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Monotonicity of 'ranking by choosing': A progress report

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Abstract. Procedures designed to select alternatives on the basis of the results of pairwise contests between them have received much attention in literature. The particular case of tournaments has been studied in depth. More recently weak tournaments and valued generalizations thereof have been investigated.The purpose of this paper is to investigate to what extent these choice procedures may be meaningfully used to define ranking procedures via their repeated use, i.e. when the equivalence classes of the ranking are determined by successive applications of the choice procedure. This is what we call "ranking by choosing". As could be expected, such ranking procedures raise monotonicity problems. We analyze these problems and show that it is nevertheless possible to isolate a large class of well-behaved choice procedures for which failures of monotonicity are not overly serious. The hope of finding really attractive ranking by choosing procedures is however shown to be limited. Our results are illustrated on the case of tournaments.

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1 Introduction

In many different contexts, it is necessary to make a choice between alternatives on the sole basis of the results of several kinds of pairwise contests between these alternatives. Among the many possible examples, let us mention:

– Sports leagues (games usually involve two teams).
– Social choice theory, via the use of $C1$ or $C2$ Social Choice functions (as defined by Fishburn, 1977), in view of the well-known results in McGarvey (1953) and Debord (1987b).
– Multiple criteria decision making using "ordinal information" (see Arrow and Raynaud 1986; Roy 1991) in view of the results in Bouyssou (1996).
– Psychology with, e.g., the study of binary choice probabilities (see Luce 1959; Suppes et al. 1989).

This problem has received close attention in recent years most particularly when the result of the pairwise contests may be summarized by a *tournament* (an excellent account of this literature may be found in Laslier 1997) and much is known on the properties and interrelations of such choice procedures. This line of research has been recently extended to weak tournaments (ties are allowed, see Peris and Subiza 1999; Schwartz 1986) and valued generalizations of (weak) tournaments (intensity of preference or number of victories may be taken into account, see Barett et al. 1990; Basu et al. 1992; Dutta et al. 1986; Dasgupta and Deb 1991; Dutta and Laslier 1999; de Donder et al. 2000; Fodor and Roubens 1994; Kitainik 1993; Litvakov and Vol'skiv 1986; Nurmi and Kacprzyk 1991; Pattanaik and Sengupta 2000; Roubens 1989).

The related problem of *ranking* alternatives on the basis of the results of pairwise contests between these alternatives has comparatively received much less attention in recent years (see, however, Henriet 1985; Rubinstein 1980), although it generated classical studies (see Kemeny 1959; Kemeny and Snell 1962; Slater 1961) and is clearly in the spirit of Social Welfare Functions *à la* Arrow (Arrow 1963). This is a pity since most classical applications of choice procedures are also potential applications for ranking procedures. This is, e.g., clearly the case for sports since most leagues want to rank order teams at the end of season and not only to select the winner(s). This also the case in the many situations in which, although a choice between alternatives is to be made, alternatives may disappear (e.g., candidates for a position may withdraw), so that there is a necessity of building a waiting list.

The problem of devising sound ranking procedures for such situations can be studied without explicit reference to choice procedures (see Bouyssou 1992b; Bouyssou and Perny 1992; Bouyssou and Pirlot 1997; Bouyssou and Vincke 1997; Henriet 1985; Rubinstein 1980; Vincke 1992). This is in line with the advice in Moulin (1986) to clearly distinguish the question of ranking alternatives from the one of selecting winners.

We shall be concerned in this paper with quite a different approach to ranking on the basis of pairwise contests that is intimately connected with choice procedures. Several authors have indeed suggested (see Arrow and Raynaud 1986; Roy 1991) that a ranking procedure could well be devised by *successive applications* of a choice procedure. The most natural way to do so goes as follows:

– Apply the choice procedure to the whole set of alternatives. Define the first equivalence class of the ranking as the chosen elements in the whole set.
– Remove the chosen elements from the set of alternatives.
– Apply the choice procedure to the reduced set. Define the second equivalence class of the ranking as the chosen elements in the reduced set.
– Repeat the above two steps to define the following equivalence classes of the ranking until there are no more alternatives to rank.

This is what we call “ranking by choosing”. An example may help clarify the process.

Example 1 (Ranking by choosing with Copeland). Let $X = \{a, b, c, d, e, f, g\}$. Consider the tournament T on X defined by:

$$\begin{gathered} aTb, aTf, \\ bTc, bTd, bTe, bTf, \\ cTa, cTe, cTf, cTg, \\ dTa, dTc, dTe, dTf, dTg \\ eTa, eTf, eTg, \\ fTg, \\ gTa, gTb. \end{gathered}$$

Suppose that you want to use the Copeland choice procedure $Cop(A, T)$ selecting the elements in A having a maximal Copeland score (i.e. maximal outdegree) in T restricted to A as a basis for ranking alternatives.

Applying the above ranking by choosing algorithm successively leads to:

$$\begin{gathered} Cop(X, T) = \{d\}, \\ Cop(X \backslash \{d\}, T) = \{c\}, \\ Cop(X \backslash \{d, c\}, T) = \{e\}, \\ Cop(X \backslash \{d, c, e\}, T) = \{a, g\}, \\ Cop(X \backslash \{a, d, c, e, g\}, T) = \{b\}, \\ Cop(X \backslash \{a, b, c, d, e, g\}, T) = \{f\}. \end{gathered}$$

Hence we obtain the ranking (using obvious notation): $d \succ c \succ e \succ [a \sim g] \succ b \succ f$. This result is clearly different from the one that we would have obtained ranking alternatives using their Copeland scores in X, i.e.:

$$d \succ [b \sim c] \succ e \succ [a \sim g] \succ f,$$

although both rankings clearly coincide on their first equivalence class.

Using ranking by choosing, we may associate a well-defined ranking procedure to every choice procedure. A natural question arises. If the choice procedure has “nice properties”, will it also be the case for the induced ranking procedure? This is the subject of this paper.

Most ranking procedures that are used in practice are not of this ranking by choosing type. Most often (take the example of most sports leagues) they are rather based on some kind of *scoring function* that aggregates into a real number the results of the various pairwise contests, e.g. one may rank alternatives according to their Copeland scores.

Although ranking procedures induced by choice procedures may seem complex when compared to those based on scoring functions, several authors have forcefully argued in favor of their reasonableness (see Arrow and Raynaud 1986; Roy 1991) and many of them were proposed (see Arrow and Raynaud 1986; Debord 1987a; Matarazzo 1990; Roy 1978). They are, in general, easy to compute and rather easy to explain. They are—structurally—insensitive to a possible withdrawal of (all) best ranked alternatives (see Vincke 1992). Furthermore, if the answer to the preceding question were to be positive, there would be a clear interest in using well-behaved choice procedures as a basis for ranking procedures.

The situation is however more complex.The potential drawbacks of these ranking by choosing procedures should be obvious: their very conception implies the existence of discontinuities together with a progressive impoverishment of information from one iteration to another. This is likely to create difficulties with most wanted normative properties like monotonicity as was forcefully shown by Perny (1992).The purpose of this paper is to explore the extent of these difficulties concentrating on monotonicity. An example will clarify how bad the situation can be.

Example 1 (continued). Consider the tournament V identical to T except that aVd. We now have $Cop(X, V) = \{b, c, d\}$. We had $a \succ b$ with T. We now obtain $b \succ a$ with V, while the position of a has clearly improved when going from T to V. This is a serious monotonicity problem.

The problem studied in this paper is reminiscent of the well-known monotonicity problems encountered in electoral procedures with “run-offs”, e.g., the French system of plurality with run-off, the Hare, Coombs and Nansson procedures (see Fishburn 1977; Moulin 1988) that also involve discontinuities. It is well-known that they often have a disappointing behavior with respect to monotonicity (see Fishburn 1977, 1982; Moulin 1988; Saari 1994; Smith 1973). Although these difficulties are linked with our problem, electoral procedures with run-offs are choice procedures and not ranking procedures. Hence the problem studied here has distinctive characteristics.

