Soc Choice Welfare (2004) 22: 253–266
DOI: 10.1007/s00355-003-0285-z

Social Choice and Welfare

Freedom of choice and diversity of options: Some difficulties

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Abstract. This paper explores the possibility for a (non-preference-based) freedom ranking of opportunity sets that is sensitive to the diversity of the options. It turns out that how distances between sets and alternatives are measured is crucial to the derivation of such a ranking. Several proposals are examined, each of which is shown to lead to impossibility results.

1 Introduction

Although the concept of freedom has been an important subject of study for a very long time, substantially new approaches to the topic are still being developed. Particularly interesting in this respect is the recent work on the part of welfare economists and social choice theorists on ranking opportunity sets.[1] An opportunity set is a set of alternatives, usually interpreted as bundles of goods, from which an individual can choose an element. The question is how to compare different opportunity sets in terms of the amount of freedom of choice they provide.

Earlier versions of this paper were presented at conferences in Kortrijk, Osnabrück, Oisterwijk, Caen and Pavia. I thank the participants at these conferences for their comments. I would also like to thank Steven Hartkamp, Theo Kuipers, Clemens Puppe and especially Kotaro Suzumura for their helpful comments. Furthermore, I am very grateful to Somdeb Lahiri for pointing out a mistake in a previous version of this paper.

[1] These new developments can be seen as evolving from a more general interest in the analysis of rights and freedom in a rational choice setting. This interest was, of course, sparked off by Amartya Sen's celebrated work on the liberal paradox (Sen 1970).

A seminal contribution in this respect was Pattanaik and Xu's axiomatic characterisation of a particular opportunity set ranking (Pattanaik and Xu 1990). Their first axiom, *Indifference Between No-choice Situations*, states that two singleton sets always yield an equal amount of freedom. The underlying idea is that such sets do not offer any freedom of choice at all: if I am offered only one option, then I have no real choice. The second axiom is a weak axiom of monotonicity. This axiom, *Strict Monotonicity*, states that a set consisting of two elements always offers strictly more freedom than any of the elements on its own. If the set of alternatives expands, my range of action expands and my freedom of choice therefore also increases. Finally, their third axiom, *Independence*, states that the ranking between two sets A and B does not change if an element x that does not belong to A or B is added to both sets: the set A gives at least as much freedom as B if and only if the union of A and $\{x\}$ gives at least as much freedom as the union of B and $\{x\}$. It was subsequently shown that the three axioms together yield a unique and rather simple way of ranking opportunity sets: the cardinality rule. According to this rule, the freedom of choice offered by an opportunity set depends completely on its number of elements – the larger an opportunity set is, the more freedom of choice it provides.

Pattanaik and Xu themselves conceived of this result as a sort of impossibility theorem. In their view, the rule that is characterised by the three axioms is so naive that some of the axioms must be rejected or weakened. They suggested that the independence axiom has to be dropped since it does not take account of differences between alternatives. If we have two sets A and B that represent an equal amount of freedom and if we have an alternative x that differs greatly from all the elements of A, while being almost identical to one of the elements of B, then we would be inclined to say that the opportunity set that results from adding x to A yields more freedom of choice than the one resulting from adding x to B. Pattanaik and Xu give the following example. Suppose the alternatives are modes of transportation. By the first axiom the singleton set {train} gives the same amount of freedom as {blue car} and the axiom of Independence thus implies that the sets {train, red car} and {blue car, red car} also represent the same degree of freedom. However, it seems obvious that the freedom of choice given by {blue car, red car} is smaller than the freedom given by {train, red car}.

The literature that developed in response to Pattanaik and Xu's paper falls roughly into two categories. It has been argued, firstly, that the counterintuitive implications outlined above are less the result of the axiom of Independence than of a more fundamental flaw in the framework as such, viz. the neglect of preferences (Sen 1990). In this view, one's freedom of choice depends at least partly on one's preferences. As a result several authors, including Pattanaik and Xu themselves, have proposed freedom rankings which do depend on individual preferences (Sen 1991; Arrow 1995; Puppe 1996; Pattanaik and Xu 1998; Sugden 1998). I have argued elsewhere that such preference-based approaches to freedom are important insofar as the *value* of freedom is concerned, but that in order to determine the *contents* of the concept as such, no references to individual preferences is needed (Van Hees and Wissenburg 1999; Van Hees 2000; see also Carter 1999). Since I am here primarily interested in the contents of the freedom concept, I shall not follow this first line of research but instead focus on a second line of research. This second line embraces the suggestion to abandon the axiom of Independence and to take account of the differences between the elements of opportunity sets. In this approach, alternatives are usually viewed as points located in a certain space. Marlies Ahlert (Klemish-Ahlert 1993), for example, has proposed to let the freedom of choice of a set of elements in $\mathbb{R}^n$ depend on the convex hull of that set: the larger the convex hull of an opportunity set is, the more freedom of choice the set provides. In a somewhat similar spirit, Rosenbaum (2000) takes the maximum distance that exists between a pair of elements belonging to a set as being indicative of the freedom of choice the set provides. These rankings, however, lead to counterintuitive results. Assume, for instance, that the alternatives represent opinions that an individual is allowed to express and which can be described as scores on a single dimension - say, a left-right scale. Now compare a situation in which we are only allowed to express the extremes of the scale to a situation in which we can express not only the extremes, but also many of the intermediate positions. It is odd to say that the two opportunity sets yield an equal amount of freedom of speech, yet this is precisely what Ahlert claims and what also follows from Rosenbaum's ranking.[2]

