtain

$$W'_\phi(x,y) = \phi^{-1}(\min(\phi(x) + \phi(y), 1)).$$

A *t*-norm T is *Archimedean* if $T(x,x) < x$ for all $x \in (0,1)$. A *t*-norm T has *zero divisors* if there exist $x,y \in (0,1)$ such that $T(x,y) = 0$.

A *fuzzy binary relation Q on A* is a fuzzy subset of $A \times A$. The value $\mu_Q(a,b)$ will be denoted by $Q(a,b)$. If $Q(a,b) \in \{0,1\}$ for all $a,b \in A$ then Q is an ordinary binary relation. In this case, aQb denotes $Q(a,b) = 1$.

Given a fuzzy binary relation Q on A, the fuzzy binary relations Q^{-1} and Q^c are defined by $Q^{-1}(a,b) = Q(b,a)$ and $Q^c(a,b) = 1 - Q(a,b)$, for all $a,b \in A$.

Given a t-norm T and its dual t-conorm S, a fuzzy binary relation Q on A is:

1. *Reflexive* if $Q(a,a) = 1$ for all $a \in A$.
2. *Irreflexive* if $Q(a,a) = 0$ for all $a \in A$.
3. *Symmetric*[1] if $Q(a,b) = Q(b,a)$ for all $a,b \in A$.
4. *Asymmetric* if $T(Q(a,b),Q(b,a)) = 0$ for all $a,b \in A$.
5. *Complete* if $S(Q(a,b),Q(b,a)) = 1$ for all $a,b \in A$.

The above definitions of asymmetry and completeness are not standard. Some authors, such as Dutta [12], Banerjee [2] and Richardson [24], call asymmetry antisymmetry, and they consider the following definitions:

1. *Asymmetry*: $Q(a,b) > 0 \Rightarrow Q(b,a) = 0$, for all $a,b \in A$.
2. *Completeness*: $Q(a,b) + Q(b,a) \geq 1$ for all $a,b \in A$.

However, this definition of asymmetry is a very strong condition since $Q(b,a)$ has to be null even though $Q(a,b)$ is very small. In this respect, Blin [8] and Barrett and Pattanaik [4] suggest that the vagueness "arises through the multiplicity of dimensions underlying preferences", and consequently, both $Q(a,b) > 0$ and $Q(b,a) > 0$ would be possible (see the mentioned authors for more details).

Moreover, the definitions of asymmetry and completeness in fuzzy set theory depend on the properties that we use for these concepts in the ordinary framework. If we consider $Q \cap Q^{-1} = \emptyset$ and $Q \cup Q^{-1} = A \times A$, respectively, then $Q \cap Q^{-1} = \emptyset$ results in

$$Q(a,b) > 0 \Rightarrow Q(b,a) = 0, \text{ for all } a,b \in A,$$

only for t-norms without zero divisors, so Lukasiewicz's t-norm should not be used. But if the t-norm has no zero divisors and we utilize its dual t-conorm for the union of fuzzy sets, then $Q \cup Q^{-1} = A \times A$ implies

$$Q(a,b) = 1 \text{ or } Q(b,a) = 1, \text{ for all } a,b \in A,$$

which is a strong restriction.

On the other hand, we can make use of properties which are independent of t-norms and t-conorms. For instance, $Q^c \subseteq Q^{-1}$ is equivalent to $Q \cup Q^{-1} = A \times A$ in the ordinary framework, which allows us to obtain

$$Q(a,b) + Q(b,a) \geq 1 \text{ for all } a,b \in A,$$

as the definition of completeness. But, in this case, $Q^{-1} \subseteq Q^c$ is also equivalent to $Q \cap Q^{-1} = \emptyset$ and we should use

$$Q(a,b) + Q(b,a) \leq 1 \text{ for all } a,b \in A,$$

as the definition of asymmetry.

[1] In the ordinary framework $Q = Q^{-1}$.

In fuzzy set theory completeness and asymmetry do not imply reflexivity and irreflexivity, respectively. Therefore, a fuzzy binary relation R on A is a *fuzzy weak preference relation* if it is reflexive and complete. Given a fuzzy weak preference relation R we say that (P, I) is an *axiomatic factorization of R* if the fuzzy binary relations P and I satisfy the following properties:

1. P is irreflexive and asymmetric.
2. I is reflexive and symmetric.
3. $P \cap I = \emptyset$.
4. $R = P \cup I$.

The third assumption gives us another reason for rejecting $P(a,b) > 0 \Rightarrow P(b,a) = 0$. If this condition stems from $P \cap P^{-1} = \emptyset$, i.e. the asymmetry of P, then $P(a,b) > 0 \Rightarrow I(a,b) = 0$ should also be considered since $P \cap I = \emptyset$.

In the fuzzy framework $P = R \cap (R^{-1})^c$ is not equivalent to $P = (R^{-1})^c$. Moreover, we obtain different results when we consider that (P, I) is an axiomatic factorization of R and when we suppose that $P = R \cap (R^{-1})^c$ (or $P = (R^{-1})^c$) and $I = R \cap R^{-1}$. Therefore, there exist two courses of action in order to analyze fuzzy preferences: axiomatic factorizations and definitions of P and I. These courses of action will be studied in the following sections.

To conclude this section we are going to show some of the most usual factorizations from the literature.

1. The following factorization was given by Orlovsky [19] and afterwards characterized by Richardson [24]:

$$P(a,b) = \max(R(a,b) - R(b,a), 0),$$
$$I(a,b) = \min(R(a,b), R(b,a)).$$

2. Ovchinnikov [20] gave this factorization and later it was characterized by Dutta [12]:

$$P(a,b) = \begin{cases} R(a,b), & \text{if } R(a,b) > R(b,a), \\ 0, & \text{otherwise,} \end{cases}$$
$$I(a,b) = \min(R(a,b), R(b,a)).$$

3. This one was given by Dutta [12] and also by Roubens and Vincke [25] (with incomparability relations). Afterwards it was characterized by Banerjee [2]:

$$P(a,b) = 1 - R(b,a),$$
$$I(a,b) = \min(R(a,b), R(b,a)).$$

4. Barrett and Pattanaik [4] gave the following factorization with incomparability relations:

$$P(a,b) = 1 - R(b,a),$$
$$I(a,b) = R(a,b) + R(b,a) - 1.$$

We would like to emphasize that in this factorization we have

$$R(a,b) + R(b,a) = 1 \Rightarrow \begin{cases} I(a,b) = 0, \\ P(a,b) = R(a,b). \end{cases}$$

In this case, $R(a,b)$ represents the intensity with which a is preferred to b. This is the interpretation that Bezdek et al. [7], Nurmi [18], Tanino [27], Nakamura [17] and García-Lapresta and Llamazares [15], among others, have used for the modelization of individuals' preferences. Therefore, the factorization given by Barrett and Pattanaik [4] generalizes the framework utilized by the previous authors.