Although many ranking by choosing procedures have been suggested, their study has received limited attention so far. Perny (1992) showed that most procedures of this type proposed in the literature violate monotonicity. He suggested to study the problem more in depth. Shortly after, we proposed in Bouyssou (1995) some results in that direction (since more powerful results appear difficult to obtain, this text is a revised and simplified version of Bouyssou 1995). More recently, the problem was tackled in Durand (2001) and Juret (2001) in a Social Choice context.

We show here that, rather surprisingly, there are non-trivial and rather well-behaved choice procedures leading to ranking by choosing procedures satisfying a weak form of monotonicity. The hope of finding really attractive ranking by choosing procedures is however shown to be limited.
The paper is organized as follows. The next section introduces our main definitions and elucidates our notation. Our results are collected in Sect.3. We apply our results to the classical case of tournaments in Sect.4. A final section discusses our findings.

2 The setting

Throughout the paper, X will denote a *finite* set with $|X| = m \geq 1$ elements. Elements of X will be interpreted as alternatives that are to be compared on the basis the results of several kinds of pairwise contests. We denote by $\mathscr{P}(X)$ the set of all nonempty subsets of X.

2.1 Pairwise contests between alternatives

Pairwise contests between alternatives arise in many different contexts. Therefore, it is not surprising that many different models have been proposed to summarize them. The most simple ones consist of binary relations: tournaments (see Laslier 1997; Moulin 1986), weak tournaments (see Peris and Subiza 1999), reflexive binary relations (see Vincke 1992). More sophisticated models use real-valued functions on X^2: weighted tournaments (see de Donder et al. 2000), comparison functions (see Dutta and Laslier 1999) or general valued relations (see Kitainik 1993; Fodor and Roubens 1994; Roubens 1989). Many of these models can be justified by results saying that some type of aggregation methods lead to all (or nearly all) instances of these models (see Bouyssou 1996; Deb 1976; Debord 1987b; McGarvey 1953).

Although our results can be extended to more general cases (see Bouyssou 1995), we use throughout the paper the comparison function model presented in Dutta and Laslier (1999). It is sufficiently flexible to include:

– all complete binary relations and, hence, to deal with all $C1$ social choice functions in the sense of Fishburn (1977), i.e., all social choice functions based on the simple majority relation of some profile of linear orders and
– all 0-weighted tournaments, as defined in de Donder et al. (2000) and, hence, to deal with most (in fact with what de Donder et al. (2000) called $C1.5$ social choice functions) $C2$ social choice functions in the sense of Fishburn (1977), i.e. social choice functions that are based on a matrix giving for each ordered pair (x, y) of alternatives the number $n(x, y)$ being the difference between the number of linear orders in the profile for which x is ahead of y minus the number of linear orders for which y is ahead of x.

These two examples are detailed below. We refer to Dutta and Laslier (1999) for more possible interpretations.

A *comparison function* π on X is a skew-symmetric real-valued function on X^2 (i.e. such that $\pi(x,y) = -\pi(y,x)$, for all $x, y \in X$). The set of all comparison functions on X is denoted $\mathscr{G}(X)$. We denote by $\pi|_A$ the restriction of π on $A \subseteq X$, i.e. the function $\pi|_A$ on A such that $\pi|_A(x,y) = \pi(x,y)$, for all $x, y \in A$.

Example 2 (weak tournaments). A *weak tournament* V on X is a complete (xVy or yVx, for all $x \in X$) binary relation[1] on X. A *tournament* is an antisymmetric (xVy and $yVx \Rightarrow x = y$, for all $x, y \in X$) weak tournament. We denote $\mathscr{T}(X)$ (resp. $\mathscr{W}\mathscr{T}(X)$) the set of all tournaments (resp. weak tournaments) on X. A transitive tournament (resp. weak tournament) is a linear order (resp. weak order). We note $\mathscr{W}\mathscr{O}(X)$ the set of all weak orders on X.

The interest in weak tournaments is explained by McGarvey's theorem (see McGarvey, 1953) ensuring that any $V \in \mathscr{W}\mathscr{T}(X)$ is the simple majority relation of some profile of linear orders.
Note that any comparison function $\pi \in \mathscr{G}(X)$ induces a weak tournament $V \in \mathscr{W}\mathscr{T}(X)$ letting $xVy \Leftrightarrow \pi(x,y) \geq 0$. Conversely, any weak tournament $V \in \mathscr{W}\mathscr{T}(X)$ induces a comparison function $\pi_V \in \mathscr{G}(X)$ defined letting, for all $x, y \in X$,

$$\pi_V(x,y) = \begin{cases} 1 & \text{if } xVy \text{ and } Not[yVx], \\ 0 & \text{if } xVy \text{ and } yVx, \\ -1 & \text{if } yVx \text{ and } Not[xVy]. \end{cases} \tag{1}$$

We sometimes abuse notation in the sequel writing V instead of π_V when dealing with weak tournaments.

Example 3 (0-weighted tournaments). A 0-weighted tournament (de Donder et al. 2000) on X is a complete digraph which set of vertices is X and in which each arc (x,y) has a skew symmetric *integer* valuation $n(x,y)$. Using Debord's theorem (see Debord 1987b), any 0-weighted tournament with all $n(x,y)$ having the same parity is the net preference matrix of some profile of linear orders on X, i.e., there is a profile of linear orders such that $n(x,y)$ is the number of linear orders in the profile for which $x > y$ minus the number of linear orders in the profile for which $y > x$. Clearly the set of comparison functions includes all 0-weighted tournaments.

[1] We follow here the widely used terminology of Moulin (1986) and Peris and Subiza (1999) although the term *match* suggested by Monjardet (1978) and Ribell (1973) seems more satisfactory. Note that we work here, for commodity, with *reflexive* (weak) tournaments although most authors prefer the asymmetric version (see Laslier 1997). This has no consequences in what follows.

Definition 1 (Improving the position of an alternative). *Let π and π' be two comparison functions on X. We say that π' improves $x \in X$ w.r.t. π if for all $y, z \in X \backslash \{x\}$,*

$$\pi'(y,z) = \pi(y,z) \quad \text{and} \quad \pi'(x,y) \geq \pi(x,y).$$

We often denote $\pi^{x\uparrow}$ a comparison function improving $x \in X$ w.r.t. π.

Let $\pi \in \mathscr{G}(X)$, $A \subseteq X, x, y \in A$. We say that *x covers y* in A if $\pi(x,y) > 0$ and, for all $z \in A \backslash \{x,y\}, \pi(x,z) \geq \pi(y,z)$. It is clear that the covering relation thus defined is asymmetric and transitive. Hence it has maximal elements. We denote $UC(A,\pi) \subseteq A$ the set of maximal elements of the covering relation in A. This definition, due to Dutta and Laslier (1999), extends to comparison functions a well-known concept due to Fishburn (1977) and Miller (1977, 1980).

We say that *x sign-covers y* in A for π if it covers y for the comparison function π_{sign} defined by:

$$\pi_{sign}(x,y) = \begin{cases} 1 & \text{if } \pi(x, y) > 0, \\ 0 & \text{if } \pi(x, y) = 0, \\ -1 & \text{if } \pi(x, y) < 0, \end{cases}$$

for all $x, y \in X$. It is clear that the sign covering relation is asymmetric and transitive and, therefore, has maximal elements. We denote $SUC(A,\pi) \subseteq A$ the set of maximal elements of the sign covering relation in A. It is easy to see that $SUC(A,\pi) \subseteq UC(A,\pi)$, while the two sets coincide for weak tournaments.

A *Condorcet winner* in $A \in \mathscr{P}(X)$ for a comparison function $\pi \in \mathscr{G}(X)$ is an alternative x that defeats all other alternatives in A in pairwise contests, i.e. such that $\pi(x,y) > \pi(y,x)$, for all $y \in A \backslash \{x\}$. It is clear that the set of Condorcet Winners $Cond(A,\pi)$ is either empty or is a singleton.

Remark 1. When there is a Condorcet winner, it is clear that $Cond(X,\pi) = SUC(A,\pi)$ and, hence, $Cond(X,\pi) \subseteq UC(A,\pi)$. The uncovered set $UC(A,\pi)$ may however contain other alternatives.