In a later paper, Pattanaik and Xu (2000) presented an extension of their cardinality-based ordering which does incorporate information about the similarity or dissimilarity of alternatives. For this purpose, they assume that a (reflexive and symmetric) relation over the set of all feasible alternatives, X, is given which can be interpreted as 'is similar to'. They then axiomatise an ordering based of the cardinality of the smallest similarity-based partition of opportunity sets. A disadvantage of this approach is that no account is taken of the different *degrees* to which alternatives can be said to be dissimilar to each other. Obviously, such differences matter: a blue train differs less from a red bus than from a glass of red wine. Pattanaik and Xu do not consider these

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[2] Ahlert and Rosenbaum themselves discuss this type of example also. Rosenbaum acknowledges that it forms a problem and that it might seem to necessitate a shift in the direction of a measurement of freedom that not only takes the maximum distance as a criterion but also the cardinality of a set (Rosenbaum 2000, p 223). For Ahlert, examples like these do not seem to pose a problem. As she puts it (Klemish-Ahlert 1990, p. 197): 'For the right of free speech the extreme opinions that are allowed to be expressed (...) are the main factors that determine this kind of freedom. The fact that certain normal opinions or mixed opinions between extreme ones are additionally admissible does not enlarge the freedom of choice granted by this right.' In fact, if we consider the situation in many Western countries, in which any position can be expressed except extreme right-wing ones, we would have to say on the basis of Ahlert's (as well as Rosenbaum's) ranking that an individual has *less* freedom of speech in these Western societies, than in societies in which an individual is only allowed to express the extreme positions.

differences and it is thus a consequence of their approach that, if all alternatives are dissimilar to each other, the new ordering coincides with the simple cardinality rule. The same intuitive problems arise: the opportunity set {blue train, red bus} then gives an equal amount of freedom of choice as {blue train, red wine}.[3]

The main object of this paper is to explore the possibility of extending Pattanaik and Xu's framework in order to take account of the different degrees to which alternatives can be said to be similar (or dissimilar) to each other. However, it should be noted that the view that the diversity of one's options affects one's freedom is not uncontested. Indeed, in his important study of freedom, Ian Carter (1999) argues against this view. However, I shall ignore this topic here, simply presupposing that diversity should indeed play an independent role in the derivation of a measurement of freedom.

I shall assume that a distance function (or metric) which describes the degree of dissimilarity between alternatives is given. However, to reformulate the independence axiom we not only need information about the degree of dissimilarity between alternatives but also between alternatives and *sets* of alternatives. It is not altogether clear how such distances should be determined; in the literature one can find several ways of extending a 'normal' distance function to a function that also measures distances between sets and alternatives. I shall analyse several of these extensions and show that impossibility results arise regardless of which extension one chooses. Although the results are thus mainly negative, they, like any impossibility result, can be interpreted positively by saying that they underpin the assumptions that underlie alternative approaches that do yield possibility results. In this respect, two recent important contributions to the theory of diversity deserves special mentioning: Bossert et al. (2001) and Nehring and Puppe (2003).

2 Notation and definitions

Let X be the non-empty (but not necessarily finite) set of feasible alternatives and let W denote the set of non-empty and finite subsets of X. The elements of W are called *opportunity sets*. A particular example is the case in which the alternatives can be described as vectors in Euclidean space and where W is the set of non-empty finite subsets of $\mathbb{R}^n$.

