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Strong equilibrium outcomes of voting games are the generalized Condorcet winners

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Abstract. We consider voting games induced by anonymous and top-unanimous social choice functions. The class of such social choice functions is quite broad, including every "t-refinement" of the Plurality Rule, Plurality with a Runoff, the Majoritarian Compromise and the Single Transferable Vote, i.e., any selection from either of these social choice rules which is obtained via tie-breaking among candidates according to any total order t on the set of alternatives. As announced in our title, the strong equilibrium outcomes of the voting games determined by such social choice functions turn out to be nothing but generalized Condorcet winners, namely the "(n,q)-Condorcet winners". In the case of social choice functions (such as those just listed) which are furthermore "top-majoritarian", they coincide with the classical Condorcet winners.

The non-alphabetic ordering of names has a reason deserving to be explained. Murat R. Sertel, this man from the Bosphorus whose contributions put Turkey on the academic map of Economic Theory, passed away on January 25, 2003, while this paper was in print. His untimely death was met with deep sorrow by his colleagues, friends and students all over the world. Murat R. Sertel was one of the forerunners of the Age of Economic Design. He was a genuine scholar, brilliant intellectual, excellent teacher, beloved friend, a wonderful company, a devoted son and an affectionate father. May he rest in peace.

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1 Introduction

We know, thanks to Gibbard (1973) and Satterthwaite (1975), that there exists no singleton-valued, strategy-proof and non-dictatorial social choice rule with a range containing at least three alternatives. From Dasgupta, Hammond and Maskin (1979) we also know that if a singleton-valued social choice rule is Nash-implementable on the full domain of preferences, then it must also be truthfully implementable in dominant strategies. Combining these two results leads to the conclusion that no singleton-valued social choice rule whose range contains at least three alternatives can be Nash implemented unless it is dictatorial, a result which was also established by Muller and Satterthwaite (1977).

So, in a real sense, many social choice rules are quite manipulable, and it is impossible to implement them via dominant strategy or Nash equilibria by using direct mechanisms. Regarding implementability via indirect mechanisms, we have many positive results which partially or fully characterize implementable social choice correspondences.[1]

Alternatively, one could inquire into the "performance" (see, e.g., Hurwicz and Sertel 1999) of a voting rule derived from a social choice rule. Nash and strong equlibrium outcomes of direct mechanisms associated with many social choice rules have been studied after the pioneering work of Hurwicz (1979). In particular, Thomson (1984) determines Nash equilibria of manipulation games associated with resource allocation mechanisms. Tadenuma and Thomson (1995) consider the problem of fairly allocating an indivisible good and study the direct revelation games associated with subsolutions of the no-envy solution. Otani and Sicilian (1982, 1990) characterize equilibrium outcomes of Walras preference games induced by a direct mechanism as well as by a mechanism asking the agents to reveal their demand maps. Sertel and Sanver (1999) characterize equilibrium outcomes of Lindahl endowment pretension games[2]. Roth (1984) characterizes Nash equilibrium outcomes when stable matching rules are instituted as a direct mechanism in the marriage market.

The "performance" of an outcome function is especially important in the context of voting: If we want to institute a social choice rule as a voting rule, realistically speaking, it has to be through a direct mechanism where people

[1] Among these, we have Maskin (1977, 1999), Danilov (1992) for Nash implementation; Maskin (1979), Dutta and Sen (1991) for strong implementation; Moore and Repullo (1988), Abreu and Sen (1990) for subgame perfect implementation; Palfrey and Srivastava (1991), Jackson, Palfrey and Srivastava (1994) for undominated Nash implementation; Jackson (1991), Dutta and Sen (1994) for Bayesian implementation. So we know a lot about what is implementable, and what is not, via various solution concepts. In particular, Maskin (1979) shows that strong implementability of a social choice correspondence is inconsistent with the weak no veto power condition.

[2] Sanver (2002) offers a related analysis regarding the Nash equilibrium allocations of an allocation rule with wealth-regressive tax rates.

indicate their preferences. Artificial message spaces and indirect mechanisms based on these seem to be out of the question, except in our ivory towers of theory. As a result, we have to reckon with what will happen under "strategic voting" with free coalescing, which we interpret as meaning strong equilibrium. Hence we ask the following question: If we were to declare a singleton-valued social choice rule as the outcome function and naively ask for the agents' preferences through a "direct" mechanism, what would we actually end up implementing?

A main motivation for studying the strong equilibrium outcomes of the direct mechanism determined by a social choice rule may be found in the following historical procedure, related by Lakeman and Lambert (1959, p.101):

> "Rowland Hill … records that, when he was teaching in his father's school, his pupils were asked to elect a committee by standing beside the boy they liked best. This first produced a number of unequal groups, but soon the boys in the largest groups came to the conclusion that not all of them were actually necessary for the election of their favourite and some moved on to help another candidate, while on the other hand a few supporters of an unpopular boy gave him up as hopeless and transferred themselves to the candidate they considered the next best. The final result was that a number of candidates equal to the number required for the committee were each surrounded by the same number of supporters, with only two or three boys left over who were dissatisfied with all those elected."

Lakeman and Lambert (1959) consider this as "an admirable example of the use of the Single Transferable Vote."[3] As shown in further detail by Sertel and Kalaycıoğlu (1995), it is actually unclear how well the outcome of this procedure would be approximated by the Single Transferable Vote, but in electing a single-member committee by a small number of voters, Hill's procedure would stop, if it stops, at a strong equilibrium of the voting game induced by instituting the Plurality Rule as the outcome function (a boy receiving the maximal number of votes wins).[4]

Will this procedure stop, and if so at what outcome? Are there social choice rules other than Plurality which would lead to the same outcome when instituted as the outcome function? In investigating such questions, we allow a social choice rule to act as an outcome function, and we study its "performance correspondence" i.e., the correspondence which, at each preference profile, gives the set of outcomes under the outcome function defined by this social choice rule when each agent is to reveal his preference and the game so determined is resolved at a strong equilibrium.