3 Results about axiomatic factorizations

In this section we focus our attention on the consequences which are derived when we suppose that (P,I) is an axiomatic factorization of R. Obviously, our analysis depends on the t-norm T and its dual t-conorm S used for representing the intersection and the union of fuzzy sets. In this section we consider only continuous t-norms. When the t-norm has no zero divisors, then R, P and I are ordinary binary relations. If the t-norm has zero divisors we distinguish between non-Archimedean and Archimedean t-norms. In the first case we obtain that there exists a threshold so that the fuzzy weak preference relation cannot take non-zero values less than this threshold. In the second one, Ovchinnikov and Roubens [22] have proven that the t-norm is a ϕ–transform of Lukasiewicz's t-norm. When we consider reciprocal order automorphisms and we include the requirements $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$ (or $P = (R^{-1})^c$), then we obtain generalizations of the factorizations given by Orlovsky [19] and Barrett and Pattanaik [4], respectively.

First of all, we are going to show the conditions that a fuzzy weak preference relation and an axiomatic factorization have to hold.

Remark 1. If R is a fuzzy weak preference relation and (P,I) is an axiomatic factorization of R, then for all $a,b \in A$ the following statements hold:

1. $R(a,a) = 1$ and $S(R(a,b),R(b,a)) = 1$.
2. $P(a,a) = 0$ and $T(P(a,b),P(b,a)) = 0$.
3. $I(a,a) = 1$ and $I(a,b) = I(b,a)$.
4. $T(P(a,b),I(a,b)) = 0$.
5. $R(a,b) = S(P(a,b),I(a,b))$.

Our analysis starts with continuous t-norms without zero divisors.

Theorem 1. *Let R be a fuzzy weak preference relation and (P, I) an axiomatic factorization of R. If T is a continuous* t-norm without zero divisors, then *R is an ordinary weak preference relation.*

Proof. Given $a, b \in A$ we are going to prove that $R(a, b) \in \{0, 1\}$. Since

$$T(1 - R(a, b), 1 - R(b, a)) = 1 - S(R(a, b), R(b, a)) = 0$$

and T is a t-norm without zero divisors, then we obtain $R(a, b) = 1$ or $R(b, a) = 1$. If $R(b, a) = 1$, then

$$T(1 - P(b, a), 1 - I(b, a)) = 1 - S(P(b, a), I(b, a)) = 1 - R(b, a) = 0.$$

Again, since T has no zero divisors we have $P(b, a) = 1$ or $I(b, a) = 1$. We study these cases:
1. If $P(b, a) = 1$, then

$$P(a, b) = T(P(a, b), 1) = T(P(a, b), P(b, a)) = 0,$$
$$I(a, b) = T(1, I(a, b)) = T(P(b, a), I(b, a)) = 0.$$

Therefore, $R(a, b) = S(P(a, b), I(a, b)) = S(0, 0) = 0$.

2. If $I(b, a) = 1$, then

$$R(a, b) = S(P(a, b), I(a, b)) = S(P(a, b), 1) = 1.$$ ■

Remark 2. It is easy to check that P and I are also ordinary binary relations on A.

In view of the previous result, we focus our analysis on continuous t-norms with zero divisors. Continuous non-Archimedean t-norms with zero divisors have been studied by Van de Walle et al. [28]. They obtain, among others, the following result.

Proposition 1. *Let T be a continuous non-Archimedean* t-norm with zero divisors and *S its dual t-conorm. Then there exist $\theta \in (0, 1)$ and an order automorphism φ such that for all $(x, y) \in [0, 1]^2$,*

$$S(x, y) = 1 \Leftrightarrow \begin{cases} x = 1, \mathit{or} \\ y = 1, \mathit{or} \\ (1 - x, 1 - y) \in (0, \theta)^2 \ \mathit{and}\ \varphi(\frac{1-x}{\theta}) + \varphi(\frac{1-y}{\theta}) \leq 1. \end{cases}$$

Remark 3. Van de Walle et al. [28] obtain in their work that $(1 - x, 1 - y) \in (0, \theta]^2$. However, it is easy to check that $1 - x = \theta$ or $1 - y = \theta$ are not possible.

In the following theorem we prove that for continuous non-Archimedean t-norms with zero divisors there exists a threshold so that the fuzzy weak preference relation cannot take non-zero values less than the threshold. The proof is similar to that provided by Van de Walle et al. [28] in the factorization of fuzzy preferences with incomparability relations.

Theorem 2. *Let R be a fuzzy weak preference relation and (P, I) an axiomatic factorization of R. If T is a continuous non-Archimedean t*-norm with zero divisors, then there exists $\theta \in (0,1)$ *such that* $R(a,b) \in \{0,1\}$ or $R(a,b) > 1-\theta$, *for all* $a, b \in A$.

Proof. Given $a, b \in A$, we have $S(R(a,b), R(b,a)) = 1$. By Proposition 1 there exist $\theta \in (0,1)$ and an order automorphism φ such that one of the following statements holds:

1. $R(a,b) = 1$.
2. $R(b,a) = 1$.
3. $(1-R(a,b), 1-R(b,a)) \in (0,\theta)^2$ and $\varphi\left(\frac{1-R(a,b)}{\theta}\right) + \varphi\left(\frac{1-R(b,a)}{\theta}\right) \leq 1$.