Since I shall need information about the similarity or dissimilarity between different alternatives, I assume that a *distance function* or *metric d* exists which

[3] Bavetta and Del Seta (2001) present two rankings of opportunity sets on the basis of a set's so-called rough set approximation. Although the freedom rankings take account of the differences between the alternatives, they ignore information about the extent to which alternatives differ, similar to the way the similarity-based ordering of Pattanaik and Xu does.

assigns a real number to any pair of alternatives, describing their degree of similarity. For all $x, y, z \in X$, a distance function satisfies

(i) $d(x,x) = 0$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,z) \leq d(x,y) + d(y,z)$;
(iv) if $x \neq y$, $d(x,y) > 0$.

For the case in which $X = \mathbb{R}^n$, it is assumed that Euclidean distance describes the distance between alternatives.

As indicated in the introduction, we not only need to know the distances between the elements of X, but also the distances between sets of alternatives on the one hand, and alternatives on the other. An *extended distance function* or *extended metric* Δ is a distance function which describes these distances as well. It is thereby assumed that for all $x, y \in X$,

(v) $\Delta(\{x\}, y) = d(x,y)$

To simplify notation I shall write $\Delta(x,y)$ rather than $\Delta(\{x\},y)$. Furthermore, I shall write 'distance function' whenever it is clear that I refer to an extended distance function Δ. An extended distance function in Euclidean space is called a *Euclidean* distance function if it is an extension of Euclidean distance.

Most of the results presented in this paper assume the existence of a particular type of sets, viz. *strings*. A string is an opportunity set A, the elements of which can be ordered in such a way that (a) the distance between alternatives adjacent to each other is always the same, and (b) the distance between any two alternatives belonging to the string is the sum of the distances between the adjacent alternatives that lie between them.

Definition 1. *Given a distance function* Δ, *an opportunity set* $A = \{x_1, \ldots, x_k\}$ *is called a string* (*with length* k) *in* X *if*
(a) $\Delta(x_i, x_{i+1}) = \Delta(x_j, x_{j+1}) \qquad (1 \leq i, j < k)$
(b) $\Delta(x_i, x_j) = \Delta(x_i, x_{i+1}) + \Delta(x_{i+1}, x_{i+2}) + \cdots + \Delta(x_{j-1}, x_j) \qquad (1 \leq i < j \leq k)$.

Examples of strings are easily found if the alternatives can be described numerically (or at least partly so). Take, for instance, an opportunity set consisting of possible vacations, some of which only differ in terms of their duration, say, one can go to Greece for one, two, three or four weeks.

Now let $\succeq$ be a binary relation over W that can be interpreted as 'offers at least as much freedom as' and let $\succ$ be its asymmetric part ('offers more freedom than') and $\sim$ its symmetric part ('offers the same degree of freedom as'). The relation is called a quasi-ordering if it is transitive and reflexive, and an ordering if it is complete as well. We shall discuss the following conditions that can be imposed on $\succeq$:

Axiom 1 (Indifference between no-choice situations). *For all* $x, y \in X$, $\{x\} \sim \{y\}$.
Axiom 2 (Weak Monotonicity). *For all distinct* $x, y \in X$, $\{x, y\} \succ \{y\}$.

Axiom 3 (Strong Monotonicity). *For all* $A \in W$ *and all* $x \in X - A$, $A \cup \{x\} \succ A$.
Axiom 4 (Restricted Independence). *For all* $A, B \in W$ *and all* $x \in X - (A \cup B)$,

If $A \sim B : A \cup \{x\} \succeq B \cup \{x\} \Leftrightarrow \Delta(A,x) \geq \Delta(B,x)$, *and*
If $A \succ B : \Delta(A,x) \geq \Delta(B,x) \Rightarrow A \cup \{x\} \succ B \cup \{x\}$.

Axiom 5 (Weak Restricted Composition). *For all* $A \in W$, *and all* $x, y \notin A$,
$A \cup \{x\} \succeq A \cup \{y\} \Leftrightarrow \Delta(A,x) \geq \Delta(A,y)$.
Axiom 6 (Restricted Composition). *For all* $A, B \in W$, *and all* $x \notin A$ *and* $y \notin B$,

If $A \sim B : A \cup \{x\} \succeq B \cup \{y\} \Leftrightarrow \Delta(A,x) \geq \Delta(B,y)$, *and*
If $A \succ B : \Delta(A,x) \geq \Delta(B,y) \Rightarrow A \cup \{x\} \succ B \cup \{y\}$.

Axioms 1 and 2 were used by Pattanaik and Xu in their characterisation of the cardinality ranking and have already been discussed in the introduction. Axiom 2 is here called Weak Monotonity rather than Strict Monotonicity, in order to distinguish it clearly from Axiom 3, the axiom of Strong Monotonicity. Strong Monotonicity implies that adding an extra element always leads to an increase in freedom.