Identifying outcome functions with their respective mechanisms, it is, of course, quite possible that the social choice rule as a mechanism does not implement itself, as may happen when truth-telling is not an equilibrium of

[3] For a comprehensive study of the Single Transferable Vote, see Tideman (1995).
[4] Thus, there may be ties.

the revelation game determined by the mechanism. Indeed, we characterize the strong equilibrium outcomes under a wide class of voting rules as generalized Condorcet winners. This coincidence is also a positive result for implementation by use of direct mechanisms, somewhat in contrast with the generally negative results in this area. We also explore the extent to which the notion of a Condorcet winner needs to be generalized in order to ensure its existence at all preference profiles.

In Sect. 2 we give the preliminaries. Section 3 presents two notions of effectivity and formulates certain properties for social choice rules. Section 4 contains the main characterization results. Section 5 gives the "necessary and sufficient" generalization of Condorcet winners so that they always exist. Section 6 is devoted to the application of our results to known voting rules. Section 7 records some closing remarks.

2 Preliminaries

Taking any integer $n \geq 2$, we consider a set $\mathbf{N} = \{1,\ldots,n\}$ of voters, confronting a non-empty finite set of candidates $\mathbf{A}$ with $|\mathbf{A}| = m \geq 2$. We denote $\mathbf{W}$ for the set of all complete pre-orders[5] on $\mathbf{A}$, and imagine that each voter $i \in \mathbf{N}$ has a preference $R_i \in \mathbf{W}$, so that we are always given some "preference profile" $R = (R_1,\ldots,R_n) \in \mathbf{W}^N$. Given any coalition $K \subset \mathbf{N}$ of voters, we denote $R_K \in \mathbf{W}^K$ for a preference profile restricted to K. Given any $i \in \mathbf{N}$ and any $R_i \in \mathbf{W}$, P_i will stand for the strict counterpart and I_i for the indifference counterpart of R_i.[6]

By a *voting rule* we mean any function $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$.[7] For every preference profile $R \in \mathbf{W}^N$, a voting rule $\mathbf{V}$ induces a (normal form) "voting game" $\Gamma(\mathbf{V}, R) = \{(M_i,u_i)\}_{i\in N}$, where $M_i = \mathbf{W}$ is the message space of the generic agent $i \in \mathbf{N}$, and u_i is a real-valued utility function, defined on the message profile space $\mathbf{M} = \Pi_{i\in N} M_i$, representing the preference of voter $i \in \mathbf{N}$ on $\mathbf{M}$ as follows:

$$u_i(\mathbf{m}) \geq u_i(\mathbf{m}') \Leftrightarrow \mathbf{V}(\mathbf{m})R_i\mathbf{V}(\mathbf{m}') \text{ for all } \mathbf{m},\mathbf{m}' \in \mathbf{M}.$$

Given any voting rule $\mathbf{V}$, and denoting $\boldsymbol{\sigma}$ for the strong equilibrium[8] solution concept for normal form games, at any preference profile R, we

[5] A pre-order is a reflexive and transitive binary relation. Thus, a complete pre-order is a complete (hence also reflexive) and transitive binary relation.

[6] Thus, for every $x, y \in \mathbf{A}$, we have $x\, P_i\, y$ iff $x\, R_i\, y$ holds and $y\, R_i\, x$ fails. Similarly, $x\, I_i\, y$ iff $x\, R_i\, y$ and $y\, R_i\, x$.

[7] Thus, a voting rule can be considered as a selection from (i.e., singleton-valued refinement of) a social choice rule ("correspondence") via some tie-breaking rule.

[8] We say that a joint strategy $\mathbf{m} = \{m_i\}_{i \in N} \in \mathbf{M}$ is a *strong equilibrium* of the game $\Gamma(\mathbf{V}, R) = (\{M_i,u_i\}_{i\in N})$ if and only if, given any coalition $K \subset \mathbf{N}$, there is no $\mathbf{m}' = \{m'_i\}_{i \in N} \in \mathbf{M}$ with $m'_j = m_j$ for every $j \in \mathbf{N} \setminus K$ such that $u_i(\mathbf{m}') > u_i(\mathbf{m})$ for each $i \in K$.

denote $\boldsymbol{\sigma}\,[\Gamma(\mathbf{V}, \mathrm{R})] \subseteq \mathbf{W}^{\mathrm{N}}$ for the set of strong equilibria of the game $\Gamma(\mathbf{V}, \mathrm{R})$ and

$$\mathbf{V}^{\sigma}(\mathrm{R}) = \{\mathbf{V}(\mathbf{m}) \in \mathbf{A} | \mathbf{m} \in \boldsymbol{\sigma}[\Gamma(\mathbf{V}, \mathrm{R})]\}$$

for the set of strong equilibrium outcomes of the voting rule **V**.

We characterize the correspondence $\mathbf{V}^{\sigma}: \mathbf{W}^{\mathrm{N}} \to 2^{\mathbf{A}}$ in terms of "(n,q)-Condorcet winners", a generalization of the classical notion of a Condorcet winner. To formulate this notion, we first define a useful "dominance" relation on **A**:

Given any $q \in \mathbf{N} \cup \{0, n+1\}$, any $R \in \mathbf{W}^{\mathrm{N}}$ and any $x, y \in \mathbf{A}$, we say that x (n,q)-dominates y (according to R), and we write x D(R;n,q) y, if and only if $|\{i \in \mathbf{N} \mid x\, P_i\, y\}| \geq q$.

The undominated alternatives according to the binary relation D(R;n,q) constitute the (n,q)-Condorcet winners according to R. Given any $R \in \mathbf{W}^{\mathrm{N}}$, by the *(n,q)-Condorcet Set*, denoted by C(R;n,q), we mean the set of (n, q)-Condorcet winners according to R, i.e., alternatives which are undominated according to D(R;n,q). Thus, $C(R;n,q) = \{x \in \mathbf{A} \mid [y, z \in \mathbf{A} \text{ and } y\, D(R;n,q)\, z] \Rightarrow z \neq x\}$.