2.2 Ranking procedures

A ranking procedure (for comparison functions on X) $\succsim$ associates with each comparison function π on X a weak order $\succsim(\pi) \in \mathscr{WO}(X)$, i.e. is a function from $\mathscr{G}(X)$ into $\mathscr{WO}(X)$. The asymmetric (resp. symmetric) part of $\succsim(\pi)$ is denoted $\succ(\pi)$ (resp. $\sim(\pi)$).

Example 4 (Ranking procedures induced by a scoring function). Many ranking procedures are based on *scoring functions* on X. A simple[2] scoring function associates with each $\pi \in \mathscr{G}(X)$, each $A \subseteq X$ and each $x \in A$ a real number

[2] More general scoring functions can be defined having for argument the whole comparison function π, as in methods based on Markov chains or on eigenvalues (see Laslier 1997). We do not envisage them here and, hence, we omit "simple" in what follows.

$Score_F(x, A, \pi) = F_{|A|}(\pi(x, y)_{y \in A \setminus \{x\}})$, where $F_{|A|}$ is a real-valued function on $\mathbb{R}^{|A|-1}$ being *symmetric* in its arguments and *nondecreasing* in all its arguments. The ranking procedure $\succsim_F$ associated to $Score_F$ ranks alternatives in X according to their score $Score_F(x, X, \pi)$, i.e.,

$$x \succsim_F (\pi) y \Leftrightarrow Score_F(x, X, \pi) \geq Score_F(y, X, \pi), \tag{2}$$

for all $x, y \in X$ and all $\pi \in \mathcal{G}(X)$.
Two scoring functions that are often used are:
– the *Copeland* score in which $F = \sum$ and
– the *Kramer* score in which $F = \min$.

Note that using the Copeland score on a 0-weighted tournament corresponding to a net preference matrix of a profile of linear orders amounts to ranking alternatives according to their Borda score (see e.g., Young 1974).

By definition, the function $F_{|X|}$ used to compute $Score_F(x, X, \pi)$ is independent of x and symmetric in its arguments. Therefore, such ranking procedures do not depend on a particular labeling of the alternatives. Furthermore, since $F_{|X|}$ have been supposed to be nondecreasing in all its arguments, the ranking will respond in the expected direction to an improvement of x in π. This is formalized below.

Let $\Sigma(X)$ be the set of all one-to-one functions on X (i.e. permutations). Given a comparison function π and a permutation $\sigma \in \Sigma(X)$, we define, π^σ as the comparison function defined letting, for all $x, y \in X$, $\pi^\sigma(\sigma(x), \sigma(y)) = \pi(x, y)$.

Definition 2 (Neutral ranking procedures). *A ranking procedure $\succsim$ on X is said to be neutral if, for all for all $\pi \in \mathcal{G}(X)$ and all $\sigma \in \Sigma(X), x \succsim (\pi) y \Leftrightarrow \sigma(x) \succsim (\pi^\sigma) \sigma(y)$.*

Observe that with a neutral ranking procedure, if the comparison function is totally indecisive, i.e. if $\pi(x, y) = \pi(y, x) = 0$, for all $x, y \in X$, then this indecisivity is reflected in the weak order $\succsim(\pi)$, i.e. $x \succsim (\pi) y$, for all $x, y \in X$.

Definition 3 (Monotonic ranking procedure). A ranking procedure $\succsim$ on X is said to be:

– strictly monotonic *if*
  $x \succsim (\pi) y \Rightarrow x \succ (\pi') y$,
– monotonic, *if*
  $x \succsim (\pi) y \Rightarrow x \succsim (\pi') y$ and
  $x \succ (\pi) y \Rightarrow x \succ (\pi') y$,
– weakly monotonic *if*
  $x \succsim (\pi) y \Rightarrow x \succsim (\pi') y$,
– very weakly monotonic *if*
  $x \succ (\pi) y \Rightarrow x \succsim (\pi') y$,

for all $x, y \in X$ and all $\pi, \pi' \in \mathscr{G}(X)$ such that $\pi \neq \pi'$ and π' improves x w.r.t. π (see Definition 1).

Strict monotonicity requires that any improvement of the position of an alternative is sufficient to break ties in $\succsim$. This is a very strong condition, although it proves useful to characterize ranking procedures based on scoring functions $F_{|X|}$ that are increasing in all arguments (see Bouyssou 1992b; Henriet 1985; Rubinstein 1980). Monotonicity implies weak monotonicity which in turn implies very weak monotonicity. As already observed, it is easy to build a monotonic ranking procedure using a scoring function. This will clearly be more difficult with ranking by choosing procedures in view of Example 1. In a weakly monotonic ranking procedure, "efforts do not hurt", since the position of the improved alternative cannot deteriorate: it may only happen that beaten alternatives now tie with the improved one. Very weak monotonicity only forbids strict reversals in $\succsim$ after an improvement. Although this is a very weak condition, Example 1 shows that it can be violated with seemingly reasonable ranking by choosing procedures.

Remark 2. Durand (2001), in a classic social choice context, proves a negative result on the existence of strictly monotonic ranking by choosing procedure. His use of *strict* monotonicity tends to limit the scope of this result however.

Consider a weak order $W \in \mathscr{W}\mathscr{O}(X)$ and its associated comparison function π_W as defined by (1). Since W is a weak order, it seems obvious to require that any reasonable ranking procedure should not alter this ranking.

Definition 4 (Faithful ranking procedure). *A ranking procedure $\succsim$ on X is said to be faithful if, for all weak orders $W \in \mathscr{W}\mathscr{O}(X)$ and all $x, y \in X$, $x \succsim(\pi_W) y \Leftrightarrow xWy$. A ranking procedure is said to be faithful for linear orders if the above condition holds for antisymmetric weak orders, i.e., linear orders.*

Many other conditions can obviously be defined for ranking procedures (for an overview, see Bouyssou and Vincke 1997; Henriet 1985; Rubinstein 1980; Vincke 1992). They will not be useful here. The analysis of ranking by choosing procedures clearly calls now for a closer look at choice procedures.

2.3. Choice procedures

A choice procedure (for comparison functions on X) $\mathscr{S}$ associates with each comparison function $\pi \in \mathscr{G}(X)$ and each nonempty subset $A \in \mathscr{P}(X)$ a non-empty set of chosen[3] alternatives included in A. More formally, a *choice procedure* $\mathscr{S}$ on X is a function from $\mathscr{P}(X) \times \mathscr{G}(X)$ into $\mathscr{P}(X)$ such that, for all $A \in \mathscr{P}(X)$ and all $\pi \in \mathscr{G}(X)$, $\mathscr{S}(A, \pi) \subseteq A$. Given two choice procedures $\mathscr{S}'$ and $\mathscr{S}$, we say that $\mathscr{S}'$ *refines* $\mathscr{S}$ if, for all $A \in \mathscr{P}(X)$ and all $\pi \in \mathscr{G}(X)$, $\mathscr{S}'(A, \pi) \subseteq \mathscr{S}(A, \pi)$.

[3] We use the term *chosen* even if there may be more than one alternative in $\mathscr{S}(A, \pi)$

Example 5 (Choice procedures induced by scoring functions). Like with ranking procedures, many choice procedures are based on simple scoring functions (again, we do not envisage here scoring functions that depend on the entire comparison function π). Using the notation introduced in Example 4, we simply have, for all $A \in \mathscr{P}(X)$ and all $x \in A$,

$$x \in \mathscr{S}_F(A,\pi) \Leftrightarrow Score_F(x,A,\pi) \geq Score_F(y,A,\pi), \text{for all } y \in A, \tag{3}$$

such choice procedures are clearly independent of the labeling of alternative and have obvious monotonicity properties. Furthermore, the chosen elements in A only depends on the restriction $\pi|_A$ of π to A. We formalize these properties below.

