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# Renormalization group analysis of a confined, interacting Bose gas

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**ABSTRACT** The renormalization group is not only a powerful method for describing universal properties of phase transitions, but it is also useful for evaluating non-universal thermodynamic properties beyond mean-field theory. In this contribution we concentrate on these latter aspects of the renormalization group approach. We introduce its main underlying ideas in the familiar context of the ideal Bose gas and then apply them to the case of an interacting, confined Bose gas within the framework of the random phase approximation. We model confinement by periodic boundary conditions and demonstrate how confinement modifies the flow equations of the renormalization group, thus changing the thermodynamic properties of the gas.

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## 1 Introduction

The recent experimental realization of Bose–Einstein condensation has renewed the interest in the study of trapped, dilute, weakly interacting Bose gases. Despite the long history of this subject there still exist unresolved theoretical problems for the interacting Bose gas even without the presence of a trap. Here we discuss some of these problems, their origins, and possible ways of overcoming them.

A trivial example of Bose–Einstein condensation occurs, of course, in the non-interacting (ideal) case. What is of real physical interest though is the interacting case. This is so not only because the ideal gas is, as the name suggests, an idealized approximation to reality but also because it is the interactions that render the condensate a true superfluid with remarkable properties [1]. The presence of interactions, even weak ones, complicates the mathematical treatment of the problem and calls for approximate schemes. The most well known is the Bogoliubov approach [2] and, though it was proposed more than 50 years ago, it remains the approach used most frequently in the literature of the subject.

The first step of the Bogoliubov approach is to simplify the interparticle interactions. We consider that the origin of the interactions is two-body collisions only. The gas is assumed to be very dilute, which is equivalent to assuming that the interactions are very weak. This means that the interacting gas differs only slightly from the ideal gas, and consequently most particles in the interacting ground state should have zero momentum, as they do in the ideal case. In this limit of low momenta, one can show that the interactions are dominantly of s-wave type. Thus they can be characterized by a scattering length, $a$, which is positive for repulsive interactions.

The second step is to replace the creation and annihilation operators of particles in the zero-momentum state $a_0^+$ and $a_0$, respectively, by their expectation values. The reason is that we will want to treat the interaction term perturbatively. However, for bosons, the usual form of perturbation theory fails, because neither $a_0^+$ nor $a_0$ annihilate the interacting ground state. This means that it is impossible to define normal-ordered products with a vanishing ground-state expectation value, and consequently Wick's theorem, a cornerstone of conventional perturbation theory, cannot be applied in a straightforward way [1]. This problem is fixed if one replaces $a_0^+ \to \langle a_0^+ \rangle = \sqrt{N_0}$ and $a_0 \to \langle a_0 \rangle = \sqrt{N_0}$, where $N_0$ is the average number of condensate atoms, and reinterprets $a_0^+$, $a_0$ as ordinary numbers that commute. Approximating these operators this way is justified because their commutator is of order $1/\sqrt{N_0}$ with respect to their individual matrix elements and because we assume that the condensate density $n_0 = N_0/V$, $V$ being the volume of the system, remains finite as $N_0, V \to \infty$ in the thermodynamic limit. In fact, this replacement leads to the so-called Hartree–Fock–Bogoliubov (HFB) theory [3]. One can do better than this by replacing the operators with their average value (which is an ordinary number) plus a small fluctuation (which is an operator), i.e. $a_0^+ \to \sqrt{N_0} + \delta a_0^+$, $a_0 \to \sqrt{N_0} + \delta a_0$, and minimizing the Hamiltonian with respect to the number of condensate atoms $N_0$. This leads to the Bogoliubov (B) theory proper [3], whose results, of course, reduce to those of HFB theory when the fluctuations are set to zero [1, 4]. From these considerations it is apparent that the Bogoliubov theory is of mean-field type, so one could in principle improve upon it by using more sophisticated techniques. One promising possibility, near the critical region, is the renormalization group [5–8].

What makes the use of beyond-mean-field-theory methods imperative though is that the Bogoliubov theory has other shortcomings in addition to its mean-field character [9]. In the critical region the Bogoliubov theory simply does not work, because there are fluctuations around the mean field

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that cannot be treated perturbatively. This happens because, as the temperature approaches the critical temperature, $T_c$, the thermal cloud density develops an infrared singularity and thus diverges as the momentum tends to zero [3, 6]. This really calls for Wilsonian renormalization treatment [10] of the problem around the critical region. It is well known that renormalization techniques circumvent such infrared singularities because, to put it simply, one arrives at the flow equations without having to integrate out all momenta down to zero [11].

Furthermore, within the Bogoliubov theory, it is usual to make a further approximation. One assumes that the presence of the condensate and of the thermal cloud does not modify significantly the effective interaction between two colliding atoms from its vacuum value. This is the Bogoliubov–Popov (BP) theory [12]. To avoid this approximation, which is particularly unjustified in two-dimensional cases or when the interactions are attractive, one has to use many-body T-matrix theory [13, 14]. However, this approach also runs into infrared singularity problems in the critical region. Therefore renormalization techniques would have to be employed to take correctly into account the many-body effect on two-body collisions near the critical region.

However, the renormalization group is not only a powerful method for describing interacting quantum systems close to a phase transition, but it is also useful for evaluating thermodynamic properties at arbitrary temperatures, thereby transcending mean-field theory. In this contribution we concentrate on these latter aspects of the renormalization group approach by introducing its main underlying ideas in the familiar context of the ideal Bose gas and by applying them to the case of an interacting, confined Bose gas within the framework of the random phase approximation.

This paper is organized as follows: In Sect. 2, we present an introduction to renormalization group methods in the familiar context of the ideal Bose gas, in an arbitrary number of spatial dimensions. In Sect. 3 we apply the renormalization group to the realistic case of an interacting, confined Bose gas whose confinement can be modelled by periodic boundary conditions. Comparisons with the ideal gas are made where appropriate. We analyse the resulting critical fixed point and its associated critical exponent characterizing the scaling of the correlation length in three spatial dimensions. We also investigate a particular non-universal property which is accessible to experimental observation, namely the second-order coherence factor, and discuss the effect of one-dimensional confinement on this physical quantity.

## 2 Renormalization group and the ideal Bose gas

Before treating the interacting Bose gas case, we illustrate the basic ideas of the renormalization group in the familiar context of the ideal Bose gas.

