Experiments in Fluids 35 (2003) 188–192
DOI 10.1007/s00348-003-0656-3

Extended proper orthogonal decomposition: a tool to analyse correlated events in turbulent flows

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Abstract A tool to analyse correlated events in turbulent flows based on an extended proper orthogonal decomposition (POD) is proposed in this paper. A general definition of extended POD modes is presented and their properties are demonstrated. If the initial POD analysis in a spatio-temporal domain *S* concerns, for example, velocity—the concept of extended modes can be applied to study the correlation of any physical quantity in any domain Ω with the projection of the velocity field on POD modes in *S*. The link with particular associations of POD and linear stochastic estimation (LSE) recently proposed is demonstrated at the end of the paper. The method is believed to provide a valuable tool to extend the well-documented POD analysis of eddy structures in turbulent flows, for example, in boundary layers or free shear flows. If extended modes are velocity modes, spatial and temporal interactions between eddy structures can be detected and studied. The rapid development of experimental diagnostic techniques now permit measurements of the concentration in the domain, the velocity of a dispersed phase in the domain or the static pressure at the boundary together with the fluid velocity field. Using this method we are then able to extract objectively the link between the representative groups of velocity modes and the correlated part of the concentration, particle motion or pressure signals.

1 Introduction

The proper orthogonal decomposition (POD) was proposed by Lumley (1967) as an unbiased method for extracting structures in a turbulent flow. The POD is optimal as far as the kinetic energy contained in the successive modes is concerned. This basic statistical tool is now widely applied for analysing experimental data obtained with PIV (particle image velocimetry) or rakes of hot wires. It can also be useful for extracting relevant information from the large amount of data obtained with direct numerical simulation or large eddy simulation.

The POD has been applied to many flow configurations. Berkooz et al. (1993) provide a comprehensive survey of the early applications. A recent review dedicated to turbulent free shear flows is proposed by Bonnet et al. (2002). In stationary quasi-parallel turbulent flows—such as jets, mixing layers and boundary layers—the main goals are to identify and to analyse the spatial and dynamic properties of coherent structures embedded in the turbulence and to construct low-dimensional models that exhibit most of the coherent properties of the flow (Aubry et al. 1988; Ukeiley et al. 2001). The use of POD combined with linear stochastic estimation (LSE) (Adrian 1975) to obtain relevant boundary conditions for unsteady computations from a limited number of measurement points is an important recent development of the technique presented by Bonnet and Delville (2001) and Bonnet et al. (2002). A particularly challenging application of the POD is to derive practical flow sensing and control strategies. The recent contributions of Picard and Delville (2000) and Taylor and Glauser (2002) show that a particular combination of POD and LSE techniques allows the remote sensing of the velocity field via pressure signals at the wall or at the free boundary of a round jet, as described at the end of the paper. The POD analysis is now applied to study the correlation of any physical quantities such as pressure, concentration or temperature with the coherent properties of a given flow.

An extended POD was first defined and used in Maurel (2001) and Maurel et al. (2001) to study a jet–vortex interaction in a model representative of internal combustion engine flows (Borée et al. 2002). Extended modes were introduced using the snapshot method (Sirovich 1987). The POD modes were first computed in only one subdomain *S* of the measurement domain Ω. *S* was located on the vortex core containing just 3% of the total kinetic energy in Ω. For any mode, we were then able to educe in all Ω the particular part of the velocity field correlated with the projection of the velocity on this mode in *S*.

The goal of this paper is to provide a more general definition of extended POD modes, independent of the particular technique used to obtain physical data or to educe POD decomposition in *S*. If the POD analysis in *S* concerns, for example, velocity, the concept of extended modes can be applied to study the correlation of any physical quantity with the velocity field, as will be

Received: 24 September 2002 / Accepted: 12 May 2003
Published online: 12 July 2003

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The author addresses a special acknowledgement to Dr. S. Maurel and warmly thanks Professors J.L .Lumley and J.C. Valière, Drs. J. Delville and J.P. Bonnet for their comments about this work.

demonstrated. Finally, we will discuss some possible applications to problems of fluid mechanics of turbulent flows and the link with particular associations of POD and linear stochastic estimation (LSE) recently proposed.

