Room assignment-rent division: A market approach

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Abstract. A group of friends consider renting a house but they shall first agree on how to allocate its rooms and share the rent. We propose an auction mechanism for room assignment-rent division problems which mimics the market mechanism. Our auction mechanism is efficient, envy-free, individually-rational and it yields a non-negative price to each room whenever that is possible with envy-freeness.

1 Introduction

A group of friends rent a house and they shall allocate its rooms and share the rent. Alternatively a group of friends consider renting a house but they shall first agree on how to allocate its rooms and share the rent. They will rent the house only if they can find a room assignment-rent division which appeals to each of them.

In this paper we propose an auction mechanism for room assignment-rent division problems which mimics the market mechanism. In order to do that, a

key first step is formulating a notion of an overdemand. A well-known result in discrete mathematics -*Hall's Theorem*- provides an important hint concerning how this shall be done. Hall's Theorem suggests that the room market clears at price p if and only if for any group of agents the number of different rooms collectively demanded by the group is no less than the size of the group. Motivated by Hall's Theorem, Demange, Gale and Sotomayor (1986) define a set Q of objects to be *minimally overdemanded* at price p if (i) the number of agents demanding only objects in Q at price p is greater than the number of objects in Q and (ii) no strict subset of Q has the same property. Demange et al. (1986) introduce this notion in the context of a closely related two-sided matching market and for their purposes it is sufficient to find an arbitrary minimally overdemanded set of objects. We, on the other hand, need to construct the entire set of overdemanded rooms and merely considering minimally overdemanded sets do not suffice. We iteratively apply Demange, Gale and Sotomayor idea in order to find the *full set of overdemanded rooms*.

Once the full set of overdemanded rooms is formulated the rest is an application of the well-known tâtonnement procedure: We initially set the prices equal and find the full set of overdemanded rooms. If it is empty then the procedure stops, each agent is assigned a room in her demand and she pays an even share of the rent. Otherwise we continuously increase prices of all rooms in the full set of overdemanded rooms and continuously decrease prices of remaining rooms such that

(i)   prices of all rooms in the full set of overdemanded rooms are increased at the same rate,
(ii) prices of all remaining rooms are decreased at the same rate, and
(iii) summation of prices stay constant at rent.

(Note that unless the full set of overdemanded rooms consists of half of the rooms, the rate of increase in prices of overdemanded rooms is different than the rate of decrease in remaining rooms.) At each instant the full set of overdemanded rooms is calculated using the updated prices and the price of a room increases at a given instant if and only if the room is overdemanded at that instant. The procedure stops when the full set of overdemanded rooms is empty, each agent is assigned a room in her demand and she pays the final price of her assignment. We refer to this tâtonnement procedure as the *continuous-price auction*.

The only instances that are crucial in the continuous price auction are those when some agent's demand set changes. It is only at those instances the full set of overdemanded rooms may change. We can analytically derive these instances using individual valuations and this observation allows us to formulate an equivalent *discrete-price auction*.

While our auction is dynamic, to be realistic for its real-life consumption it is more appropriate to interpret it as a sealed-bid auction where each agent reports her valuations for the rooms and a computer determines a room assignment together with a rent division via our auction.

Our continuous-price auction (or its discrete equivalent) can be useful only if it converges. Throughout the paper we assume that individual utilities are quasi-linear in prices and in Theorem 1 we show that our discrete-price auction (and hence our continuous-price auction as well) converges. We prove this result by showing that the summation of indirect utilities strictly decreases at each step of the discrete-price auction until it converges to a feasible level in finite steps.

Recently Brams and Kilgour (2001) and Haake et al. (2002) introduce other mechanisms for room assignment-rent division problems. So why shall one care for one additional mechanism? All three mechanisms are efficient so one cannot compare these mechanisms based on efficiency. *Envy-freeness* is widely considered the central notion of fairness in the context of room assignment-rent division problems. It can also be interpreted as a stability requirement since it is difficult to sustain envious allocations in real-life applications. In such situations there are agents who are eager to pay more than their occupants for some of the rooms. If the house is not rented yet, it will most likely not be rented unless the agents agree on an envy-free allocation. Based on these points we believe envy-freeness is essential for room assignment-rent division problems. In addition to envy-freeness, a mechanism should charge a non-negative price to each of the rooms for otherwise agents who are having a positive share of the rent will benefit by leaving the negative priced rooms empty. Unfortunately there exists situations where these two essential objectives cannot be met simultaneously. That is, there exists situations where at least one of the rooms has a negative price at each envy-free allocation. In these situations no matter what allocation is chosen someone will be upset. If agents have not already rented the house, they will either not rent it or they will not rent it altogether.

Brams and Kilgour (2001) observe this difficulty and they propose a mechanism which always charges a non-negative price to each of the rooms. A difficulty with their mechanism is that its outcome may be envious even in problems where there exists envy-free allocations with non-negative prices. Haake et al. (2002), on the other hand, propose an envy-free mechanism but a difficulty with their mechanism is that it may charge negative prices to some of the rooms even in problems where there exists envy-free allocations with non-negative prices. Our auction mechanism is envy-free (Corollary 1) and it charges each room a non-negative price unless there exists no envy-free allocation with non-negative prices (Theorem 2). We obtain this result by relating our auction to the well-known Demange et al. (1986) exact auction that yields the buyer-optimal competitive price for a related class of two-sided matching markets.

There are two additional papers which are closely related to our paper. Alkan et al. (1991) and Su (1999) analyze the structure of envy-free allocations for room-assignment-rent division problems. In addition Svensson (1983), Maskin (1987), Tadenuma and Thomson (1991), Aragones (1995) and Klijn (2000) analyze a closely related fair division problem where a number of indivisible goods together with some money shall be fairly

allocated to a number of agents. Envy-freeness plays the key role in each of these papers.

The rest of the paper is organized as follows: In Sect. 2 we introduce the model. In Sect. 3 we formulate the notion of the full set of overdemanded rooms and introduce our continuous-price auction as well as its discrete equivalent. In Sect. 4 we show that our auction converges and in Sect. 5 we show that it is efficient, envy-free and individually rational. In Sect. 6 we relate our auction to Demange, Gale and Sotomayor exact auction and show that our auction yields non-negative prices whenever there exists envy-free allocations with non-negative prices. Finally we conclude in Sect. 7.

2 The model

A group of friends consider renting a house but they shall first agree on how to allocate its rooms and share the rent. Formally a **room assignment-rent division problem** is a four-tuple $\langle I, R, V, c \rangle$ where

1. $I = \{i_1, \ldots, i_n\}$ is a set of agents,
2. $R = \{r_1, \ldots, r_n\}$ is a set of rooms,
3. $V = [v_r^i]_{i \in I, r \in R}$ is a value matrix where v_r^i denotes the value of room r for agent i, and
4. $c \in \mathbb{R}_{++}$ is the rent of the house.

Following Su (1999), Brams and Kilgour (2001), Haake et al. (2002) we assume that $\sum_{r \in R} v_r^i \geq c$ for each agent $i \in I$. Note that if this assumption fails for an agent that means the agent does not think that the house is worth the rent and hence it is not unreasonable to assume that such an agent will not rent the house. Throughout the paper we fix a problem.

A **matching** μ is an assignment of rooms to agents such that each agent is assigned one and only one room. Let μ_i denote the room assignment of agent i under μ and let $\mathscr{M}$ denote the set of matchings.

A **price** is a vector $p \in \mathbb{R}^n$. A price p is **feasible** if $\sum_{r \in R} p_r = c$. Let

$$\mathscr{P} = \left\{ p \in \mathbb{R}^n : \sum_{r \in R} p_r = c \right\}$$

denote the set of feasible prices.

An **allocation** is a matching-feasible price pair $(\mu, p) \in \mathscr{M} \times \mathscr{P}$. Here agent $i \in I$ is assigned room μ_i and her share of the rent is p_{μ_i}.

Each agent $i \in I$ is endowed with a utility function $u_i : R \times \mathbb{R} \to \mathbb{R}$ which is quasi-linear in rent:

$$u_i(r, p_r) = v_r^i - p_r.$$

We assume that $u_i = 0$ is the reservation utility for each agent and it corresponds to the utility of outside options.