Note that, for any $q, r \in \mathbf{N} \cup \{0, n+1\}$, we have $q \leq r \Rightarrow C(R;n,q) \subseteq C(R;n,r)$. So, given any size n of the electoral population, the proportion of the theoretically possible n-long preference (complete preorder) profiles at which the set of (n,q)-Condorcet winners is non-empty will be a non-decreasing function of $q \in \{0, 1,\ldots,n, n+1\}$. In particular, taking any positive integer $n \geq 2$, there is no (n,0)-Condorcet winner, the set of (n,1)-Condorcet winners are the alternatives unanimously considered as best,…, the (n,n)-Condorcet winners are the alternatives which are weakly Pareto dominated by no alternative (i.e., the set of alternatives compared with which we can find no alternative which everybody finds strictly superior), and of course the set of (n,n+1)-Condorcet winners is the entire set of alternatives. Note also that, defining $n^{**} = \min\{q \in \{0, 1,\ldots,n\} \mid q > n - q\}$, $C(R;n,n^{**})$ coincides with the set of Condorcet winners in the classical sense.[9]

At any preference profile R, the set $\mathbf{V}^{\sigma}(\mathrm{R})$ of strong equilibrium outcomes of a voting rule **V** can be expressed in terms of C(R;n,q) for a specific value of q derived from the distribution of "effectivity" over the coalitions $K \subseteq \mathbf{N}$, as described in the following section.

[9] When n is even, we need to distinguish between strong and weak Condorcet winners. A weak Condorcet winner is an alternative which is found to be at least as good as any other alternative by at least n/2 people. A strong Condorcet winner is an alternative which is found to be at least as good as any other alternative by a strict majority, thus by more than n/2 people. Hence, $C(R;n,n^{**})$ is the set of weak Condorcet winners. This distinction vanishes when n is odd.

3 Effectivity of voting rules

Given a voting rule $\mathbf{V}$, we will consider the ability of a coalition to get an alternative chosen under $\mathbf{V}$. Following Moulin and Peleg (1982), Moulin (1983), Peleg (1984), and Ichiishi (1986), this ability is generally referred to as the "effectivity" of a coalition. We give two versions of it.

Definition 3.1.[10] Given a voting rule $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$, we say that a coalition $K \subseteq \mathbf{N}$ of voters is *β-effective for* $x \in \mathbf{A}$ if and only if $\forall\ R_{N\backslash K} \in \mathbf{W}^{N\backslash K}$, there exists some $R_K \in \mathbf{W}^K$ such that $\mathbf{V}(R_K, R_{N\backslash K}) = x$.

We denote $\beta_v^+(x)$ for the set of β-effective coalitions for the alternative x under the voting rule $\mathbf{V}$. We let $b_v^+(x)$ stand for the cardinality of any minimal cardinality coalition belonging to $\beta_v^+(x)$. By convention, for any $x \in \mathbf{A}$, we set $b_v^+(x) = n+1$ whenever $\beta_v^+(x)$ is empty. We define $b_V^+ = \max_{x\in A}\{b_v^+(x)\}$.

Definition 3.2. Given a voting rule $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$, we say that a coalition $K \subseteq \mathbf{N}$ of voters is *weakly β-effective for* $x \in$ A if and only if there exists an alternative $y \in \mathbf{A}\backslash\{x\}$ such that for some profile $R_{N\backslash K} \in \mathbf{W}^{N\backslash K}$ of the complementary coalition $\mathbf{N}\backslash K$ with $\{y\} = \text{argmax } R_i$ at every $i \in \mathbf{N}\backslash K$, the coalition K has a profile $R_K \in \mathbf{W}^K$ such that $x = \mathbf{V}(R_K, R_{N\backslash K})$.

We denote $\beta_v^-(x)$ for the set of weakly β-effective coalitions for the alternative x under the voting rule $\mathbf{V}$. We let $b_v^-(x)$ stand for the cardinality of any minimal cardinality coalition belonging to $\beta_v^-(x)$. By convention, for any $x \in \mathbf{A}$, we set $b_v^-(x) = n+1$ whenever $\beta_v^-(x)$ is empty. We define $b_V^- = \min_{x\in A}\{b_v^-(x)\}$.

Remark that, for any voting rule $\mathbf{V}$, we always have $b_v^- \leq b_v^+$, as $\beta_v^+(x) \subseteq \beta_v^-(x)$ for every $x \in \mathbf{A}$. Moreover, for every $x \in \mathbf{A}$ we have $[K \subseteq K' \subseteq \mathbf{N}$ and $K \in \beta_v^-(x)] \Rightarrow K' \in \beta_v^-(x)$ and $[K \subseteq K' \subseteq \mathbf{N}$ and $K \in \beta_v^+(x)] \Rightarrow K' \in \beta_v^+(x)$.

4 Strong equilibrium outcomes of voting games and (n,q)-Condorcet winners

Now we examine some relations between the (n,q)-Condorcet sets and the set of strong equilibria, $\mathbf{V}^\sigma$, of voting games. Our first result, for top-unanimous[11] voting rules, connects the (n,q)-Condorcet sets to the strong equilibrium outcomes:

Theorem 4.1. *If* $V: W^N \to A$ *is an top-unanimous voting rule, then at every* $R \in W^N$*, we have* $C(R;n,q) \subseteq V^\sigma(R)$ *for* $q = b_v^-$.

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[10] We thank Bezalel Peleg for bringing to our attention that our Definition 3.1 was the well-known definition of β-effectivity.

[11] A voting rule $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$ is said to be *top-unanimous* if and only if for every $x \in \mathbf{A}$, we have $\mathbf{V}(R) = x$ whenever $\{x\} = \text{argmax } R_i$ for every $i \in \mathbf{N}$.