Let us start from the path-integral representation of the grand-canonical partition function of the ideal Bose gas in $D$ dimensions [15], i.e.

$$Z(\mu,\beta,V) \equiv \mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})} = \int \delta[\varphi,\varphi^*]\mathrm{e}^{-S[\varphi,\varphi^*]}\,,$$

with the (dimensionless) action

$$S[\varphi,\varphi^*] = \frac{1}{\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\int_V \mathrm{d}^D\boldsymbol{x} \times\left[\varphi^*(\tau,\boldsymbol{x})\left(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu\right)\varphi(\tau,\boldsymbol{x})\right]. \quad (1)$$

The boson mass is denoted by $m$, $\mu$ is the chemical potential and $\beta = 1/(k_\mathrm{B}T)$ is the inverse temperature with Boltzmann's constant $k_\mathrm{B}$. Due to the trace operation involved in the definition of the partition function the complex-valued field $\varphi(\tau,\boldsymbol{x})$ has to fulfill periodic boundary conditions with respect to the imaginary time $\tau$, i.e. $\varphi(\tau,\boldsymbol{x}) = \varphi(\tau+\hbar\beta,\boldsymbol{x})$. Thermodynamic properties of the non-interacting Bose gas can be evaluated from the extensive quantity

$$\begin{aligned}F(\mu,\beta,V) &= -\ln Z(\mu,\beta,V) \equiv \beta\Omega(\mu,\beta,V)\\ &= \sum_i \ln\left\{1-\exp\left[-\beta\left(\frac{\hbar^2\boldsymbol{k}_i^2}{2m}-\mu\right)\right]\right\}\,, \end{aligned}\quad (2)$$

with $V$ denoting the volume of the system. $\Omega(\mu,\beta,V)$ is the grand-canonical thermodynamic potential. If we model confinement by periodic boundary conditions with spatial periods $L_i$ ($i \in \{1\ldots D\}$), the corresponding possible wave numbers are given by $\boldsymbol{k}_i = 2\pi n_i/L_i$, with $n_i$ being an integer.

If we want to evaluate the thermodynamic function of (2) numerically, it is convenient to introduce first of all an ultraviolet momentum cutoff, $|\hbar\boldsymbol{k}_\mathrm{max}| \equiv \hbar\Lambda$, which constitutes an upper limit for the summation over all states of the ideal Bose gas. In order to ensure convergence of the partition function, this momentum cutoff has to be much larger than all the physical momenta characterizing the problem, i.e. $(\hbar\Lambda)^2/(2m) \gg 1/\beta, \mu$. Introducing this ultraviolet momentum cutoff and the dimensionless scale parameter $l$ according to the relation $|\boldsymbol{k}| = \Lambda\mathrm{e}^{-l}$, in the continuum limit, i.e. $L_i \to \infty$, (2) can be approximated by

$$F(M_0,b_0,V) \equiv \int_0^\infty \mathrm{d}l'\,\mathcal{N}\mathrm{e}^{-Dl'}\ln\left(1-\mathrm{e}^{-b(l')\left[\frac{1}{2}-M(l')\right]}\right)\,, \quad (3)$$

with the dimensionless, rescaled thermodynamic parameters

$$\begin{aligned} b(l) &= b_0\mathrm{e}^{-2l} \equiv (\beta/\beta_\Lambda)\mathrm{e}^{-2l}\,,\\ M(l) &= M_0\mathrm{e}^{2l} \equiv (\mu\beta_\Lambda)\mathrm{e}^{2l}\,, \end{aligned}\quad (4)$$

and with the cutoff-dependent scale factor $\beta_\Lambda = m/(\hbar\Lambda)^2$. The quantity $\mathcal{N}\mathrm{d}l' = [V\Lambda^D\Omega_D/(2\pi)^D]\mathrm{d}l'$ denotes the number of states in volume $V$ within the infinitesimal momentum shell $\hbar\Lambda(1-\mathrm{d}l') \le |\hbar\boldsymbol{k}| \le \hbar\Lambda$. $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of a $D$-dimensional hypersphere of unit radius, and $\Gamma(x)$ denotes the Gamma function [16].

The main idea underlying the renormalization group approach is to perform the integration over all states involved in the evaluation of the partition function or of the thermodynamic function of (3) not in one step but in small, successive steps. Integrating out some of the states in one of these small steps yields scaling relations from which one can determine the non-analytic behaviour of thermodynamic functions at a phase transition. Successive applications of this partial decimation procedure eventually yields the partition function.

In order to derive such a scaling relation for the thermodynamic function of (3) let us integrate out a small momentum shell corresponding to the interval $l' \in [0, l]$, for example. For the ideal Bose gas this yields the result

$$\begin{aligned} F(M_0, b_0, V) &= \left( \int_0^l \mathrm{d}l' + \int_l^\infty \mathrm{d}l' \right) \\ &\quad \times \left[ \mathcal{N} \mathrm{e}^{-Dl'} \ln \left( 1 - \mathrm{e}^{-b(l')[\frac{1}{2} - M(l')]} \right) \right] \\ &= \int_0^l \mathrm{d}l' \mathcal{N} \mathrm{e}^{-Dl'} \ln \left( 1 - \mathrm{e}^{-b(l')[\frac{1}{2} - M(l')]} \right) \\ &\quad + \mathrm{e}^{-Dl} F\big(M(l), b(l), V\big) \, . \end{aligned} \tag{5}$$

The last equality of (5) is obtained by shifting the integration variable $l'$ in the second integral according to $l' \to l' - l$. The two characteristic steps of Wilsonian renormalization are apparent in the derivation of this scaling relation. The first step, the Kadanoff transformation, involves a partial decimation of some states of the Bose gas. In the second step the original ultraviolet momentum cutoff $\hbar\Lambda$ is re-established by rescaling momenta according to the transformation $|\hbar \boldsymbol{k}| \to |\hbar \boldsymbol{k}| \mathrm{e}^l$. In our case, this latter trivial rescaling is performed by the translation $l' \to l' - l$. In configuration space this trivial rescaling implies that we are increasing the minimum distance of resolution or effective block size from $1/\Lambda$ to $1/(\Lambda \mathrm{e}^{-l})$ so that with increasing values of $l$ the physical system is described on larger and larger length scales. Thus, expressed in terms of this minimum distance of resolution any physical length $L$ shrinks with increasing values of $l$ according to $L \to L\mathrm{e}^{-l}$. Scaling relations of the form of (5) are useful for investigating the non-analytic scaling behaviour of thermodynamic quantities close to a second-order phase transition [9].