3 A market approach

How shall one determine an allocation for a given problem? In this paper we propose an auction which mimics the market mechanism.

Given an agent $i \in I$ and a price $p \in \mathbb{R}^n$, define the **demand** of agent i at price p as

$$D_i(p) = \{r \in R : u_i(r, p_r) \geq u_i(s, p_s) \text{ for all } s \in R\}.$$

Let $D(p) = (D_i(p))_{i \in I}$ denote the list of individual demands at price p.

Given an agent $i \in I$ and a price $p \in \mathbb{R}^n$, define the **indirect utility** of agent i at price p as

$$\tilde{u}_i(p) = \max_{r \in R} u_i(r, p_r)$$

Given a price vector $p \in \mathbb{R}^n$, when can we find a matching which assigns each agent a room in her demand? The answer of this question is given by Hall (1935).

Hall's Theorem. Let $p \in \mathbb{R}^n$. There exists a matching $\mu \in \mathcal{M}$ with $\mu_i \in D_i(p)$ for each $i \in I$ if and only if

$$\forall J \subseteq I \quad \left| \bigcup_{i \in J} D_i(p) \right| \geq |J|$$

Hall's Theorem suggests that the room market clears at price p if and only if the cardinality of the union of demands of any group of agents is at least as big as the size of the group. Hall's Theorem is key to define the set of over-demanded rooms at price p.

3.1 Overdemanded rooms

Motivated by Hall's Theorem and following Demange et al. (1986) define a set of rooms to be **overdemanded** at price p if the number of agents demanding only rooms in this set is greater than the number of the rooms in the set. Formally $S \subset R$ is overdemanded if $|\{i \in I : D_i(p) \subseteq S\}| > |S|$. Note that this definition allows a room to be overdemanded even though it is not demanded by any agent. For example suppose $D_i(p) = D_j(p) = D_k(p) = \{s\}$. Clearly the singleton $\{s\}$ is an overdemanded set. The difficulty is that $\{s, r\}$ is overdemanded as well for any $r \in R$ regardless of the demands. This observation motivates the following definition: A set of rooms is a **minimal overdemanded set** if it is overdemanded and none of its proper subsets is overdemanded.

Demange et al. (1986) introduce these definitions in the context of multi-unit auctions and at each step of their auction they increase prices of objects in an arbitrary minimal overdemanded set by one unit. Since prices of the rooms shall add up to rent in the present context, we will increase prices of all "excessively demanded" rooms simultaneously and reduce prices of the remaining rooms. As the following example shows merely considering mini-mal overdemanded sets may not be sufficient for our purposes. Let

$I = \{i, j, k, l\}$ and $R = \{a, b, c, d\}$. Suppose that the value matrix V and price p induce the following demands:

$$D_i(p) = D_j(p) = \{a\}, \quad D_k(p) = \{b\}, \quad D_l(p) = \{a, b\}$$

Here the only minimal overdemanded set is $\{a\}$. Nevertheless there is a clear "excess demand" for room b as well.

Motivated by this observation iteratively define the **full set of overdemanded rooms** at price p as follows: Given p find all minimal overdemanded sets. Remove these rooms from the demand of each agent and find the minimal overdemanded sets for the modified demand profiles. Proceed in a similar way until there is no minimal overdemanded set for the modified demand profiles. The full set of overdemanded rooms is the union of each of the sets encountered in the procedure.

In the earlier example the singleton $\{a\}$ is a minimal overdemanded set. Once room a is removed from the demand of each agent we have $D_k(p)\backslash\{a\} = D_l(p)\backslash\{a\} = \{b\}$ and hence room b is also included in the full set of overdemanded rooms at price p.

Let $OD(p)$ denote the full set of overdemanded rooms at price p. The following lemma will be useful to define our auction.

Lemma 1. *Let $p \in \mathbb{R}^n$.*

$$OD(p) = \emptyset \Longleftrightarrow \left| \bigcup_{i \in J} D_i(p) \right| \geq |J| \text{ for all } J \subseteq I$$

Proof. Let $p \in \mathbb{R}^n$ and suppose that $OD(p) = \emptyset$. By definition of $OD(p)$ there is no minimal overdemanded set at price p and thus there is no overdemanded set either. Then for each $J \subseteq I$ we have $|\bigcup_{i \in J} D_i(p)| \geq |J|$ for otherwise $S = \bigcup_{i \in J} D_i(p)$ would be an overdemanded set.

Conversely suppose that for every $J \subseteq I$ we have $|\bigcup_{i \in J} D_i(p)| \geq |J|$. Then there are no overdemanded sets and hence $OD(p) = \emptyset$. ∎

3.2 The continuous-price auction

We are now ready to propose an auction to find a "market" allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$:

1. Set initially the price of each room to c/n. That is, set $p = (\frac{c}{n}, \ldots, \frac{c}{n})$.
2. (a) If $OD(p) = \emptyset$ then by Lemma 1 and Hall's Theorem there exists a matching μ such that $\mu_i \in D_i(p)$ for each agent $i \in I$. Terminate the procedure.
    (b) If $OD(p) \neq \emptyset$ then continuously increase prices of all rooms in $OD(p)$ equally by $dx \to 0$ and continuously decrease prices of all remaining rooms (i.e. rooms in $R\backslash OD(p)$) equally by $dy \to 0$ such that $|OD(p)|dx = (n - |OD(p)|)dy$. This ensures that summation of the prices of the rooms is equal to rent throughout the procedure.
    (c) Return to beginning of Step 2 with the updated price vector.

Note that the auction procedure terminates when we reach a price vector $p \in \mathscr{P}$ with $OD(p) = \emptyset$. In the next section we show that our continuous-price auction converges.

4 Convergence

In the continuous-price auction the only instances that are crucial are those instances where some agent's demand changes. At these instances the full set of overdemanded rooms may possibly change. Between two such instances prices of overdemanded rooms increase uniformly and prices of the remaining rooms decrease uniformly in such a way that the sum of the prices stay constant at c. This observation allows us to formulate the following discrete equivalent of our continuous-price auction.

4.1 An equivalent discrete-price auction

In order to introduce the discrete equivalent of our auction we need additional notation.

Let $p \in \mathscr{P}$ be a price vector that is obtained at an instant of our continuous-price auction. Define

$$J(p) = \{i \in I : D_i(p) \subseteq OD(p)\}.$$

That is, $J(p)$ is the set of agents each of whom only demand rooms in the full set of overdemanded rooms.

Lemma 2. *For each $p \in \mathscr{P}$ with $OD(p) \neq \emptyset$ we have $|J(p)| > |OD(p)|$.*

Proof. Let $p \in \mathscr{P}$ with $OD(p) \neq \emptyset$. Consider the construction of the full set of overdemanded rooms. Let S_1 be an arbitrary minimal overdemanded set and let $J_1 = \{i \in I : D_i(p) \subseteq S_1\}$. By definition we have $S_1 \subseteq OD(p)$ and $|J_1| > |S_1|$. If $S_1 = OD(p)$ then we are done. Otherwise remove rooms in S_1 from the demand of each agent and let S_2 be an arbitrary minimal overdemanded set for the modified market. Let $J_2 = \{i \in I : D_i(p)\backslash S_1 \subseteq S_2$ and $D_i(p)\backslash S_1 \neq \emptyset\}$. Note that J_1 and J_2 are disjoint sets. By definition we have $S_2 \subseteq OD(p)$ and $|J_2| > |S_2|$. Proceeding in a similar way we obtain $|J(p)| = |\cup J_k| > |\cup S_k| = |OD(p)|$. ∎

Let p' be the first price vector obtained in the continuous-price auction after price p where an agent's demand set gets larger. Such an agent is necessarily a member of $J(p)$. That is because (i) each agent in $I\backslash J(p)$ demands a room in $R\backslash OD(p)$ and prices of these rooms have been falling uniformly, and (ii) utilities are quasi-linear in prices. Therefore the full set of overdemanded rooms may only change when an agent in $J(p)$ demands a room in $R\backslash OD(p)$. Define

$$x(p) = \begin{cases} \min_{j \in J(p)} \left(\tilde{u}_j(p) - \max_{s \in R \setminus OD(p)} u_j(s, p_s) \right) & \text{if } OD(p) \neq \emptyset \\ 0 & \text{if } OD(p) = \emptyset \end{cases}$$

Note that $x(p) > 0$ if and only if $OD(p) \neq \emptyset$.

