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Optics of moving media

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Abstract. Light experiences a moving medium as an effective gravitational field. In the limit of low medium velocities the medium flow plays the role of a magnetic vector potential. We review the background of our theory [U. Leonhardt and P. Piwnicki, Phys. Rev. A **60**, 4301 (1999); Phys. Rev. Lett. **84**, 822 (2000)], including our proposal of making optical black holes.

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Light propagation in moving media is an old topic. The first relevant result was achieved by Fresnel in 1818 [1], long before the theory of relativity or Maxwell's theory of electricity and magnetism had been developed. He used the ether model of light propagation to obtain

$$v = \frac{c}{n} \pm \left(1 - \frac{1}{n^2}\right) u \tag{1}$$

for the velocity of a light ray propagating in a moving transparent medium. Here v is the velocity of the light as seen from the laboratory system, c the velocity of light in vacuum, n the medium's index of refraction and u the velocity of the medium. The factor in parentheses is usually called Fresnel's drag coefficient (Mitführungskoeffizient), as according to Fresnel's interpretation this coefficient defines the fraction of the ether that is dragged by the medium. The sign in the coefficient depends on whether light and medium are moving in the same or in opposite directions. Fresnel's result was verified experimentally by Fizeau in 1851 [2]. As long as we are considering the effects only to the first order in u/c and light and medium are propagating in parallel directions, (1) gives the correct expression even in the light of modern physics. It is fairly easy to obtain this formula using the expression for the relativistic velocity addition,

$$\boldsymbol{v} = \frac{\boldsymbol{u} + \boldsymbol{v}'}{1 + (\boldsymbol{u} \cdot \boldsymbol{v}')/c^2}\,, \tag{2}$$

which is valid when $\boldsymbol{u}$ and $\boldsymbol{v}'$ are parallel. As the question of velocity transformations is a problem of only geometrical transformations, the light in the medium can be simply seen as "something" moving with the velocity c/n. The actual nature of the moving signal is of no importance. In the same sense wave fronts and elementary waves transform simply according to normal Lorentz transformations:

$$x' = (x - ut)\gamma\,, \quad y' = y\,, \quad z' = z\,, \quad t' = (t - ux/c^2)\gamma\,, \tag{3}$$

with

$$\gamma = \frac{1}{\sqrt{1 - (u/c)^2}}\,. \tag{4}$$

Thus light in a dielectric is a very special thing. On one hand it has an important feature of light in vacuum – a fixed velocity $c' = c/n$; on the other hand, as it is not propagating with the vacuum velocity of light, it is subject to the normal Lorentz transformation used for massive particles. Consequently, the velocity as seen in the laboratory system depends on the velocity of the medium, and elementary waves are Lorentz contracted. But note that due to the fixed velocity of the light in the medium frame the velocities in the laboratory system are fixed as well. They now depend on the direction of propagation, but as soon as the direction is fixed, the modulus of the velocity is known. The explicit formulae allowing these velocities to be calculated are rather cumbersome, as the velocity transformation in its full three-dimensional form is needed.

Let us consider an elementary wave emitted at some point within the medium. In the medium's rest frame this elementary wave is obviously a sphere fulfilling

$$x'^2 + y'^2 + z'^2 - c'^2 t'^2 = 0\,, \tag{5}$$

where the primed quantities are those in the rest frame of the medium. Using the Lorentz transformation (3) we can transform this equation to the laboratory frame to obtain

$$(x - at)^2/b + y^2 + z^2 - bc'^2 t^2 = 0\,, \tag{6}$$

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with

$$a = v\frac{1-1/n^2}{1-u^2/(c^2n^2)} \quad \text{and} \quad b = \frac{1-(u/c)^2}{1-[u/(cn)]^2}\,. \tag{7}$$

Thus an elementary wave that is emitted at time t at some point $P=(x_0, y_0)$ in the xy-plane of the laboratory system will at time $t+\Delta t$ have a curve of intersection with the xy-plane that is described by

$$(x-x_0-a\Delta t)^2/b+(y-y_0)^2-bc'^2\Delta t^2=0\,. \tag{8}$$

This is an ellipse with its center at the point $Q=(x_0+a\Delta t, y_0)$ and its short semiaxis in the direction of the medium's motion. The elementary waves are thus dragged by the medium and contracted in the direction of motion (see Fig. 1).