Definition 5 (Properties of a choice procedure). *A choice procedure $\mathscr{S}$ on X is said to be:*

- neutral *if*
  $x \in \mathscr{S}(A,\pi) \Leftrightarrow \sigma(x) \in \mathscr{S}(A,\pi^\sigma)$,
- local *if*
  $[\pi|_A = \pi'|_A] \Rightarrow \mathscr{S}(A,\pi) = \mathscr{S}(A,\pi')$,
- *Condorcet if*
  $Cond(A,\pi) \neq \emptyset \Rightarrow \mathscr{S}(A,\pi) = Cond(A,\pi)$,
- monotonic *if*
  $x \in \mathscr{S}(A,\pi) \Rightarrow x \in \mathscr{S}(A,\pi^{x\uparrow})$,
- properly monotonic *if it is monotonic and*
  $[x \neq y \text{ and } y \notin \mathscr{S}(A,\pi)] \Rightarrow y \notin \mathscr{S}(A,\pi^{x\uparrow})$,

for all $\pi,\pi' \in \mathscr{G}(X)$, all $A \in \mathscr{P}(X)$, all $\sigma \in \Sigma(X)$, all $x,y \in X$ and all $\pi^{x\uparrow} \in \mathscr{G}(X)$, with $\pi^{x\uparrow} \neq \pi$, improving x w.r.t. π.

We refer to de Donder et al. (2000), Dutta and Laslier (1999), Henriet (1985), Laslier (1997), Moulin (1986) and Peris and Subiza (1999) for a thorough overview of the variety and the properties of neutral, local, *Condorcet* and monotonic choice procedures. An example of such procedures is $SUC(A,\pi)$ (see Dutta and Laslier 1999) as defined above.

Remark 3. Note that, with the question of ranking by choosing procedures in mind, only *local* choice procedures raise problems. Using a non local choice procedure, e.g., the one selecting in all $A \in \mathscr{P}(X)$ alternatives of maximal Copeland score in X, instead of A, it is easy to obtain a monotonic ranking by choosing procedure.

Choice procedures may be viewed as associating a *choice function* (see Moulin 1985) on X to every comparison function π defined on X. Hence, when π is kept fixed, classical properties of choice functions may be transferred to choice procedures. We recall some of them below, referring the reader to Aizerman (1985), Aizerman and Aleskerov (1995), Malishevski (1993), Moulin (1985) and Sen (1997) for a detailed study of these conditions and their relations to the classical one guaranteeing that a choice functions can be rationalized, i.e. that there is a complete binary relation on X such that chosen elements in any subset are the greatest elements of this binary relation restricted to that subset.

Definition 6 (Choice functions properties of choice procedures). *A choice procedure $\mathscr{S}$ on X is said to satisfy:*

– Strong Superset Property (*SSP*) *if*

$[\mathscr{S}(A,\pi) \subseteq B \subseteq A] \Rightarrow \mathscr{S}(B,\pi) = \mathscr{S}(A,\pi)$,

– Aizerman if

$[\mathscr{S}(A,\pi) \subseteq B \subseteq A] \Rightarrow \mathscr{S}(B,\pi) \subseteq \mathscr{S}(A,\pi)$,

– Idempotency *if*

$\mathscr{S}(\mathscr{S}(A,\pi),\pi) = \mathscr{S}(A,\pi)$,

– β^+ *if*

$[A \subseteq B$ and $A \cap \mathscr{S}(B,\pi) \neq \emptyset] \Rightarrow \mathscr{S}(A,\pi) \subseteq \mathscr{S}(B,\pi)$,

for all $\pi \in \mathscr{G}(X)$ and all $A, B \in \mathscr{P}(X)$.

Remark 4. We follow here the terminology of Moulin (1995) that has gained wide acceptance. Let us however observe that the name *Aizerman*, is especially unfortunate since, in fact, M.A. Aizerman and his collaborators apparently *never* used this condition in their classical works on choice functions; on the contrary, they made central use of *SSP* under the name *Outcast* (see Aizerman and Malihevski 1981; Aizerman 1985; Aizerman and Aleskerov 1995). We follow Sen (1977) for β^+.

Let us observe that *SSP* clearly implies both *Aizerman* and *Idempotency*. The reverse implication is also true (see Aizerman and Aleskerov 1995; Dutta and Laslier 1999; Moulin 1985). On the other hand, *SSP* and β^+ are independent conditions (see Aizerman 1985; Aizerman and Aleskerov 1995; Malishevski 1993; Sen 1977). Clearly, none of these conditions is sufficient to imply that the choice function can be rationalized (for such conditions, see Aizerman and Aleskerov 1995; Moulin 1985; Sen 1977).

Remark 5 (Refining choices). Let $\mathscr{S}$ be a choice procedure on X and define $\mathscr{S}^1 = \mathscr{S}$. For all integers $k \geq 2$, we define $\mathscr{S}^k$ and $\mathscr{S}^\infty$ letting, for all $A \in \mathscr{P}(X)$ and all $\pi \in \mathscr{G}(X)$,

$$\mathscr{S}^k(A,\pi) = \mathscr{S}(\mathscr{S}^{k-1}(A,\pi),\pi) \text{ and}$$
$$\mathscr{S}^\infty(A,\pi) = \bigcap_{k \geq 1} \mathscr{S}^k(A,\pi).$$

It is clear that $\mathscr{S}^k$ and $\mathscr{S}^\infty$ are choice procedures. They are obtained by successive refinements of $\mathscr{S}$. It is well-known that when $\mathscr{S}$ is monotonic but not idempotent, it may happen that $\mathscr{S}^\infty$ is not monotonic. This the case with *SUC* (see Laslier 1997).

An apparently open question is to find necessary and sufficient conditions on $\mathscr{S}$ so that this is the case. This problem is clearly related to the alreadymentioned monotonicity problems encountered in electoral procedures with runoffs. We do not study it here.

2.4. Ranking procedures induced by choice procedures

Ranking by choosing procedures build a weak order by successive applications of a choice procedure, its first equivalence class consisting of the elements chosen in X, the second equivalence class of the elements chosen after the elements chosen at the first step are removed from X and so on. We need some more notation in order to formalize this idea. Let W be a weak order on a set Y. We denote by $Cl_k(Y, W)$ (where k is an integer ≥ 1) the elements in the k-th equivalence class of W, i.e. $Cl_1(Y, W) = \{x \in Y : xWy, \forall y \in Y\}$ and, for all $k \geq 2$,

$$Cl_k(Y, W) = \{x \in Z_{k-1} = Y \backslash [\bigcup_{\ell=1}^{k-1} Cl_\ell(Y, W)] : xWy, \forall y \in Z_{k-1}\}.$$

Note that $Cl_1(Y, W)$ is always nonempty and that a weak order is clearly uniquely defined by its ordered set of equivalence classes.

Similarly, we denote $R_k(X, \mathscr{S}, \pi)$, the unchosen elements in X with π after $k \in \mathbb{N}$ applications of $\mathscr{S}$, i.e.

$$R_0(X, \mathscr{S}, \pi) = X,$$
$$R_k(X, \mathscr{S}, \pi) = R_{k-1}(X, \mathscr{S}, \pi) \backslash \mathscr{S}(R_{k-1}(X, \mathscr{S}, \pi), \pi),$$

with the understanding that $\mathscr{S}(\emptyset, \pi) = \emptyset$. Note that $R_0(X, \mathscr{S}, \pi)$ is nonempty by construction.

Definition 7 (Ranking procedure induced by a choice procedure). *Let $\mathscr{S}$ be a choice procedure on X. The ranking procedure $\succsim_{\mathscr{S}}$ induced by $\mathscr{S}$ is the ranking procedure such that, for all $\pi \in \mathscr{G}(X)$ and all integers $k \geq 1$,*

$$Cl_k(X, \succsim_{\mathscr{S}}) = \mathscr{S}(R_{k-1}(X, \mathscr{S}, \pi), \pi).$$

Some properties of $\mathscr{S}$ are easily transferred to $\succsim_{\mathscr{S}}$.