Consider any pair of rooms r, s such that $r \in OD(p)$ and $s \in R \setminus OD(p)$. The price differential $(p_r - p_s)$ increases at the same rate for any pair of such rooms until the full set of overdemanded rooms changes. As we have already mentioned this may only happen when an agent in $J(p)$ demands a room in $R \setminus OD(p)$ and $x(p)$ is the minimum price differential needed for that to happen. When price of room $r \in OD(p)$ increases to $p_r + \frac{n - |OD(p)|}{n} x(p)$ and price of room $s \in R \setminus OD(p)$ reduces to $p_s - \frac{|OD(p)|}{n} x(p)$, the price differential $(p_r - p_s)$ reaches $x(p)$. For each $r \in R$ define

$$f_r(p) = \begin{cases} p_r - \frac{|OD(p)|}{n} x(p) & \text{if } r \notin OD(p) \\ p_r + \frac{n - |OD(p)|}{n} x(p) & \text{if } r \in OD(p) \end{cases}$$

By construction $p' = (f_r(p))_{r \in R}$ is the first price vector obtained in the continuous-price auction where an agent's demand set gets larger. We are now ready to introduce the discrete equivalent of our continuous-price auction:

Step 0. Set $p^0 = (\frac{c}{n}, \ldots, \frac{c}{n})$. If $OD(p^0) = \emptyset$ then find a matching μ such that $\mu_i \in D_i(p^0)$ for each $i \in I$, set $p = p^0$ and terminate the procedure. If $OD(p^0) \neq \emptyset$ then proceed to Step 1.

In general,

Step t. Set $p_r^t = f_r(p^{t-1})$ for all $r \in R$. If $OD(p^t) = \emptyset$ then find a matching μ such that $\mu_i \in D_i(p^t)$ for each $i \in I$, set $p = p^t$ and terminate the procedure. If $OD(p^t) \neq \emptyset$ then proceed to Step $t + 1$.

Before we show that the discrete-price auction converges, we give a detailed example which illustrates the dynamics of the discrete-price auction.

Example. Let the set of agents be $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, the set of rooms be $R = \{a, b, c, d, e, f\}$, the valuation matrix be

		a	b	c	d	e	f
	i_1	15	18	10	15	24	28
	i_2	18	24	3	18	25	15
$V =$	i_3	6	25	15	18	18	25
	i_4	18	5	18	12	9	25
	i_5	6	22	5	5	10	12
	i_6	6	9	2	21	25	9

and the rent be 60.

Step 0. $p^0 = (10, 10, 10, 10, 10, 10)$. In order to obtain the demand of each agent at p^0, we shall find utilities of agents over rooms at p^0. In the following utility matrix indirect utilities of agents are given in bold:

$$[u_i(r, p_r^0)]_{i \in I,\, r \in R} = $$

	a	b	c	d	e	f
i_i	5	8	0	5	14	**18**
i_2	8	14	−7	8	15	5
i_3	−4	15	5	8	8	15
i_4	8	−5	8	2	−1	15
i_5	−4	12	−5	−5	0	2
i_6	−4	−1	−8	11	15	−1

Therefore the demand of each agent at p^0 is as follows:

$$D_{i_1}(p^0) = \{f\} \quad D_{i_3}(p^0) = \{b, f\} \quad D_{i_5}(p^0) = \{b\}$$
$$D_{i_2}(p^0) = \{e\} \quad D_{i_4}(p^0) = \{f\} \quad D_{i_6}(p^0) = \{e\}$$

Next we find the full set of overdemanded rooms at p^0:

Iteration 1. $S_1 = \{f\}$ is minimally overdemanded since each of the agents in $J_1 = \{i_1, i_4\}$ demand only room f. $S_2 = \{e\}$ is minimally overdemanded since each of the agents in $J_2 = \{i_2, i_6\}$ demand only room e. There are no other minimal overdemanded sets. Remove $S_1 \cup S_2$ from the demand of each agent.

Iteration 2. $S_3 = \{b\}$ is minimally overdemanded once rooms f and e are removed from the demands. That is because agents in $J_3 = \{i_3, i_5\}$ demand only room b once rooms f and e are removed from the demands. (That is, $D_{i_3}(p^0)\backslash(S_1 \cup S_2) = D_{i_5}(p^0)\backslash(S_1 \cup S_2) = \{b\}$.) S_3 is the unique minimal overdemanded set once rooms e and f are removed from the demands. Remove S_3 from the demand of each agent.

Iteration 3. There are no minimal overdemanded sets once rooms f, e and b are removed from demands.

Therefore $OD(p^0) = S_1 \cup S_2 \cup S_3 = \{b, e, f\}$ and $J(p^0) = J_1 \cup J_2 \cup J_3 = \{i_1, i_2, i_3, i_4, i_5, i_6\}$. Since $OD(p^0) \neq \emptyset$ we proceed with Step 1.
Step 1. We determine p^1 as follows: $x(p^0) = \tilde{u}_{i_6}(p^0) - u_{i_6}(d, p_d^0) = 15 - 11 = 4$ and $|OD(p^0)| = 3$. Therefore for each $r \in R$, we have

$$p_r^1 = f_r(p^0) = \begin{cases} p_r^0 + 2 & \text{if } r \in OD(p^0) \\ p_r^0 - 2 & \text{otherwise} \end{cases}$$

and hence $p^1 = (8, 12, 8, 8, 12, 12)$. Utility matrix at p^1 is given as follows:

$$[u_i(r, P_r^1)]_{i \in I, r \in R} = $$

	a	b	c	d	e	f
i_i	7	6	2	7	12	**16**
i_2	10	12	−5	10	**13**	3
i_3	−2	**13**	7	10	6	13
i_4	10	−7	10	4	−3	13
i_5	−2	**10**	−3	−3	−2	0
i_6	−2	−3	−6	**13**	13	−3

Therefore the demand of each agent at p^1 is as follows:

$$D_{i_1}(p^1) = \{f\} \quad D_{i_3}(p^1) = \{b,f\} \quad D_{i_5}(p^1) = \{b\}$$
$$D_{i_2}(p^1) = \{e\} \quad D_{i_4}(p^1) = \{f\} \quad D_{i_6}(p^1) = \{d,e\}$$

Next we find the full set of overdemanded rooms at p^1:

Iteration 1. $S_1 = \{f\}$ is minimally overdemanded since each of the agents in $J_1 = \{i_1, i_4\}$ demand only room f. S_1 is the unique minimal overdemanded set. Remove S_1 from the demand of each agent.

Iteration 2. $S_2 = \{b\}$ is minimally overdemanded once room f is removed from the demands. That is because agents in $J_2 = \{i_3, i_5\}$ demand only room b once room f is removed from the demands. S_2 is the unique minimal overdemanded set once room f is removed from the demands. Remove S_2 from the demand of each agent.

Iteration 3. There are no minimal overdemanded sets once rooms f and b are removed from demands.

Therefore $OD(p^1) = S_1 \cup S_2 = \{b,f\}$ and $J(p^1) = J_1 \cup J_2 = \{i_1, i_3, i_4, i_5\}$. Since $OD(p^1) \neq \emptyset$ we proceed with Step 2.