One more important point should be noted here. In the case of a light ray propagating in a moving medium, the velocity vector $\boldsymbol{v}$ and the wave vector $\boldsymbol{k}$ are not parallel, except for the cases when they are parallel to the velocity of the medium or the light ray has the velocity c, i.e. the medium has no influence on the light or is simply not present. In a technical sense this is due to the fact that these vectors are parallel in the rest frame of the medium but transform to the laboratory frame according to different laws. The actual calculation is straightforward but complicated. To understand the physical background, recall the interpretation of the two vectors. The velocity vector is tangent to the ray, whereas the wave vector is orthogonal to the wave fronts of the light wave. The fact that they are not parallel means that the velocity has a component that is parallel to the wave fronts.

Being a wave phenomenon, light has to fulfill a wave equation. When the medium is at rest we can write the equation as

$$\left(\nabla^2-\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)F_{\mu\nu}=0\,, \tag{9}$$

where $F_{\mu\nu}$ is the field strength tensor of the light field (40). This equation differs from the equation for light in vacuum only by the factor n^2 that multiplies the time-derivative term.

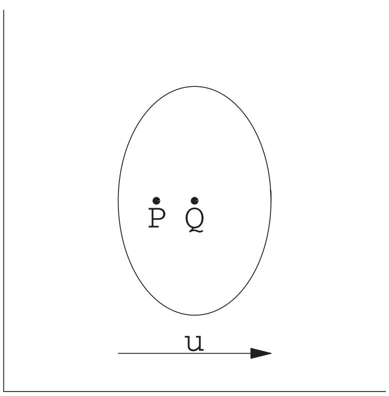

Fig. 1. An elementary wave in a moving dielectric. The medium is moving to the right with the velocity u. A light wave is emitted at point P and after having propagated the elementary wave has the form of an ellipse centered at Q. Thus the wave is dragged by the medium and simultaneously Lorentz contracted

The equation can be rewritten in the form

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{n^2-1}{c^2}\frac{\partial^2}{\partial t^2}\right)F_{\mu\nu}=0\,. \tag{10}$$

The first two terms in the parenthesis of (10) form the d'Alembert operator that is invariant under Lorentz transformations. As the medium is assumed to be at rest, its four velocity

$$u^\mu=\gamma\left(1, \frac{\boldsymbol{u}}{c}\right) \tag{11}$$

is of the form

$$u^\mu=(1, \mathbf{0})\,. \tag{12}$$

This allows us to interpret the time derivative as a term of the form $\partial_\mu u^\mu$ with the four gradient

$$\partial_\mu=\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)\,, \tag{13}$$

and thus we obtain a covariant form of the wave equation, because we can rewrite it in the form

$$\left[\partial_\alpha\partial^\alpha+(n^2-1)(u^\alpha\partial_\alpha)^2\right]F_{\mu\nu}=0\,. \tag{14}$$

Here and throughout the paper we apply Einstein's summation convention. When transforming the equation into a new frame we also have to transform the field strength tensor. But in this case the Lorentz transformation does not depend on space and time, and hence there are no additional contributions from differentiating the transformation included in the field strength tensor.