By applying the scaling relation of (5) repeatedly, the evaluation of the thermodynamic function $F(M_0, b_0, V)$ can be reduced to the solution of a system of ordinary differential equations, i.e.

$$\begin{aligned} \frac{\mathrm{d}F(l)}{\mathrm{d}l} &= \mathcal{N} \mathrm{e}^{-Dl} \ln \left( 1 - \mathrm{e}^{-b(l)[\frac{1}{2} - M(l)]} \right) , \\ \frac{\mathrm{d}M(l)}{\mathrm{d}l} &= 2M(l) \, , \\ \frac{\mathrm{d}b(l)}{\mathrm{d}l} &= -2b(l) \, , \end{aligned} \tag{6}$$

which have to be solved with the initial conditions $M(l = 0) = M_0 \ll 1$ and $b(l = 0) = b_0 \gg 1$ consistent with the choice of the ultraviolet cutoff. Thereby, the thermodynamic function $F(M_0, b_0, V) \equiv F(l \to \infty)$ is obtained from the solution of (6) with the additional initial condition $F(l = 0) = 0$. We will be referring to (6) as the renormalization group (RG) equations. We note that the equation for $F(l)$ depends on the solutions $M(l)$ and $b(l)$, but $F(l)$ itself does not couple back to the equations of motion for these two quantities. Thus, one can first solve the two equations of motion for the scaled thermodynamic parameters $M(l)$ and $b(l)$ and insert the resulting solutions into the equation of motion for $F(l)$. In the case of the ideal Bose gas the RG equations for $M(l)$ and $b(l)$ are simple and describe the trivial scaling of these scaled thermodynamic parameters. This example demonstrates the basic concepts involved in the evaluation of thermodynamic partition functions by RG methods.

The system of (6) is autonomous with an unstable, fixed point at $(b_*, M_*) \equiv (0, 0)$. The parametric plot of a typical trajectory $M(l)$ versus $b(l)$ is given in Fig. 1 together with the fixed point and its associated stable (s) and unstable (u) manifolds. It is well known that in the grand-canonical ensemble the phase transition of the ideal Bose gas occurs at zero chemical potential, i.e. at $M_0 = 0$. According to (6) this reflects the fact that at the critical point the thermodynamic properties of an ideal Bose gas are determined by the stable manifold of the unstable fixed point $(b_*, M_*)$. The unstable manifold of this fixed point governs the behaviour of the ideal Bose gas close to criticality. The eigenvalues of the scaled thermodynamic parameters corresponding to the stable and unstable manifolds are given by $\lambda_- = -2$ and $\lambda_+ = 2$ [compare with (6)]. In particular, the positive eigenvalue $\lambda_+ = 2$, which is associated with the (one-dimensional) unstable manifold, determines the critical exponents governing the scaling relations of singular parts of physical quantities near the critical point [9, 11]. As an example, let us consider the correlation length $\xi_0$ of the ideal Bose gas which changes under renormalization according to $\xi(l) = \xi_0 \mathrm{e}^{-l}$. Furthermore, according to (6), the relevant variable scales as $(M(l) - M_*) = (M_0 - M_*)\mathrm{e}^{\lambda_+ l}$. If the system is close to the critical point, i.e. $| M_0 - M_* | \ll 1$, and if we iterate the RG equations up to a point $l_0 \gg 1$ far away from this critical point, say with $M(l_0) - M_* = 1$, we find the scaling relation

$$\xi_0 = \xi(l_0)\mathrm{e}^{l_0} = \xi(l_0)(M_0 - M_*)^{-1/\lambda_+} \equiv \xi(l_0)(M_0 - M_*)^{-\nu} \, . \tag{7}$$

Thus, the divergence of the correlation length at the critical point is governed by the critical exponent $\nu = 1/\lambda_+$. With

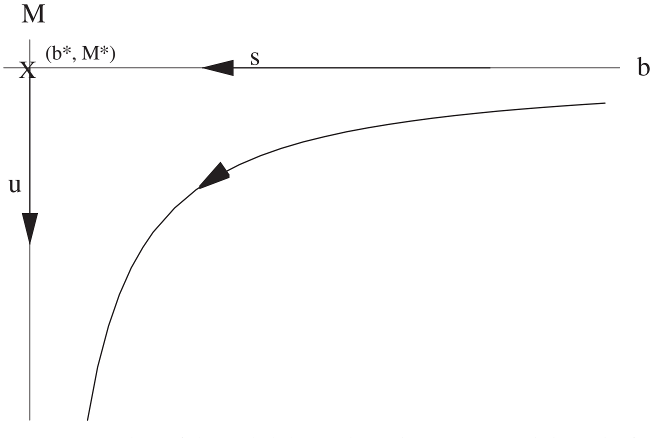

**FIGURE 1** Flow of the scaled thermodynamic parameters $(b(l), M(l))$ for the ideal Bose gas according to the RG equations (6). With increasing renormalization, i.e. increasing values of $l$, an initial point $(b_0, M_0)$ with $b_0 = b(l = 0) \gg 1$ and $M_0 = M(l = 0) < 0$ in the thermodynamic parameter space is driven to higher effective temperatures, i.e. to smaller values of $b(l)$, and to more negative effective chemical potentials. The unstable fixed point $(b_*, M_*) = (0, 0)$ is indicated by a *cross*

the help of scaling relations such as (5), one may also relate the characteristic eigenvalue $\lambda_+$ of the unstable manifold to the critical exponents of other physical quantities of interest. Let us consider the thermodynamic function $F(M_0, b_0, V)$ as a further example. Whereas for any finite value $l_0$ the integral in the last line of (5) is a smooth function of the intensive thermodynamic parameters $\mu$ and $\beta$, the second term of this last line gives rise to non-analytical behaviour in the neighbourhood of the critical point. If we are close to the critical point and choose $l_0$ again so large that $(M(l_0) - M_*) = 1$ and $b(l_0) \ll 1$, we find that the singular part of this thermodynamic function scales as

$$\begin{aligned} F_\mathrm{s}(M_0, b_0, V) &= \mathrm{e}^{-Dl_0} F_s(1, 0, V) \\ &\equiv (M_0 - M_*)^{D/\lambda_+} F_\mathrm{s}(1, 0, V)\,. \end{aligned} \tag{8}$$