Step 2. We determine $p^2 = f(p^1)$ as follows: $x(p^1) = \widetilde{u}_{i_3}(p^1) - u_{i_3}(d, p_d^1) = 13 - 10 = 3$ and $\left|OD(p^1)\right| = 2$. Therefore for each $r \in R$ we have

$$p_r^2 = f_r(p^1) = \begin{cases} p_r^1 + 2 & \text{if } r \in OD(p^1) \\ p_r^1 - 1 & \text{otherwise} \end{cases}$$

and hence $p^2 = (7, 14, 7, 7, 11, 14)$. Utility matrix at p^2 is given as follows:

	a	b	c	d	e	f
i_1	8	4	3	8	13	**14**
i_2	11	10	−4	11	**14**	1
i_3	−1	**11**	8	**11**	7	**11**
i_4	**11**	−9	**11**	5	−2	**11**
i_5	−1	**8**	−2	−2	−1	−2
i_6	−1	−5	−5	**14**	**14**	−5

$$\left[u_i(r, P_r^2)\right]_{i \in I, r \in R} =$$

Therefore the demand of each agent at p^2 is as follows:

$$D_{i_1}(p^2) = \{f\} \quad D_{i_3}(p^2) = \{b,d,f\} \quad D_{i_5}(p^2) = \{b\}$$
$$D_{i_2}(p^2) = \{e\} \quad D_{i_4}(p^2) = \{a,c,f\} \quad D_{i_6}(p^2) = \{d,e\}$$

Next we find the full set of overdemanded rooms at p^2:

Iteration 1. $S_1 = \{b,d,e,f\}$ is minimally overdemanded since each of the agents in $J_1 = \{i_1, i_2, i_3, i_5, i_6\}$ demands only rooms from S_1 and no other

subset of S_1 is overdemanded. S_1 is the unique minimal overdemanded set. Remove S_1 from the demand of each agent.

Iteration 2. There are no minimal overdemanded sets once rooms b, d, e, and f are removed from demands.

Therefore $OD(p^2) = S_1 = \{b, d, e, f\}$ and $J(p^2) = J_1 = \{i_1, i_2, i_3, i_5, i_6\}$. Since $OD(p^2) \neq \emptyset$ we proceed with Step 3.

Step 3. We determine p^3 as follows: $x(p^2) = \tilde{u}_{i_3}(p^2) - u_{i_3}(c, p_c^2) = 14 - 11 = 3$ and $|OD(p^2)| = 4$. Therefore for each $r \in R$ we have

$$p_r^3 = f_r(p^2) = \begin{cases} p_r^2 + 1 & \text{if } r \in OD(p^2) \\ p_r^2 - 2 & \text{otherwise} \end{cases}$$

and hence $p^3 = (5, 15, 5, 8, 12, 15)$. Utility matrix at p^3 is given as follows:

	a	b	c	d	e	f
i_i	10	3	5	7	12	**13**
i_2	**13**	9	−2	10	**13**	0
i_3	1	**10**	**10**	**10**	6	**10**
i_4	**13**	−10	**13**	4	−3	10
i_5	1	**7**	0	−3	−2	−3
i_6	1	−6	−1	**13**	**13**	−6

$$[u_i(r, P_r^3)]_{i \in I, r \in R} =$$

Therefore the demand of each agent at p^3 is as follows:

$$D_{i_1}(p^3) = \{f\} \quad D_{i_3}(p^3) = \{b, c, d, f\} \quad D_{i_5}(p^3) = \{b\}$$
$$D_{i_2}(p^3) = \{a, e\} \quad D_{i_4}(p^3) = \{a, c\} \quad D_{i_6}(p^3) = \{d, e\}$$

Next we find the full set of overdemanded rooms at p^3:

Iteration 1. There are no minimal overdemanded sets.

Therefore $OD(p^3) = \emptyset$ and hence we terminate the procedure. We have $\mu_i \in D_i(p^3)$ for all $i \in I$ for

$$\mu \in \{\mu_1, \mu_2\} = \left\{ \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ f & e & c & a & b & d \end{pmatrix}, \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ f & a & d & c & b & e \end{pmatrix} \right\}$$

and therefore either of the allocations (μ_1, p^3) or (μ_2, p^3) can be obtained as an outcome of our auction. ∎

We next show that our discrete-price auction converges which in turn implies that our continuous-price auction converges as well.

Theorem 1. *Let* $\{p^t\}$ *be the price sequence in the discrete-price auction. There exists finite T such that $OD(p^T) = \emptyset$.*

Proof. Let $\{p^t\}$ be the price sequence in the discrete-price auction. Here is our proof strategy: We will first show that the sum of indirect utilities are

bounded below. Then we will show that the sum of indirect utilities $\sum_{i \in I} \tilde{u}_i(p^t)$ strictly decreases at each step as long as $OD(p)$ is non-empty. This implies that (i) the sum of indirect utilities will converge and (ii) for any two steps t, u with $OD(p^t) \neq \emptyset$ and $OD(p^u) \neq \emptyset$ we have $D(p^t) \neq D(p^u)$. Since there are only finite combinations of demand profiles we obtain the desired convergence result in finite steps.

Claim 1. For all $p \in \mathscr{P}$ we have $\sum_{i \in I} \tilde{u}_i(p) \geq \frac{1}{n} \left(\sum_{i \in I} \sum_{r \in R} v_r^i \right) - c$.

Proof of Claim 1. Let $p \in \mathscr{P}$. For all $i \in I$ and $r \in R$ we have $\tilde{u}_i(p) \geq u_i(r, p_r)$. This implies for all $i \in I$ we have

$$\tilde{u}_i(p) \geq \frac{\sum_{r \in R} u_i(r, p_r)}{n} = \frac{\sum_{r \in R} v_r^i - c}{n}$$

Hence we obtain $\sum_{i \in I} \tilde{u}_i(p) \geq \frac{1}{n} \left(\sum_{i \in I} \sum_{r \in R} v_r^i \right) - c$.

Claim 2. For all $t \geq 0$ we have $\sum_{i \in I} \tilde{u}_i(p^t) \geq \sum_{i \in I} \tilde{u}_i(p^{t+1}) + x(p^t)$.

Proof of Claim 2. Let $t \geq 0$ be such that $OD(p^t) \neq \emptyset$. We consider agents in $J(p^t)$ and agents in $I \setminus J(p^t)$ separately.

1. Let $i \in J(p^t)$ and $a \in D_i(p^t)$. By construction of $J(p^t)$ we have $a \in OD(p^t)$ and by construction of $f_a(p^t)$ we have

$$\begin{aligned}
u_i(a, p_a^{t+1}) &= v_a^i - p_a^{t+1} \\
&= v_a^i - \left(p_a^t + \frac{n - |OD(p^t)|}{n} x(p^t) \right) \\
&= u_i(a, p_a^t) - \frac{n - |OD(p^t)|}{n} x(p^t) \\
&= \tilde{u}_i(p^t) - \frac{n - |OD(p^t)|}{n} x(p^t)
\end{aligned}$$

Next consider rooms in $R \setminus D_i(p^t)$. We will show that $u_i(a, p_a^{t+1}) \geq u_i(r, p_r^{t+1})$ for all $r \in R \setminus D_i(p^t)$ which in turn shows that $\tilde{u}_i(p^{t+1}) = u_i(a, p_a^{t+1})$. We consider rooms in $R \setminus OD(p^t)$ and rooms in $OD(p^t) \setminus D_i(p^t)$ separately.

(a) Let $r \in R \setminus OD(p^t)$. By construction of $x(p^t)$ we have

$$\tilde{u}_i(p^t) - u_i(r, p_r^t) \geq x(p^t) = \min_{j \in J(p^t)} \left(\tilde{u}_j(p^t) - \max_{s \in R \setminus OD(p^t)} u_j(s, p_s^t) \right) \tag{1}$$

and by construction of $f_r(p^t)$ we have

$$\begin{aligned}
u_i(r, p_r^{t+1}) &= v_r^i - p_r^{t+1} \\
&= v_r^i - \left(p_r^t - \frac{|OD(p^t)|}{n} x(p^t) \right) \\
&= u_i(r, p_r^t) + \frac{|OD(p^t)|}{n} x(p^t)
\end{aligned}$$

Hence

$$u_i(a,p_a^{t+1}) - u_i(r,p_r^{t+1}) = \widetilde{u}_i(p^t) - \frac{n-|OD(p^t)|}{n}x(p^t) - u_i(r,p_r^t) - \frac{|OD(p^t)|}{n}x(p^t)$$
$$= \widetilde{u}_i(p^t) - u_i(r,p_r^t) - x(p^t)$$

and therefore $u_i(a,p_a^{t+1}) \geq u_i(r,p_r^{t+1})$ for all $r \in R \setminus OD(p^t)$ by Relation 1.