Lemma 1. (Transferring properties from choice procedures to ranking procedures)

- If $\mathscr{S}$ is neutral then $\succsim_{\mathscr{S}}$ is neutral.
- If $\mathscr{S}$ is Condorcet then $\succsim_{\mathscr{S}}$ is faithful for linear orders.
- If $\mathscr{S}$ is based on a scoring function with all functions $F_{|A|}$ being increasing in all arguments then $\succsim_{\mathscr{S}}$ is faithful.
- If $\mathscr{S}$ is a local, neutral, Aizerman and refines UC then $\succsim_{\mathscr{S}}$ is faithful.

Proof. The first three assertions are immediate from the definitions. Let us prove the last one. Suppose that W is a weak order. It is clear that $UC(X, W) = Cl_1(X, W)$. Since $\mathscr{S}$ refines UC we must have $\mathscr{S}(X, W) \subseteq Cl_1(X, W)$. We have $\mathscr{S}(X, W) \subseteq Cl_1(X, W) \subseteq X$. Hence, since $\mathscr{S}$ is *Aizerman*, $\mathscr{S}(Cl_1(X, W), W) \subseteq \mathscr{S}(X, W)$. Since $\mathscr{S}$ is local and neutral, we know that $\mathscr{S}(Cl_1(X, W), W) = Cl_1(X, W)$. Hence, $\mathscr{S}(X, W) = Cl_1(X, W)$. The conclusion follows from a repetition of this argument. ■

Unfortunately, as shown in Example 1 above, monotonicity is not transferred as easily from choice procedures to ranking procedures. Since monotonicity seems to be a vital condition for the reasonableness of a ranking procedure, we investigate below which choice procedures $\mathscr{S}$ have an associated ranking procedure $\succsim_{\mathscr{S}}$ that is monotonic or weakly monotonic.

Remark 6. It should be observed that given a scoring function $Score_F$ the ranking procedures $\succsim_F$ and $\succsim_{\mathscr{S}_F}$ may have quite different properties. Considering for instance the Kramer score $Score_{\min}$ and its extension to choice procedures, it is easy to see that $\succsim_{\min}$ is not faithful (since all alternatives not belonging to the first equivalence of a weak order are tied with $\succsim_{\min}$). On the contrary, it is clear that that $\succsim_{\mathscr{S}\min}$ is indeed faithful.

3. Results

3.1 Weak monotonicity

Our aim is to find conditions on choice procedures that would guarantee that the ranking procedures they induce are weakly monotonic. As already shown by Example 1, there are choice procedures $\mathscr{S}$ that are neutral, local, (properly) monotonic and *Condorcet* while $\succsim_{\mathscr{S}}$ is not even very weakly monotonic. Guaranteeing that $\succsim_{\mathscr{S}}$ is weakly monotonic is therefore not as trivial a task as it might appear at first sight.

Our central result in this section says that any local and monotonic choice procedure satisfying *SSP* generates a ranking procedure that is weakly monotonic.

Proposition 1 (SSP and weak monotonicity). *If $\mathscr{S}$ is local, monotonic and satisfies SSP then $\succsim_{\mathscr{S}}$ is weakly monotonic.*

Proof. Suppose that $\mathscr{S}$ is local, monotonic and satisfies *SSP* and that $\succsim_{\mathscr{S}}$ is not weakly monotonic. By definition this implies that for some $\pi \in \mathscr{G}(X)$, some $x, y \in X$ and some $\pi^{x\uparrow}$ improving $x \in X$ w.r.t. π, we have $x \succsim_{\mathscr{S}}(\pi) y$ and $y \succ_{\mathscr{S}} (\pi^{x\uparrow})x$.

Since $\mathscr{S}$ is monotonic, it is impossible that $x \in Cl_1(X, \succsim_{\mathscr{S}}(\pi)) = \mathscr{S}(X, \pi)$ since this would imply $x \in Cl_1(X, \succsim_{\mathscr{S}}(\pi^{x\uparrow})) = \mathscr{S}(X, \pi^{x\uparrow})$, which violates $y \succ_{\mathscr{S}} (\pi^{x\uparrow})x$. By construction, we know that $x \notin Cl_1(X, \succsim_{\mathscr{S}}(\pi^{x\uparrow})) = \mathscr{S}(X, \pi^{x\uparrow})$. Let $Z = X \setminus \{x\}$. We have $\pi|_Z = \pi^{x\uparrow}|_Z$. Since $\mathscr{S}$ is local, this implies $\mathscr{S}(Z, \pi) = \mathscr{S}(Z, \pi,^{x\uparrow})$.

Since $x \notin \mathscr{S}(X, \pi)$, we have $\mathscr{S}(X, \pi) \subseteq Z \subseteq X$ and *SSP* implies $\mathscr{S}(Z, \pi) = \mathscr{S}(Z, \pi)$. Similarly, we know that $x \notin \mathscr{S}(X, \pi^{x\uparrow})$ so that $\mathscr{S}(X, \pi^{x\uparrow}) \subseteq Z \subseteq X$ and *SSP* implies $\mathscr{S}(X, \pi^{x\uparrow}) = \mathscr{S}(Z, \pi^{x\uparrow})$. Because $\mathscr{S}(Z, \pi) = \mathscr{S}(Z, \pi^{x\uparrow})$, we have $\mathscr{S}(X, \pi^{x\uparrow}) = \mathscr{S}(X, \pi)$ and, hence, $Cl_1(X, \succsim_{\mathscr{S}}(\pi)) = Cl_1(X, \succsim_{\mathscr{S}}(\pi^{x\uparrow}))$. Note, in particular that $y \notin Cl_1(X, \succsim_{\mathscr{S}}(\pi^{x\uparrow}))$.

It is now impossible that $x \in Cl_2(X, \succsim_{\mathscr{S}}(\pi))$. Indeed this would imply that $x \in \mathscr{S}(R_1(X, \mathscr{S}, \pi), \pi)$, so that, using the monotonicity of $\mathscr{S}$,

$x \in \mathcal{S}(R_1(X, \mathcal{S}, \pi), \pi^{x\uparrow})$. Since $R_1(X, \mathcal{S}, \pi) = R_1(X, \mathcal{S}, \pi^{x\uparrow})$, this would imply $x \in Cl_2(X, \succsim_{\mathcal{S}}(\pi^{x\uparrow}))$, which would contradict $y \succ_{\mathcal{S}} (\pi^{x\uparrow})x$.

Because $\mathcal{S}$ is local, the above reasoning can now be applied to $R_1(X, \mathcal{S}, \pi) = R_1(X, \mathcal{S}, \pi^{x\uparrow})$. As above, this leads to $Cl_2(X, \succsim_{\mathcal{S}}(\pi)) = Cl_2(X, \succsim_{\mathcal{S}}(\pi^{x\uparrow}))$ and $y \notin Cl_2(X, \succsim_{\mathcal{S}}(\pi^{x\uparrow}))$.

Iterating the above reasoning easily leads to a contradiction. ■

Let us note that in the literature on tournaments it is possible to find rather well-behaved choice procedures that are neutral, local, monotonic while satisfying *SSP* (e.g., *MCS*, *BP*, as defined below, see Laslier 1997). For general comparison functions, Dutta and Laslier (1999) also present several such procedures. Proposition 1, therefore shows that there are many well-behaved weakly monotonic ranking procedures induced by choice procedures. Let us give an example of such a procedure.

Example 6 (sign essential set). The bipartisan set *BP* defined for tournaments (see Laffond et al. 1993a) has recently been generalized to comparison functions (see Dutta and Laslier 1999; de Donder et al. 2000). Observe that any comparison function π induces a symmetric two-person zero-sum game (in which each of the two players have the set of strategies X and the payoff functions are given by $\pi(x, y)$ and $\pi(y, x)$). The same is clearly true for π_{sign}.

It is well-known that all such games have Nash equilibria in mixed strategies (see von Neumann and Morgenstern 1947). The Sign Essential Set (*SES*) consists in all pure strategies that are played with strictly positive probability in one of the Nash equilibria in the symmetric two-person zero-sum game induced by π_{sign}.