Axiom 4, Restricted Independence, weakens Pattanaik and Xu's independence axiom by making it dependent on the degree of similarity or dissimilarity between the alternatives. First, it states that if two opportunity sets A and B are ranked equally, then the ranking of the sets resulting from adding an alternative x to A and B is completely determined by the distance between x on the one hand, and A and B on the other. Secondly, in case an opportunity set A gives strictly more freedom than B and x is at least as distant from A as it is from B, it demands that $A \cup \{x\}$ gives more freedom of choice than $B \cup \{x\}$.

The same idea that motivates Axiom 4 motivates Axiom 5, the axiom of Weak Restricted Composition. Given some set A which can be enlarged by either x or y, the axiom demands that the ranking of $A \cup \{x\}$ versus $A \cup \{y\}$ depends completely on the distances between A on the one hand, and x and y on the other. If x is more distant from A than y is, then $A \cup \{x\}$ gives more freedom than $A \cup \{y\}$. If they are equally distant, the resulting opportunity sets are assumed to give an equal amount of freedom of choice.

Axiom 6, the Axiom of Restricted Composition, is stronger than Axioms 4 and 5 but is based on the same intuition. It is a weaker version of a composition axiom used by Sen (1991).[4]

3 The simple cardinality rule

Before applying these axioms to some nontrivial distance functions, I will discuss how they relate to the simple cardinality rule of Pattanaik and Xu.

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[4] It relaxes Sen's axiom in two respects. First of all, it incorporates information about the degree of similarity between the sets and the elements which are added to them. Secondly, it only considers combining sets with singleton sets, whereas Sen's axiom also combines sets with non-singleton sets.

Using # to denote the cardinality of a set, this cardinality rule is formally defined as follows.

Definition 2. *For all* A, $B \in W$:

$$A \succeq_{\#} B \Leftrightarrow \#A \geq \#B.$$

A particular group of distance functions can be called *trivial* in the sense defined below.

Definition 3. *A trivial distance function* Δ_t *is a distance function such that for all* $A, B \in W$ *and all* $x \in X - A$ *and all* $y \in X - B$

$$\Delta_t(A, x) = \Delta_t(B, y).$$

In particular, the definition implies that all distinct singleton sets, and thus by property (v) also all distinct alternatives, are equally similar to each other.

Although Axiom 4 is weaker than Pattanaik and Xu's independence axiom even in case $\Delta = \Delta_t$, we can present the following result as a reformulation of their theorem.

Proposition 1. (Pattanaik and Xu 1990) *Let* $\Delta = \Delta_t$. *A quasi-ordering* $\succeq$ *satisfies Axioms* 1, 2 *and* 4 *if and only if* $\succeq \, = \, \succeq_{\#}$.

Using Axiom 5 rather than Axiom 4 and replacing Weak Monotonicity by Strong Monotonicity we get the following alternative characterisation of the simple cardinality rule.

Proposition 2. *Let* $\Delta = \Delta_t$. *A quasi-ordering* $\succeq$ *satisfies Axioms* 1, 3 *and* 5 *if and only if* $\succeq \, = \, \succeq_{\#}$.

Proof. I prove only that the axioms yield the cardinality rule. First I show that $A \sim B$ for all A, B of equal cardinality. If A and B are singletons, the result follows directly from Axiom 1. Assume therefore that A and B contain at least two alternatives. Let $A - B = \{a_1, \ldots, a_k\}$ and $B - A = \{b_1, \ldots, b_k\}$. Adding a_1 and b_1 to $A - \{a_1\}$, respectively, leads by Axiom 5 to

$$A \sim (A - \{a_1\}) \cup \{b_1\}.$$

Next, adding a_2 and b_2, respectively, to $(A - \{a_1, a_2\}) \cup \{b_1\}$ yields by Axiom 5

$$(A - \{a_1\}) \cup \{b_1\} \sim (A - \{a_1, a_2\}) \cup \{b_1, b_2\}.$$

Proceeding in this way we eventually get

$$(A - \{a_1, \ldots, a_{k-1}\}) \cup \{b_1, \ldots, b_{k-1}\} \sim (A - \{a_1, \ldots, a_k\}) \cup \{b_1, \ldots, b_k\}.$$

Since $(A - \{a_1, \ldots, a_k\}) \cup \{b_1, \ldots, b_k\} = B$ it follows by transitivity that $A \sim B$.

Now assume A is larger than B. Let A^* be a subset of A with equal cardinality to that of B. In the previous step it was shown that $A^* \sim B$. Repeated application of Axiom 3 yields $A \succ A^*$ from which $A \succ B$ follows by transitivity of $\succeq$. ■

Thus, if a distance function is trivial, Axioms 1, 2 and 4, or 1, 3 and 5 are sufficient and necessary conditions for yielding the cardinality rule. It does not yet follow, however, that the cardinality rule presupposes a trivial metric. The following proposition makes clear what the cardinality ranking exactly presupposes about the distances between the alternatives.