(b) Let $r \in OD(p^t) \setminus D_i(p^t)$. By construction of $f_r(p^t)$ we have

$$u_i(r,p_r^{t+1}) = v_r^i - p_r^{t+1}$$
$$= v_r^i - \left(p_r^t + \frac{n-|OD(p^t)|}{n}x(p^t)\right)$$
$$= u_i(r,p_r^t) - \frac{n-|OD(p^t)|}{n}x(p^t)$$

Hence

$$u_i(a,p_a^{t+1}) - u_i(r,p_r^{t+1}) = u_i(a,p_a^t) - u_i(r,p_r^t) = \widetilde{u}_i(p^t) - u_i(r,p_r^t) > 0$$

and therefore $u_i(a,p_a^{t+1}) \geq u_i(r,p_r^{t+1})$ for all $r \in OD(p^t) \setminus D_i(p^t)$ as well.

Therefore $u_i(a,p_a^{t+1}) \geq u_i(r,p_r^{t+1})$ for all $r \in R \setminus D_i(p^t)$ and hence $a \in D_i(p^{t+1})$. This in turn implies

$$\widetilde{u}_i(p^{t+1}) = u_i(a,p_a^{t+1}) = \widetilde{u}_i(p^t) - \frac{n-|OD(p^t)|}{n}x(p^t) \qquad (2)$$

2. Let $i \in I \setminus J(p^t)$ and $a \in D_i(p^t) \setminus OD(p^t)$. Since $i \notin J(p^t)$, such a room necessarily exists. By construction of $f_a(p^t)$ we have

$$u_i(a,p_a^{t+1}) = v_a^i - p_a^{t+1}$$
$$= v_a^i - \left(p_a^t - \frac{|OD(p^t)|}{n}x(p^t)\right)$$
$$= u_i(a,p_a^t) + \frac{|OD(p^t)|}{n}x(p^t)$$
$$= \widetilde{u}_i(p^t) + \frac{|OD(p^t)|}{n}x(p^t)$$

Next consider rooms in $R \setminus D_i(p^t)$. Let $r \in R \setminus D_i(p^t)$. Since prices of rooms have either decreased by $\frac{|OD(p^t)|}{n}x(p^t)$ or increased by $\frac{n-|OD(p^t)|}{n}x(p^t)$, we have

$$u_i(r,p_r^{t+1}) = v_r^i - p_r^{t+1}$$
$$\leq v_r^i - \left(p_r^t - \frac{|OD(p^t)|}{n}x(p^t)\right)$$
$$= u_i(r,p_r^t) + \frac{|OD(p^t)|}{n}x(p^t)$$
$$< u_i(a,p_a^t) + \frac{|OD(p^t)|}{n}x(p^t) = u_i(a,p_a^{t+1})$$

Therefore $u_i(a, p_a^{t+1}) > u_i(r, p_r^{t+1})$ for all $r \in R \setminus D_i(p^t)$ and hence $a \in D_i(p^{t+1})$. This in turn implies

$$\widetilde{u}_i(p^{t+1}) = u_i(a, p_a^{t+1}) = \widetilde{u}_i(p^t) + \frac{|OD(p^t)|}{n} x(p^t) \tag{3}$$

We are now ready to complete the proof of Claim 2 which in turn completes the proof of the theorem. By Lemma 2 we have $|J(p^t)| \geq |OD(p^t)| + 1$ and $|I \setminus J(p^t)| = n - |J(p^t)| \leq n - |OD(p^t)| - 1$. These together with Equations 2 and 3 imply

$$
\begin{aligned}
\sum_{i \in I} \widetilde{u}_i(p^{t+1}) &= \sum_{i \in J(p^t)} \widetilde{u}_i(p^{t+1}) + \sum_{i \in I \setminus J(p^t)} \widetilde{u}_i(p^{t+1}) \\
&= \sum_{i \in J(p^t)} \left(\widetilde{u}_i(p^t) - \frac{n - |OD(p^t)|}{n} x(p^t) \right) + \sum_{i \in I \setminus J(p^t)} \left(\widetilde{u}_i(p^t) + \frac{|OD(p^t)|}{n} x(p^t) \right) \\
&= \sum_{i \in I} \widetilde{u}_i(p^t) - \frac{|J(p^t)|(n - |OD(p^t)|)}{n} x(p^t) + \frac{(n - |J(p^t)|)|OD(p^t)|}{n} x(p^t) \\
&\leq \sum_{i \in I} \widetilde{u}_i(p^t) - \frac{(|OD(p^t)| + 1)(n - |OD(p^t)|)}{n} x(p^t) + \frac{(n - |OD(p^t)| - 1)|OD(p^t)|}{n} x(p^t) \\
&= \sum_{i \in I} \widetilde{u}_i(p^t) - x(p^t)
\end{aligned}
$$

completing the proof of Claim 2 as well as the theorem.   ∎

5 Efficiency, envy-freeness and individual rationality

Efficiency and fairness often play key roles in evaluation of mechanisms for various resource allocation problems. Envy-freeness (Foley (1967)) is widely considered the central notion of fairness in the context of room assignment-rent division problems.

An allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ is **envy-free** if and only if $u_i(\mu_i, p_{\mu_i}) \geq u_i(r, p_r)$ for all $i \in I$ and $r \in R$.

An allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ is **efficient** if and only if

$$\sum_{i \in I} u_i(\mu_i, p_{\mu_i}) \geq \sum_{i \in I} u_i(\eta_i, q_{\eta_i}) \quad \text{for all } \eta \in \mathcal{M} \text{ and } q \in \mathcal{P}.$$

Since utilities are quasi-linear in prices, an allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ is efficient if and only if $\sum_{i \in I} v_{\mu_i}^i \geq \sum_{i \in I} v_{\eta_i}^i$ for all $\eta \in \mathcal{M}$. Therefore prices have no significance for efficiency considerations.

Proposition 1. *An allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ is envy-free if and only if $\mu_i \in D_i(p)$ for each agent $i \in I$.*

Proof. Let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be such that $\mu_i \in D_i(p)$ for each agent $i \in I$. Then $u_i(\mu_i, p_{\mu_i}) \geq u_i(r, p_r)$ for all $i \in I$ and $r \in R$. Therefore (μ, p) is envy-free.

Conversely let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be envy-free. Then $u_i(\mu_i, p_{\mu_i}) \geq u_i(r, p_r)$ for all $i \in I$ and $r \in R$. Therefore $\mu_i \in D_i(p)$ for all $i \in I$.   ∎

Corollary 1. *The outcome of our auction is envy-free.*

Svensson (1983) show that envy-freeness implies efficiency in the context of room assignment-rent division problems.

Proposition 2 (Svensson 1983). *Let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be an envy-free allocation. Then (μ, p) is efficient.*

Proof. Let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be an envy-free allocation and consider an allocation $(\eta, q) \in \mathcal{M} \times \mathcal{P}$. Since $u_i(\mu_i, p_{\mu_i}) \geq u_i(r, p_r)$ for all $i \in I$ and $r \in R$ we have

$$\sum_{i \in I} u_i(\mu_i, p_{\mu_i}) \geq \sum_{i \in I} u_i(\eta_i, p_{\eta_i}) = \sum_{i \in I} v^i_{\eta_i} - \sum_{i \in I} p_{\eta_i}$$
$$= \sum_{i \in I} v^i_{\eta_i} - c = \sum_{i \in I} v^i_{\eta_i} - \sum_{i \in I} q_{\eta_i}$$
$$= \sum_{i \in I} u_i(\eta_i, q_{\eta_i})$$

and therefore allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ is efficient. ∎

Corollary 2. *The outcome of our auction is efficient.*

Since our auction mimics the market mechanism, Corollary 2 can be interpreted as a *First Fundamental Theorem of Welfare Economics* in the present context.

A mechanism should ensure that each agent receives a non-negative utility in order to sustain the stability of its outcome. Otherwise agents may opt-out and receive their reservation utilities each of which is 0.

An allocation $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ is **individually rational** if $u_i(\mu_i, p_{\mu_i}) \geq 0$ for all $i \in I$.