Dutta and Laslier (1999) show that *SES* defines a choice procedure that is monotonic, *Condorcet* and satisfies *SSP*, on top of being clearly local and neutral. It is not difficult to show that it refines *UC* (as well as several other reasonable choice procedures). Hence, using lemma 1 and proposition 1, we know that $\succsim_{SES}$ is a neutral, faithful and weakly monotonic ranking procedure. It therefore qualifies as a very reasonable ranking by choosing procedure.

Remark 7 (Aizerman cannot be substituted to SSP). The above proposition does not hold if *Aizerman* is substituted to *SSP*. It is well-known that *SUC* is monotonic and satisfies *Aizerman* but violates *SSP* (see Laslier 1997). The following example shows that $\succsim_{SUC}$ is not even very weakly monotonic.

Example 7 ($\succsim_{SUC}$ is not very weakly monotonic). Let $X = \{a, b, c, d, e, f, g\}$. Consider the tournament T on X defined by:

$$\begin{array}{c} aTb, aTd, aTe, aTf, aTg, \\ bTc, bTd, bTe, bTf, bTg, \\ cTa, cTe, cTf, cTg, \\ dTc, dTe, \\ eTf, \\ fTd, fTg, \\ gTd, gTe. \end{array}$$

It is easy to check, using the comparison function defined by (1), that $SUC(X,T) = \{a,b,c\}, SUC(X\backslash\{a,b,c\},T) = \{e,f,g\}$. Hence, we have $f \succ_{SUC} (T)d$.

Consider now the tournament V identical to T except that eVb. We have: $SUC(X,V) = \{a,b,c,d\}$, so that $d \succ_{SUC} (V)e$. This shows that $\gtrsim_{SUC}$ is not very weakly monotonic.

Remark 8 (Monotonicity is not implied). It is clearly tempting to look for a result similar to Proposition 1 involving the monotonicity of $\gtrsim_{\mathscr{S}}$. This problem is far more difficult than with weak monotonicity and we only have negative results on that point. Proposition 2 below implies that Proposition 1 is no longer true if monotonicity is substituted to weak monotonicity.

Remark 9 (SSP is not necessary). For local and monotone choice procedures π, *SSP* is a sufficient condition for $\gtrsim_{\mathscr{S}}$ to be weakly monotonic. It is not necessary however, even when attention is restricted to the well-structured case of tournaments. Let us consider this case and show that there are, on some sets X, choice procedures violating *SSP* while being weakly monotonic. We abuse notation in the sequel and write T instead of π_T.

Suppose that $|X| = 5$. The following example shows that *SUC* may violate *SSP*.

Example 8 (SUC violates SSP when $|X| = 5$). Let $X = \{a,b,c,d,e\}$. Consider the tournament T on X defined by:

$$aTb, aTd,$$
$$bTc, bTe,$$
$$cTa, cTd, cTe,$$
$$dTb, dTe,$$
$$eTa.$$

We have $SUC(X,T) = \{a,b,c,d\}$ (e is covered by c) and $SUC(\{a,b,c,d\},T) = \{a,b,c\}$ (d is covered by a). This violates *SSP* since $SUC(X,T) \subseteq \{a,b,c,d\} \subseteq X$ but $SUC(\{a,b,c,d\},T) = \{a,b,c\} \neq SUC(X,T) = \{a,b,c,d\}$.

Let us now show that, when $|X| \leq 5$, $\gtrsim_{SUC}$ is weakly monotonic. It clearly suffices to show that weak monotonicity holds when an alternative is improved w.r.t. a single other alternative. The proof uses the following well-known facts on uncovered elements in a tournament.

Lemma 2. *(Miller 1977, 1980; Moulin 1986)*

1. *$x \in SUC(A,T)$ iff for all $y \in A\backslash\{x\}$, either xTy or $[xTz$ and $zTy]$, for some $z \in A$ (2-step principle).*
2. *$SUC(A,T) = \{x\}$ iff xTy for all $y \in A\backslash\{x\}$.*
3. *If $|SUC(A,T)| \neq 1$ then $|SUC(A,T)| \geq 3$ and we have Cond $(SUC(A,T), T|_{SUC(A,T)}) = \varnothing$.*

Lemma 3. *If $|X| \leq 4$, $\succsim_{SUC}$ is weakly monotonic.*

Proof. If $|X| \leq 3$, the proof easily follows from Lemma 2 and the monotonicity of SUC. If $|X| = 4$, three cases arise by Lemma 2.

1. If $|Cl_1(X, \succsim_{SUC}(T))| = 1$. Let $\{a\} = Cl_1(X, \succsim_{SUC}) = SUC(X, T)$. Since a is a Condorcet winner in X, it is impossible to improve a. If any $b \neq a$ is improved w.r.t. a, it becomes uncovered, using Lemma 2, and weak monotonicity of $\succsim_{SUC}$ cannot possibly be violated. If $b \neq a$ is improved w.r.t. an alternative different from a, then a remains the Condorcet winner and it is clear that weak monotonicity of $\succsim_{SUC}$ cannot possibly be violated.
2. If $|Cl_1(X, \succsim_{SUC}(T))| = 3$ and therefore $|Cl_2(X, \succsim_{SUC}(T))| = 1$. Weak monotonicity of $\succsim_{SUC}$ can only be violated if an element in $Cl_1(X \succsim_{SUC}(T)) = SUC(X, T)$ is improved. Since SUC is monotonic, this improved element will remain uncovered in X. Thus, weak monotonicity cannot possibly be violated.
3. If $|Cl_1(X, \succsim_{SUC}(T))| = 4$, weak monotonicity of $\succsim_{SUC}$ follows from the monotonicity of SUC.

Lemma 4. *If $|X| = 5$, $\succsim_{SUC}$ is weakly monotonic.*

Proof. Four cases arise by Lemma 2.

1. If $|Cl_1(X, \succsim_{SUC}(T))| = 1$. Let $\{a\} = Cl_1(X, \succsim_{SUC}) = SUC(X, T)$. Since a is a Condorcet winner in X, it is impossible to improve a. If an alternative not in $Cl_1(X, \succsim_{SUC}(T))$ is improved w.r.t. a, it becomes uncovered, because of part 1 of Lemma 2. Thus weak monotonicity cannot be violated. If an alternative not in $Cl_1(X, \succsim_{SUC}(T))$ is improved w.r.t. another alternative not in $Cl_1(X, \succsim_{SUC}(T))$, it is clear that after the improvement a remains a Condorcet winner and, thus, chosen alone in X. In view of Lemma 3, weak monotonicity cannot possibly be violated.
2. If $|Cl_1(X, \succsim_{SUC}(T))| = 3$ and, therefore, $|Cl_2(X, \succsim_{SUC}(T))| = 1$ and $|Cl_3(X, \succsim_{SUC}(T))| = 1$. Let $X = \{a, b, c, d, e\}$ and suppose w.l.o.g. that $Cl_1(X, \succsim_{SUC}(T)) = SUC(X, T) = \{a, b, c]$, $Cl_2(X, \succsim_{SUC}(T)) = \{d\}$ and $Cl_3(X, \succsim_{SUC}(T)) = \{e\}$. We know from Lemma 2 that there is a circuit linking a, b and c and that dTe. We suppose w.l.o.g. that the circuit is aTb, bTc, cTa.

   It is impossible to improve e and to violate weak monotonicity. In view of part 1 of Lemma 2, observe that d can beat at most one alternative in $\{a, b, c\}$ because we know that $d \notin SUC(X, T)$. If d beats exactly one alternative in $\{a, b, c\}$ any improvement of d will make it uncovered. Hence, weak monotonicity cannot be violated. Suppose therefore that d does not beat any alternative in $\{a, b, c\}$. Because $e \notin SUC(X, T)$, e can beat at most one alternative in $\{a, b, c\}$.

   Suppose first that e does not beat any alternative in $\{a, b, c\}$. In any T' improving d, it is not difficult to check that $\succsim_{SUC}(T') = \succsim_{SUC}(T)$ and no violation of weak monotonicity can occur.