Proof. Take any top-unanimous voting rule $\mathbf{V}$, and any $R \in \mathbf{W}^N$. Consider the game $\Gamma(\mathbf{V}, R)$. For any $x \in C(R;n, b_V^-)$ regard any message profile $\mathbf{m}$ where $\{x\} = \text{argmax } m_i$ for each $i \in \mathbf{N}$. As $\mathbf{V}$ is top-unanimous, $\mathbf{V}(\mathbf{m}) = x$. It suffices to show that $\mathbf{m} \in \sigma\, [\Gamma(\mathbf{V}, R)]$. To this end, take any $y \in \mathbf{A} \setminus \{x\}$. As $x \in C(R;n, b_v^-)$, we have $|\{i \in \mathbf{N} \mid y\, P_i\, x\}| < b_v^- \leq b_v^-(y)$ and so $\{i \in \mathbf{N} \mid y\, P_i\, x\} \notin \beta_v^-(y)$. This shows that $\mathbf{m} \in \sigma\, [\Gamma(\mathbf{V}, R)]$ (hence, $x \in \mathbf{V}^\sigma(R)$). ■

Remark 4.2. The reverse of the inclusion established in Theorem 4.1 does not hold. Consider the top-unanimous voting rule $\mathbf{V}$: $\mathbf{W}^N \to \mathbf{A}$ where $\mathbf{V}(R) = x$ whenever argmax $R_i = \{x\}$ for each $i \in \mathbf{N}$ and $\mathbf{V}(R) = x^*$ otherwise for some fixed $x^* \in \mathbf{A}$. Note that $b_v^-(x) = n$ for every $x \in \mathbf{A} \setminus \{x^*\}$ and $b_v^-(x^*) = 1$, implying that $b_v^- = 1$. Consider the case of a doubleton $\mathbf{A} = \{x, x^*\}$ and $\mathbf{N} = \{1, 2\}$. Take a profile $R = (R_1, R_2)$ where $x\, P_1\, x^*$ and $x^*\, P_2\, x$. Now consider the game $\Gamma(\mathbf{V}, R)$. Take the message profile $\mathbf{m}$ with argmax $m_i = \{x^*\}$ for each $i \in \{1, 2\}$. Clearly $\mathbf{m} \in \sigma\, [\Gamma(\mathbf{V}, R)]$. Thus, $x^* \in \mathbf{V}^\sigma(R)$, although $x\, P_1\, x^*$ and so $x^* \notin C(R; n, b_v^-)$.

Nevertheless, a parallel of the reverse of the inclusion established in Theorem 4.1 holds for anonymous voting rules, as our next result tells us.

Theorem 4.3. *If V: $W^N \to A$ is an anonymous*[12] *voting rule, then at every $R \in W^N$ we have $V^\sigma(R) \subseteq C(R; n, q)$ for $q = b_v^+$.*

Proof. Take any anonymous voting rule $\mathbf{V}$. Thus, for any $x \in \mathbf{A}$, the family of β-effective coalitions will be of the form $\beta_v^+(x) = \{K \subseteq \mathbf{N} \mid |K| \geq k\}$ for some $k \in \mathbf{N}$. Now take any $R \in \mathbf{W}^N$ and consider any $x \in \mathbf{V}^\sigma(R)$. Thus, there exists $\mathbf{m} \in \mathbf{M}$ with $\mathbf{V}(\mathbf{m}) = x$ and $\mathbf{m} \in \sigma\, [\Gamma(\mathbf{V}, R)]$. Hence, for any $y \in \mathbf{A} \setminus \{x\}$, $\{i \in \mathbf{N} \mid y\, P_i\, x\} \notin \beta_v^+(y)$. As $\beta_v^+(y)$ is of the form $\{K \subseteq \mathbf{N} \mid |K| \geq k\}$ for some $k \in \mathbf{N}$, any $K \in \beta_v^+(y)$ has a cardinality greater than the cardinality of $\{i \in \mathbf{N} \mid y\, P_i\, x\}$. Thus, $|\{i \in \mathbf{N} \mid y\, P_i\, x\}| < b_v^+(y) \leq b_v^+$. So, there exists no $y \in \mathbf{A}$ with $y\, D(R; n, b_v^+)\, x$, implying that $x \in C(R; n, b_v^+)$, and thereby showing that $\mathbf{V}^\sigma(R) \subseteq C(R;n, b_v^+)$, as desired. ■

Remark 4.4. The reverse of the inclusion in Theorem 4.3 does not hold. Consider the constant voting rule $\mathbf{V}$: $\mathbf{W}^N \to \mathbf{A}$ (with $|\mathbf{A}| > 1$) where for each $R \in \mathbf{W}^N$ we have $\mathbf{V}(R) = x^*$ for some fixed $x^* \in \mathbf{A}$. Clearly $\mathbf{V}$ is anonymous. Note that $b_v^+(x) = n + 1$ for every $x \in \mathbf{A} \setminus \{x^*\}$ and $b_v^+(x^*) = 0$ implying that $b_V^+ = n + 1$. Thus, for any profile $R \in \mathbf{W}^N$ we have $C(R;n, b_V^+) = \mathbf{A}$, whereas $\mathbf{V}^\sigma(R) = \{x^*\}$.

Thus, we know by Theorems 4.1 and 4.3 that, for any top-unanimous and anonymous voting rule $\mathbf{V}$, $C(R;n, b_v^-) \subseteq \mathbf{V}^\sigma(R) \subseteq C(R;n, b_v^+)$ holds at every $R \in \mathbf{W}^N$. Of course, the tighter the interval $[b_v^-, b_v^+]$, the stronger is our result.

[12] We have the usual definition of anonymity, i.e., we say that a voting rule $\mathbf{V}$: $\mathbf{W}^N \to \mathbf{A}$ is *anonymous* if and only if, given any permutation ρ: $\mathbf{N} \to \mathbf{N}$ of voters and any $(R_i)_{i \in N} \in \mathbf{W}^N$, we have $\mathbf{V}((R_i)_{i \in N}) = \mathbf{V}((R_{\rho(i)})_{i \in N})$.

In particular, writing $n^* = n^{**} - 1$, in view of Theorems 4.1 and 4.3, it is of interest to see for which voting rules **V** we have $\{b_v^-, b_v^+\} \subseteq \{n^*, n^{**}\}$. For the strong equilibrium outcomes of the voting game induced by any such rule would coincide with the Condorcet winners in the classical sense.[13] To explore this case, we focus on a class of voting rules which we call top-majoritarian and define as follows:

Definition 4.5. A voting rule $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$ is said to be *top-majoritarian* if and only if given any $x \in \mathbf{A}$ and any $R \in \mathbf{W}^N$ with $|\{i \in \mathbf{N}: \{x\} = \text{argmax } R_i\}| \geq n^{**}$, we have $\mathbf{V}(R) = x$.

The following lemma characterizes the size of the β-effective and weakly β-effective coalitions for top majoritarian voting rules.