## 3 The interacting Bose gas

When we treat two-body collisions between bosons in the low-momentum or s-wave approximation, the path-integral representation of the partition function of the homogeneous interacting Bose gas is given by

$$Z(\mu, \beta, V, g) \equiv \mathrm{Tr}\, \mathrm{e}^{-\beta\left(\hat{H}-\mu\hat{N}\right)} = \int \delta[\varphi, \varphi^*] \mathrm{e}^{-S[\varphi,\varphi^*]}\,, \tag{9}$$

with the (dimensionless) action

$$\begin{aligned} S[\varphi, \varphi^*] = \frac{1}{\hbar} \int_0^{\hbar\beta} \mathrm{d}\tau \int_V \mathrm{d}^D\boldsymbol{x} \Bigg[ &\varphi^*(\tau, \boldsymbol{x}) \\ &\times \left( \hbar \frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu \right) \varphi(\tau, \boldsymbol{x}) + \frac{1}{2} g |\varphi(\tau, \boldsymbol{x})|^4 \Bigg]\,. \end{aligned} \tag{10}$$

In the low-momentum approximation the interparticle interaction can be described by the zero-momentum component of the Fourier transform of the two-body interaction potential. Thus, within this approximation a repulsive, short-range potential can be characterized by a positive interaction strength, $g$. In three spatial dimensions, for example, this interaction strength is related to the positive scattering length $a$ of the interparticle interaction by the familiar relation $g = 4\pi\hbar^2 a/m$.

In the rest of this paper we will be interested not only in the case of an unconfined interacting Bose gas but also in cases where at least one of the spatial degrees of freedom is confined by a potential. In the simplest approximation such a confinement in the 3- or $z$-direction, for example, can be described by a periodic boundary condition of the form

$$\varphi(\tau, \boldsymbol{x}) = \varphi(\tau, \boldsymbol{x} + L_\mathrm{z} \boldsymbol{e}_\mathrm{z})\,,$$

with $L_\mathrm{z}$ denoting the characteristic length of confinement. Within such a description of confinement the translationally invariant character of the problem is conserved.

Before addressing the evaluation of the partition function of (9) with the help of the RG let us briefly summarize its approximate evaluation within the framework of mean-field theory. In mean-field theory one approximates the partition function of (9) by saddle-point integration [17]. For this purpose one determines first of all the most probable configuration, $\overline{\varphi}(\boldsymbol{x})$, by minimizing the action of (10). This way one arrives at the Gross–Pitaevski equation [1]:

$$\left( \hbar \frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu + g|\overline{\varphi}(\boldsymbol{x})|^2 \right) \overline{\varphi}(\boldsymbol{x}) = 0\,. \tag{11}$$

In the homogeneous case that we are examining, the most probable static and space-independent configuration is given by

$$\overline{\varphi} = \sqrt{\mu/g}\,. \tag{12}$$

In a second step one expands the action of (10) around this most probable configuration up to the second order, assuming that the fluctuations around this most probable configuration are small. The resulting Gaussian integrations can be easily performed, thus yielding the well-known mean-field approximation for the partition function [18].

### 3.1 *Renormalization group approach*

Using the RG it is possible to improve on this mean-field approximation. However, in contrast to the case of the ideal Bose gas, in the presence of interparticle interactions the first step of the RG method, namely the Kadanoff transformation, can be implemented only approximately. A frequently employed approximation in this context is the random phase approximation [19], whose resulting RG equations will be discussed in the following.

*Kadanoff transformation.* As we want to take into account the fluctuations of the complex-valued field $\varphi(\tau, \boldsymbol{x})$ beyond mean-field theory, it is convenient to separate this field according to

$$\varphi(\tau, \boldsymbol{x}) \to \overline{\varphi} + \varphi(\tau, \boldsymbol{x})\,. \tag{13}$$

Thus, we obtain the following symmetry broken form of the (dimensionless) action

$$\begin{aligned} S[\varphi, \varphi^*] = &-\beta V \left( \mu n_0 - \frac{n_0^2 g}{2} \right) \\ &+ \frac{1}{\hbar} \int \mathrm{d}x \varphi^*(x) \left( \hbar \frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu + 2gn_0 \right) \varphi(x) \\ &+ \frac{gn_0}{2\hbar} \int \mathrm{d}x \left[ \varphi^*(x)\varphi^*(x) + \varphi(x)\varphi(x) \right] \\ &+ \frac{g\overline{\varphi}}{\hbar} \int \mathrm{d}x \left[ \varphi^*(x)\varphi^*(x)\varphi(x) + \varphi^*(x)\varphi(x)\varphi(x) \right] \\ &+ \frac{g}{2\hbar} \int \mathrm{d}x \varphi^*(x)\varphi^*(x)\varphi(x)\varphi(x)\,, \end{aligned} \tag{14}$$

with the condensate density $n_0 = |\overline{\varphi}|^2 = \mu/g$. The U(1) global symmetry of the action of (10) has been broken spontaneously by the introduction of the most probable configuration, $\overline{\varphi}$. In order to simplify the notation we have taken $x$ to represent both time and coordinates, so, for example, $\varphi(x) \equiv \varphi(\tau, \boldsymbol{x})$ and $\mathrm{d}x \equiv \mathrm{d}\tau \mathrm{d}^D\boldsymbol{x}$.