2
Definition of POD modes

We consider $\mathbf{U}(x,y,z,t)=\mathbf{U}(\mathbf{X})$ to be a set of realisations of the velocity field in the spatio-temporal domain S. The fields here are all real. The POD modes are obtained by searching for the deterministic function $\Phi(\mathbf{X})$ that is most similar to the members of $\mathbf{U}(\mathbf{X})$ on average. We define the inner product $(\mathbf{U},\mathbf{V})$ and the norm $\|\mathbf{U}\|=(\mathbf{U},\mathbf{U})^{1/2}$ by:

$$(\mathbf{U},\mathbf{V})=\sum_{i=1}^{n_c}\int_S U_i(\mathbf{X})V_i(\mathbf{X})\mathrm{d}\mathbf{X} \tag{1}$$

where n_c is the number of components of $\mathbf{U}$.

Looking at the function that has the largest mean square projection is a maximisation problem that leads to a Fredholm integral value problem, where the kernel is the two-point correlation tensor $R_{ij}(\mathbf{X},\mathbf{X}')=\langle U_i(\mathbf{X})U_j(\mathbf{X}')\rangle$. The average operator $\langle - \rangle$ can be temporal, spatial, ensemble or phase average depending on the approach used.

We only recall here the major properties of POD decomposition—see Berkooz et al. (1993) and Delville et al. (1998) for more details. The integral equation has a discrete set of solutions $\Phi^{(n)}(\mathbf{X})$ and $\lambda^{(n)}$ where (n) is the order of the orthogonal decomposition. $\lambda^{(n)}$ and $\Phi^{(n)}(\mathbf{X})$ are respectively the eigenvalues and eigenfunctions of the two-point correlation tensor. All the quantities are real in the situation discussed here. The eigenvalues are positive and $\lambda^{(n)}>\lambda^{(n+1)}$. The eigenfunctions are chosen to be orthonormal with:

$$\left(\Phi^{(n)},\Phi^{(p)}\right)=\delta_{np}. \tag{2}$$

Each realisation of the random velocity field $\mathbf{U}(\mathbf{X})$ may be reproduced by a decomposition in the eigenfunction:

$$i=1,\ldots,n_c \quad U_i(\mathbf{X})=\sum_n a^{(n)}\Phi_i^{(n)}(\mathbf{X}). \tag{3}$$

An important property of the decomposition is that the random coefficients $a^{(n)}=(\mathbf{U},\Phi^{(n)})$ are uncorrelated with:

$$\left\langle a^{(n)}a^{(p)}\right\rangle=\lambda^{(n)}\delta_{np}. \tag{4}$$

This general statistical theory is very well adapted to problems in turbulence study as it provides: (i) an optimal decomposition of the kinetic energy integrated over the domain S with

$$(\mathbf{U},\mathbf{U})=\sum_n \lambda^{(n)};$$

(ii) a diagonal decomposition of the correlation tensor

$$R_{ij}(\mathbf{X},\mathbf{X}')=\sum_n \lambda^{(n)}\Phi_i^{(n)}(\mathbf{X})\Phi_j^{(n)}(\mathbf{X}').$$

The following expression (5) for $\Phi^{(p)}(\mathbf{X})$ is derived from these general properties and will be useful for the definition of extended modes. By multiplying (3) with the random coefficient $a^{(p)}$ associated with the projection of $\mathbf{U}(\mathbf{X})$ on mode $\Phi^{(p)}(\mathbf{X})$ and by averaging, one obtains:

$$\left\langle a^{(p)}\mathbf{U}(\mathbf{X})\right\rangle=\left\langle a^{(p)}\sum_n a^{(n)}\Phi^{(n)}(\mathbf{X})\right\rangle$$
$$=\sum_n\left\langle a^{(n)}a^{(p)}\right\rangle\Phi^{(n)}(\mathbf{X})=\lambda^{(p)}\Phi^{(p)}(\mathbf{X}).$$

Therefore,

$$\Phi^{(p)}(\mathbf{X})=\frac{\langle a^{(p)}\mathbf{U}(\mathbf{X})\rangle}{\lambda^{(p)}}. \tag{5}$$

3
Definition of extended POD modes

We consider $\alpha(\mathbf{X}')$ to be a set of physical quantities (scalar or vectors) in a domain Ω (Ω can be equal to S, can contain S or not). The dimension of Ω is not necessarily the same as the dimension of S, for example, if $\alpha(\mathbf{X}')$ is a pressure signal acquired along a boundary. Each realisation of $\alpha(\mathbf{X}')$ is associated with a realisation of the velocity field $\mathbf{U}(\mathbf{X})$ in S.