Proposition 3. *If the cardinality rule* $\succeq_\#$ *satisfies Axiom* 4, 5, *or* 6, *then* $\Delta(x,y) = \Delta(v,w)$ *for all* x, y, v, w $(x \neq y, v \neq w)$.

Proof. Trivial.

We explained in the introduction that Pattanaik and Xu argued that the cardinality ranking is counterintuitive because it does not incorporate information about the differences between the various choice options. Proposition 3 shows that the cardinality rule does indeed presuppose that all alternatives are 'equally similar' to each other. In the rest of this paper I shall examine what happens if this assumption is dropped.

4 Minimum and maximum distance

Obviously, in most circumstances alternatives are not equally similar to each other: the degree of similarity between 'red car' and 'blue car' is not the same as the degree of similarity between 'red train' and 'blue car'. Given our axioms, we need some way of extending a distance function in order to describe the distance between a set of alternatives and an alternative. Probably the most familiar way of doing this is to take the minimum distance between the elements of A and x:

$$\Delta_{\min}(A,x) = \min_{y \in A} d(y,x).$$

Using this extension, however, readily leads to impossibility results.

Proposition 4. *If* $\Delta = \Delta_{\min}$ *and* X *contains a string with a length of 4 or more, then there exists no* $\succeq$ *satisfying Axioms 1 and 4.*

Proof. Let $\Delta = \Delta_{\min}$. If X contains a string with a length of 4 or more, then it also contains a string with length 4. Let $\{x_1, \ldots, x_4\}$ be such a string. By definition of a string we have $\Delta(x_1,x_2) = \Delta(x_2,x_3) = \Delta(x_3,x_4) < \Delta(x_1,x_3) = \Delta(x_2,x_4)$. By Axiom 1 we have $\{x_1\} \sim \{x_4\}$. Since $\Delta(x_1,x_3) > \Delta(x_3,x_4)$, adding x_3 to x_1 and x_4, respectively, leads by Axiom 4 to $\{x_1,x_3\} \succ \{x_3,x_4\}$. By the minimum distance property we have $\Delta(\{x_1,x_3\},x_2) = \Delta(x_2,x_3) = \Delta(\{x_3,x_4\},x_2)$. If we subsequently add x_2 to $\{x_1,x_3\}$ and $\{x_3,x_4\}$, respectively, we get by Axiom 4 $\{x_1,x_2,x_3\} \succ \{x_2,x_3,x_4\}$. In the same way, by adding x_2 to

$\{x_1\}$ and $\{x_4\}$ we get $\{x_2, x_4\} \succ \{x_1, x_2\}$, and by subsequently adding x_3 we derive $\{x_2, x_3, x_4\} \succ \{x_1, x_2, x_3\}$, which is a contradiction. ■

Note that the proposition does not assume that the binary relation is transitive or complete. The following proposition shows that, if one assumes that it satisfies transitivity, Axiom 5 is in itself already sufficient to generate an impossibility result.

Proposition 5. *If* $\Delta = \Delta_{\min}$ *and* X *contains a string with a length of 4 or more, then there exists no transitive* $\succeq$ *satisfying Axiom 5.*

Proof. Let $\{x_1, \ldots, x_4\}$ be a string with length 4. Adding x_2 and x_3 to $\{x_1, x_4\}$, respectively, leads by Axiom 5 to $\{x_1, x_2, x_4\} \sim \{x_1, x_3, x_4\}$. Similarly, adding x_1 and x_2 to $\{x_3, x_4\}$, respectively, leads by Axiom 5 to $\{x_1, x_3, x_4\} \succ \{x_2, x_3, x_4\}$. Hence, by transitivity of $\succeq$ we get $\{x_1, x_2, x_4\} \succ \{x_2, x_3, x_4\}$. However, since we see that $\Delta_{\min}(\{x_2, x_4\}, x_1) = \Delta_{\min}(\{x_2, x_4\}, x_3)$ we must also have $\{x_1, x_2, x_4\} \sim \{x_2, x_3, x_4\}$, which is a contradiction. ■

The structure of the proofs makes it clear that the impossibilities arise because, in establishing the distance between a set A and an alternative x, the function $\Delta_{\min}$ focuses only on the distance between x and one particular element of A, viz. the element in A that minimises the distance; the function discards information about the distances between x and other elements of A altogether. It should therefore not come as a surprise that similar impossibility results can be obtained if one uses a distance function which suffers from the same flaw. One such distance function proposed in the literature takes the *maximum* rather than the minimum distance as describing the distance between A and x:

$$\Delta_{\max}(A, x) = \max_{y \in A} d(x, y).$$

And, indeed, it leads to an impossibility comparable to Proposition 4:

Proposition 6. *If* $\Delta = \Delta_{\max}$ *and* X *contains a string with a length of 4 or more, then there exists no* $\succeq$ *satisfying Axioms 1 and 4.*

Proof. Let $\Delta = \Delta_{\max}$. Let $\{x_1, \ldots, x_4\}$ be a string with length 4. We then have $\Delta(x_1, x_2) = \Delta(x_3, x_4) < \Delta(x_1, x_3) = \Delta(x_2, x_4)$. By Axiom 1 we have $\{x_2\} \sim \{x_3\}$. *Since* $\Delta(x_1, x_3) > \Delta(x_1, x_2)$, we get $\{x_1, x_3\} \succ \{x_1, x_2\}$ by Axiom 4. Furthermore, by the maximum distance function we have $\Delta(\{x_1, x_3)\}, x_4) = \Delta(x_1, x_4) = \Delta(\{x_1, x_2\}, x_4)$. Hence adding x_4 to $\{x_1, x_3\}$ and $\{x_1, x_2\}$, respectively, leads by Axiom 4 to $\{x_1, x_3, x_4\} \succ \{x_1, x_2, x_4\}$. Similarly, by adding x_4 to $\{x_2\}$ and $\{x_3\}$ we get $\{x_2, x_4\} \succ \{x_3, x_4\}$ and by subsequently adding x_1 we get $\{x_1, x_2, x_4\} \succ \{x_1, x_3, x_4\}$, which is a contradiction. ■

Using information about only the minimum or only the maximum distance thus turns out to be inadequate. This does not imply that the minimum and maximum distances are not relevant for assessing the distance between a set and an alternative. Consider Fig. 1, for instance.

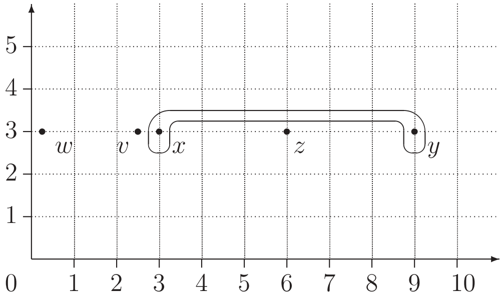

Fig. 1.

It can be maintained quite plausibly that z is more distant from $\{x, y\}$ than v because the minimum distance between z and $\{x, y\}$ is larger than the minimum distance between v and $\{x, y\}$. The fact that the maximum distance is smaller does not seem to matter much. However, with respect to the distance between w and $\{x, y\}$ one could argue that w is more distant from $\{x, y\}$ than z is, even though the minimum distance from z to $\{x, y\}$ is larger. The reason, of course, is the substantially larger maximum distance between w and $\{x, y\}$. This seems to outweigh the fact that the minimum distance is smaller.

Instead of using either the minimum distance or the maximum distance one could therefore argue that a combination of the two distance functions should be used:

$$\Delta^{\gamma}_{\mathrm{mm}}(A, x) = \gamma\Delta_{\min}(A, x) + (1 - \gamma)\Delta_{\max}(A, x) \quad (0 < \gamma < 1)$$

This distance function surely is more sophisticated than either the minimum or the maximum distance functions themselves. It may avoid the counterintuitive results associated with applying these functions to situations such as those described by Figure 1. Unfortunately, such a more sophisticated distance function also leads to impossibilities.

Proposition 7. *Let $\Delta = \Delta^{\gamma}_{\mathrm{mm}}$. If X contains a string with a length of 5 or more, then there exists no $\succeq$ satisfying Axioms 1 and 6.*

Proof. Let $\{x_1, \ldots, x_5\}$ be a string with length 5. By Axiom 1 we have $\{x_1\} \sim \{x_2\}$. Adding x_4 and x_5 leads by Axiom 6 to $\{x_1, x_4\} \sim \{x_2, x_5\}$, respectively. Similarly, adding x_2 and x_3 implies $\{x_1, x_2, x_4\} \sim \{x_2, x_3, x_5\}$, respectively. Finally, adding x_5 and x_1 yields $\{x_1, x_2, x_4, x_5\} \sim \{x_1, x_2, x_3, x_5\}$.