Suppose then that e beats one alternative in $\{a, b, c\}$ and suppose w.l.o.g. that eTa. If T' improves d w.r.t. a, we still have $\succsim_{SUC}(T') = \succsim_{SUC}(T)$. If T' improves d w.r.t. b then $SUC(X, T') = \{a, b, c, d\}$ so that no violation of weak monotonicity can occur. If T' improves d w.r.t. c then $SUC(X, T') = \{a, b, c\}$ so that no violation of weak monotonicity can occur.

3. If $|Cl_1(X, \succsim_{SUC}(T))| = 4$. We have $|Cl_2(X, \succsim_{SUC}(T))| = 1$. Weak monotonicity of $\succsim_{SUC}$ can only violated if an element in $Cl_1(X, \succsim_{SUC}(T)) = SUC(X, T)$ is improved. Since SUC is monotonic, this improved element will remain uncovered in X. Thus, weak monotonicity cannot possibly be violated.
4. If $|Cl_1(X, \succsim_{SUC}(T))| = 5$, weak monotonicity of $\succsim_{SUC}$ follows from the monotonicity of SUC. ■

Remark 10. As conjectured by Perny (1995), it is possible to show that if $\mathscr{S}$ is monotonic and satisfies β^+ then $\succsim_{\mathscr{S}}$ is weakly monotonic. This offers alternative sufficient conditions on $\mathscr{S}$ guaranteeing the weak monotonicity of $\succsim_{\mathscr{S}}$ (since there are local, monotonic choice procedures satisfying *SSP* but violating β^+, e.g. *SES*, it is clear that β^+ is not a necessary condition for weak monotonicity). It should nevertheless be observed that:

– The result does not make use of the locality of $\mathscr{S}$, whereas the question of the monotonicity of ranking by choosing procedures is only of particular interest if $\mathscr{S}$ is local,
– it is well-known (see Moulin 1986; Sen, 1977) that β^+ is a very strong condition. For instance, in the case of tournaments, any choice procedure $\mathscr{S}$ satisfying β^+ and *Condorcet* must include the top cycle *TC*, i.e. the choice procedure selecting in A the maximal elements of the asymmetric part of the transitive closure on A of T. Clearly, such choice procedures are highly undiscriminating.

Therefore, although β^+ and *SSP* are independent conditions, we do not pursue this point here and leave to the interested reader the easy proof of the above claim (see http://www.lamsade.dauphine.fr/~bouyssou/).

Remark 11. It is not difficult to observe that the proof of Proposition 1 makes no use of the skew-symmetry property of comparison functions (when weak monotonicity is properly redefined). It can therefore be easily extended to cover more general cases (see Bouyssou 1995) e.g., general valued (or fuzzy) binary relations (see Barrett et al. 1990; Bouyssou 1992a; Bouyssou and Pirlot 1997). We do not explore this point here.

3.2 Monotonicity

Let us consider the case of tournaments (see Laslier 1997; Moulin 1986). There are neutral, monotonic and *Condorcet* choice procedures $\mathscr{S}$ such that $\succsim_{\mathscr{S}}$ is monotonic. This is clearly the case for *TC* which satisfies both *SSP* and β^+. We already observed that *TC* is a very undiscriminating choice procedure for tournaments. It would therefore be of interest to find more discriminating choice procedures $\mathscr{S}$ so that $\succsim_{\mathscr{S}}$ is monotonic. As show below, this proves difficult however.

Proposition 2 (Covering compatibility and Aizerman). *Let $\mathscr{S}$ be a local, neutral and monotonic choice procedure satisfying Aizerman. If $\mathscr{S}$ refines UC then $\succsim_{\mathscr{S}}$ is not monotonic.*

Proof. A necessary condition for $\succsim_{\mathscr{S}}$ to be monotonic is that $\mathscr{S}$ is properly monotonic. Indeed, suppose that that, for some X, some $\pi \in \mathscr{G}(X)$, some $x, y \in X$ with $x \neq y$, we have $y \notin \mathscr{S}(A, \pi)$ and $y \in \mathscr{S}(A, \pi^{x\uparrow})$, where $\pi^{x\uparrow}$ improves x w.r.t. π. This would imply $x \succ_{\mathscr{S}} (\pi) y$ and $x \sim_{\mathscr{S}} (\pi^{x\uparrow}) y$, violating monotonicity.

Thus, the claim will be proved if we can show that, for all neutral and monotonic choice procedures refining *UC* and satisfying *Aizerman*, there is a comparison function π such that $a \in \mathscr{S}(X, \pi)$, $b \notin \mathscr{S}(X, \pi)$ and $b \in \mathscr{S}(X, \pi^{a\uparrow})$, i.e. that $\mathscr{S}$ is not properly monotonic. The following example suffices.

Example 9. Let $X = \{a, b, c, d, e\}$. Consider the tournament T on X defined by:

$$\begin{array}{c} aTd, aTe, \\ bTa, \\ cTa, cTb, \\ dTb, dTc, dTe, \\ eTb, eTc. \end{array}$$

We have $UC(X, T) = \{a, c, d\}$ and aTd, dTc and cTa. Therefore, since $\mathscr{S}$ refines UC, we have $\mathscr{S}(X, T) \subseteq \{a, c, d\} \subseteq X$. Since $\mathscr{S}$ satisfies *Aizerman*, $\mathscr{S}(\{a, c, d\}, T) \subseteq \mathscr{S}(X, T)$.

Because $\mathscr{S}$ is local and neutral, we know that $\mathscr{S}(\{a, c, d\}, T) = \{a, c, d\}$. Hence we must have $\mathscr{S}(X, T) = \{a, c, d\}$.

Consider now the tournament V identical to T except that aVc. Using the same reasoning as above, it is easy to check that $\mathscr{S}(X, V) = UC(X, V) = \{a, b, d\}$. Hence b enter the choice set while a is improved and $\mathscr{S}$ is not properly monotonic. ∎

Remark 12. Perny (1998, 2000) has proposed a different negative result using a "positive discrimination" condition on choice procedures that, in our framework, says that, starting with any comparison function, it is always possible to obtain any alternative as the unique choice provided this alternative is "sufficiently" improved. This negative result only deals with *weak* monotonicity of $\succsim_{\mathscr{S}}$ however.

Remark 13. In a classic social choice context, Juret (2001, Theorem 1) shows that monotonic and *rationalizable* choice procedures induce monotonic ranking by choosing procedures. This positive result seem to contrast with Proposition 2. Let us however observe that, when $|X| \geq 3$, it easily follows from Moulin (1986) that there is no local and *Condorcet* choice procedure satisfying *Chernoff*, i.e., for all $\pi \in \mathscr{G}(X)$ and all $A, B \in \mathscr{P}(X)$, $[A \subseteq B] \Rightarrow \mathscr{S}(B, \pi) \cap A \subseteq \mathscr{S}(A, \pi)$. Indeed suppose that $\{x, y, z\} \subseteq X$ and consider any $\pi \in \mathscr{G}(X)$ such that $\pi(x,y) = 1$, $\pi(y,z) = 1$ and $\pi(z,x) = 1$. If $\mathscr{S}$ is local and *Condorcet* then we must have $\mathscr{S}(\{x,y\},\pi) = \{x\}$, $\mathscr{S}(\{y,z\},\pi) = \{y\}$ and $\mathscr{S}(\{z,x\},\pi) = \{z\}$. Using *Chernoff* implies that $\mathscr{S}(\{x,y,z\},\pi) = \emptyset$, a contradiction.

Since *Chernoff* is a necessary condition for $\mathscr{S}$ to be rationalized and given the correspondence noted above between our setting and $C1$ and $C2$ social choice functions, in the sense of Fishburn (1977), this limits the scope of the result in Juret (2001) either to $C1$ and $C2$ choice procedures that violate locality or *Condorcet* or to $C3$ choice procedures, i.e. procedures that are neither $C1$ (not based on the simple majority relation) nor $C2$ (not based on the 0-weighted tournament based on the profile).

