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Quantum nondemolition measurements on two-level atomic systems and temporal Bell inequalities

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Abstract. The evolution of a two-level system subjected to stimulated transitions which is undergoing a sequence of measurements of the level occupation probability is evaluated. Its time correlation function is compared to the one obtained through the pure Schrödinger evolution. Systems of this kind have been recently proposed for testing the quantum mechanical predictions against those of macrorealistic theories, by means of temporal Bell inequalities. The classical requirement of noninvasivity, needed to define correlation functions in the realistic case, finds a quantum counterpart in the quantum nondemolition condition. The consequences on the observability of quantum mechanically predicted violations to temporal Bell inequalities are drawn and compared to the already dealt case of the rf-SQUID dynamics.

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The validity of quantum mechanics at the macroscopic level is still an open question crucial to understand why a particular limit of it, *classical mechanics*, works so well in a wide variety of situations visible to our eyes. Leggett and Garg have challenged this question by proposing laboratory tests aimed at comparing, in a macroscopic domain, the predictions of a set of theories incorporating realism and noninvasivity, two properties manifestly not shared by quantum mechanics, and quantum mechanics itself [1]. In analogy to the well-known *spatial* Bell inequalities [2], already tested [3] and making light on the ultimate contrast of quantum mechanics with locality at the microscopic level, Leggett and Garg have shown that certain relations among the correlation probabilities - called *temporal* Bell inequalities - which hold in realistic theories, are instead violated, with a proper choice of the measurement times, by the coherent evolution of the state dictated by quantum mechanics. The ingredients of temporal Bell inequalities, regardless of the concrete scheme used, are different-time correlation probabilities between subsequent measurements of a two-valued (dichotomic) observable. However, the quantum mechanical predictions discussed so far do not consider the effect of the Heisenberg principle on consecutive measurements of the same observable of the monitored system. In this paper we discuss this effect in the exactly solvable case of two-level systems, which have been recently proposed to experimentally test temporal Bell inequalities. The first proposal is based upon three two-level systems coupled through optical pulses [4], one of which is monitored and the other two are treated as nondissipative memories which register the state of the first one at given times. The second proposal is based upon a Rydberg atom interacting with a single quantized mode of a superconducting resonant cavity [5]. While Leggett and Garg claim that their proposed experiment gives insights on the validity of quantum mechanics at the macroscopic level [1], the proposal in [4] is in a purely microscopic framework and the one in [5] is located in between, an atomic system being involved, although in a large quantum number state, and interacting with a single *mesoscopic* mode of a QED cavity. In all these cases it turns out that the concept of quantum nondemolition (QND) measurements [6, 7] and its refinement to nearly QND measurements play a key role for understanding if violations to temporal Bell inequalities can be observed when somebody looks at them.

Temporal Bell inequalities are based upon different-time correlation functions, calculable either in a classical (realistic) or in a quantum context. The different-time correlation function for a generic observable $Q(t)$ can be written as [4]

$$K(t_1,t_2) \stackrel{\text{def}}{\equiv} \int \mathscr{D}[Q(t)]P[Q(t)]Q(t_1)Q(t_2) \tag{1}$$

where the information about the dynamics of the system is expressed through the probability functional $P[Q(t)]$, which selects the $Q(t)$ allowed by the dynamical evolution, possibly including the effect of the measurement. This last can be easily taken into account by means of the concept of projection of the state [8]. Indeed, a quantum observable can be written in terms of its eigenvalues $q \in \mathrm{Sp}(\hat{Q})$, and the related projectors $\hat{P}_q$ (such that $\hat{P}_q^2 = \hat{P}_q$), as $\hat{Q} = \sum_q q\hat{P}_q$ which implies

$$Q(t) \stackrel{\text{def}}{\equiv} \frac{\langle\psi(t)|\hat{Q}|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} = \sum_q q\frac{\langle\psi(t)|\hat{P}_q|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle}. \tag{2}$$

Before a measurement performed at time $\tilde{t}$,

$$Q(\tilde{t}^-) = \sum_q q \frac{\langle\psi(\tilde{t}^-)|\hat{P}_q|\psi(\tilde{t}^-)\rangle}{\langle\psi(\tilde{t}^-)|\psi(\tilde{t}^-)\rangle}; \tag{3}$$

and, if the measurement result is $\tilde{q}$, after it we get

$$|\psi(\tilde{t}^+)\rangle = \hat{P}_{\tilde{q}}|\psi(\tilde{t}^-)\rangle \tag{4}$$

and therefore

$$Q_{\tilde{q}}(\tilde{t}^+) = \tilde{q}\frac{\langle\psi(\tilde{t}^-)|\hat{P}_{\tilde{q}}|\psi(\tilde{t}^-)\rangle}{\langle\psi(\tilde{t}^-)|\psi(\tilde{t}^-)\rangle}. \tag{5}$$