Proposition 3. *The outcome of our auction is individually rational.*

Proof. Let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be the outcome of our auction. Suppose $u_i(\mu_i, p_{\mu_i}) < 0$ for some agent $i \in I$. Since $\mu_i \in D_i(p)$ we have $0 > u_i(\mu_i, p_{\mu_i}) \geq u_i(r, p_r) = v^i_r - p_r$ for all $r \in R$ and therefore $\sum_{r \in R} v^i_r < \sum_{r \in R} p_r = c$. But by assumption we have $\sum_{r \in R} v^i_r \geq c$ for each agent $i \in I$ yielding the desired contradiction. ∎

6 Envy-free allocations at non-negative prices

An important difficulty about envy-freeness is that for some problems at least one of the prices is negative at each envy-free allocation: For example let

$I = \{i, j\}$, $R = \{a, b\}$, $c = 10$ and let the valuations be $(v_a^i, v_b^i) = (v_a^j, v_b^j) = (15, 1)$. Here, even if $p_a = c = 10$ and $p_b = 0$, both agents still prefer room a to room b. We shall have $p_a = 12$ and $p_b = -2$ in order to ensure envy-freeness.

Brams and Kilgour (2001) give up envy-freeness based on this difficulty and their mechanism always gives non-negative prices. A disadvantage of their mechanism is that it may still choose envious allocations even though there exists envy-free allocations with non-negative prices.

We believe that cases like the earlier example are rather unlikely in real-life applications by the nature of the problem. It is unlikely that agents i and j will jointly rent this house when both are eager to pay more than the rent for room a and almost nothing for room b. In this case whoever has the lease will most likely find another roommate or rent the house alone.

Haake et al. (2002) insist on envy-freeness but their mechanism may yield negative prices even though there exists envy-free allocations with non-negative prices. By Corollary 1 our auction is envy-free as well. Therefore it admits negative prices if there exists no envy-free allocation with non-negative prices. However unlike the mechanism of Haake et al. (2002), our auction mechanism yields an envy-free allocation with non-negative prices whenever such an allocation exists. We shall introduce a related model and relate our auction to the well-known auction of Demange et al. (1986) in order to prove this result.

6.1 A two-sided matching model

We next turn our attention to a two-sided matching model analyzed by Demage et al. (1986).[1]

Let $I = \{i_1, \ldots, i_n\}$ be a set of buyers and $R = \{r_1, \ldots, r_n\}$ be a set of objects. Each buyer has use for one and only one object and $V = [v_r^i]_{i \in I, r \in R}$ is a value matrix where $v_r^i \geq 0$ denotes the value of object r for buyer i.

A matching μ is an assignment of objects to buyers such that each buyer is assigned one and only one object. A price is a vector $p \in \mathbb{R}^n$. Let the reservation price of each object be 0. Therefore in the present context we only consider non-negative prices. Each buyer $i \in I$ is endowed with a utility function $u_i : R \times \mathbb{R}_+ \to \mathbb{R}$ which is quasi-linear in prices:

$$u_i(r, p_r) = v_r^i - p_r.$$

Note that the key difference between the two models is the following: The prices shall add up to c in room assignment-rent division problems whereas the only constraint in the present two-sided matching model is that the price of each object shall be non-negative (i.e., no less than its reservation price).

---

[1] See Roth and Sotomayor (1990) for an extensive survey of two-sided matching models.

As in the case of room assignment-rent division problems, the demand of buyer $i \in I$ at price $p \in \mathbb{R}^n_+$ is given by

$$D_i(p) = \{r \in R : u_i(r, p_r) \geq u_i(s, p_s) \text{ for all } s \in R\}.$$

Given a price $p \in \mathbb{R}^n$, notions of overdemanded set, minimal overdemanded set, full set of overdemanded objects and the price differential $x(p)$ are defined as in room assignment-rent division problems. The price $p \in \mathbb{R}^n_+$ is **competitive** if there exists a matching μ such that $\mu_i \in D_i(p)$ for all $i \in I$. The pair (μ, p) is referred to as a **competitive equilibrium**. Shapley and Shubik (1972) show that competitive equilibria always exist and there exists a competitive price $p \in \mathbb{R}^n_+$ such that $p_r \leq q_r$ for all $r \in R$ and for any competitive price $q \in \mathbb{R}^n_+$. We refer to p as the **buyer-optimal competitive price**.

6.2 Demange, Gale and Sotomayor exact auction

Demange et al. (1986) assumes that the value matrix is integer valued and provides the following auction which yields the buyer-optimal competitive price.

Step 0. Set $p^0 = (0, \dots, 0)$. If there exists no minimal overdemanded set at p^0 then find a matching μ such that $\mu_i \in D_i(p^0)$ for each $i \in I$, set $p^{DGS} = p^0$ and terminate the procedure. Otherwise proceed to Step 1.

In general,
Step t. Pick an arbitrary minimal overdemanded set S at price p^{t-1}. Let $p^t_r = p^{t-1}_r + 1$ for each $r \in S$ and let $p^t_r = p^{t-1}_r$ for each $r \in R \setminus S$. If there exists no minimal overdemanded set at p^t then find a matching μ such that $\mu_i \in D_i(p^t)$ for each $i \in I$, set $p^{DGS} = p^t$ and terminate the procedure. Otherwise proceed to Step $t + 1$.

We refer to this auction as **DGS exact auction**.
Theorem (Demange, Gale and Sotomayor). DGS exact auction yields the buyer-optimal competitive price.

While Demange et al. (1986) assumes that valuations are integer valued, it is straightforward to extend their auction as well as their result for real-valued valuations.

6.3 The modified discrete-price auction

Consider the following price updating rule at any price $q \in \mathbb{R}^n_+$: Construct the full set of overdemanded rooms $OD(q)$ at price q and find $x(q)$. Recall that

$$x(q) = \begin{cases} \min\limits_{j \in J(q)} \left(\widetilde{u}_j(q) - \max\limits_{s \in R \setminus OD(q)} u_j(s, q_s) \right) & \text{if } OD(q) \neq \emptyset \\ 0 & \text{if } OD(q) = \emptyset \end{cases}$$

For any $r \in R$ define

$$g_r(q) = \begin{cases} q_r & \text{if } r \notin OD(q) \\ q_r + x(q) & \text{if } r \in OD(q) \end{cases}$$

We are now ready to define the **modified discrete-price auction** which will be key to relate our discrete-price auction (and hence our continuous-price auction as well) with DGS exact auction.

Step 0. Set $q^0 = (0, \ldots, 0)$. If $OD(q^0) = \emptyset$ then find a matching μ such that $\mu_i \in D_i(q^0)$ for each $i \in I$, set $q = q^0$ and terminate the procedure. If $OD(q^0) \neq \emptyset$ then proceed to Step 1.

In general,
Step t. Set $q_r^t = g_r(q^{t-1})$ for all $r \in R$. If $OD(q^t) = \emptyset$ then find a matching μ such that $\mu_i \in D_i(q^t)$ for each $i \in I$, set $q = q^t$ and terminate the procedure. If $OD(q^t) \neq \emptyset$ then proceed to Step $t+1$.

We need the following lemma in order to relate the modified discrete-price auction and the discrete-price auction.

Lemma 3. *Let $p, q \in \mathbb{R}^n$ be such that $p_r = q_r + k$ for all $r \in R$ and some $k \in \mathbb{R}$. Then*

1. $D(q) = D(p)$ *and*
2. $x(q) = x(p)$.

Proof. Fix $p, q \in \mathbb{R}^n$ and $k \in \mathbb{R}$ such that $p_r = q_r + k$ for all $r \in R$. Since utilities are quasi-linear in prices, we have $u_i(r, p_r) = v_r^i - p_r = v_r^i - (q_r + k) = u_i(r, q_r) - k$ for all $i \in I$ and $r \in R$.

1. Fix $i \in I$. For any $r \in R$ we have

$$r \in D_i(p) \Longleftrightarrow u_i(r, p_r) \geq u_i(s, p_s) \quad \text{for all } s \in R$$
$$\Longleftrightarrow v_r^i - p_r \geq v_s^i - p_s \quad \text{for all } s \in R$$
$$\Longleftrightarrow v_r^i - q_r \geq v_s^i - q_s \quad \text{for all } s \in R$$
$$\Longleftrightarrow u_i(r, q_r) \geq u_i(s, q_s) \quad \text{for all } s \in R$$
$$\Longleftrightarrow r \in D_i(q)$$

and hence $D_i(p) = D_i(q)$. Since $i \in I$ is arbitrary we have $D(p) = D(q)$.