Now add x_5 and x_1 to $\{x_1, x_4\}$ and $\{x_2, x_5\}$, respectively. We then get $\{x_1, x_4, x_5\} \sim \{x_1, x_2, x_5\}$. However, if we subsequently add x_2 and x_3, we get $\{x_1, x_2, x_4, x_5\} \succ \{x_1, x_2, x_3, x_5\}$, which is a contradiction. ■

As the structure of the proof indicates, the impossibility result arises because, even with this more sophisticated function, the distance between x and a set A is determined by focusing on the distance between x and *some* of the elements in A, thereby ignoring information about the distance between x and other elements of A. One could therefore argue that the function is still not sophisticated enough – one should have an extended distance function based on the distances between x and *all* elements of A.

5 Inclusive distance functions

Distance functions that are based on all of the distances between an alternative and the elements of an opportunity set might be called *inclusive distance functions*. Drawing on developments within the philosophy of science, I will examine the following inclusive distance functions.[5]

$$\Delta_{\text{av}}(A,x) = \frac{1}{\#A}\sum_{y\in A} d(x,y)$$

$$\Delta_{\text{sum}}(A,x) = \sum_{y\in A} d(x,y)$$

$$\Delta_{\text{ms}}^{\gamma\gamma'}(A,x) = \gamma\Delta_{\min}(A,x) + \gamma'\Delta_{\text{sum}}(A,x) \quad (\gamma > 0, \gamma' > 0)$$

The distance function Δ_{av} takes the average distance between x and the elements of A as indicative of the distance between x and A, whereas Δ_{sum} take the sum of these distances.[6] The function $\Delta_{\text{ms}}^{\gamma\gamma'}$ is a weighted distance function: the distance between x and A depends on both $\Delta_{\min}$ and Δ_{sum}.

Possibility results emerge if these distance functions are used. For instance, we can define a ranking of the opportunity sets in which an opportunity set A is said to give at least as much freedom of choice as B if, and only if, the sum of the distances between elements of A is at least as large as the sum of the distances between elements of B. If we furthermore assume that $\Delta = \Delta_{\text{sum}}$ we get a ranking that satisfies Axioms 1 through 6. Yet the ranking is not very convincing since it assumes that the addition of an alternative can lead to a substantial increase in freedom even though the alternative is almost (but not entirely) identical to one of the elements already in the opportunity set. Consider Fig. 1 again. Taking the sum of the distances as indicative of the freedom of choice offered by a set, we have to say that the set $\{x,y,v\}$ gives more freedom of choice than $\{x,y,z\}$. In fact, this conclusion follows even if we were to move v infinitely close to (but still to the left of) x. This is clearly counterintuitive: if v and x are very similar, then the addition of v would contribute less to our freedom than would the addition of z.

[5] The distance functions have been used in theories of truth-approximation to measure the distance between a set of statements ('a theory') and a singular statement ('the truth'). The distance function $\Delta_{\text{mm}}^{\gamma}$ presented in the previous section has also been derived from this literature. For an extensive survey of theories of truth-approximation, see Niiniluoto (1987).

[6] Niiniluoto defines Δ_{sum} somewhat differently because he takes the ratio between the sum of the distances between x and the elements of A on the one hand, and the sum of the distances between x and all elements of X, on the other. The results of this section are not affected by my simplification of the function, however.

It may therefore be reasonable to assume that there are threshold values limiting the degree to which the addition of an alternative can contribute to one's freedom. The following axiom represents this idea:

Axiom 7 (Insensitivity to Small Differences). *There are positive numbers* g, k $(k > g)$ *such that for all* $A \in W$ *and all* $x, y \notin A$: *If* $\Delta(y,z) \geq k$ *for all* $z \in A$, *and* $\Delta(x,z) \leq g$ *for some* $z \in A$, *then*

$$A \cup \{y\} \succeq A \cup \{x\}.$$

Indeed, assume that in Fig. 1 the distance between v and x is smaller than the threshold value g and that the distance between z and x (and hence also between z and y) is larger than k. It then follows from the axiom that the addition of z yields at least as much freedom of choice as the addition of v would.

Reasonable though this assumption may be, in combination with Axiom 5 and an inclusive distance function it leads to impossibility results.