In the first case and in the last one the result is obvious. If $R(b,a) = 1$, then $S(P(b,a), I(b,a)) = 1$. Hence, by Proposition 1 we have the following cases:

1. If $P(b,a) = 1$, then

$$P(a,b) = T(P(a,b),1) = T(P(a,b),P(b,a)) = 0,$$
$$I(a,b) = T(1,I(a,b)) = T(P(b,a),I(b,a)) = 0.$$

Therefore, $R(a,b) = S(P(a,b), I(a,b)) = S(0,0) = 0$.

2. If $I(b,a) = 1$, then $R(a,b) = S(P(a,b), I(a,b)) = S(P(a,b), 1) = 1$.

3. If $(1-P(b,a), 1-I(b,a)) \in (0,\theta)^2$ and $\varphi\left(\frac{1-P(b,a)}{\theta}\right) + \varphi\left(\frac{1-I(b,a)}{\theta}\right) \leq 1$, then

$$R(a,b) = S(P(a,b), I(a,b)) \geq S(0, I(b,a)) = I(b,a) > 1-\theta. \qquad \blacksquare$$

The following result, given by Ovchinnikov and Roubens [22], characterizes continuous Archimedean t-norms with zero divisors by means of ϕ–transforms of Lukasiewicz's t-norm.

Theorem 3. *A t*-norm *T is a continuous Archimedean t*-norm with zero divisors if and only if there exists an order automorphism ϕ *such that* $T = W_\phi$.

Theorem 3 allows us to obtain some conditions that an axiomatic factorization holds for continuous Archimedean t-norms with zero divisors.

Corollary 1. *Let R be a fuzzy weak preference relation and (P, I) an axiomatic factorization of R. If T is a continuous Archimedean t*-norm with zero divisors, then there exists an order automorphism ϕ *such that for all $a, b \in A$ the following statements hold:*

1. $\phi(1-R(a,b)) + \phi(1-R(b,a)) \leq 1$.
2. $\phi(P(a,b)) + \phi(P(b,a)) \leq 1$.
3. $\phi(P(a,b)) + \phi(I(a,b)) \leq 1$.
4. $\phi(1-R(a,b)) = \max(\phi(1-P(a,b)) + \phi(1-I(a,b)) - 1, 0)$.

Proof. It is sufficient to take into account that $T = W_\phi$ and that the conditions $R \cup R^{-1} = A \times A$, $P \cap P^{-1} = \emptyset$, $P \cap I = \emptyset$ and $R = P \cup I$ are satisfied. ■

Now we focus our analysis on reciprocal order automorphisms. This fact allows us to determine P depending on R and I.

Theorem 4. *Let R be a fuzzy weak preference relation and (P, I) an axiomatic factorization of R. If ϕ is a reciprocal order automorphism and $T = W_\phi$, then for all $a, b \in A$,*

$$P(a,b) = \phi^{-1}(\phi(R(a,b)) - \phi(I(a,b))),$$

$$\phi^{-1}\left(\frac{\phi(R(a,b)) + \phi(R(b,a)) - 1}{2}\right) \leq I(a,b) \leq \min(R(a,b), R(b,a)).$$

Proof. Given $a, b \in A$, by 3 and 4 of Corollary 1 and the reciprocity of ϕ we have

$$\begin{aligned} 1 - \phi(R(a,b)) &= \max(1 - \phi(P(a,b)) - \phi(I(a,b)), 0) \\ &= 1 - \phi(P(a,b)) - \phi(I(a,b)), \end{aligned}$$

hence, $P(a,b) = \phi^{-1}(\phi(R(a,b)) - \phi(I(a,b)))$.

Since $\phi(R(a,b)) = \phi(P(a,b)) + \phi(I(a,b))$, then $\phi(I(a,b)) \leq \phi(R(a,b))$, or equivalently

$$I(a,b) \leq R(a,b).$$

Analogously, from $\phi(R(b,a)) = \phi(P(b,a)) + \phi(I(b,a))$ we have

$$I(a,b) = I(b,a) \leq R(b,a).$$

Therefore, $I(a,b) \leq \min(R(a,b), R(b,a))$.

On the other hand, since $\phi(P(a,b)) + \phi(P(b,a)) \leq 1$ we have

$$\phi(R(a,b)) + \phi(R(b,a)) - 2\phi(I(a,b)) \leq 1,$$

or equivalently,

$$\phi(I(a,b)) \geq \frac{\phi(R(a,b)) + \phi(R(b,a)) - 1}{2}.$$ ■

Since the factorization obtained in Theorem 4 is not unique, we can impose additional conditions. Thus, in the following theorems we also consider the conditions $P = R \cap (R^{-1})^c$, $P = (R^{-1})^c$ and $I = R \cap R^{-1}$, which are satisfied in the ordinary framework.

Theorem 5. *Let R be a fuzzy weak preference relation and (P, I) an axiomatic factorization of R. If ϕ is a reciprocal order automorphism,* $T = W_\phi$ and $P = R \cap (R^{-1})^c$, *then for all $a, b \in A$,*

$$P(a,b) = \max(\phi^{-1}(\phi(R(a,b)) - \phi(R(b,a))), 0),$$
$$I(a,b) = \min(R(a,b), R(b,a)).$$

Proof. If $P = R \cap (R^{-1})^c$, then for all $a, b \in A$ the following holds

$$\begin{aligned} P(a,b) &= T(R(a,b), 1 - R(b,a)) \\ &= \phi^{-1}(\max(\phi(R(a,b)) + \phi(1 - R(b,a)) - 1, 0)) \\ &= \phi^{-1}(\max(\phi(R(a,b)) - \phi(R(b,a)), 0)) \\ &= \max(\phi^{-1}(\phi(R(a,b)) - \phi(R(b,a))), 0), \end{aligned}$$

and by Theorem 4 we have

$$\begin{aligned} \phi(I(a,b)) &= \phi(R(a,b)) - \phi(P(a,b)) \\ &= \phi(R(a,b)) - \max(\phi(R(a,b)) - \phi(R(b,a)), 0) \\ &= \phi(R(a,b)) + \min(\phi(R(b,a)) - \phi(R(a,b)), 0) \\ &= \min(\phi(R(b,a)), \phi(R(a,b))). \end{aligned}$$ ■

Remark 4. If ϕ is the identity automorphism in Theorem 5, then we obtain the factorization given by Orlovsky [19]. So, this factorization can be characterized by $P = R \cap (R^{-1})^c$ and by (P, I) being an axiomatic factorization of R for Lukasiewicz's t-norm.