Lemma 4.6. *Let* $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$ *be a top-majoritarian voting rule. We have*

(*i*) $b_v^- = b_v^+ = n^{**}$ *if n is odd;*
(*ii*) $\{b_v^-, b_v^+\} \subseteq \{n^*, n^{**}\}$ *if n is even.*

Proof. Consider any top majoritarian voting rule **V**.

ad (i): Let n be odd. Taking any $x \in \mathbf{A}$, we check that $b_v^-(x) < n^{**}$ (hence $b_v^+(x) < n^{**}$) or $b_v^+(x) > n^{**}$ (hence $b_v^-(x) > n^{**}$) would contradict that **V** is top-majoritarian. So $b_v^-(x) = b_v^+(x) = n^{**}$ holds for every $x \in \mathbf{A}$, showing that $b_v^- = b_v^+ = n^{**}$, proving (i).

ad (ii): Consider now the case where n is even. As **V** is top-majoritarian, clearly $\{b_v^-(x), b_v^+(x)\} \subseteq \{n^*, n^{**}\}$ for all $x \in \mathbf{A}$, which implies $\{b_v^-, b_v^+\} \subseteq \{n^*, n^{**}\}$ as well, showing (ii), which completes the proof. ■

The following result is a direct corollary to the conjunction of Theorem 4.1, Theorem 4.3, Lemma 4.6 and the fact that top-majoritarianism implies top-unanimity.

Theorem ("Quasi-Characterization") 4.7. *Let* $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$ *be a top-majoritarian and anonymous voting rule. For any* $R \in \mathbf{W}^N$, *we have*

(*i*) $\mathbf{V}^\sigma(R) = C(R; n, n^{**})$ *when n is odd;*
(*ii*) $C(R;n, n^*) \subseteq \mathbf{V}^\sigma(R) \subseteq C(R;n, n^{**})$ *when n is even.*

Many interesting voting rules are covered by Theorem 4.7.[14] We will give examples of these in Sect. 6. But before this, we recall the infamous common

[13] When n is even, $C(R;n,n^*)$ is the set of strong Condorcet winners and $C(R;n,n^{**})$ is the set of weak Condorcet winners. We do not have such a distinction when n is odd and $C(R; n, n^{**})$ is the set of Condorcet winners. See Footnote 9.

[14] At this point, we wish to remark top-majoritarianism is not necessary for the charaterization result of Theorem 4.7 to hold. For example, the inverse plurality rule which picks the alternatives that are considered the worse by the highest number of voters is not top-majoritarian (even not top-unanimous). However, when instituted, its strong equilibrium outcomes coincide with the Condorcet winners, just as stated in Theorem 4.7.

weakness of Condorcet winners on the one hand and strong equilibria on the other: They may fail to exist. So it is natural to search for the lowest value of q for which the existence of an (n,q)-Condorcet winner is guaranteed for every $R \in \mathbf{W}^N$. We turn to this in the next section.

5 On the existence of (n, q) – Condorcet winners

Condorcet (1785) introduced the idea of choosing an alternative which, in comparison with **every** rival alternative, receives the support of a **majority** (possibly different from one rival to another) in its own favor. When a Condorcet winner does not exist, there are two natural ways of mending this rule: lowering the hurdle according to one of the two critical words – "majority" and "every" – in the definition of a Condorcet winner. Copeland (1951) does this for "every" and proposes to choose alternatives receiving majority support against the highest number of rivals. Such an alternative, called a Copeland winner, may lose in pairwise comparisons against a rival by a huge majority. Alternatively, lowering the hurdle of "majority", one can choose the alternatives receiving the most populous support against all other candidates. This is known as Kramer's (1977) Rule.[15] The generalized (n, q)-Condorcet winner concept is in the same spirit. In fact, at any given $R \in \mathbf{W}^N$, writing q(R) for the lowest value of q ensuring the existence of (n, q)-Condorcet winner, the set of (n, q(R))-Condorcet winners coincides with the set of Kramer winners.[16]

Given n voters confronting m alternatives, define $q(n, m) = \lfloor n.(m-1)/m \rfloor + 1$, where $\lfloor s \rfloor$ shows the integral part of any real number s. We now state a proposition showing the acyclicity of the binary relation D(R;n, q) when $q \geq q(n, m)$.

Proposition (Acyclicity) 5.1. Take any $k \in \{2,\ldots,m\}$. For any $R \in \mathbf{W}^N$ we have $x_i \, D(R;n, q(n, m)) \, x_{i+1}$ for all $i \in \{1,\ldots,k-1\} \Rightarrow$ not $x_k \, D(R;n, q(n, m)) \, x_1$

Proof.[17] Take any $k \in \{2,\ldots, m\}$, any $R \in \mathbf{W}^N$ and assume that $x_i \, D(R;n, q(n, m)) \, x_{i+1}$ for all $i \in \{1,\ldots, k-1\}$. Denote $K_i = \{j \in \mathbf{N} \mid x_i \, P_j \, x_{i+1}\}$ for all $i \in \{1,\ldots, k-1\}$. First note that, by the transitivity each $R_i \in \mathbf{W}$, for every h, $h' \in \{1,\ldots, k\}$ with $h' > h$ we have $\{i \in \mathbf{N} \mid x_h \, P_i \, x_{h'}\} \supset \cap_{j \in \{h,\ldots, h'-1\}} K_j$. Now,

[15] Kramer's Rule, sometimes called Simpson's Rule, is further discussed by Moulin (1983, 1988).

[16] Kramer's Rule is the application of the compromise idea of the Majoritarian Compromise (which was originally called "Approval Voting with a Plurality Floor" by Sertel 1986) to (n,q)-Condorcet winners: Choose the alternatives which are (n-q)-Condorcet winners for the lowest possible value of q.