Proposition 8. *Let* $\Delta = \Delta_{\text{av}}$ or Δ_{sum}. *If* X *contains a string* $A = \{x_1, x_2, x_3\}$ *and a point* x_0 *such that*

1. $\Delta(x_1, x_2) = \Delta(x_2, x_3) = m \quad (m \geq k)$
2. $\Delta(x_0, x_i) = h + m(i-1) \quad (0 < h \leq g, i = 1, 2$ *or* $3)$,

then there is no binary relation $\succeq$ *satisfying Axioms 5 and 7.*

Proof. Since $\Delta(x_0, x_1) = h \leq g$ and $\Delta(x_1, x_2) = m = \Delta(x_2, x_3)$ with $m \geq k$, adding x_0 respectively x_2 to $\{x_1, x_3\}$ yields by Axiom 7 $\{x_1, x_2, x_3\} \succeq \{x_0, x_1, x_3\}$. However, since $\Delta(x_1, x_0) + \Delta(x_3, x_0) = 2m + 2h > 2m = \Delta(x_1, x_2) + \Delta(x_3, x_2)$, Axiom 5 implies $\{x_0, x_1, x_3\} \succ \{x_1, x_2, x_3\}$. ■

The following proposition shows that an impossibility can also arise if one uses $\Delta_{\text{ms}}^{\gamma\gamma'}$ rather than Δ_{av} or Δ_{sum}.

Proposition 9. *Let* $\Delta = \Delta_{\text{ms}}^{\gamma\gamma'}(A, x)$. *If* X *contains a string* $A = \{x_1, \ldots, x_{2j-1}\}$ *and a point* x_0 *such that*

1. $\Delta(x_i, x_{i+1}) = m \quad (1 \leq i < 2j-1,\ m \geq k)$;
2. $\Delta(x_0, x_i) = h + m(i-1)\ (0 < h \leq g)$;
3. j *is an integer for which it is true that* $j^2 - 3j > [\gamma(m-h)/\gamma' m] - [h(2j-2)/m] - 2$,

then there is no binary relation $\succeq$ *satisfying Axioms 5 and 7.*

Proof. Take $A - \{x_j\}$ and add x_j and x_0, respectively, to it. It follows immediately from Axiom 7 that $A \succeq (A - \{x_j\}) \cup \{x_0\}$.

From part (3) of the hypothesis, we derive:

$$m(j^2 - 3j + 2) + h(2j-2) > \gamma(m-h)/\gamma' \Rightarrow$$

$$\gamma'[m(j^2 - 3j + 2) + h(2j-2)] > \gamma(m-h) \Rightarrow$$

$$\gamma h + \gamma'[m(j^2 - 3j + 2) + h(2j - 2)] > \gamma m \Rightarrow$$

$$\gamma h + \gamma'[m(2j^2 - 4j + 2) + h(2j - 2)] > \gamma m + \gamma'(j^2 - j)m.$$

Elementary combinatorics shows that

$$\sum_{y \in A - x_j} d(x_j, y) = (j^2 - j)m$$

and that

$$\sum_{y \in A - x_j} d(x_0, y) = m(2j^2 - 4j + 2) + h(2j - 2).$$

Hence, we get:

$$\gamma h + \gamma' \sum_{y \in A - x_j} d(x_0, y) > \gamma m + \gamma' \sum_{y \in A - x_j} d(x_j, y).$$

Adding $\{x_0\}$ and $\{x_j\}$ to $A - \{x_j\}$, respectively, therefore implies by Axiom 5 that $(A - \{x_j\}) \cup \{x_0\} \succ A$, which is a contradiction. ■

Corollary 1. *Let $X = \mathbb{R}^n$ and let Δ be the Euclidean version of* Δ_{av}, Δ_{sum}, *or* $\Delta_{\mathrm{ms}}^{\gamma\gamma'}$. *There is no binary relation $\succeq$ satisfying Axioms 5 and 7.*

6 Conclusion

The central question of this paper was whether the Pattanaik and Xu approach can be reformulated to take account of the degrees of (dis)similarity between alternatives. The way how the distances between sets and alternatives are established turned out to be crucial to the analysis. I have examined several proposals in this respect, each of which leads to impossibilities. Obviously, this does not yet imply that such a reformulation is not possible at all. Indeed, in a recent unpublished paper Bossert et al. (2001) present a measure of diversity that satisfies Indifference Between No-Choice Situations and Strong Monotonicity, and that is equivalent to the measure of diversity proposed by Weitzman (1992). Although they use a distance metric that I have also applied, viz. Δ_{min}, the impossibility results are precluded since they adopt an independence-axiom that is weaker than the axiom of Restricted Independence used here. Another important recent contribution is the theory of diversity published by Nehring and Puppe (2002). In their approach, the diversity of a set depends on the given attributes of the alternatives. They subsequently examine under what circumstances their ‘multi-attribute’ approach to diversity is compatible with the view that diversity of a set can be established by aggregating the dissimilarities between the elements of that set.

Although the results of this paper are negative, they do have a positive upshot. Since one cannot incorporate diversity by adapting the original axioms of Pattanaik and Xu in the ways examined in this paper, the results established here underpin the importance of these recent approaches for assessing the relation between freedom of choice and diversity.

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