1 Geometrical optics and general relativity

In the previous section the problem was rather simple and could be handled well with the different kinds of special relativistic transformations. The calculations might become rather tedious, particularly when the general velocity transformation is involved, but everything is absolutely straightforward. For the problem we are interested in and which is discussed in [3, 4], we go one step further and allow for a dependence of the velocity on space and time. For example, imagine a transparent fluid in nonuniform motion – e.g. in the form of a vortex. This fluid can be considered to be built up of small volume elements, each of them moving at a well-defined velocity. Thus in the rest frame of every volume element, light propagates in all directions with the velocity c/n, and by using the Lorentz transformation we can find out the possible kinds of motion as seen from the laboratory for every element. But what we are interested in is the motion through several volume elements, and we have the problem that we should match smoothly the velocities of the different elements. This is not a trivial task: in every volume element there exists only one permitted value of velocity for every direction of motion and it might happen that these values "do not match" for neighboring elements and the light ray has to change its direction to allow for a smooth propagation.

Let us discuss the propagation of light rays in a medium with a slowly changing velocity profile. The assumption of a velocity profile that does not significantly change over several wavelengths is central for the approximations made in this work.

We assume the light field to be of the form

$$F_{\mu\nu} = \mathcal{F}_{\mu\nu} \exp(\mathrm{i}S) + \text{c.c.}\,, \tag{15}$$

where $\mathcal{F}_{\mu\nu}$ is a slowly varying envelope and

$$S = \int (-\omega \; \mathrm{d}t + \boldsymbol{k}\cdot \; \mathrm{d}\boldsymbol{x}) \tag{16}$$

is the action. Here we introduce the wave four vector

$$k_\nu = \left(\frac{\omega}{c}, -\boldsymbol{k}\right) \tag{17}$$

and the line element

$$\mathrm{d}x^\nu = (c \; \mathrm{d}t, \; \mathrm{d}\boldsymbol{x})\,. \tag{18}$$

Now we can apply (14) to the field (15). For a general field, we would now have to take into account the tensor nature of $\mathcal{F}_{\mu\nu}$, as it has to be transformed to different frames at different points and thus the derivatives would also operate on the velocities involved in the Lorentz transformation. But as we have assumed that the velocity profile changes slowly, we neglect all the derivatives of the profile in comparison with the quadratic wave vector contributions. Thus we arrive again at (14) as the wave equation for the field and we finally obtain

$$k_\mu k^\mu + (n^2-1)(u^\mu k_\mu)^2 = 0\,. \tag{19}$$

This equation can be rewritten in the form

$$k_\mu k_\nu g^{\mu\nu} = 0\,, \tag{20}$$

with

$$g^{\mu\nu} = \eta^{\mu\nu} + (n^2-1)u^\mu u^\nu\,, \tag{21}$$

where

$$\eta^{\mu\nu} = \mathrm{diag}\,[1, -1, -1, -1] \tag{22}$$

is Minkowski's metric tensor of special relativity. Relation (20) strongly resembles the relation for the wave vector of light used in the general theory of relativity. In that case $g^{\mu\nu}$ is the metric tensor describing the structure of space–time. In our case this tensor reflects the velocity profile of the moving medium. But the formal analogy of these two cases allows us to interpret the motion of light in a dielectric in the spirit of general relativity and to call the tensor introduced in (21) the metric tensor of the system. We can conclude that the light's propagation will formally be governed by exactly the same principles.

What should be noted here is that due to the structure of the metric tensor the spatial components of the co- and contravariant forms of the k four vector will be different; in particular they will not be parallel anymore. This takes us back to our discussion of the difference between velocity and momentum vectors in a moving medium. According to (17), the space components of the covariant wave vector k_μ form the three-dimensional wave vector, whereas the space components of the contravariant vector $k^\mu \equiv g^{\mu\nu}k_\nu$ are parallel to the velocity. Thus the difference between wave vector and velocity can be understood as following from the difference between co- and contravariant four vectors.