Now we want to integrate out large momentum fluctuations, so we also split the fluctuation $\varphi(x)$ around the most probable configuration $\overline{\varphi}$ into a long-wavelength component $\varphi_<(x)$ and a short-wavelength component $\delta\varphi_>(x)$, i.e.

$$\varphi(x) = \varphi_<(x) + \delta\varphi_>(x)\,, \tag{15}$$

with

$$\varphi_<(x) = \sum_{\boldsymbol{k_m} \in V_k - \delta V_k} \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k_m}\cdot\boldsymbol{x}}}{\sqrt{V}} \frac{\mathrm{e}^{-\mathrm{i}n\omega\tau}}{\sqrt{\hbar\beta}} \varphi_{n\boldsymbol{m}} \tag{16}$$

and

$$\delta\varphi_>(x) = \sum_{\boldsymbol{k_m}\in\delta V_k}{}' \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k_m}\cdot\boldsymbol{x}}}{\sqrt{V}} \frac{\mathrm{e}^{-\mathrm{i}n\omega\tau}}{\sqrt{\hbar\beta}} \delta\varphi_{n\boldsymbol{m}}\,, \tag{17}$$

where $\varphi_{n\boldsymbol{m}}$ ($\delta\varphi_{n\boldsymbol{m}}$) are the Fourier components of $\varphi_<(x)$ $[\delta\varphi_>(x)]$, $n\omega \equiv n(2\pi/\beta\hbar)$ are the Matsubara frequencies, and $\hbar V_k$ denotes a hypersphere of radius $\hbar\Lambda$ in $D$-dimensional momentum space. The short-wavelength fluctuations involve momentum components which are contained only in an infinitesimally thin shell in momentum space of thickness $\hbar\Lambda(1-dl) \le |\hbar\boldsymbol{k}| \le \hbar\Lambda$ and which is denoted $\hbar\delta V_k$ (compare with Fig. 2). For the sake of simplicity we will be referring to $\varphi_<(x)$ as the lower field and to $\delta\varphi_>(x)$ as the upper field.

The upper field, which we want to eliminate from the partition function, involves an infinitesimal momentum shell. Therefore, it is sufficient to expand the action of (14) up to the second order in terms of the upper field [20]. The resulting expression can be further simplified in the random phase approximation [19]. In this approximation it is assumed that the upper field is rapidly varying in space in comparison with the lower field, so that the dominant contributions to the action arise from those particular terms quadratic in the upper field which are slowly varying in space. Thus, in the random phase approximation, the action reads

$$\begin{aligned} S[\varphi,\varphi^*] = {} & S[\varphi_<,\varphi_<^*] \\ & + \frac{1}{\hbar}\int \mathrm{d}x\,\delta\varphi_>^*(x)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right)\delta\varphi_>(x) \\ & + \frac{2g}{\hbar}\left(\int \mathrm{d}x|\overline{\varphi}+\varphi_<(x)|^2\right) \\ & \quad \times\left(\frac{1}{V\hbar\beta}\int \mathrm{d}x\,\delta\varphi_>^*(x)\delta\varphi_>(x)\right) \\ & + \frac{g}{2\hbar}\left(\int \mathrm{d}x(\overline{\varphi}+\varphi_<(x))^2\right) \\ & \quad \times\left(\frac{1}{V\hbar\beta}\int \mathrm{d}x\,\delta\varphi_>^*(x)\delta\varphi_>^*(x)\right) \\ & + \frac{g}{2\hbar}\left(\int \mathrm{d}x(\overline{\varphi}+\varphi_<^*(x))^2\right) \\ & \quad \times\left(\frac{1}{V\hbar\beta}\int \mathrm{d}x\,\delta\varphi_>(x)\delta\varphi_>(x)\right)\,, \end{aligned} \tag{18}$$

where we have taken into account that terms linear in $\delta\varphi_>(x)$ and $\delta\varphi_>^*(x)$ vanish because the fluctuations are expanded around the most probable configuration $\overline{\varphi}$ which fulfills (11). Performing the Gaussian integration in terms of the upper field yields an effective action for the lower field of the form [8]

$$S_{\text{eff}}[\varphi_<,\varphi_<^*] = S[\varphi_<,\varphi_<^*] + \delta S[\varphi_<,\varphi_<^*]\,. \tag{19}$$

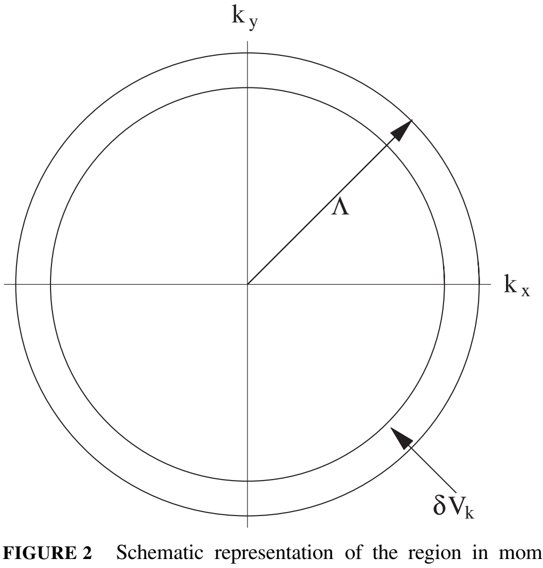

**FIGURE 2** Schematic representation of the region in momentum space which is eliminated in the Kadanoff transformation

We want to point out that, in contrast to the case of the mean-field approximation, now the contribution to the effective action, i.e. $\delta S[\varphi_<,\varphi_<^*]$, still depends on the lower field, which characterizes the long-wavelength fluctuations. Stated differently, the mean-field approximation would correspond to the replacement $\delta S[\varphi_<,\varphi_<^*] \to \delta S[\varphi_< \equiv 0, \varphi_<^* \equiv 0]$ in this step of the decimation procedure. Thus, after integration over the short-wavelength fluctuations the simplest improvement transcending the mean-field approximation is obtained by expanding $\delta S[\varphi_<,\varphi_<^*]$ up to the second order in the lower field. As apparent from (14), such a second-order expansion leads to a change of the most probable configuration, i.e. $\overline{\varphi} \to \overline{\varphi} + \delta\overline{\varphi}$, through the terms linear in $\varphi_<$ and $\varphi_<^*$ and to a change of the chemical potential through the terms quadratic in the lower field, i.e. $\mu \to \delta\mu$. Enforcing the relation (12) at each step of the renormalization procedure implies a corresponding scaling of the interparticle coupling strength $g$ according to the relation $|\overline{\varphi}+\delta\overline{\varphi}|^2 \equiv (\mu+\delta\mu)/(g+\delta g)$ [7, 8].