We define the extended mode number (p) by:

$$\mathbf{X}'\in\Omega \quad \Psi_\alpha^{(p)}(\mathbf{X}')=\frac{\langle a^{(p)}\alpha(\mathbf{X}')\rangle}{\lambda^{(p)}}. \tag{6}$$

One can note that for $\mathbf{X}$ in the domain S and if $\alpha=\mathbf{U}$, extended modes $\Psi_\mathbf{U}^{(p)}(\mathbf{X})$ restrict to POD mode $\Phi^{(p)}(\mathbf{X})$ by virtue of relation (5). If the averaging corresponds to an ensemble average of N independent samples, then $\Psi_\alpha^{(p)}(\mathbf{X}')$ is easily computed with:

$$\Psi_\alpha^{(p)}(\mathbf{X}')=\sum_{k=1}^{N}\frac{a_k^{(p)}}{\lambda^{(p)}N}\alpha_k(\mathbf{X}'). \tag{7}$$

If the averaging corresponds to a time average, then the domains S and Ω restrict obviously to spatial domains and $\Psi_\alpha^{(p)}(\mathbf{X}')$ reads:

$$\Psi_\alpha^{(p)}(\mathbf{X}')=\frac{1}{\lambda^{(p)}T}\int_0^T a^{(p)}(t)\alpha(\mathbf{X}',t)\mathrm{d}t \tag{8}$$

where T is the integration time.

For each realisation of the data set $\alpha(\mathbf{X}')$, we propose the following decomposition:

$$\alpha_C(\mathbf{X}')=\sum_n a^{(n)}\Psi_\alpha^{(n)}(\mathbf{X}')$$
$$\alpha_D(\mathbf{X}')=\alpha(\mathbf{X}')-\alpha_C(\mathbf{X}'). \tag{9}$$

We stress that $a^{(n)}$ are the random coefficients associated with the POD problem in the domain S for the velocity $\mathbf{U}$ in the present context.

The following results are then easy to demonstrate because the random coefficients $a^{(n)}$ are uncorrelated (see relation (4)):

Proposition I: *α_C is the only part of the signal α* correlated with $\mathbf{U}$.

Indeed, for $\mathbf{X}\in S$; $\mathbf{X}'\in\Omega$:

$$\langle\alpha(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle = \left\langle \alpha(\mathbf{X}')\sum_n a^{(n)}\Phi^{(n)}(\mathbf{X}) \right\rangle = \sum_n \left\langle a^{(n)}\alpha(\mathbf{X}')\right\rangle \Phi^{(n)}(\mathbf{X}).$$

Thus,

$$\langle\alpha(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle = \sum_n \lambda^{(n)}\Psi_\alpha^{(n)}(\mathbf{X}')\Phi^{(n)}(\mathbf{X})$$

by definition (6) of extended modes. Moreover,

$$\langle\alpha_C(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle = \left\langle \sum_p a^{(p)}\Psi_\alpha^{(p)}(\mathbf{X}')\sum_n a^{(n)}\Phi^{(n)}(\mathbf{X}) \right\rangle = \sum_n\sum_p \left\langle a^{(n)}a^{(p)}\right\rangle \Psi_\alpha^{(p)}(\mathbf{X}')\Phi^{(n)}(\mathbf{X}).$$

According to (4),

$$\langle\alpha_C(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle = \sum_n \lambda^{(n)}\Psi_\alpha^{(n)}(\mathbf{X}')\Phi^{(n)}(\mathbf{X}) = \langle\alpha(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle.$$

Finally $\langle\alpha_D(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle=\langle\alpha(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle-\langle\alpha_C(\mathbf{X}')\mathbf{U}(\mathbf{X})\rangle=0$ which proves proposition I.

Because $\Phi^{(n)}$ is an orthonormal basis, we also conclude that whatever the mode (n), $\langle\alpha_D(\mathbf{X}')a^{(n)}\rangle=0$. Note that the decorrelated part $\alpha_D(\mathbf{X}')$ was omitted in the analysis of Maurel et al. (2001).

Proposition II: *$a^{(n)}\Psi_\alpha{}^{(n)}(\mathrm{X}')$ is the only contribution to α_C correlated with the projection of the velocity field on mode (n) in S.*

Let us call $\mathbf{U}_n(\mathbf{X})$ the projection of $\mathbf{U}$ on mode (n) in S: $\mathbf{U}_n(\mathbf{X})=a^{(n)}\Phi^{(n)}(\mathbf{X})$. Using relations (9) and (4), we show that $\langle\alpha_C(\mathbf{X}')\mathbf{U}_n(\mathbf{X})\rangle=\lambda^{(n)}\Psi_\alpha^{(n)}(\mathrm{X}')\Phi^{(n)}(\mathbf{X})$ which proves Proposition II.