Light rays in a moving medium propagate according to the rules of general relativity with the metric (21). Since this is a central point here, let us briefly repeat the main ideas of motion in general relativity. In a flat space–time described by the Minkowski metric $\eta^{\mu\nu}$, particles will move along straight lines according to

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = 0\,, \tag{23}$$

with s being the proper time and u^μ the particle's four velocity. The proper time s can be introduced via the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\;\mathrm{d}x^\mu\;\mathrm{d}x^\nu\,, \tag{24}$$

with

$$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\,[1, -1, -1, -1] \tag{25}$$

in the flat empty space and

$$g_{\mu\nu} = \eta_{\mu\nu} + \left(\frac{1}{n^2} - 1\right) u_\mu u_\nu \tag{26}$$

in the moving medium, $g_{\mu\nu}$ being the covariant version of the space–time metric (21). One can easily check that (21) and (26) are exactly inverse. In this form the equation is only valid for massive particles because the four velocity is not defined and the line element $\mathrm{d}s$ vanishes in the case of a massless particle.

Equation (23) simply means that we require the four velocity to have the same coordinates at all points in a coordinate system that seems to be "natural" in a sense as the Cartesian coordinates seem to be a "natural" choice in the case of a flat space. The situation becomes much more involved as soon as we go to a curved space. In this case we can still attach a tangent space to our space–time at every point and require the velocity vectors to be elements of these spaces. But the tangent spaces at different points are not parallel anymore, and thus there is no natural way of saying that two vectors at two different points are "the same".[1] Here one introduces the notion of parallel transport, a method of defining corresponding vectors in different tangent spaces. It is obviously not a good idea to require the coordinates of the vector to be constant, as we can arbitrarily choose the systems of coordinates in the different spaces and thus this requirement would not be covariant; the equation would not have the same form in all systems of coordinates. In this sense not even (23) is a good equation, as it will not keep its shape when transformed into curvilinear coordinates. The solution is to generalize the notion of a derivative by introducing covariant derivatives. We will only state the result for the covariant derivative, a derivation can be found in the book by Landau and Lifshitz [5] or for a more mathematical discussion see the book by Wald [6].

[1] "All tangent spaces are isomorphic, but there is no canonical isomorphism," as a mathematician would put it.

In the case of a contravariant vector a^μ one replaces the normal derivative in the following way:

$$\partial_\nu a^\mu \rightarrow D_\nu a^\mu = \partial_\nu a^\mu + \Gamma^\mu_{\lambda\nu} a^\lambda \,, \tag{27}$$

and analogously for covariant vectors

$$\partial_\nu a_\mu \rightarrow D_\nu a_\mu = \partial_\nu a_\mu - \Gamma^\lambda_{\mu\nu} a_\lambda \,, \tag{28}$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma} \right) . \tag{29}$$

We can use this definition to introduce the notion of the parallelly transported vector. A vector a^μ is said to be parallel transported along a curve if its directional derivative along this curve vanishes, i.e.

$$t^\nu D_\nu a^\mu = 0 \,, \tag{30}$$

where t^ν is tangent to the curve. In the physical situation of a particle moving in a curved space–time, this curve is the particle's world line. In order to generalize the equation of motion for a particle, we have to require that it moves in a way allowing its own velocity vector to be parallel transported along this world line, i.e. it is always "the same" according to the covariant derivative. We obtain the condition

$$u^\nu D_\nu u^\mu = 0 \,, \tag{31}$$

or written out in components

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\lambda\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s} = 0 \,. \tag{32}$$

This is the geodesic equation for a massive particle. When we look at it from another perspective, this formula reflects the variational principle behind general relativity: the path actually chosen by a particle is the one where the proper time becomes maximal. As mentioned above, this kind of equation cannot be applied for light. In that case we have to replace the four velocity u^μ by the contravariant wave vector k^μ, which is also tangent to the world line. We then obtain for the geodesic equation

$$\frac{\mathrm{d}k^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\lambda\nu} k^\nu k^\lambda = 0 \,, \tag{33}$$

where τ is some parameter along the curve. Due to the formal equivalence this is also the condition that will define the trajectories for light rays in a moving medium.