Let us suppose, by starting from the state $|\psi(t_0)\rangle$, to measure the observable $\hat{Q}$ at N instants of time $t_1 < t_2 < \cdots < t_N$ with outcomes $\{q_i\}_{1\leq i\leq N}$. Thus the successive evolutions of the state can be recursively written as

$$\begin{aligned} |\psi(t_i^-)\rangle &= U(t_i - t_{i-1})|\psi(t_{i-1}^+)\rangle; \\ |\psi(t_i^+)\rangle &\overset{(4)}{=} \hat{P}_{q_i}U(t_i - t_{i-1})|\psi(t_{i-1}^+)\rangle. \end{aligned} \tag{6}$$

The time-dependent expectation value $Q_{\{q_i\}}(t)$, including the effect of the N measurements, can be calculated through Eqs. (5), (6). The different times product of Q's required to define correlation functions as the (1), is written as

$$\begin{aligned} &Q_{\{q_i\}}(t_1)Q_{\{q_i\}}(t_2)\cdots Q_{\{q_i\}}(t_N) = \\ &\quad q_1q_2\cdots q_N\langle\psi(t_N^+)|\psi(t_N^+)\rangle. \end{aligned} \tag{7}$$

The N-times correlation function in the presence of measurements (in the case dealt here, $N = 2$) can be evaluated by summing on the $Q_{\{q_i\}}(t)$ rather than on the $Q(t)$. The presence of measured trajectories $Q_{\{q_i\}}(t)$ replacing the $Q(t)$ is not the only effect of the measurement process in Eq. (1). The trajectory weigth $P[Q(t)]$ is also affected, since the need to measure in an actual experiment these correlations as joint probabilities for finding the system in *definite* states [1], "requires the obtained data to be purged by throwing away all the information coming from channels in which the state of the first memory changes" [4]. The amount of off-line data processing and selection is therefore expressed as $\Delta = 1 - \langle\psi(t_1)|\hat{P}_{q_1}|\psi(t_1)\rangle$. When $\Delta \neq 0$, $Q_{\{q_i\}}(t) \neq Q(t)$ for $t > t_1$, the only exception in which $Q_{\{q_i\}}(t) = Q(t)$ being the impulsive QND case, when the system at the measurement time is already in the observed eigenstate. More in general ideal QND stroboscopic measurements are obtained if they are performed each time interval corresponding to a complete reconstruction of the state (a complete revival of the wavefunction after the collapse induced by the measurement), as discussed in [9] for the case of position measurements on a generic nonlinear system. On the other hand, if $\Delta \neq 0$, the probability functional should select the $Q_{\{q_i\}}(t)$ for which $1 - \Delta \geq \varepsilon$, where ε is a distinguishability threshold which expresses the degree of reliability for the measurement to indicate a definite state of the system. This selection leads to the *selective* correlation function $K_\varepsilon(t_1, t_2)$ which in the limit of complete selection becomes null. It follows that $K_\varepsilon(t_1, t_2)$ cannot violate temporal Bell inequalities if ideal QND measurements are required. Violations could be found for $\varepsilon = 0$; however in this case the correlation function does not allow to distinguish the two eigenstates. We are looking for an intermediate regime in which a well-defined selective correlation function will show detectable violations to temporal Bell inequalities.

For a system with two energy levels $|+\rangle$ and $|-\rangle$, $\hat{Q} = \hat{P}_+ - \hat{P}_- = |+\rangle\langle+| - |-\rangle\langle-|$ and

$$Q(t) = |\langle+|\psi(t)\rangle|^2 - |\langle-|\psi(t)\rangle|^2 \tag{8}$$

which holds both for a spin-$\frac{1}{2}$ system [4] and for an atom coupled to a single mode of a resonant cavity [5], provided that $|+\rangle$ and $|-\rangle$ stand for the excited and the ground state respectively. If the system is harmonically oscillating between the two states, the matrix elements involved in the evaluation of Eq. (7) are

$$\begin{aligned} \langle+|U(t-t_0)|+\rangle &= \cos\omega(t-t_0) \\ \langle+|U(t-t_0)|-\rangle &= -\sin\omega(t-t_0) \\ \langle-|U(t-t_0)|+\rangle &= \sin\omega(t-t_0) \\ \langle-|U(t-t_0)|-\rangle &= \cos\omega(t-t_0). \end{aligned} \tag{9}$$