[17] The proof can be omitted by referring to Austen-Smith and Banks (1999) who show the same result in a slightly different context.

since $|K_j| \geq q(n, m)$ for each $j \in \{h,\ldots, h' - 1\}$, we can compute that $|\cap_{j \in \{h,\ldots, h'-1\}} K_j| \geq (h'-h)\, q(n, m) - (h' - h - 1)\, q(n, m)$. Thus, $|\{i \in \mathbf{N} \mid x_h P_i x_{h'}\}| \geq (h' - h)\, q(n, m) - (h' - h - 1)\, q(n, m)$. Setting $h = 1$ and $h' = k$, we have $|\{i \in \mathbf{N} \mid x_1 P_i x_k\}| \geq (k - 1)\, q(n, m) - (k - 2)\, q(n, m)$ and so $|\{i \in \mathbf{N} \mid x_k R_i x_1\}| \leq (k - 1)\, (n - q(n, m))$. The proof is completed by checking that $(k - 1)\, (n - q(n, m)) < q(n, m)$ for any $k \in \{2,\ldots,m\}$. ■

The following theorem is a direct corollary to Proposition 5.1.

Theorem 5.2. *For every* $R \in \mathbf{W}^N$ *we have* $C(R;n, q(n, m)) \neq \emptyset$.

It is easy to check that $q(n, m)$ is minimal for Theorem 5.2 to hold, i.e., there exists no $q < q(n, m)$ for which $C(R;n, q) \neq \emptyset$ would hold at every $R \in \mathbf{W}^N$.

Theorem 5.2 tells that $(n, q(n, m))$ Condorcet winners always exist. Thus, for $m = 4$ we have 75%, for $m = 5$ we have 80%, etc., as "benchmark" majorities, guaranteeing the existence of a generalized $(n, q(n, m))$-Condorcet winner, i.e., an alternative which is regarded as worse than no alternative by more than this majority of voters.

6 Applications

Write **T** for the set of total orders[18] over **A**. We say that a voting rule $\mathbf{V}: \mathbf{W}^N \to \mathbf{A}$ is a t-*refinement* of a social choice correspondence $\mathbf{F}: \mathbf{W}^N \to 2^A \setminus \{\emptyset\}$ if and only if there exists some "tie-breaking rule" $\mathbf{t} \in \mathbf{T}$ such that, at any $R \in \mathbf{W}^N$, $\mathbf{V}(R) = \text{argmax}_{F(R)}\, \mathbf{t}$. Clearly, every t-refinement of every anonymous social choice correspondence **F** will be anonymous.

6.1 Refinements of anonymous top-majoritarian social choice correspondences

We say that a social choice correspondence $\mathbf{F}: \mathbf{W}^N \to 2^A \setminus \{\emptyset\}$ is *top-majoritarian* if and only if the following two conditions hold for all $x \in \mathbf{A}$:

(i) $\mathbf{F}(R) = \{x\}$ whenever $\{x\} = \text{argmax}\, R_i\ \forall\, i \in K$ for some $K \subset \mathbf{N}$ with $|K| \geq n^{**}$

(ii) $x \in \mathbf{F}(R)$ whenever $\{x\} = \text{argmax}\, R_i\ \forall\, i \in K$ for some $K \subset \mathbf{N}$ with $|K| = n^*$, if n is even.

Proposition 6.1. Any t-refinement of an anonymous top-majoritarian social choice correspondence is anonymous and top-majoritarian.

We omit the proof which is obvious. Nevertheless, we wish to note that Plurality, Plurality with a Runoff, the Single Transferable Vote, the Majoritarian Compromise (Sertel 1986, Sertel and Yılmaz 1999) are all anonymous,

[18] We recall that a *total order* is a complete, transitive and antisymmetric binary relation.

top-majoritarian social choice correspondences. So, thanks to Proposition 6.1, we know that any t-refinement of these will be covered by Theorem 4.7, whereby the strong equilibrium outcomes of their voting game will coincide, at any preference profile, with the Condorcet winners in the classical sense.

6.2 Refinements of Borda's rule

It is easy to construct examples where Borda's Rule $\mathbf{B}: \mathbf{W}^{N} \rightarrow 2^{A} \setminus \{\emptyset\}$ selects some singleton $x \in \mathbf{A}$, although n is odd and n^{**} of the voters $i \in \mathbf{N}$ have $\{y\} = \operatorname{argmax}_A R_i$ for some $y \in \mathbf{A}\setminus\{x\}$. Thus, $\mathbf{B}$ has no top-majoritarian t-refinement and escapes Theorem 4.7. Of course, for any voting rule $\mathbf{V}$ equated to a t-refinement of $\mathbf{B}$, by Theorems 4.1 and 4.3, we have $C(R;n, b_V^-) \subseteq \mathbf{V}^{\sigma}(R) \subseteq C(R;n, b_v^+)$, since Borda's Rule (hence $\mathbf{V}$) is anonymous and top-unanimous. But b_v^- and b_v^+ may be quite disparete in this case, as we see from the next proposition.

For any real number s, let $\langle s \rangle$ stand for the smallest integer which is no less than s.

Proposition 6.2. Let $\mathbf{V}: \mathbf{W}^{N} \rightarrow \mathbf{A}$ be a t-refinement of $\mathbf{B}$. We have

(i) $b_v^+ = n^{**}$ when n is odd;
(ii) $b_v^+ \in \{n^*, n^{**}\}$ when n is even;
(iii) $b_v^- = \langle (1/m)n \rangle$ if n is not divisible by m;
(iv) $b_v^- \in \{\langle (1/m)n \rangle, \langle (1/m)n \rangle + 1\}$ if n is divisible by m.