Proposition III:

$$\langle\alpha(\mathbf{X}')\alpha(\mathbf{X}')\rangle = \langle\alpha_D(\mathbf{X}')\alpha_D(\mathbf{X}')\rangle + \sum_n \lambda^{(n)}\Psi_\alpha^{(n)}(\mathbf{X}')\Psi_\alpha^{(n)}(\mathbf{X}').$$

The proof of Proposition III is the result of an easy computation. The energy of the correlated part of the signal α is therefore decomposed as a sum of contributions related to each extended modes. Because a POD problem was solved in the domain S only, this decomposition in Ω is not optimal.

4
Comments and concluding remarks

Following the present derivation if we suppose that a mode or a group of modes in S is associated with a given physical process, then, for any signal α in the domain Ω, we provide an objective way to educe the only part and all the part of α correlated with this physical process.

If α is a velocity field, this technique can be used, for example, to study the spatial and temporal interactions between coherent structures in a flow. For this purpose, the POD analysis should be restricted to one sub-region of the measured or computed flow field and extended modes should be deduced in another region, eventually with a varying time lag. A basic example is boundary layers where the spatial and temporal interactions between bursts are not fully understood. There is a major effort to apply this knowledge to the optimal control under given constraints (Aubry et al. 1996). For example, if a low-dimensional model is identified in a given sub-region of the flow field, then the evolution of the correlated part is immediately deduced when extended modes are known.

In free turbulent shear flows, one major challenge (Fiedler 1998) is to analyse, predict and eventually control the role of coherent structures on (i) entrainment and mixing of external fluid, (ii) sound generation, (iii) transport of a dispersed solid or liquid phase (Longmire and Eaton 1992). For the experimentalists, it is difficult to provide simultaneous measurements of various quantities in a domain, but there have been rapid recent developments in Laser diagnostic techniques. Suppose that realisations of the concentration field in S, of the motion of particles in S or of the static pressure distribution at the boundary ∂S of the domain have been recorded simultaneously with the velocity field $\mathbf{U}$. On the basis of well-documented analysis of eddy structures in such flows using POD (Glauser and George 1987; Delville et al. 1998; Citriniti and George 2000), we are then able to extract in an objective way the correlation between the representative groups of velocity modes and the correlated concentration, particle motion or pressure extended modes.

We stated in the introduction that a very important application of POD studies is devoted to flow sensing and control. Picard and Delville (2000) with an axisymmetric jet and Taylor and Glauser (2002), using a backward facing step with adjustable flap, have shown recently that if a data base of simultaneous pressure and velocity measurements is initially obtained, then it is possible to perform a remote sensing of the velocity field by using LSE (Adrian 1975) to estimate the instantaneous velocity field from instantaneous pressure information. The links with the tool presented here are established to conclude this paper.

A modified stochastic approach is proposed by Taylor and Glauser (2002) to deduce estimated random POD coefficients $\tilde{a}^{(n)}$ of the velocity field. The interest of this method, stressed by the authors, is that the surface pressure is then viewed as an indicator of the presence of the conditional structure associated with the projection of the velocity field on POD mode number (n). The LSE of $a^{(n)}$ then reads

$$\tilde{a}^{(n)} = \sum_{j=1}^{q} b_{nj}p_j$$

where p_j (j=1,...,q) is the value of the instantaneous pressure at the probe number j. The coefficients b_{nj} are obtained by solving the linear system of equations:

$$\sum_{j=1}^{q} \langle p_i p_j\rangle b_{nj} = \left\langle a^{(n)}p_i\right\rangle. \quad \text{for } \mathrm{i}=1,\cdots,\mathrm{q}$$

The correlation between the surface pressure and the POD expansion coefficients is therefore used for the remote sensing of a velocity field. The properties of this correlation have been demonstrated in the present work. The determination of b_{nj} depends on the distribution of the extended pressure mode number (n) with

$$\sum_{j=1}^{q} \langle p_i p_j \rangle b_{nj} = \lambda^{(n)} \Psi_{pi}^{(n)}.$$

Moreover, by definition of the LSE of $a^{(n)}$, it is easy to show that

$$\left\langle \tilde{a}^{(n)} p_i \right\rangle = \left\langle \sum_{j=1}^{q} b_{nj} p_j p_i \right\rangle = \sum_{j=1}^{q} b_{nj} \langle p_j p_i \rangle = \left\langle a^{(n)} p_i \right\rangle.$$

This means that the extended pressure modes

$$\tilde{\Psi}_{pi}^{(n)} = \left\langle \tilde{a}^{(n)} p_i \right\rangle / \lambda^{(n)}$$

of the estimated velocity field are strictly identical to the extended pressure modes $\Psi_{pi}^{(n)} = \langle a^{(n)} p_i \rangle / \lambda^{(n)}$ of the true velocity field. This is believed to be an important property and strength of the methodology proposed by Taylor and Glauser (2002).