Proposition 2 is fairly negative as long as *Aizerman* and the refinement of *UC* are considered important properties. When this is not the case, it is possible to envisage several choice procedures inducing a monotonic ranking by choosing procedure. As an example, consider the well known TC^* choice procedure (see Schwartz 1986) for weak tournaments selecting in any subset, the maximal elements of the asymmetric part of the transitive closure (on that subset) of the asymmetric part of the weak tournament. Simple examples show that TC^* violates *Aizerman* and does not refine *UC*. Vincke (1992) proves that $\succsim_{TC^*}$ is monotonic (see also Juret 2001). It should however be noticed that $\succsim_{TC^*}$ is a very particular ranking by choosing procedure since the transitive closure operation has a clearly global character, in spite of the progressive restriction on the set of alternatives. This type of ranking by choosing procedures is studied in Juret (2001).

4 Application: The case of tournaments

In this section we apply the above results and observations to the case of tournaments, i.e., we only consider choice procedures defined for comparisons functions derived from tournaments. This case is of particular interest because such choice procedures have been analyzed in depth and, in spite of the restrictiveness of the antisymmetry hypothesis, the underlying choice problem is encountered in many different and important settings.

Laslier (1997) studies in detail seven[4] different choice procedures. We briefly present them below referring the reader to Laslier (1997), Laffond et al. (1995) and Moulin (1986) for precise definitions and results:

Top Cycle *TC*. Selecting in A the element of the first equivalence class of the weak order being the transitive closure of T on A.

[4] Since it is not known whether the Tournament Equilibrium Set introduced in Schwartz (1990) is a monotonic choice procedure, we do not envisage it here. We refer the reader to Laffond et al. (1993b) for a thorough analysis of the many open problems concerning this choice procedure

Copeland *Cop*. Selecting in A the alternatives with the highest Copeland score in the tournament restricted to A.

Slater *SL*. Selecting in A all alternatives having the first rank in a linear order on A at minimal distance of the restriction of T on A.

Uncovered Set *UC*. Selecting all the uncovered alternatives in A (Fishburn 1977; Miller 1977).

Banks *B*. Selecting all alternatives in A starting a maximal transitive path of T on A (Banks 1985),

Minimal Covering Set *MCS*. Selecting all alternatives in the unique covering set included in A of minimal cardinality (Dutta 1988),

Bipartisan Set *BP*. Selecting in A all alternatives in the support of the unique Nash equilibrium of the symmetric two-person zero-sum game on A induced by T (Laffond et al. 1993a).

We summarize the monotonicity properties of the ranking procedures induced by these seven choice procedures in the following:

Proposition 3. *(Ranking by choosing procedures for Tournaments)*

1. $\succsim_{TC}$ *is monotonic,*
2. $\succsim_{MCS}$ *and* $\succsim_{BP}$ *are weakly monotonic but not monotonic,*
3. $\succsim_{UC}$, $\succsim_{B}$, $\succsim_{COP}$ *and* $\succsim_{SL}$ *are not very weakly monotonic.*

Proof. Part 1 is left to reader as an easy exercise. The weak monotonicity of $\succsim_{MCS}$ and $\succsim_{BP}$ results from Proposition 1, since it is well-known that both procedures are neutral, local, monotonic and satisfy *SSP*. The fact that they are not monotonic follows from Proposition 2 since they both refine *UC*.

Part 3. We respectively showed in examples 1 and 7 that $\succsim_{Cop}$ and $\succsim_{UC}$ are not very weakly monotonic. It is easy to see that example 7 also shows that $\succsim_{B}$ is not very weakly monotonic; we have $\succsim_{UC} = \succsim_{B}$ for both tournaments used in this example. It remains to show that $\succsim_{SL}$ is not very weakly monotonic. We skip the quite cumbersome details of the computation of Slater's orders below. Details can be found at http://www.lamsade.dauphine.fr/˜bouyssou. We do not know whether this example is minimal.

Example 10 ($\succsim_{SL}$ is not very weakly monotonic). Let $X = \{a, b, c, d, e, f, g, h, i\}$. Consider the tournament T on X defined by:

$$
\begin{gathered}
aTb, aTe, aTg, aTh, aTi, \\
bTc, bTe, bTf, bTg, bTi, \\
cTa, cTd, cTe, cTf, \\
dTa, dTb, dTe, dTi, \\
eTf, eTh, \\
fTa, fTd, fTh, fTi, \\
gTc, gTd, gTe, gTf, gTh, \\
hTb, hTc, hTd, hTi, \\
iTc, iTe, iTg.
\end{gathered}
$$

Linear orders at minimal distance of T are at distance $d = 10$. There are exactly 40 such orders and we have $SL(X, T) = \{a, b, d, f, g, h\}$. It is clear that the restriction of T to $\{c, e, i\}$ is the linear order iTc, cTe, iTe. Hence, we have $i \succ_{SL} (T)c$.

Consider now the tournament V identical to T except that iVa. Again skipping details, linear orders at minimal distance of V are at distance $d = 10$. There are exactly 11 such orders.We have $SL(X, V) = \{b, g, h\}$. Similarly, we obtain $SL(X \setminus \{b, g, h\}, V) = \{c\}$. Therefore $c \succ_{SL} (V)i$. This shows that $\succsim_{SL}$ is not very weakly monotonic.

5 Discussion

Using a ranking by choosing procedure raises serious monotonicity problems. Rather surprisingly, as shown by Proposition 1, it is possible to isolate a class of well-behaved choice procedures that lead to *weakly* monotonic ranking by choosing procedures. If weak monotonicity is considered as an attractive property, these ranking procedures may well be good candidates to compete with other ranking procedures. If monotonicity is considered of vital importance, then the situation is more critical since, as shown in Proposition 2, there are no local, neutral, monotonic and *Aizerman* choice procedure that is reasonably discriminatory being included in *UC* and inducing a monotonic ranking procedure. This suggests several directions for future research.

It would clearly be interesting to look for necessary and sufficient conditions on $\mathscr{S}$ for $\succsim_{\mathscr{S}}$ to be (weakly) monotonic. In view of Remark 9, this task is likely to be complex since the repeated use of $\mathscr{S}$ in order to build $\succsim_{\mathscr{S}}$ only uses the result of the application of $\mathscr{S}$ on a relatively small number of subsets. Another intriguing problem would be to look for connections between the problem studied here and the one of finding necessary and sufficient conditions guaranteeing that $\mathscr{S}^{\infty}$ is monotonic. More research in this direction is clearly needed.

The difficulties encountered with ranking procedures induced by choice procedures may also be considered as an incentive to study ranking procedures for their own sake, i.e. independently of any choice procedure. Research in that direction has already started (see Bouyssou 1992b; Bouyssou and Perny 1992; Bouyssou and Pirlot 1997; Bouyssou and Vincke 1997; Henriet 1985; Fodor and Roubens 1994; Gutin and Yeo 1996; Kano and Sakamoto 1983; Rubinstein 1980; Vincke 1992) mainly considering ranking procedures based on scoring functions. This is at variance with the advice in Moulin (1986) to focus research on ranking procedure based on the approximation of a tournament (or a comparison function) by linear orders (or weak orders). This idea dates back at least to Barbut (1959), Kemeny (1959), Kemeny and Snell (1962) and Slater (1961). Although it raises fascinating deep combinatorial questions and difficult algorithmic problems (see Barthélémy et al. 1989; Barthélémy and Monjardet 1981, 1988; Bermond 1972; Charon

Fournier et al. 1992; Charon et al. 1996; Hudry 1989; Monjardet 1990), this line of research raises other difficulties. As argued in Perny (1992) and Roy and Bouyssou (1993),

- the choice of the distance function should be analyzed with care as soon as one leaves the easy case of a distance between tournament and linear orders (see, e.g., Roy and Słowiński 1993),
- the likely occurrence of multiple optimal solutions to the optimization problem underlying the approximation is not easily dealt with,
- the normative properties of such procedures are not easy to analyze (see, however, Young and Levenglick 1978).

Hence, studying simpler procedures, e.g. the ones based on scoring functions maybe a good starting point. In many common situations, *ranking* and not *choosing* is the central question and there is a real need for a thorough study of ranking procedures.

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