Proof. Let $\mathbf{V}$ be any t-refinement of $\mathbf{B}$. It is easy to see that for every $x \in \mathbf{A}$ we have $b_v^+(x) = n^{**}$ when n is odd and $b_v^+(x) \in \{n^*, n^{**}\}$ when n is even, which is sufficient to show that $b_v^+ = n^{**}$ when n is odd and $b_v^+ \in \{n^*, n^{**}\}$ when n is even. Now, for any $n \geq 2$, consider some $K \subseteq \mathbf{N}$ with $|K| = k$ whose members rank some $y \in \mathbf{A}$ as the best and some $x \in \mathbf{A} \setminus \{y\}$ as the second best alternative, i.e., for every $i \in K$, we have $\operatorname{argmax}_A R_i = \{y\}$ and $\operatorname{argmax}_{A\setminus\{y\}} R_i = \{x\}$. If the remaining agents $\mathbf{N}\setminus K$ wish to ensure that $x \in \mathbf{A}$ is chosen, then the best thing they can do is to rank x as the best and y as the worst, i.e., for every $i \in \mathbf{N}\setminus K$, let $\operatorname{argmax}_A R_i = \{x\}$ and $\operatorname{argmin}_A R_i = \{y\}$. In this case x will get a Borda score $B(x) = k(m - 1) + (n - k)m$ and y a Borda score $B(y) = km + (n - k)$. Thus, $B(x) \geq B(y) \Rightarrow k \leq n[(m - 1)/m] \Rightarrow n - k \geq n(1/m)$ and similarly $B(x) > B(y) \Rightarrow n - k > n(1/m)$. Thus if n is not divisible by m, for any t-refinement $\mathbf{V}$, a coalition of size $\langle n(1/m) \rangle$ will be weakly β-effective over x. If n is divisible by m, then, depending on the tie-breaking rule used in the refinements, a coalition of size $\langle n(1/m) \rangle$ or $\langle n(1/m) \rangle + 1$ will be weakly β-effective over x. This completes the proof. ■

6.3 k-majority rules

Given any integer $k > n - k$, what we mean by a k-majority rule is a voting rule $\mathbf{V}_k$: $\mathbf{W}^N \to \mathbf{A}$ which is defined as follows: For some "status quo" $s \in \mathbf{A}$, every $x \in \mathbf{A}$ and every $R \in \mathbf{W}^N$,

$$\mathbf{V}_k(R) = \begin{cases} x & \text{if there exists } K \subseteq \mathbf{N} \text{ with } |K| \geq k \text{ and } \{x\} = \operatorname{argmax} R_i \\ & \quad \text{for all } i \in K; \\ s & \text{otherwise.} \end{cases}$$

Proposition 6.3. Let $\mathbf{V}_k$: $\mathbf{W}^N \to \mathbf{A}$ (for some $k > n - k$) be a k-majority rule. Regardless of the status quo $s \in \mathbf{A}$, $\mathbf{V}_k$ is top-unanimous and anonymous (so that Theorems 4.1 and 4.3 apply). Furthermore, we have $b_v^- = n - k + 1$ and $b_v^+ = k$.

Proof. Take any integer $k > n - k$ and any status quo $s \in \mathbf{A}$. It is clear that $\mathbf{V}_k$ is top-unanimous and anonymous. One can directly check that we have $b_v^+(x) = b_v^-(x) = k$ for every $x \in \mathbf{A} \setminus \{s\}$ and $b_{\mathbf{V}}^-(s) = b_v^+(s) = n - k + 1$, thus implying $b_v^- = n - k + 1$ and $b_v^+ = k$, completing the proof. ■

Thus, given any integer $k > n - k$, we have $C(R;n, n-k+1) \subseteq \mathbf{V}^\sigma(R) \subseteq C(R;n, k)$ at every $R \in \mathbf{W}^N$ when a k-majority rule $\mathbf{V}_k$ is instituted as the outcome function.

6.4 (n, q)-Condorcet winners as a social choice correspondence

We know by Theorem 5.2 that, for every integer $q \geq q(n, m)$, we have $C(R;n, q) \neq \emptyset$ at every $R \in \mathbf{W}^N$. So, taking such a $q \geq q(n, m)$, we can use $C(. ; n, q) = \mathbf{F}$ as a social choice correspondence $\mathbf{F}$: $\mathbf{W}^N \to 2^A \setminus \{\emptyset\}$. Let $\mathbf{V}$: $\mathbf{W}^N \to \mathbf{A}$ be any t-refinement of such a social choice correspondence $\mathbf{F}$. One can check that $\mathbf{V}$ is top-unanimous and anonymous, so that Theorems 4.1 and 4.3 apply. In fact, we have $b_v^+(x) = b_v^-(x) = n - q + 1$ for every $x \in \mathbf{A}$, thus implying $b_{\mathbf{V}}^- = b_v^+ = n - q + 1$. Hence, given any integer $q \geq q(n, m)$, instituting any t-refinement of $C(R;n, q)$ as outcome function, we have $\mathbf{V}^\sigma(R) = C(R;n, n-q+1)$ at every $R \in \mathbf{W}^N$.

Given any positive integer $q < q(n, m)$, from $C(R; n, q)$ we can still derive a social choice correspondence $\mathbf{F}$: $\mathbf{W}^N \to 2^A \setminus \{\emptyset\}$ by setting $\mathbf{F}(R) = s$ for some previously fixed $s \in \mathbf{A}$ whenever $C(R;n, q)$ is empty. Let $\mathbf{V}$: $\mathbf{W}^N \to \mathbf{A}$ be any t-refinement of such a social choice correspondence $\mathbf{F}$. One can check that $\mathbf{V}$ is unanimous and anonymous, whereby Theorems 4.1 and 4.3 apply. One can directly verify that $b_v^+(x) = b_v^-(x) = n - q + 1$ for every $x \in \mathbf{A} \setminus \{s\}$ and $b_v^-(s) = b_v^+(s) = q$, thus implying $b_v^- = q$ and $b_v^+ = n - q + 1$. Thus, given any positive integer $q < q(n, m)$, we have $C(R;n,q) \subseteq \mathbf{V}^\sigma(R) \subseteq C(R;n, n-q+1)$ at every $R \in \mathbf{W}^N$ when we institute any t-refinement $\mathbf{V}$ of $\mathbf{F}$ as outcome function.

7 Closing remarks

In this paper we characterized the strong equilibrium outcomes of certain voting games. It turns out, for many well-known social choice rules, that if we refine them to singleton-valued functions by use of tie-breaking rules and employ them in the role of the outcome function of a direct mechanism, at the strong equilibria of the voting games so determined, we obtain the Condorcet winners when they exist. It may be regarded as a somewhat striking irony that many social choice rules which themselves fail to be Condorcet-consistent, turn out to induce a Condorcet-consistent outcome at the strong equilibria of the voting games they determine.[19]

Our results also afford an interpretation in the spirit of implementing the classical Condorcet social choice rule on a restricted domain of preference profiles where it is non-empty-valued, e.g. where preferences are single-peaked. On such a domain, we implement the Condorcet social choice rule via strong equilibrium by use of direct mechanisms employing outcome functions which are refinements obtained through tie-breaking rules of certain well-known social choice rules: Theorem 4.7 is a full implementation result via top-majoritarian social choice rules, while Theorem 4.3 is a partial implementation result via anonymous social choice rules. This should constitute a certain rehabilitation of Condorcet's social choice rule: In particular, when $q = n^{**}$, the Condorcet Rule implements itself under strong equilibrium on the domain of preference profiles where a Condorcet winner exists.