When ϕ is a reciprocal order automorphism, $T = W_\phi$ and (P, I) is an axiomatic factorization of R, then the conditions $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$ are incompatible, as we will see in the next section. However, under the same suppositions, the conditions $P = (R^{-1})^c$ and $I = R \cap R^{-1}$ are equivalent.

Theorem 6. *Let R be a fuzzy weak preference relation and (P, I) an axiomatic factorization of R. If ϕ is a reciprocal order automorphism and $T = W_\phi$, then $I = R \cap R^{-1}$ is equivalent to $P = (R^{-1})^c$ and, when one of these relationships is satisfied, we have*

$$P(a,b) = 1 - R(b,a),$$
$$I(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1),$$

for all $a, b \in A$.

Proof. If $I = R \cap R^{-1}$, then for all $a, b \in A$ the following holds

$$\begin{aligned} I(a,b) &= T(R(a,b), R(b,a)) \\ &= \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)) \\ &= \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1), \end{aligned}$$

and by Theorem 4 we have

$$\begin{aligned}\phi(P(a,b)) &= \phi(R(a,b)) - \phi(I(a,b)) = 1 - \phi(R(b,a))\\ &= \phi(1 - R(b,a)).\end{aligned}$$

Therefore, $P = (R^{-1})^c$.

Reciprocally, if $P = (R^{-1})^c$, then for all $a, b \in A$ the following holds

$$P(a,b) = 1 - R(b,a),$$

and again by Theorem 4 we have

$$\begin{aligned}\phi(I(a,b)) &= \phi(R(a,b)) - \phi(P(a,b))\\ &= \phi(R(a,b)) + \phi(R(b,a)) - 1.\end{aligned}$$

Therefore, $I = R \cap R^{-1}$. ■

Remark 5. The factorization given in Theorem 6 can also be obtained when we replace $I = R \cap R^{-1}$ or $P = (R^{-1})^c$ by the following conditions, which are also satisfied in the ordinary framework:

1. $I = (P \cup P^{-1})^c$.
2. $R \cap P^{-1} = \emptyset$ and $R \cup P^{-1} = A \times A$.
3. $\phi(P(a,b)) + \phi(P(b,a)) + \phi(I(a,b)) = 1$ for all $a, b \in A$.

Remark 6. If ϕ is the identity automorphism in Theorem 6, then we obtain the factorization given by Barrett and Pattanaik [4]. So, this factorization can be characterized by $I = R \cap R^{-1}$ and by (P, I) being an axiomatic factorization of R for Lukasiewicz's t-norm.

Remark 7. It is possible to make a geometric interpretation of the factorizations obtained in Theorems 5 and 6. Given a and b, two different elements of A, and R a fuzzy weak preference relation, by Theorem 4 we have

$$\phi^{-1}\left(\frac{\phi(R(a,b)) + \phi(R(b,a)) - 1}{2}\right) \leq I(a,b) \leq \min(R(a,b), R(b,a)).$$

Since $P(a,b) = \phi^{-1}(\phi(R(a,b)) - \phi(I(a,b)))$, when $I(a,b)$ falls in the range between the previous numbers we obtain the possible values for $P(a,b)$ and $P(b,a)$. So, the points $(P(a,b), P(b,a))$ are in a curve with endpoints $X_1 = (x_{11}, x_{12})$ and $X_2 = (x_{21}, x_{22})$, where

$$\begin{aligned}x_{11} &= \phi^{-1}\left(\frac{\phi(R(a,b)) - \phi(R(b,a)) + 1}{2}\right),\\ x_{12} &= \phi^{-1}\left(\frac{\phi(R(b,a)) - \phi(R(a,b)) + 1}{2}\right),\\ x_{21} &= \max(\phi^{-1}(\phi(R(a,b)) - \phi(R(b,a))), 0),\\ x_{22} &= \max(\phi^{-1}(\phi(R(b,a)) - \phi(R(a,b))), 0).\end{aligned}$$

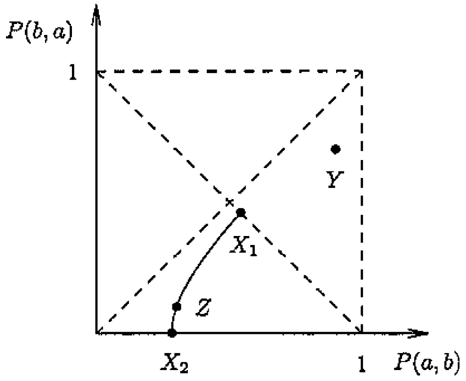

Fig. 1.

It is easy to check that the point X_1 is in the straight line $P(a,b) + P(b,a) = 1$. Moreover, the point X_2 corresponds to the factorization given in Theorem 5. If we represent $Y = (R(a,b), R(b,a))$ in the same plane as $P(a,b)$ and $P(b,a)$, the point of the previous curve that is symmetric to Y with respect to the straight line $P(a,b) + P(b,a) = 1$ is $Z = (1 - R(b,a), 1 - R(a,b))$, which corresponds to the factorization given in Theorem 6 (see Fig. 1).

4 Results based on definitions of *P* and *I*

In the ordinary framework there exists a unique factorization of a weak preference relation R into a strict preference relation P and an indifference relation I. This factorization is given by $P = R \cap (R^{-1})^c$, which is equivalent to $P = (R^{-1})^c$, and $I = R \cap R^{-1}$. Since in fuzzy logic these definitions of P are not equivalent, in this section we consider both definitions together with $I = R \cap R^{-1}$ and we analyze when (P, I) is an axiomatic factorization of R.

According to the t-norm and the definition of P that we have chosen, we obtain different factorizations of R. For instance, we can point out that when we use the minimum as the t-norm, we obtain the factorization given by Dutta [12] and Roubens and Vincke [25], whether $P = R \cap (R^{-1})^c$ or whether $P = (R^{-1})^c$.