2. Since $D(p) = D(q)$ we have $OD(p) = OD(q)$, $J(p) = J(q)$ and $\widetilde{u}_i(p) = \widetilde{u}_i(q) - k$ for all $i \in I$. Therefore

$$x(p) = \begin{cases} \min_{j \in J(p)} \left(\widetilde{u}_j(p) - \max_{r \in R \setminus OD(p)} u_j(r, p_r) \right) & \text{if } OD(p) \neq \emptyset \\ 0 & \text{if } OD(p) = \emptyset \end{cases}$$

$$= \begin{cases} \min_{j \in J(q)} \left(\tilde{u}_j(q) - k - \max_{r \in R \setminus OD(q)} \left(u_j(r, q_r) - k \right) \right) & \text{if } OD(q) \neq \emptyset \\ 0 & \text{if } OD(q) = \emptyset \end{cases}$$

$$= \begin{cases} \min_{j \in J(q)} \left(\tilde{u}_j(q) - \max_{r \in R \setminus OD(p)} u_j(r, q_r) \right) & \text{if } OD(q) \neq \emptyset \\ 0 & \text{if } OD(q) = \emptyset \end{cases}$$

$$= x(q) \hspace{8cm} \blacksquare$$

We are ready to relate the modified discrete-price auction and the discrete-price auction. Next we will relate the modified discrete-price auction and DGS exact auction clarifying the relation between our discrete-price auction and DGS exact auction.

Proposition 4. *Let $\{p^t\}_{t=0}^{T}$ be the price sequence obtained by the discrete price auction and $\{q^t\}$ be the price sequence obtained by the modified discrete-price auction. Then the modified discrete-price auction converges in T steps and*

$$p_r^T = q_r^T + \frac{c - \sum\limits_{s \in R} q_s^T}{n} \quad \text{for all } r \in R.$$

Proof. Let $\{p^t\}_{t=0}^{T}$ be the price sequence obtained by the discrete price auction and $\{q^t\}$ be the price sequence obtained by the modified discrete-price auction.

Claim. For each $t \leq T$, we have

(i) $q_r^t = p_r^t + \dfrac{\left(\sum\limits_{t-1 \geq u \geq 0} |OD(p^u)| x(p^u) \right) - c}{n}$ for all $r \in R$,

(ii) $x(q^t) = x(p^t)$, and
(iii) $D(q^t) = D(p^t)$.

Proof of the Claim. We will prove the claim by induction. By construction of the modified discrete-price auction we have $q_r^0 = p_r^0 - \frac{c}{n}$. Therefore, since $\sum\limits_{t-1 \geq u \geq 0} |OD(p^u)| x(p^u) = 0$ for $t = 0$, Claim (i) holds for $t = 0$. Moreover by Lemma 3 we have $D(q^0) = D(p^0)$ and $x(q^0) = x(p^0)$. Hence the Claim holds for $t = 0$.

Next assume that $0 \leq t^* \leq T$ and the Claim holds for $t < t^*$. We shall prove that the Claim holds for $t = t^*$ as well. First note that the assumption ensures that $OD(q^t) = OD(p^t) \neq \emptyset$ for any $t < t^*$. Moreover by construction of the price updating rule g_r and since the Claim holds for $t = t^* - 1$, for any $r \in R$ we have

$$q_r^{t^*} = g_r(q^{t^*-1}) = \begin{cases} q_r^{t^*-1} + x(q^{t^*-1}) & \text{if } r \in OD(q^{t^*-1}) \\ q_r^{t^*-1} & \text{if } r \notin OD(q^{t^*-1}) \end{cases}$$

$$= \begin{cases} p_r^{t^*-1} + \dfrac{\left(\sum\limits_{t^*-2 \geq u \geq 0} |OD(p^u)|x(p^u)\right) - c}{n} + x(p^{t^*-1}) & \text{if } r \in OD(p^{t^*-1}) \\ p_r^{t^*-1} + \dfrac{\left(\sum\limits_{t^*-2 \geq u \geq 0} |OD(p^u)|x(p^u)\right) - c}{n} & \text{if } r \notin OD(p^{t^*-1}) \end{cases}$$

$$= \begin{cases} p_r^{t^*-1} + \dfrac{\left(\sum\limits_{t^*-2 \geq u \geq 0} |OD(p^u)|x(p^u)\right) - c}{n} + \dfrac{n - |OD(p^{t^*-1})|}{n} x(p^{t^*-1}) + \dfrac{|OD(p^{t^*-1})|}{n} x(p^{t^*-1}) \\ \quad \text{if } r \in OD(p^{t^*-1}) \\ p_r^{t^*-1} + \dfrac{\left(\sum\limits_{t^*-2 \geq u \geq 0} |OD(p^u)|x(p^u)\right) - c}{n} \\ \quad \text{if } r \notin OD(p^{t^*-1}) \end{cases}$$

$$= \dfrac{\left(\sum\limits_{t^*-2 \geq u \geq 0} \sum |OD(p^u)|x(p^u)\right) - c}{n} + \dfrac{|OD(p^{t^*-1})|x(p^{t^*-1})}{n}$$

$$+ \begin{cases} p_r^{t^*-1} + \dfrac{n - |OD(p^{t^*-1})|}{n} x(p^{t^*-1}) & \text{if } r \in OD(p^{t^*-1}) \\ p_r^{t^*-1} - \dfrac{|OD(p^{t^*-1})|}{n} x(p^{t^*-1}) & \text{if } r \notin OD(p^{t^*-1}) \end{cases}$$

$$= \dfrac{\left(\sum\limits_{t^*-1 \geq u \geq 0} |OD(p^u)|x(p^u)\right) - c}{n} + p_r^{t^*}$$

and hence Claim (i) holds for $t = t^*$. Moreover by Lemma 3 we have $D(q^{t^*}) = D(p^{t^*})$ as well as $x(q^{t^*}) = x(p^{t^*})$ completing the proof of the Claim. ∎

We are now ready to complete the proof of Proposition 4. Let

$$\sigma = \sum_{T-1 \geq u \geq 0} |OD(p^u)|x(p^u).$$

By the Claim we have $q_r^T = p_r^T + \frac{\sigma-c}{n}$ for all $r \in R$. This together with equality $\sum\limits_{s \in R} p_s^T = c$ imply that for any room $r \in R$ we have

$$q_r^T + \frac{c - \sum\limits_{s \in R} q_s^T}{n} = p_r^T + \frac{\sigma - c}{n} + \frac{c - \sum\limits_{s \in R} \left(p_s^T + \frac{\sigma-c}{n}\right)}{n}$$

$$= p_r^T + \frac{\sigma - c}{n} + \frac{c - c - n\frac{(\sigma-c)}{n}}{n}$$

$$= p_r^T$$

showing the desired equality. Moreover by the Claim we have $D(q^T) = D(p^T)$ and therefore $OD(p^T) = \emptyset$ implies $OD(q^T) = \emptyset$. Hence the modified discrete-price auction converges in T steps as well. ∎

6.4 Equivalence of DGS exact auction and the modified discrete price auction

We are now ready to relate the modified discrete-price auction and DGS exact auction: Both auctions yield the buyer-optimal competitive price.

Proposition 5. *Let $\{q^t\}_{t=0}^T$ be the price sequence obtained by the modified discrete-price auction and let p^{DGS} be the final price obtained by DGS exact auction. We have $q^T = p^{DGS}$.*

Proof. We will consider a general format of DGS exact auction where prices of rooms in a minimal overdemanded set are increased by a sufficiently small increment. Let $\{q^t\}_{t=0}^T$ be the price sequence obtained by the modified discrete-price auction. Recall that initial price vector is $(0, \dots, 0)$ for the modified discrete-price auction as well as DGS exact auction. We will show that by an appropriate choice of

1. the order of minimal overdemanded sets and
2. the price increments,

price q^1 can be reached by DGS exact auction. Iteration of the same argument shows that prices $q^2, \dots, q^T$ can as well be reached by DGS exact price auction. Once q^T has been reached, $OD(q^T) = \emptyset$ implies that there exists no minimal overdemanded set and hence DGS exact auction terminates yielding $p^{DGS} = q^T$.