In the actual calculation [3] we have chosen a Hamiltonian approach instead of explicitly solving the geodesic equation. We started from the wave equation (19) in the explicit form

$$\omega^2 - c^2 k^2 + (n^2 - 1)\gamma^2 (\omega - \boldsymbol{u} \cdot \boldsymbol{k})^2 = 0 \tag{34}$$

and defined ω to be the Hamiltonian. The details of the calculations can be found in [3].

In order to get an impression of what happens to a light ray in a moving dielectric we have chosen a concrete example of a velocity profile and calculated the trajectories of light rays for this case. The velocity profile we used was that of a vortex. We have slightly modified the profile in comparison to the usual vortex [7], denoted in polar coordinates

$$\boldsymbol{u} = \frac{\mathcal{W}}{r} \boldsymbol{e}_\varphi \,, \tag{35}$$

in order to avoid the velocity of the flow becoming superluminal. The profile we used had the form

$$\boldsymbol{u} = \frac{\mathcal{W}}{\gamma r} \boldsymbol{e}_\varphi \,, \tag{36}$$

with

$$\gamma = \sqrt{1 + \frac{\mathcal{W}^2}{c^2 r^2}} \,. \tag{37}$$

Here $\mathcal{W}$ denotes the vorticity: a constant fixing the velocity of the vortex. This profile is similar to the usual velocity profile of a vortex in the sense that the space components of the velocity four vector are equal to the velocity three vector of the nonrelativistic vortex. For low velocities, i.e. for large distances from the vortex's core, the two kinds of vortices can be considered equal.

Let us sum up the results in a few words. The main result is that the vortex attracts light rays passing by. Light rays that are far away are deflected, rays coming very close to the vortex's core can even fall into it. In this sense a vortex acts like a black hole. But only light rays propagating in the direction opposite to the flow will fall into the black hole, whereas rays propagating with the flow can always escape. Furthermore one should note that every motion with light falling into the vortex has a time-reversed counterpart with light coming out of the vortex and vanishing to infinity. The trajectories corresponding to these motions are mirror pictures of each other. The full Schwarzschild metric of general relativity does not only contain a black hole, but also a white hole. The white hole is a time-reversed version of the black hole; no particles can enter the event horizon and every particle present within the horizon has to leave it. A comprehensible discussion of the full Schwarzschild metric and its two singularities can be found in the book by Misner et al. [8]. Using the concept of a white hole, we might interpret the situation encountered in our model as a black and a white hole in one. One should emphasize that it is a risky enterprise to interpret our model in the spirit of "real" black holes. Up to now we did not investigate the relation between our metric and the metrics of the different kinds of black holes. But there is an intuitive argument showing why our metric does not describe a black hole [9]. The medium flow has no radial component. A real event horizon would be created if the flow velocity would have a radial component larger than the velocity of light in the medium. In our model there is always a direction the light can take in order to come to regions with a lower flow velocity.

2 Gordon's approach

The idea of using a general relativistic description for light in a moving medium was discussed for the first time by Gordon in his 1923 paper *Zur Lichtfortpflanzung in der Relativitätstheorie* [10]. His approach was different from ours, but he arrived at the same results for the metric tensor. As his calculations show the theory from a different perspective, it is worthwhile to briefly present the calculations that led Gordon to his results. In his calculations he started from the classical description of electric and magnetic fields in matter.