For the spin system considered in [4], ω is the frequency of Rabi oscillations Ω_R, whereas for the atom-cavity system [5] the Jaynes-Cummings evolution gives $\omega = \Omega_R\sqrt{n+1}$ (where n is the principal quantum number of the Rydberg excited state, for an experimental demonstration on single atoms see [10]).
The initial state of the system $|\psi(t_0)\rangle = c_+(t_0)|+\rangle + c_-(t_0)|-\rangle$ (with $|c_+(t_0)|^2 + |c_-(t_0)|^2 = 1$) can be parametrized with $c_+(t_0) = \cos\omega(t-t')$, $c_-(t_0) = \sin\omega(t-t')$. From Eq. (9) then follows $Q(t) = \cos 2\omega(t-t')$, where t' is the only free parameter (representing some arbitrary instant at which the system was in $|+\rangle$) on which one should integrate for evaluating the correlation function:

$$\begin{aligned} K(t_1, t_2) &= \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} \cos 2\omega(t'-t_1)\cos 2\omega(t'-t_2)dt' \\ &= \cos 2\omega(t_1 - t_2), \end{aligned} \tag{10}$$

which depends only upon the time difference $t_2 - t_1$. The distinguishability condition is expressed as

$$\langle\psi(t_1)|\hat{P}_{q_1}|\psi(t_1)\rangle \geq \varepsilon. \tag{11}$$

The propagator including the effect of the measurement is calculated by summing over n_1 and n_2 and integrating on t_0 under the condition (11), obtaining a factorized form

$$\begin{aligned} K_\varepsilon(t_1, t_2) &= \frac{1}{\pi}\left[2\sqrt{\varepsilon(1-\varepsilon)} + \arccos(2\varepsilon - 1)\right]K(t_2 - t_1) \\ &\overset{\text{def}}{\equiv} A_\varepsilon K(t_2 - t_1), \end{aligned} \tag{12}$$

which shows the correct limits $K_0(t_1, t_2) = K(t_1, t_2)$ and $K_1(t_1, t_2) = 0$.

A generic temporal Bell inequality involves a combination ΔK of two-time correlation functions $K(t_i, t_j)$ with some coefficients κ_{ij} and an upper bound B:

$$\Delta K = \sum_{i\neq j=1}^{N} \kappa_{ij}K(t_i, t_j) \leq B. \tag{13}$$

which can be violated for some values of $\{(t_i, t_j)\}_{i\neq j=1,\ldots,N}$ by the quantum mechanical predictions, with a maximum $\Delta K_{\max} \overset{\text{def}}{\equiv} \max_{\{(t_i,t_j)\}} \Delta K > B$. For the system described in [4], $B = 2$ and

$$\begin{aligned} \Delta K &= |K(t_1, t_2) + K(t_2, t_3) + K(t_3, t_4) - K(t_1, t_4)| \\ \Delta K_{\max} &= 2\sqrt{2} \end{aligned} \tag{14}$$

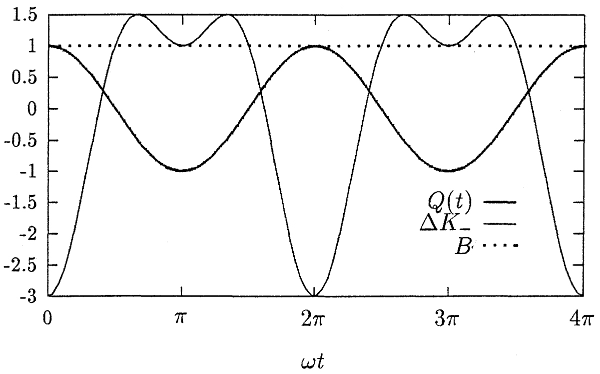

Fig. 1. Time evolution of the two-level system in [5]. The time-dependent expectation value of the observable $Q(t)$ *(thick line)* and the temporal Bell inequality parameter $\Delta K(t)$ *(thin line)* with its upper bound B *(dotted line)* are represented on the same scale. Ideal QND measurements correspond to time intervals integer multiples of π/ω, for which the inequality is not violated; however, small violations are compatible with quasi QND measurements around odd-integer multiples of π/ω