For any $p \in \mathbb{R}^n$, recall the construction of $OD(p)$: We find all minimal overdemanded sets. We remove these rooms from the demand of each agent and find the minimal overdemanded sets for the modified demand profiles. We proceed in a similar way until there exists no minimal overdemanded set for the modified demand profiles. The full set of overdemanded rooms $OD(p)$ is the union of each of the sets encountered in the procedure.

For $p = q^0 = (0, \dots, 0)$ let $S_1^1, \dots, S_{m_1}^1$ be minimal overdemanded sets, let $S_1^2, \dots, S_{m_2}^2$ be minimal overdemanded sets once $\bigcup_{\alpha=1}^{m_1} S_\alpha^1$ has been removed from the demands,..., let $S_1^k, \dots, S_{m_k}^k$ be the last group of minimal overdemanded sets once $\bigcup_{\beta=1}^{k-1} \left(\bigcup_{\alpha=1}^{m_\beta} S_\alpha^\beta \right)$ has been removed from the demands. Define

$$y(q^0) = \min_{i \in I, r \in R \setminus D_i(q^0)} \left(\tilde{u}_i(q^0) - u_i(r, q_r^0) \right)$$

Note that $y(q^0) > 0$ provided that $OD(q^0) \neq \emptyset$ and moreover

$$y(q^0) \leq x(q^0) = \min_{i \in J(q^0)} \left(\tilde{u}_i(q^0) - \max_{s \in R \setminus OD(q^0)} u_i(s, q_s^0) \right).$$

Pick an integer ℓ^0 such that $\frac{x(q^0)}{\ell^0} < y(q^0)$ and let $\epsilon^0 = \frac{x(q^0)}{\ell^0}$. The following pair of observations will be key to our proof.

Observation 1. Consider an increase in some of the prices while the remaining prices stay put. A minimal overdemanded set S remains minimal overdemanded provided that prices of the rooms in S stay put.

Observation 2. Suppose prices of all rooms in $\bigcup_{\alpha=1}^{m_1} S_\alpha^1$ increase by ϵ^0 and the remaining prices stay put. Then each of $S_1^2, \ldots, S_{m_2}^2$ becomes a minimal over-demanded set at updated prices. Similarly if prices of all rooms in $\bigcup_{\beta=1}^{2} \left(\bigcup_{\alpha=1}^{m_\beta} S_\alpha^\beta \right)$ increase by ϵ^0 while the remaining prices stay put then each of $S_1^3, \ldots, S_{m_3}^3$ becomes a minimal overdemanded set at updated prices, and so on.

Consider DGS exact price auction and initially set the price at $p = q^0 = (0, \ldots, 0)$: S_1^1 is a minimal overdemanded set, increase prices of all rooms in S_1^1 by ϵ^0. Next consider S_2^1 which was a minimal overdemanded set at q^0 and which remains a minimal overdemanded set at updated prices by Observation 1. Increase prices of all rooms in S_2^1 by ϵ^0 as well. Similarly consider each of the sets $S_3^1, \ldots, S_{m_1}^1$ one at a time and increase prices of all rooms in these sets by ϵ^0 one set at a time. At this point prices of all rooms in each of the minimally overdemanded sets at q^0 is increased by ϵ^0 and by Observation 2 each of $S_1^2, \ldots, S_{m_2}^2$ became a minimally overdemanded set at the updated prices. Similarly consider each of the sets $S_1^2, \ldots, S_{m_2}^2$ one at a time and increase prices of all rooms in these sets by ϵ^0 one set at a time. Following in a similar way we will reach a price vector p via DGS exact auction where $p_r = \epsilon^0$ for $r \in OD(q^0)$ and $p_r = 0$ for $r \notin OD(q^0)$.

Here the key observation is the following: Since $\epsilon^0 < x(q^0)$ and since the price of each room in $OD(q^0)$ has only increased by ϵ^0 we have $OD(p) = OD(q^0)$. (Recall that $x(q^0)$ is the minimum price differential needed for the full set of overdemanded rooms to change). Therefore we can replicate the same sequence of price increases $\ell^0 - 1$ additional times through DGS exact auction. When we do that we reach to a price p with $p_r = \ell^0 \epsilon^0 = x(q^0) = q_r^1$ for each $r \in OD(q^0)$ and $p_r = 0 = q_r^1$ for each $r \notin OD(q^0)$. Hence we reach $p = q^1$ via DGS exact auction.

Next construct $OD(q^1)$, define

$$y(q^1) = \min_{i \in I, r \in R \setminus D_i(q^1)} \left(\widetilde{u}_i(q^1) - u_i(r, q_r^1) \right),$$

let the integer ℓ^1 be such that $\frac{x(q^1)}{\ell^1} < y(q^1)$ and let $\epsilon^1 = \frac{x(q^1)}{\ell^1}$. Iterating the earlier arguments we can first increase prices of all rooms in $OD(q^1)$ by ϵ^1 and replicate this an additional $\ell^1 - 1$ times to reach $p = q^2$ via DGS exact auction. Proceeding in a similar way we can reach $p = q^T$ via DGS exact auction. Once we reach $p = q^T$, since $OD(q^T) = \emptyset$ there are no minimal overdemanded sets and hence DGS exact auction terminates. Therefore $q^T = p^{DGS}$.   ∎

6.5 If there are envy-free allocations at non-negative prices then our auction will find one

We are finally ready to show that if there exists envy-free allocations with non-negative prices, then our auction yields such an allocation.

Theorem 2. *Let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be the outcome of our continuous-price auction (or equivalently the discrete-price auction). We have $p_r \geq 0$ for all*

$r \in R$ if and only if there exists an envy-free allocation with non-negative prices.

Proof. Let $\langle I, R, V, c \rangle$ be a room assignment-rent division problem and let $(\mu, p) \in \mathcal{M} \times \mathcal{P}$ be the outcome of our continuous-price auction. By Corollary 1, (μ, p) is envy-free and therefore the only if part of the theorem holds immediately.

Conversely suppose that there exists an envy-free allocation $(\eta, q) \in \mathcal{M} \times \mathcal{P}$ with $q_r \geq 0$ for all $r \in R$. Consider the two-sided matching market $\langle I, R, V \rangle$. By Proposition 1 we have $\eta_i \in D_i(q)$ for all $i \in I$ and therefore q is a competitive price. Let q^T be the price obtained by the modified discrete-price auction and let p^{DGS} be the price obtained by DGS exact auction. By Proposition 5 we have $p^{DGS} = q^T$ and since p^{DGS} is the buyer optimal competitive price we have $q_r^T \leq q_r$ for all $r \in R$. Therefore $\sum\limits_{s \in R} q_s^T \leq \sum\limits_{s \in R} q_s = c$. By Proposition 4 we have

$$p_r^T = q_r^T + \frac{c - \sum\limits_{s \in R} q_s^T}{n}$$

for all $r \in R$ and therefore $p_r^T \geq q_r^T \geq 0$ for all $r \in R$ completing the proof. ∎

7 Conclusion

In this paper we propose an efficient auction for room assignment-rent division problems. Our auction is inspired by the market mechanism and it has two key advantages over existing mechanisms: (i) its outcome is always envy-free and (ii) it yields non-negative prices unless there exists no envy-free allocation with non-negative prices. Based on these properties we believe our auction mechanism can be used in real-life applications.

An important limitation of our mechanism is its vulnerability to strategic preference manipulation.[2] That is, our mechanism is not strategy-proof. Alkan et al. (1991) show that there exists no mechanism which is both envy-free and strategy-proof.[3] Hence one cannot insist on both envy-freeness and strategy-proofness. Analyzing equilibria of preference manipulation games induced by our auction mechanism is an important and interesting exercise but this is beyond the scope of the current paper.

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[2]The market mechanism is typically vulnerable to preference manipulation in most resource allocation problems. An important exception was shown by Roth (1982) in the context of housing markets (Shapley and Scarf 1974).

[3]It is possible to construct strategy-proof mechanisms by giving up envy-freeness. For example one can fix a rent division, fix an initial matching to be interpreted as an initial endowment and find the competitive allocation of the induced housing market. See Miyagawa (2001) and Svensson and Larsson (2002) for a similar approach in housing markets with monetary transfers.

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