In a situation without free charges the fields obey Maxwell's equations in the form

$$\nabla \cdot \boldsymbol{B} = 0\,, \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{D} = 0\,, \qquad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}\,. \tag{38}$$

In a medium at rest we have the additional constitutive equations

$$\boldsymbol{D} = \varepsilon\varepsilon_0 \boldsymbol{E}, \qquad \boldsymbol{B} = \frac{\mu}{\varepsilon_0 c^2}\boldsymbol{H} \tag{39}$$

describing the properties of the medium. In order to be able to consider fields in a moving medium we have to rewrite the theory in a covariant form. In relativity the electric and magnetic fields can be combined in the form of antisymmetric tensors. The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ form the tensors

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}\,,$$
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}\,. \tag{40}$$

The tensors $H_{\mu\nu}$ and $H^{\mu\nu}$ are defined in an analogous way, with $\boldsymbol{E}$ replaced by $\boldsymbol{D}$ and $c\boldsymbol{B}$ by $\boldsymbol{H}/c$. With these definitions Maxwell's equations (38) can be written in the covariant form

$$\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} = 0 \tag{41}$$

for the first pair of equations and

$$\frac{\partial H^{\lambda\mu}}{\partial x^{\mu}} = 0 \tag{42}$$

for the second one. Considering a medium that moves at the three velocity $\boldsymbol{u}$, i.e. the four velocity

$$u^{\mu} = \gamma\left(1, \frac{\boldsymbol{u}}{c}\right)\,, \tag{43}$$

we can rewrite (39) in the form

$$H^{\lambda\varrho}u_{\varrho} = \varepsilon\varepsilon_0 F^{\lambda\varrho}u_{\varrho}\,,$$
$$F_{\lambda\varrho}u_{\nu} + F_{\varrho\nu}u_{\lambda} + F_{\nu\lambda}u_{\varrho} = \frac{\mu}{\varepsilon_0}\left(H_{\lambda\varrho}u_{\nu} + H_{\varrho\nu}u_{\lambda} + H_{\nu\lambda}u_{\varrho}\right)\,. \tag{44}$$

In the case of a medium at rest, i.e. for $u^{\mu} = (1, \mathbf{0})$, this is simply (39). But, as (44) is valid in one frame of reference and is written in a covariant form, it is valid in all systems. It thus yields a connection between the tensors $F^{\varrho\nu}$ and $H^{\varrho\nu}$ that is correct irrespective of the relative velocity of the observer and the medium. These equations were published by H. Minkowski in 1908 [11]. They form the starting point for the investigations made by W. Gordon.

Using (41), (42) and (44) and the fact that

$$u^{\varrho}u_{\varrho} = 1\,, \tag{45}$$

we can reexpress the field $H_{\varrho\nu}$ in terms of $F_{\varrho\nu}$,

$$\frac{\mu}{\varepsilon_0}H_{\varrho\nu} = F_{\varrho\nu} + (1-\varepsilon\mu)(u_{\varrho}F_{\nu\lambda}u^{\lambda} - u_{\nu}F_{\varrho\lambda}u^{\lambda})\,, \tag{46}$$

or in contravariant notation

$$\frac{\mu}{\varepsilon_0}H^{\varrho\nu} = F^{\varrho\nu} + (1-\varepsilon\mu)(u^{\varrho}F^{\nu\lambda}u_{\lambda} - u^{\nu}F^{\varrho\lambda}u_{\lambda})\,. \tag{47}$$

This result can be rewritten in the form

$$\frac{\mu}{\varepsilon_0}H^{\varrho\nu} = F_{\lambda\iota}\left[\left(\eta^{\lambda\varrho}\eta^{\nu\iota}\right) + \left(n^2-1\right)\left(\eta^{\lambda\varrho}u^{\nu}u^{\iota} + \eta^{\nu\iota}u^{\lambda}u^{\varrho}\right)\right]\,, \tag{48}$$

where we have used the identity $\varepsilon\mu = n^2$. We are free to add the term $(n^2-1)^2u^{\lambda}u^{\varrho}u^{\nu}u^{\iota}F_{\lambda\iota}$. It vanishes being simultaneously symmetric and antisymmetric when the indices λ and ι are interchanged. Now we can write (48) in the form

$$\frac{\mu}{\varepsilon_0}H^{\varrho\nu} = \left(\eta^{\lambda\varrho} + \left(n^2-1\right)u^{\lambda}u^{\varrho}\right)\left(\eta^{\nu\iota} + \left(n^2-1\right)u^{\nu}u^{\iota}\right)F_{\lambda\iota}\,. \tag{49}$$