In this section we prove that if $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$, then no t-norms exist so that (P, I) is an axiomatic factorization of R. However, if we consider $P = (R^{-1})^c$ and $I = R \cap R^{-1}$, then we establish that (P, I) is an axiomatic factorization of R only for ϕ–transforms of Lukasiewicz's t-norm, where the order automorphism ϕ is reciprocal. In this case, we obtain a generalization of the factorization given by Barrett and Pattanaik [4].

Again, we suppose that the intersection and the union of fuzzy sets are defined by means of a t-norm T and its dual t-conorm S, respectively. We begin showing some properties of $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$. These properties are necessary in order that (P, I) be an axiomatic factorization of R.

Proposition 2. *Let R be a fuzzy weak preference relation, $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$. Then the following statements hold:*

1. P is irreflexive.
2. P is asymmetric.
3. I is reflexive.
4. I is symmetric.

Proof. 1. $P(a,a) = T(R(a,a), 1 - R(a,a)) = T(1,0) = 0$ for all $a \in A$.

2. Given $a, b \in A$, the following is satisfied:

$$T(1 - R(a,b), 1 - R(b,a)) = 1 - S(R(a,b), R(b,a)) = 0.$$

Then, by the commutativity and the associativity of T we have

$$\begin{aligned} T(P(a,b), P(b,a)) &= T(T(R(a,b), 1 - R(b,a)), T(R(b,a), 1 - R(a,b))) \\ &= T(T(R(a,b), R(b,a)), T(1 - R(b,a), 1 - R(a,b))) \\ &= T(T(R(a,b), R(b,a)), 0) = 0. \end{aligned}$$

3. $I(a,a) = T(R(a,a), R(a,a)) = T(1,1) = 1$ for all $a \in A$.
4. Given $a, b \in A$, we have

$$I(a,b) = T(R(a,b), R(b,a)) = T(R(b,a), R(a,b)) = I(b,a).$$ ■

Now we study the requirements $P \cap I = \emptyset$ and $R = P \cup I$. In the following remark we give a condition that the t-norm T has to fulfill so that $P \cap I = \emptyset$ is satisfied for any fuzzy weak preference relation R.

Remark 8. Given a fuzzy weak preference relation R, if $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$, then $P \cap I = \emptyset$ holds if and only if

$$T(P(a,b), I(a,b)) = T(T(R(a,b), 1 - R(b,a)), T(R(a,b), R(b,a))) = 0,$$

for all $a, b \in A$. If $x = R(a,b)$ and $y = R(b,a)$, then the previous condition is satisfied for any fuzzy weak preference relation R if and only if

$$T(T(x, 1 - y), T(x,y)) = 0,$$

for all $x, y \in [0,1]$ such that $S(x,y) = 1$.

The t-norms that satisfy the last condition are also characterized through the following condition.

Proposition 3. *Let T be a t-*norm. Then the following statements are equivalent:

1. $T(T(x, 1 - y), T(x,y)) = 0$ for all $x, y \in [0,1]$ such that $S(x,y) = 1$.
2. $T(x, 1 - x) = 0$ for all $x \in [0,1]$.

Proof. $1 \Rightarrow 2$: If $x = 1$, then for all $y \in [0,1]$ we have $S(1,y) = 1$ and

$$T(y, 1-y) = T(1-y, y) = T(T(1, 1-y), T(1, y)) = 0.$$

$2 \Rightarrow 1$: Given $x, y \in [0,1]$, by the commutativity and the associativity of T we have

$$\begin{aligned} T(T(x, 1-y), T(x,y)) &= T(T(x,x), T(1-y, y)) \\ &= T(T(x,x), 0) = 0. \end{aligned}$$ ■

When we consider continuous t-norms, the following result, given by Fodor and Roubens [14, p. 10], allows us to characterize the fuzzy binary relations P and I that satisfy the requirement $P \cap I = \emptyset$.

Proposition 4. *Let T be a continuous* t-norm. Then the following conditions are equivalent:

1. $T(x, 1-x) = 0$ *for all* $x \in [0,1]$.
2. *There exists an order automorphism* ϕ *such that* $T = W_\phi$ and $\phi(1-x) \leq 1 - \phi(x)$ *for all* $x \in [0,1]$.

Corollary 2. *Let R be a fuzzy weak preference relation, T a continuous* t-norm, $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$. *Then the following conditions are equivalent:*

1. $P \cap I = \emptyset$.
2. *There exists an order automorphism* ϕ satisfying $\phi(1-x) \leq 1 - \phi(x)$ *for all* $x \in [0,1]$ *such that* $T = W_\phi$ *and, consequently, for all* $a, b \in A$,
$P(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi(1 - R(b,a)) - 1, 0))$,
$I(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0))$.

Proof. It is sufficient to take into account Remark 8 and Propositions 3 and 4. ■

Next we focus our attention on $R = P \cup I$. In the following remark we give a condition that the t-norm T and its dual t-conorm S have to fulfill in order for this requirement to be satisfied for any fuzzy weak preference relation R.

Remark 9. Given a fuzzy weak preference relation R, if $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$, then $R = P \cup I$ holds if and only if

$$\begin{aligned} R(a,b) &= S(P(a,b), I(a,b)) \\ &= S(T(R(a,b), 1 - R(b,a)), T(R(a,b), R(b,a))), \end{aligned}$$

for all $a, b \in A$. If $x = R(a,b)$ and $y = R(b,a)$, then the previous condition is satisfied for any fuzzy weak preference relation R if and only if

$$S(T(x, 1-y), T(x,y)) = x,$$

for all $x, y \in [0,1]$ such that $S(x,y) = 1$.

In this context we can point out the result given by Alsina [1] (see also Fodor and Roubens [14, p. 73]).

Theorem 7. *There exists no* t-norm T *with dual t-conorm* S *such that*

$$S(T(x, 1-y), T(x,y)) = x,$$

for all $x, y \in [0,1]$.

The same result is obtained when we consider only the elements of $[0,1]$ which fulfill $S(x,y)=1$. In order to prove this, we give a proof similar to that provided by Fodor and Roubens [14, p. 73]. Based on this result, we obtain that $R = P \cup I$ is not satisfied for any t-norm T.