This positive implementation result loses some of its bite when we allow an unrestricted domain of preferences, as it is based on our equivalence result between the set of generalized Condorcet winners and the set of strong equilibrium outcomes of the voting game determined by instituting a social choice function as the outcome function of a mechanism. For this equivalence result may turn out to be the trivial equivalence of the empty set to the empty set when there are no generalized Condorcet winners at a given preference profile.[20] So one may ask how far we should generalize (and weaken) the notion of a "Condorcet winner" in order to guarantee its existence independent of the preference profile. We know that an (n, q)-Condorcet winner always exists whenever $q \geq \lfloor n. (m - 1) / m \rfloor + 1$. Since

[19] Neither of the social choice rules (Plurality, Plurality with a Runoff, the Majoritarian Compromise and the Single Transferable Vote) which we found in Sect. 6 to give Condorcet winners at the strong equilibrium of the voting game they determine is Condorcet-consistent in the sense of always choosing a Condorcet winner when it exists. Furthermore, Plurality and the Majoritarian Compromise are both capable of choosing a Condorcet loser (i.e., Condorcet winner with all preferences reversed) even in the presence of a Condorcet winner. See Sertel and Yılmaz (1997).

[20] In fact, one can check the existence of strong equilibria of the voting game at some preference profile just by checking the existence of (n,q)-Condorcet winners, for the appropriate q, at that preference profile.

n(m − 1)/m approaches n as the cardinality m of our set of alternatives increases, however, this generalization may have gone too far when we have a large number of alternatives. Indeed, the set of generalized Condorcet winners converges to the set of weakly Pareto optimal outcomes as the cardinality of the set of alternatives increases beyond bound.

To sum up, we show, for a large class of social choice rules (including Plurality, Plurality with a Runoff, the Majoritarian Compromise and the Single Tranferable Vote), that when strong equilibrium is used as the solution concept, the social choice rule employed in the role of outcome function of a direct mechanism implements on the domain of profiles with Condorcet winners, not necessarily itself[21], but a Condorcet social choice rule.[22] These results, although simple and intuitive[23], can be considered as positive from an implementation viewpoint, in some contrast with the prevalence of impossibility results in this field.

Research questions: It is natural to question for which shrinkings and for which swellings (expansions) of the strong equilibrium solution concept our characterization results, Theorems 4.1, 4.3 and 4.7, remain valid. The strong equilibrium being hard to shrink, we address the swelling question.[24] At the extreme, it is easy to see that under every social choice function which satisfies the weak no veto power condition, the entire set **A** of alternatives is attainable as outcomes of our voting games under Nash equilibrium. As to the case of coalition-proof Nash equilibrium, a generally smaller swelling of the strong equilibrium, the question becomes more interesting.[25]

Here we focus on the central question of whether, in the case of voting games, all coalition-proof Nash equilibrium outcomes can actually be matched in welfare by strong equilibrium outcomes. The answer is negative, for reason of the following simple example:

[21] In fact, Condorcet-consistent social choice correspondences are self-implementable on this domain of preference profiles with Condorcet winners.

[22] It is not surprising that Borda's Rule is not in this list.

[23] Regarding the intuitiveness of our results, some caution may be advisable. For it is possible to think, as we admit having done ourselves at the outset, that the Condorcet consistency of the outcomes of a voting game under strong equilibrium is more or less a direct consequence of definitions. This paper is witness to our having educated ourselves into realizing that this was far from being so. In fact, Borda's Rule and k-majority rules defy the easy credo which we had to abandon.

[24] A natural shrinking is to allow blocking coalitions whose members weakly improve, as long as one of these members strictly improves. For Theorems 4.1 and 4.7 to be valid under this stronger definition of equilibrium, we need to adopt the corresponding stronger definition of an (n, q)-Condorcet winner: An alternative x is an (n, q)-Condorcet winner if and only if there exist no other alternative y such that the number of agents who find y at least as good as x is no less than q while at least one agent strictly prefers y to x.

[25] We thank an anonymous referee for leading us to this question.

In the context of **A** = {x, y, z} and **N** = {1, 2, 3}, given a preference profile R with

R_1	R_2	R_3
x	y	z
y	x	x
z	z	y

under Plurality refined by any tie-breaking rule with z beating x and y, regarding any declaration where all three agents find y better than both x and z, such a declaration will be a coalition-proof Nash equilibrium, resulting in the outcome y, while y cannot be attained as a strong equilibrium outcome here. The strong equilibrium outcome here is x, of course, and the welfare of x and y differ for our three-agent society.

It is to be noted, furthermore, that while strong equilibria of a voting game exist precisely when a Condorcet winner exists, coalition-proof Nash equilibria may exist even in the absence of Condorcet winners. The example of

R_1	R_2	R_3
x	y	z
y	z	x
z	x	y

where we use Plurality refined by any tie-breaking rule with y beating x and z suffices to show this, as here we have no Condorcet winner but any declaration where all three agents find x better than both y and z is a coalition-proof Nash equilibrium of the associated voting game, resulting in the outcome x.

Summarily, the strong equilibrium concept gives a rather perfect fit with Condorcet's notion while the coalition-proof Nash equilibrium concept is too "fat".[26] Are there, then, interesting shrinkings or swellings of the strong equilibrium for which our characterization results, Theorems 4.1, 4.3 and 4.5, are preserved?

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[26] This is even so to the point of denying coalition-proof Nash equilibrium outcomes to be contained in C(. , n, q) for anonymous social choice functions, e.g., for selections of Plurality, so that even a counterpart of Theorem 4.3 fails to hold when we replace the strong equilibrium concept by the coalition-proof Nash equilibrium.

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