*Trivial rescaling.* The second step of the renormalization procedure is to recast the effective action of (19) in terms of the new chemical potential $\mu_{\text{eff}} = \mu + \delta\mu$ and the new interparticle coupling strength $g_{\text{eff}} = g + \delta g$ in the form of the original action by re-establishing the original momentum cutoff, $\hbar\Lambda$. For this purpose we have to rescale momenta according to $|\boldsymbol{k}| \to |\boldsymbol{k}(l)| = |\boldsymbol{k}|\mathrm{e}^{l}$. As in the ideal gas case, this transformation and the demand that the action remains formally the same after each renormalization step induce the appearance of rescaled parameters in the action, i.e.

$$\begin{aligned}
&V \to V(l) = V\mathrm{e}^{-Dl}\,,\\
&\beta \to \beta(l) = \beta\mathrm{e}^{-2l}\,,\\
&\varphi \to \varphi(l) = \varphi\mathrm{e}^{-l}\,,\\
&\mu \to \mu(l) = (\mu+\delta\mu)\mathrm{e}^{2l}\,,\\
&g \to g(l) = (g+\delta g)\mathrm{e}^{(2-D)l}\,.
\end{aligned} \tag{20}$$

These scaling relations reveal two differences from the corresponding scaling relations of the ideal gas. The first is that there is, of course, an additional equation for the rescaling of the coupling constant. The second is that the rescaled chemical potential contains an additional correction $\delta\mu$, which itself depends on $l$ and is therefore a non-trivial contribution to the scaling of $\mu$. This non-trivial part was not present in the ideal gas case. We also note that the coupling constant equation contains a non-trivial part, $\delta g$, as well.

*The RG equations.* Because we have taken the renormalization step $\mathrm{d}l$ to be infinitesimal, the corrections for the chemical potential and the coupling constant take the form of differential equations with respect to the continuous parameter $l$. Introducing the dimensionless, scaled coupling constant

$$G(l) = \beta_\Lambda \Lambda^D g(l) \tag{21}$$

and the scaled thermodynamic parameters of (4), we obtain the RG equations for the interacting Bose gas [8]:

$$\begin{aligned}
\frac{\mathrm{d}F(l)}{\mathrm{d}l} &= V\Lambda^D d(l)\mathrm{e}^{-Dl}\\
&\qquad \left(\ln\{2\sinh[\lambda(l)/2]\} - b(l)(\varepsilon_> - M_0\mathrm{e}^{2l})/2\right),\\
\frac{\mathrm{d}M(l)}{\mathrm{d}l} &= 2M(l) + d(l)G(l)A[M(l),b(l)]\,,\\
\frac{\mathrm{d}G(l)}{\mathrm{d}l} &= -(D-2)G(l) + d(l)[G(l)]^2 B[M(l),b(l)]\,,\\
\frac{\mathrm{d}b(l)}{\mathrm{d}l} &= -2b(l)\,,
\end{aligned} \tag{22}$$

with

$$\begin{aligned}
A[M(l),b(l)] &= b(l)\frac{\coth[\lambda(l)/2]}{2\lambda(l)}[2M(l)-2\varepsilon_>]\\
&\quad - [b(l)]^3\frac{M(l)}{2\lambda(l)^2}\left(\frac{1}{2\sinh^2(\lambda(l)/2)}\right.\\
&\quad \left. + \frac{\coth[\lambda(l)/2]}{\lambda(l)}\right)[2\varepsilon_> + M(l)]^2\,,\\
B[M(l),b(l)] &= 3b(l)\frac{\coth[\lambda(l)/2]}{2\lambda(l)} - [b(l)]^3\frac{1}{2\lambda(l)^2}\\
&\quad \times\left(\frac{1}{2\sinh^2[\lambda(l)/2]} + \frac{\coth[\lambda(l)/2]}{\lambda(l)}\right)\\
&\quad \times[2\varepsilon_> + M(l)]^2
\end{aligned} \tag{23}$$

and with the scaled cutoff energy $\varepsilon_> = 1/2$. The quantity

$$\lambda(l) = b(l)\sqrt{\varepsilon_>\,[\varepsilon_> + 2M(l)]} \tag{24}$$

indicates that at each step of the renormalization procedure energy and momentum are related by the Bogoliubov dispersion relation.

In contrast to the corresponding RG equations of the ideal gas [compare with (6)] (22) involve two nonlinear coupled equations of motion for the scaled parameters $M(l)$ and $G(l)$. The corresponding mean-field results would be obtained by setting the fluctuations around the mean-field equal to zero. In this case, both $A[M(l),b(l)]$ and $B[M(l),b(l)]$ vanish and the equations of motion for $M(l)$ and $G(l)$ decouple. Furthermore, comparison with (6) reveals that in the mean-field approximation the equation for the scaled chemical potential $M(l)$ is identical to the corresponding equation for the ideal gas, which just describes the trivial scaling of $M(l)$. It should be mentioned that, in contrast to the trivial scaling of $M(l)$, the trivial scaling of the dimensionless coupling parameter $G(l)$ depends on the dimensionality of the problem.

The quantity $d(l)$ appearing in these RG equations describes effects of confinement as long as they can be modelled by periodic boundary conditions. These periodic boundary conditions lead to a quantization of the momentum components in the confined directions. Thus the free-space integrals over momentum which appear in the continuum limit are replaced by sums of the form

$$\sum_{\boldsymbol{m}\in\delta V_{\boldsymbol{k}}}{}^{\prime} = V(\Lambda\mathrm{e}^{-l})^D d(l)\mathrm{d}l + O[(\mathrm{d}l)^2]\,, \tag{25}$$

where $d(l)$ characterizes the number of momentum states which are contained in an infinitesimal momentum shell whose thickness is proportional to $\mathrm{d}l$ (compare with Fig. 2). For example, in the case of confinement of one spatial direction, say the $z$-direction, we obtain in the case of a $D$-dimensional problem the expression

$$d(l) = \frac{\Omega_D}{(2\pi)^D}\frac{\pi + 2\pi[L_z\mathrm{e}^{-l}\Lambda/(2\pi)]}{L_z\mathrm{e}^{-l}\Lambda}\,, \tag{26}$$

where $L_z$ denotes the length of confinement. ($[x]$ denotes the largest integer which is less than or equal to $x$.) Taking the limit $L_z \to \infty$, (26) yields $d(l) = \Omega_D/(2\pi)^D$, and we obtain again the free-space RG equations. For the special case of three spatial dimensions these free-space results reduce to those of Bijlsma and Stoof [7].