Theorem 8. *There exists no* t-norm T *with dual t-conorm* S *such that*

$$S(T(x, 1-y), T(x,y)) = x,$$

for all $x, y \in [0,1]$ *such that* $S(x,y) = 1$.

Proof. The theorem is proven by contradiction. Suppose there exist T and S that satisfy the previous condition. If we consider $x = 1$, then for all $y \in [0,1]$ we have $S(1,y) = 1$ and

$$T(y, 1-y) = 1 - S(1-y, y) = 1 - S(T(1, 1-y), T(1,y)) = 1 - 1 = 0.$$

On the other hand, if $x = y = 0.5$, then $S(0.5, 0.5) = 1 - T(0.5, 0.5) = 1$ and we have

$$0.5 = S(T(0.5, 0.5), T(0.5, 0.5)) = S(0,0) = 0,$$

a contradiction. ■

Corollary 3. *There is no* t-norm T *with dual t-conorm* S *such that for any fuzzy weak preference relation* R *the condition* $R = P \cup I$ *holds if* $P = R \cap (R^{-1})^c$ *and* $I = R \cap R^{-1}$.

Proof. It is an inmediate consequence of Remark 9 and Theorem 8. ■

So, given a fuzzy weak preference relation R and any t-norm T, if $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$, then (P, I) cannot be an axiomatic factorization of R. Since $P = (R^{-1})^c$ is also satisfied in the ordinary framework, next we consider this definition. In this case, the value $P(a,b) = 1 - R(b,a)$ does not depend on the choice of the t-norm. Moreover, given $B \subseteq A$, we would like to emphasize that the *greatest set*[2] in B defined by Basu [5] and the set of *nondominated elements*[3] defined by Orlovsky [19] are equal using this

[2] $G(B,R)(a) = \min_{b \in B} R(a,b)$.
[3] $M(B,R)(a) = 1 - \max_{b \in B} P(b,a)$.

definition of P. These sets, as Basu [5] points out, are a formalization of the idea of a set of best elements.

Firstly we prove that $P = (R^{-1})^c$ is irreflexive and asymmetric.

Proposition 5. *Let R be a fuzzy weak preference relation. If $P = (R^{-1})^c$, then the following statements hold:*

1. *P is irreflexive.*
2. *P is asymmetric.*

Proof. 1. $P(a,a) = 1 - R(a,a) = 0$ for all $a \in A$.
2. Given $a, b \in A$, we have

$$T(P(a,b), P(b,a)) = T(1 - R(b,a), 1 - R(a,b)) = 1 - S(R(b,a), R(a,b)) = 0.$$

■

Again, with the new definition of P, we analyze the requirements $P \cap I = \emptyset$ and $R = P \cup I$. Analogously to Remark 8, we show a condition that the t-norm T has to fulfill so that $P \cap I = \emptyset$ is satisfied for any fuzzy weak preference relation R.

Remark 10. Given a fuzzy weak preference relation R, if $P = (R^{-1})^c$ and $I = R \cap R^{-1}$, then $P \cap I = \emptyset$ holds if and only if

$$T(P(a,b), I(a,b)) = T(1 - R(b,a), T(R(a,b), R(b,a))) = 0,$$

for all $a, b \in A$. If $x = R(a,b)$ and $y = R(b,a)$, then the previous condition is satisfied for any fuzzy weak preference relation R if and only if

$$T(1 - y, T(x,y)) = 0,$$

for all $x, y \in [0,1]$ such that $S(x,y) = 1$.

Next we give the same requisite as in Proposition 3 in order to characterize the above condition. Therefore, since the t-norms are equal, the family of relations $I = R \cap R^{-1}$ obtained under the assumption $P \cap I = \emptyset$ coincides for $P = (R^{-1})^c$ and $P = R \cap (R^{-1})^c$. Moreover, when the t-norm is continuous, we know it explicitly.

Proposition 6. *Let T be a t-*norm. Then the following conditions are equivalent:

1. *$T(1 - y, T(x,y)) = 0$ for all $x, y \in [0,1]$ such that $S(x,y) = 1$.*
2. *$T(x, 1 - x) = 0$ for all $x \in [0,1]$.*

Proof. $1 \Rightarrow 2$: If $x = 1$, then for all $y \in [0,1]$ we have $S(1,y) = 1$ and

$$T(y, 1 - y) = T(1 - y, y) = T(1 - y, T(1,y)) = 0.$$

$2 \Rightarrow 1$: Given $x, y \in [0,1]$, by the commutativity and the associativity of T we have

$$T(1-y, T(x,y)) = T(T(1-y,y),x) = T(0,x) = 0. \qquad \blacksquare$$

Corollary 4. *Let R be a fuzzy weak preference relation, T a continuous t-norm, $P = (R^{-1})^c$ and $I = R \cap R^{-1}$. Then the following conditions are equivalent:*

1. *$P \cap I = \emptyset$.*
2. *There exists an order automorphism ϕ satisfying $\phi(1-x) \leq 1 - \phi(x)$ for all $x \in [0,1]$ such that $T = W_\phi$ and, consequently, for all $a, b \in A$,*
   $P(a,b) = 1 - R(b,a)$,
   $I(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)).$

Proof. It is sufficient to take into account Remark 10 and Propositions 4 and 6. $\blacksquare$

Now we study $R = P \cup I$. Firstly we give a condition that the t-norm T has to fulfill in order for this requirement to be satisfied for any fuzzy weak preference relation R.

Remark 11. Given a fuzzy weak preference relation R, if $P = (R^{-1})^c$ and $I = R \cap R^{-1}$ then $R = P \cup I$ holds if and only if

$$\begin{aligned} R(a,b) &= S(P(a,b), I(a,b)) = 1 - T(1 - P(a,b), 1 - I(a,b)) \\ &= 1 - T(R(b,a), 1 - T(R(a,b), R(b,a))), \end{aligned}$$

for all $a, b \in A$. If $x = R(a,b)$ and $y = R(b,a)$, then the previous condition is satisfied for any fuzzy weak preference relation R if and only if

$$T(y, 1 - T(x,y)) = 1 - x,$$

for all $x, y \in [0,1]$ such that $S(x,y) = 1$.