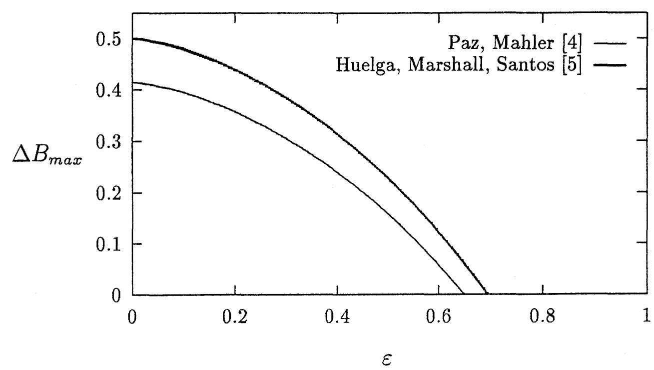

Fig. 2. Observability of violations to temporal Bell inequalities. The dependence of the maximal violation $\Delta B_{\max}$ upon the distinguishability ε is depicted for both the experiments proposed in [4] and [5]. Violations disappear in the QND limit ($\varepsilon \to 1$), and are present only for $\varepsilon \leq 0.649$ [4] and $\varepsilon \leq 0.693$ [5]

whereas in [5] $B = 1$ and

$$\begin{aligned} \Delta K_- &= -K(t_1,t_2) - K(t_2,t_3) - K(t_1,t_3) \quad \Delta K_{\max} = 3/2 \\ \Delta K_+ &= -K(t_1,t_3) + K(t_1,t_2) + K(t_2,t_3) \quad \Delta K_{\max} = 3/2. \end{aligned} \tag{15}$$

$\Delta K_\pm$ can be simplified by introducing the so-called stationarity assumption [5], namely $t_2 - t_1 = t_3 - t_2 = t$, corresponding to stroboscopic (equally spaced) measurements, such that ΔK depends only on t. For instance, Fig. 1 shows on the same scale the behavior of $\Delta K_-(t)$ and of $Q(t)$. The upper bound for the fulfilment of the corresponding temporal Bell inequality is also depicted. The impossibility of simultaneously having ideal QND measurements of the occupation number, corresponding to a complete revival of the initial state, and maximal violations to temporal Bell inequalities is evidenced. To refine in a quantitative way the possibility of coexistence of unoptimal violations to the temporal Bell inequalities and quasi QND measurements one can use the distinguishability level ε. The measurement effect is represented, as in Eq. (12), by a factor A_ε multiplying ΔK. It is worth noting that A_ε is independent upon time and thus leaves unchanged the correlation times for which ΔK is maximal. The maximal violation in percentage under the effect of the measurements is then expressed as $\Delta B_{\max} = (A_\varepsilon \Delta K_{\max} - B)/B$. Figure 2 shows $\Delta B_{\max}$ versus the distinguishability level ε. As already observed in [5] without measurement effects, we confirm here that even in presence of measurements, by assuming the same distinguishability level, the proposed inequalities are more violated than in [4]. As expected, the violations disappear for nearly QND measurements, but are present for $\varepsilon \leq 0.693$. So the proposed systems could be used for testing the predictions of quantum mechanics against those of a realistic theory, as claimed in [4, 5], only if one requires a reliability, for the detection of distinct states, not greater than 70%.

The examined experiments should be performed by looking at the correlation functions of an already-dichotomic variable, the state in two-level systems; the one discussed in [1, 11] deals with the reduction of the spectrum of a *continuous* observable, the magnetic flux ϕ trapped in a SQUID, into a dichotomic variable, its sign $\hat{\phi}/|\hat{\phi}|$. This proposal has been already discussed in [12, 13] by also including the effect of the Heisenberg principle, showing that for selective measurements there is incompatibility between violation of the inequalities and distinguishability of the dichotomic variable. Work is in progress to repeat the same analysis developed here for the case of the rf-SQUIDs dynamics. In this case the projectors on states of definite sign are $\hat{P}_\pm = \Theta(\pm\hat{\phi})$: from Eq. (2) then follows

$$Q(t) = \int_0^\infty |\psi(\phi)|^2 d\phi - \int_{-\infty}^0 |\psi(\phi)|^2 d\phi \tag{16}$$

(if $\psi(\phi)$ is normalized to 1) and $K_\varepsilon(t_1,t_2)$ can be calculated by Eq. (12), as we will report in a forthcoming paper [14].

In conclusion, our result can be simply summarized as follows: the observability of violations depends critically on the statistical criterion adopted for defining the resolution of distinct states. In other words, violations to temporal Bell inequalities can be detected only in a *probabilistic* way, unlike the spatial case. This is due to the fact that the time-dependent correlation probabilities, unlike the space-dependent ones, involve measurements on the *same* observable of the *same* system, and therefore the quantum predictions are characterized by an uncertainty dictated by the Heisenberg principle, which makes ambiguous the definition of the violations themselves.

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