From a different point of view, Picard and Delville (2000), with similar pressure and velocity data, first used the POD in order to analyse the pressure signal and then the LSE to construct estimated velocity modes for each pressure POD mode. They showed that the use of LSE technique provides a physical interpretation, in terms of structures, of the pressure POD modes. These structures were also found to be responsible for the far-field noise emission of the round jet. In order to show that the LSE estimation of the velocity field (say $\tilde{\mathbf{U}}_n$) from the pressure mode number (n) (say p_n) is strictly identical to $a^{(n)} \Psi_U^{(n)}$ (see Proposition II, Sect. 3), let us call

$$p_{ni} = p_n(\mathbf{X}_i, t) = a^{(n)}(t) \Phi_p^{(n)}(\mathbf{X}_i) = a^{(n)} \Phi_{pi}^{(n)}$$

the value of the projection of the instantaneous pressure field $p_i = p(\mathbf{X}_i, t)$ on the pressure POD mode number (n) at the pressure probe $\mathbf{X}_i$ of the boundary ∂S. The LSE of the velocity field at point $\mathbf{X}$ of the domain Ω then reads:

$$\tilde{\mathbf{U}}_n(\mathbf{X}, t) = \sum_{j=1}^{q} \mathbf{b}_j(\mathbf{X}) p_{nj}$$

In Picard and Delville (2000), the vectors $\mathbf{b}_j(\mathbf{X})$ are obtained by solving the linear system of equation

$$\sum_{j=1}^{q} \langle p_i p_j \rangle \mathbf{b}_j = \langle p_i \mathbf{U} \rangle$$

or, by definition of the POD expansion of the pressure field:

$$\sum_{j=1}^{q} \left(\sum_n \lambda^{(n)} \Phi_{pi}^{(n)} \Phi_{pj}^{(n)} \right) \mathbf{b}_j = \sum_n \lambda^{(n)} \left(\sum_{j=1}^{q} \Phi_{pj}^{(n)} \mathbf{b}_j \right) \Phi_{pi}^{(n)}$$
$$= \sum_n \left\langle a^{(n)} \mathbf{U} \right\rangle \Phi_{pi}^{(n)}$$

$\Phi_p^{(n)}$ being a basis, we conclude that:

$$\lambda^{(n)} \left(\sum_{j=1}^{q} \Phi_{pj}^{(n)} \mathbf{b}_j \right) = \left\langle a^{(n)} \mathbf{U} \right\rangle = \lambda^{(n)} \Psi_U^{(n)}.$$

Finally, for any realisation of the pressure field,

$$\tilde{\mathbf{U}}_n = \sum_{j=1}^{q} \mathbf{b}_j p_{nj} = a^{(n)} \sum_{j=1}^{q} \mathbf{b}_j \Phi_{pj}^{(n)} = a^{(n)} \Psi_U^{(n)}.$$

Moreover, the LSE of the velocity field from the complete pressure field reads:

$$\tilde{\mathbf{U}} = \sum_{j=1}^{q} \mathbf{b}_j p_j = \sum_{j=1}^{q} \mathbf{b}_j \left(\sum_n a^{(n)} \Phi_{pj}^{(n)} \right)$$
$$= \sum_n a^{(n)} \left(\sum_{j=1}^{q} \Phi_{pj}^{(n)} \mathbf{b}_j \right) = \sum_n a^{(n)} \Psi_U^{(n)}.$$

According to Propositions I and II, Sect. 3, the LSE estimation of the velocity field from the pressure mode number (n) is therefore strictly identical to the contribution of the extended velocity mode number (n). Moreover, the LSE of the velocity field from the complete pressure field is the sum of the contribution of the extended velocity modes.

The advantage of the present technique is to provide a decomposition of the LSE. Moreover, the computation is direct when the POD decomposition of the pressure is known. This approach can thus be used to build a low-dimensional correlated velocity field for remote sensing and control of the flow.

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