In the following theorem we characterize the continuous t-norms that satisfy the preceding condition by means of ϕ–transforms of Lukasiewicz's t-norm, where ϕ is reciprocal.

Theorem 9. *Let T be a continuous* t-norm. Then the following conditions are equivalent:

1. *$T(y, 1 - T(x,y)) = 1 - x$ for all $x, y \in [0,1]$ such that $S(x,y) = 1$.*
2. *There exists a reciprocal order automorphism ϕ such that $T = W_\phi$.*

Proof. $1 \Rightarrow 2$: If $x = 1$ then for all $y \in [0,1]$ we have $S(1,y) = 1$ and

$$T(y, 1-y) = T(y, 1 - T(1,y)) = 0.$$

By Proposition 4 there exists an order automorphism ϕ such that $T = W_\phi$ and $\phi(1-x) \leq 1 - \phi(x)$ for all $x \in [0,1]$. In order to obtain the remaining inequality for all $x \in (0,1)$ (if $x = 0$ or $x = 1$ it is obvious that $\phi(1-x) \geq 1 - \phi(x)$), firstly we show that $\phi(x) + \phi(y) > 1$ for all $x, y \in (0,1]$ such that $y > 1 - x$. This is proven by contradiction. Suppose it were otherwise. Then

$$T(x,y) = \phi^{-1}(\max(\phi(x) + \phi(y) - 1, 0)) = 0.$$

Since $S(x,y) \geq S(x, 1-x) = 1 - T(1-x, x) = 1$ we have

$$T(y, 1 - T(x,y)) = T(y,1) = y > 1 - x,$$

which contradicts the hypothesis. Finally, given $x \in (0,1)$, let $\{y_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $(0,1]$ converging to $1-x$. Since $y_n > 1-x$, then $\phi(x) + \phi(y_n) > 1$ for all $n \in \mathbb{N}$. By the continuity of ϕ we have

$$\phi(x) + \phi(1-x) = \lim_{n\to\infty} (\phi(x) + \phi(y_n)) \geq 1.$$

$2 \Rightarrow 1$: If $S(x,y) = 1$ then $T(1-x, 1-y) = 1 - S(x,y) = 0$. On the other hand,

$$\begin{aligned} T(1-x, 1-y) &= \phi^{-1}(\max(\phi(1-x) + \phi(1-y) - 1, 0)) \\ &= \phi^{-1}(\max(1 - \phi(x) - \phi(y), 0)). \end{aligned}$$

So, $\phi(x) + \phi(y) \geq 1$ and, consequently,

$$T(x,y) = \phi^{-1}(\max(\phi(x) + \phi(y) - 1, 0)) = \phi^{-1}(\phi(x) + \phi(y) - 1).$$

By the reciprocity of ϕ^{-1} we have $1 - T(x,y) = \phi^{-1}(2 - \phi(x) - \phi(y))$. Therefore, for all $x, y \in [0,1]$ such that $S(x,y) = 1$ the following holds

$$\begin{aligned} T(y, 1 - T(x,y)) &= \phi^{-1}(\max(\phi(y) + \phi(1 - T(x,y)) - 1, 0)) \\ &= \phi^{-1}(\max(1 - \phi(x), 0)) = \phi^{-1}(1 - \phi(x)) \\ &= 1 - x. \end{aligned}$$

■

In the following corollary we show the fuzzy binary relations $P = (R^{-1})^c$ and $I = R \cap R^{-1}$ for which $R = P \cup I$ holds.

Corollary 5. *Let R be a fuzzy weak preference relation, T a continuous t-norm, $P = (R^{-1})^c$ and $I = R \cap R^{-1}$. Then the following conditions are equivalent:*

1. *$R = P \cup I$.*
2. *There exists a reciprocal order automorphism ϕ such that $T = W_\phi$ and, consequently, for all $a, b \in A$,*

$$\begin{aligned} &P(a,b) = 1 - R(b,a), \\ &I(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1). \end{aligned}$$

Proof. 1 ⇒ 2: Given $a, b \in A$, by Remark 11 and Theorem 9 the following holds

$$P(a,b) = 1 - R(b,a),$$
$$I(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)).$$

Since ϕ is reciprocal and R is complete we have $\phi(R(a,b)) + \phi(R(b,a)) \geq 1$ for all $a, b \in A$; and consequently

$$I(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1).$$

2 ⇒ 1: Obvious. ■

Finally, when we require that $P = (R^{-1})^c$, $I = R \cap R^{-1}$ and that (P, I) should be an axiomatic factorization of R, we again obtain a generalization of the factorization given by Barrett and Pattanaik [4].

Corollary 6. *Let R be a fuzzy weak preference relation, T a continuous t-norm, $P = (R^{-1})^c$ and $I = R \cap R^{-1}$. Then the following conditions are equivalent:*

1. *(P, I) is an axiomatic factorization of R.*
2. *There exists a reciprocal order automorphism ϕ such that $T = W_\phi$ and, consequently, for all $a, b \in A$,*

$$P(a,b) = 1 - R(b,a),$$
$$I(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1).$$

Proof. It is an immediate consequence of Propositions 2 and 5 and Corollaries 4 and 5. ■

5 Conclusion

In this paper we study the factorization of fuzzy weak preference relations by means of two courses of action: axioms and definitions of strict preference and indifference. In both ways we conclude that the generalizations of the factorization given by Barrett and Pattanaik [4], i.e.,

$$P(a,b) = 1 - R(b,a),$$
$$I(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1),$$

where ϕ is a reciprocal order automorphism, seem the best because they satisfy the following properties, which are also verified in the ordinary framework:

1. P is irreflexive and asymmetric.
2. I is reflexive and symmetric.
3. $P \cap I = \emptyset$.
4. $R = P \cup I$.

5. $P = (R^{-1})^c$.
6. $I = R \cap R^{-1}$.
7. $I = (P \cup P^{-1})^c$.
8. $R \cap P^{-1} = \emptyset$.
9. $R \cup P^{-1} = A \times A$.
10. $\phi(P(a,b)) + \phi(P(b,a)) + \phi(I(a,b)) = 1$ for all $a, b \in A$.