### 3.2 *Universal critical properties*

For an investigation of the universal critical properties of the interacting Bose gas we have to determine the unstable fixed point of the RG equations (22) and its corresponding linearized flow equations. For this purpose we have to study (22) in the limit $l \to \infty$ in which the effective temperatures are large, i.e. $\beta(l) = \beta_0\mathrm{e}^{-2l} \ll 1$. In this limit the quantities $A[M(l),b(l)]$ and $B[M(l),b(l)]$ simplify to the expressions

$$\begin{aligned}
A[M(l),b(l)] &\to -2\frac{\varepsilon_>^3 + 5M(l)\varepsilon_>^2 + 2M^2(l)\varepsilon_> + 2M^3(l)\varepsilon_>}{b(l)\varepsilon_>^2[\varepsilon_> + 2M(l)]^2}\,,\\
B[M(l),b(l)] &\to -\frac{5\varepsilon_>^2 + 2\varepsilon_> M(l) + 2M^2(l)}{b(l)\varepsilon_>^2[\varepsilon_> + 2M(l)]^2}\,.
\end{aligned} \tag{27}$$

Therefore, the equation of motion for $M(l)$ does not couple directly to $G(l)$ but to the modified, scaled coupling $\overline{G}(l) \equiv$

$d(l)[G(l)/b(l)]$. In the absence of any confinement, i.e. for $d(l) = \Omega_D/(2\pi)^D$, the trivial scaling of the scaled coupling is given by $\overline{G}(l) = \overline{G}(0)\mathrm{e}^{-(D-4)l}$ so that this quantity is relevant for dimensions $D < 4$, in contrast to the variable $G(l)$ itself, whose trivial scaling indicates that it is relevant for dimension $D < 2$. However, in the case of confinement this trivial scaling changes. If one degree of freedom is confined, for example, (26) implies that for sufficiently large values of $l$ the trivial scaling of the scaled coupling is given by $\overline{G}(l) = \overline{G}(0)\mathrm{e}^{-(D-4-1)l}$, which typically has a significant influence on the position of the unstable fixed point of the RG equations.

Let us consider the physically important case of $D = 3$ in the absence of confinement in more detail. It is straightforward to determine the unstable fixed point for the three-dimensional case, i.e.

$$\begin{aligned} &M_* = 1/2\,, \\ &\tilde{G}_* \equiv (G/b)_* = \pi^2/2\,, \end{aligned} \tag{28}$$

with $\tilde{G}(l) = G(l)/b(l)$. The parametric plot of a typical trajectory $\tilde{G}(l)$ versus $M(l)$ is drawn in Fig. 3 together with the fixed point and the associated stable and unstable manifolds. The eigenvectors corresponding to the stable and unstable manifolds are obtained by linearizing the RG equations (22) around the fixed point.

$$\begin{aligned} \frac{\mathrm{d}\Delta M(l)}{\mathrm{d}l} &= 2\Delta M(l) - \frac{2}{\pi^2}\Delta\tilde{G}(l)\,, \\ \frac{\mathrm{d}\Delta\tilde{G}(l)}{\mathrm{d}l} &= -\Delta\tilde{G}(l) + \frac{2\pi^2}{3}\Delta M(l)\,, \end{aligned}$$

with $\Delta M = M(l) - M_*$ and $\Delta\tilde{G} = \tilde{G}(l) - \tilde{G}_*$. The eigenvalues and eigenvectors of this set of linear equations are given by

$$\lambda_\pm = (3 \pm \sqrt{33})/6 \tag{29}$$

and

$$(\Delta M, \Delta\tilde{G})_\pm = \left(\frac{9 \pm \sqrt{33}}{4\pi^2}, 1\right)\,. \tag{30}$$

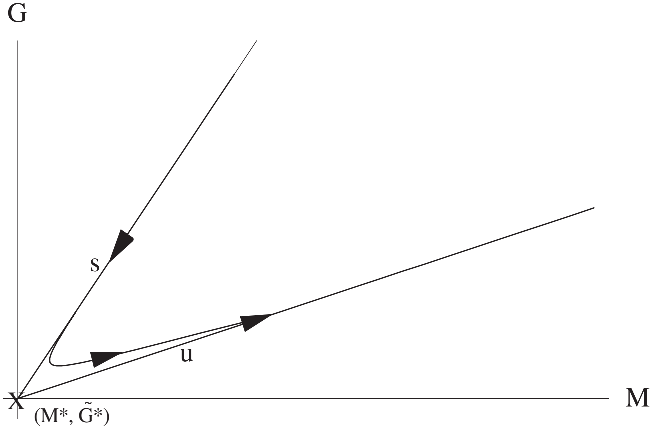

**FIGURE 3** Flow of the scaled thermodynamic parameters $\left(M(l), \tilde{G}(l)\right)$ originating from the RG equation (22) for $D = 3$. The unstable, fixed point is indicated by a *cross*, and the corresponding stable (s) and unstable (u) manifolds are indicated by *arrows*

According to (7) the positive eigenvalue $\lambda_+ = (3 + \sqrt{33})/6$ describes the rate of increase in the relevant variable and determines the critical exponent for the correlation length, i.e.

$$\nu \equiv 1/\lambda_+ = \frac{6}{3 + \sqrt{33}} \approx 0.686\,. \tag{31}$$

This critical exponent compares well with the known result of $\nu = 0.67$ [21]. The corresponding mean-field value for the critical exponent is $\nu_{\mathrm{MF}} = 1/2$, exactly the same as for the ideal gas. From these considerations it is apparent that taking into account fluctuations beyond the mean-field approach is crucial for describing the critical behaviour of the system.