Introducing here the tensor (21), allows us to write this in the final form

$$\frac{\mu}{\varepsilon_0}H^{\varrho\nu} = g^{\varrho\lambda}g^{\nu\iota}F_{\lambda\iota}\,. \tag{50}$$

Thus we can again interpret $g^{\mu\nu}$ as an effective metric tensor and $(\mu/\varepsilon_0)H^{\varrho\nu}$ as the contravariant counterpart of $F_{\varrho\nu}$. To denote this, Gordon puts parentheses around the indices whenever an index has been moved with the metric (21), and so we obtain in his notation

$$F^{(\varrho)(\nu)} = g^{\varrho\lambda}g^{\nu\iota}F_{\lambda\iota}\,. \tag{51}$$

When changing equations from special to general relativity one has to change normal derivatives to covariant ones. In the case relevant here – that of second rank tensors – one obtains

$$\partial_l A^{ik} \to D_l A^{ik} = \partial_l A^{ik} + \Gamma^{i}_{ml}A^{mk} + \Gamma^{k}_{ml}A^{im} \tag{52}$$

and

$$\partial_l A_{ik} \to D_l A_{ik} = \partial_l A_{ik} - \Gamma^{m}_{li}A_{mk} - \Gamma^{m}_{kl}A_{im}\,. \tag{53}$$

Considering Maxwell's equations in free space, i.e. for $F^{\varrho\nu} = H^{\varrho\nu}/\varepsilon_0$, we see that the introduction of covariant derivatives does not change the first set of Maxwell's equations (41). This is due to the fact that the Christoffel symbols are symmetric in the lower indices, whereas the field strength tensor is antisymmetric. The second set (42) can be rewritten in the form ([12])

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}F^{\lambda\varrho}}{\partial x^{\varrho}} = 0\,, \tag{54}$$

where g is the determinant of the metric tensor $g^{\mu\nu}$. We can obtain Maxwell's equations in the moving medium into this form by using (42) and introducing additional factors of $\sqrt{-g}$, which is permitted as long as g is a constant. Then we obtain

$$\frac{\partial F^{(\lambda)(\varrho)}}{\partial x^\varrho} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} F^{(\lambda)(\varrho)}}{\partial x^\varrho} = 0 \,. \tag{55}$$

The determinant g takes the the value $\varepsilon\mu = n^2$. That can be seen from the fact that the metric tensor in the moving medium (21) can be obtained from the metric tensor in the medium at rest $g_0^{\mu\nu} = \mathrm{diag}[n^2, -1, -1, -1]$ via a Lorentz transformation. The determinant of a Lorentz transformation is ± 1 because the Minkowski metric is Lorentz invariant. Consequently, the determinant of $g^{\mu\nu}$ does not depend on the frame of reference and is thus equal to $-n^2 = \det(g_0^{\mu\nu})$ in all frames. Thus electromagnetic fields in a moving dielectric can be described within the framework of general relativity as long as the index of refraction is constant.

3 Optical Aharonov–Bohm effect

The theory discussed up to now is valid for almost all velocities. When velocities become very high and change fast, spectacular effects such as optical black holes can be observed. However, although velocity profiles as extreme as those needed here cannot be realized in practice, there is another regime where interesting effects might be observable.