Changing the role played by P and P^{-1}, or through De Morgan laws, we can obtain other properties. Moreover, when ϕ is the identity automorphism, the last condition given above has been interpreted by Barrett and Pattanaik [4]. Thus, their factorization allows us to represent the preferences when individuals utilize several criteria with different weights to compare the alternatives.

On the other hand, Richardson [24, p. 363] points out that if we consider $P(a,b) = 1 - R(b,a)$, then $P(a,b)$ is insensitive to $R(a,b)$ (a similar argumentation is given by Dasgupta and Deb [10, pp. 492–493]). For instance, $P(a,b)$ takes the same value in the two situations $\{R(a,b) = 1, R(b,a) = 0.999\}$ and $\{R(a,b) = 0.001, R(b,a) = 0.999\}$. However, this also happens in the ordinary framework; $P(a,b)$ takes the same value when $\{R(a,b) = 1, R(b,a) = 1\}$ and $\{R(a,b) = 0, R(b,a) = 1\}$.

Finally, we also have proven that the ϕ–transforms of Lukasiewicz's t-norm are the most interesting t-norms for representing the intersection of fuzzy sets. Since in fuzzy logic transitivity is defined as $R(a,c) \geq T(R(a,b), R(b,c))$, where T is a t-norm (see Fodor and Roubens [14, p. 53]), it seems that the best-suited definition of transitivity is $R(a,c) \geq \phi^{-1}(\phi(R(a,b)) + \phi(R(b,c)) - 1)$. When ϕ is the identity automorphism we obtain the usual condition $R(a,c) \geq R(a,b) + R(b,c) - 1$. The relationships between the transitivity of R and the transitivity of P and I have been widely studied by Dasgupta and Deb [10].

References

[1] Alsina C (1985) On a family of connectives for fuzzy sets. Fuzzy Sets Syst 16: 231–235
[2] Banerjee A (1994) Fuzzy preferences and Arrow-type problems in social choice. Soc Choice Welfare 11: 121–130
[3] Barrett CR, Pattanaik PK, Salles M (1986) On the structure of fuzzy social welfare functions. Fuzzy Sets Syst 19: 1–10
[4] Barrett CR, Pattanaik PK (1989) Fuzzy sets, preference and choice: some conceptual issues. Bull Econ Res 41: 229–253
[5] Basu K (1984) Fuzzy revealed preference theory. J Econ Theory 32: 212–227
[6] Basu K, Deb R, Pattanaik PK (1992) Soft sets: An ordinal formulation of vagueness with some applications to the theory of choice. Fuzzy Sets Syst 45: 45–58
[7] Bezdek JC, Spillman B, Spillman R (1978) A fuzzy relation space for group decision theory. Fuzzy Sets Syst 1: 255–268
[8] Blin JM (1974) Fuzzy relation in group decision theory. J Cyber 4: 17–22

[9] Dasgupta M, Deb R (1996) Transitivity and fuzzy preferences. Soc Choice Welfare 13: 305–318
[10] Dasgupta M, Deb R (2001) Factoring fuzzy transitivity. Fuzzy Sets Syst 118: 489–502
[11] De Baets B, Fodor J (1997) Twenty years of fuzzy preference structures (1978–1997). Riv Mat Sci Econ Soc 20: 45–66
[12] Dutta B (1987) Fuzzy preferences and social choice. Math Social Sci 13: 215–229
[13] Fodor J (1992) An axiomatic approach to fuzzy preference modelling. Fuzzy Sets Syst 52: 47–52
[14] Fodor J, Roubens M (1994) Fuzzy preference modelling and multicriteria decision support. Kluwer Academic Publishers, Dordrecht
[15] García-Lapresta JL, Llamazares B (2000) Aggregation of fuzzy preferences: Some rules of the mean. Soc Choice Welfare 17: 673–690
[16] Llamazares B (2003) Characterization of fuzzy preference structures through Lukasiewicz triplets. Fuzzy Sets Syst 136: 217–235
[17] Nakamura K (1992) On the nature of intransitivity in human preferential judgements. In: Novák V, Ramík J et al. (eds) Fuzzy approach to reasoning and decision-making. Kluwer Academic Publishers, Dordrecht, pp 147–162
[18] Nurmi H (1981) Approaches to collective decision making with fuzzy preference relations. Fuzzy Sets Syst 6: 249–259
[19] Orlovsky S (1978) Decision-making with a fuzzy preference relation. Fuzzy Sets Syst 1: 155–167
[20] Ovchinnikov S (1981) Structure of fuzzy binary relations. Fuzzy Sets Syst 6: 169–195
[21] Ovchinnikov S (1991) Social choice and Lukasiewicz logic. Fuzzy Sets Syst 43: 275–289
[22] Ovchinnikov S, Roubens M (1991) On strict preference relations. Fuzzy Sets Syst 43: 319–326
[23] Ovchinnikov S, Roubens M (1992) On fuzzy strict preference, indifference, and incomparability relations. Fuzzy Sets Syst 49: 15–20
[24] Richardson G (1998) The structure of fuzzy preferences: Social choice implications. Soc Choice Welfare 15: 359–369
[25] Roubens M, Vincke P (1987) Fuzzy preferences in an optimization perspective. In: Kacprzyk J, Orlovski A (eds) Optimization models using fuzzy sets and possibility theory. D. Reidel Publishing Company, Dordrecht, pp 77–90
[26] Schweizer B, Sklar A (1983) Probabilistic metric spaces. Elsevier Science, New York
[27] Tanino T (1984) Fuzzy preference orderings in group decision making. Fuzzy Sets Syst 12: 117–131
[28] Van de Walle B, De Baets B, Kerre E (1998) A plea for the use of Lukasiewicz triplets in the definition of fuzzy preference structures. (I). General argumentation. Fuzzy Sets Syst 97: 349–359
[29] Van de Walle B, De Baets B, Kerre E (1998) Characterizable fuzzy preference structures. In: Bouyssou D, Vincke P (eds) Annals of operations research 80, special issue preference modeling. Baltzer Science Publishers, Bussum, pp 105–136