### 3.3 *Non-universal critical properties*

The RG equations can also be used to calculate non-universal properties of the interacting Bose gas at the critical temperature, for example. For this purpose one has to solve the RG equations of (22) along the stable manifold of the unstable, fixed point $(M_*, \tilde{G}_*)$. As an example, let us consider the critical behaviour of the (spatially averaged) second-order coherence factor $g^{(2)}$ in the physically interesting case of $D = 3$ [8]. This second-order coherence factor is well known in quantum optics from the correlation experiments of Hanburry–Brown and Twiss [22, 23] on electromagnetic radiation from distant stars. It describes the (spatially averaged) bunching properties of bosons, i.e. the tendency of bosons to be found at the same position in space, and is defined as follows:

$$g^{(2)}(0) \equiv \frac{\frac{1}{V}\int_V \mathrm{d}^3\boldsymbol{x}\langle\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\rangle}{\left[\frac{1}{V}\int_V \mathrm{d}^3\boldsymbol{x}\langle\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\rangle\right]^2}\,. \tag{32}$$

This quantity can be evaluated from the thermodynamic function $F(M_0, b_0, V)$ by appropriate derivatives with respect to the chemical potential or with respect to the coupling strength $g$ [8].

At the critical temperature the dependence of this second-order coherence factor on the scattering length $a$ is depicted in Fig. 4. It is apparent that with increasing scattering length $g^{(2)}$ decreases. This is consistent with the intuition that with increasing repulsive interaction bosons tend to avoid each other so that the probability to be found at the same position in space decreases. In the non-interacting case, i.e. for $a = 0$ and $\mu = 0$, this second-order coherence factor assumes the value of 2, consistent with the well-known behaviour of a photon gas or chaotic field.

Finally, let us consider the influence of confinement in one spatial direction on the second-order coherence factor $g^{(2)}$. For this purpose we insert the expression (26) for the density of states into the RG equations (22). The resulting dependence of $g^{(2)}$ on mean distance between the interacting bosons at fixed temperature is depicted in Fig. 5. Temperature and scattering length are chosen in such a way that $\lambda_{\mathrm{th}} = 25a$, where $\lambda_{\mathrm{th}} = \sqrt{2\pi\hbar^2/(mk_{\mathrm{B}}T)}$ denotes the thermal de Broglie wavelength. In the case of $^{87}$Rb atoms, for example, with a scattering length of $a = 5.3\,\mathrm{nm}$, this condition corresponds

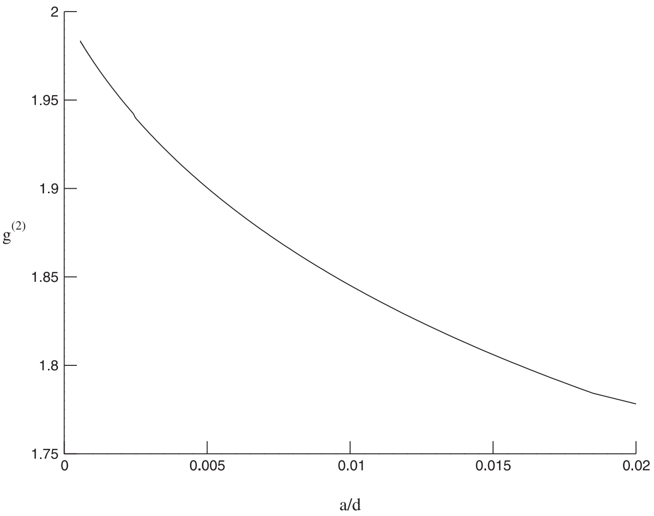

**FIGURE 4** Dependence of the second-order coherence factor $g^{(2)}$ on the scattering length $a$ at the critical temperature. The mean distance between the bosons is denoted $d \equiv (V/N)^{1/3}$

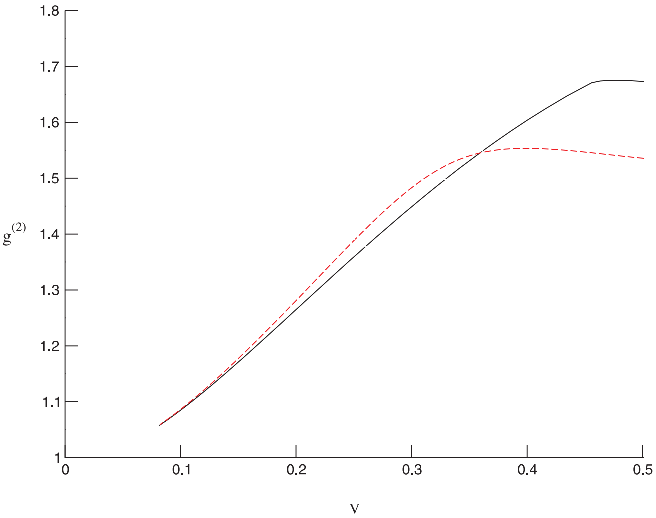

**FIGURE 5** Isothermal dependence of the second-order coherence factor $g^{(2)}$ on the scaled volume $v = (V/N)\lambda_{\mathrm{th}}^{-3}$ for a thermal de Broglie wavelength $\lambda_{\mathrm{th}} = 25a$. *Solid line*: large confinement length $L_z/\lambda_{\mathrm{th}} \gg 1$; *dashed line*: small confinement length $L_z = 1.2\lambda_{\mathrm{th}}$. For a large confinement length there is clear evidence of a second-order phase transition at $v \equiv (V/N)\lambda_{\mathrm{th}}^{-3} = 0.456$ which is smoothed out with decreasing confinement length

to a temperature of $T = 1.98\ \mu\mathrm{K}$. The full curve shows a case in which the characteristic length of confinement $L_z$ is much larger than the thermal de Broglie wavelength. In this case the characteristic signature of a second-order phase transition is apparent at a critical scaled volume of magnitude $v \equiv (V/N)\,\lambda_{\mathrm{th}}^{-3} = 0.456$. As soon as the characteristic length of confinement $L_z$ becomes comparable to the thermal de Broglie wavelength, this pronounced signature of a second-order phase transition disappears.

## 4 Summary and outlook

The renormalization group constitutes a powerful method for evaluating thermodynamic properties of interacting quantum gases beyond the limitations of mean-field theory. The previous considerations demonstrate that effects of confinement can be described by already existing momentum-space renormalization group techniques as long as confinement can be modelled by periodic boundary conditions. Within such an approach confinement leads to a modification of the density of the eliminated states in the Kadanoff transformation, thus affecting the flow equations of the renormalization group. Typically, these modifications change the fixed points of the renormalization group equations and thus influence the critical properties in a significant way. The presented treatment of effects of confinement by periodic boundary conditions constitutes a first step towards the more complicated final goal of obtaining an understanding of thermodynamic properties of trapped interacting quantum gases in realistic, smooth particle traps.

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