One can go far away from the vortex where the velocities are low. As is shown in [3], we can expand the Hamiltonian for light in the moving medium to first order in u/c to obtain

$$H = \frac{c}{n} k + \left(1 - \frac{1}{n^2}\right) \boldsymbol{u} \cdot \boldsymbol{k} \,. \tag{56}$$

Applying Hamilton's equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{k}} \,, \qquad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{57}$$

gives the velocity

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{c}{n}\frac{\boldsymbol{k}}{k} + \left(1 - \frac{1}{n^2}\right) \boldsymbol{u} \,. \tag{58}$$

We are free to rescale the velocity by multiplying it by k, the modulus of the wave vector $\boldsymbol{k}$ and thus obtain

$$\boldsymbol{w} = k\boldsymbol{v} \,. \tag{59}$$

Rewriting the equation of motion in terms of $\boldsymbol{w}$ yields

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \left(1 - \frac{1}{n^2}\right) (\nabla \times \boldsymbol{u}) \times \boldsymbol{w} \,. \tag{60}$$

This resembles strongly the equation of motion for a charged particle moving in a magnetic field. In this analogy, $\boldsymbol{u}$ corresponds to the vector potential and $\boldsymbol{w}$ to the velocity. In case we choose the vortex (35) as the velocity profile we arrive at an analogy to the Aharonov–Bohm effect [13]. Charged particles moving in an Aharonov–Bohm potential acquire only a phase shift, but the velocity is unchanged. In our case $\boldsymbol{w}$ is constant, but the modulus of the real velocity $\boldsymbol{v}$ changes. That the velocity cannot stay unchanged is already clear from Fresnel's formula (1), because the ray passes through regions with varying flow velocities. In particular, rays passing on two different sides of the vortex's core will move with different velocities, as one of them propagates with the flow and the other against it. But it is still remarkable that the rays are not bent, as would happen even in most other nontrivial velocity profiles in this approximation. As long as we are only interested in trajectories or stationary wave phenomena, light in the vortex and the Aharonov–Bohm effect are fully equivalent. It is also possible to introduce a new Hamiltonian for light in the vortex corresponding to a rescaled time parameter. In order to find this Hamiltonian we start from the wave equation (14) to the first order in u/c and choose $n^2\omega^2/c^2$ as our new Hamiltonian. We obtain

$$H = \boldsymbol{k}^2 + 2\omega \frac{n^2 - 1}{c^2} \boldsymbol{u} \cdot \boldsymbol{k} \,. \tag{61}$$

But as long as we want to be exact only to the first order in u/c, we can always add a quadratic term and obtain

$$H = \left(\boldsymbol{k} + \omega \frac{n^2 - 1}{c^2} \boldsymbol{u}\right)^2 \,, \tag{62}$$

a Hamiltonian having the same structure as that of a charged particle propagating in a magnetic field. This shows the analogy between light and charged matter waves. In writing the last expression, one should keep in mind that the quadratic term has been added only to make the expression look nicer and to show the similarity with the magnetic case. In fact, this expression is different from the correct second-order expansion of the Hamiltonian and may be used only when the second order term can be neglected.

Note that we again arrived at a situation where velocity and momentum are not parallel. Here they differ by a term proportional to the velocity of the flow.

4 Remarks

We have shown a very far-reaching formal analogy between the motion of light in a moving medium and in a gravitational field. The gain from this equivalence is twofold. On one hand we have reduced the problem of light rays in a moving medium to the known problem of light rays in gravitational fields, a topic where a substantial amount of work has been done already. On the other hand, general relativity with its ideas of curved space–time and geodesics is very abstract and hard to grasp intuitively. With the picture of a flowing medium in mind, one can understand general relativity more easily, a gain that is not at all dependent on the possibility of an experimental realization of those ideas. One should note that already ideas have been sketched to interpret general relativity in a spirit similar to our work. Harrison describes in his popular book on cosmology [14] the idea to interpret gravitation as flowing space. In this picture, elementary waves emitted from a point source are dragged by the flowing space and thus move away in relation to their source, exactly as in the case shown in Fig. 1. Because this is not a motion within space but rather the motion of space itself, velocities higher than the velocity of light would be permitted. This is exactly what will happen within the Schwarzschild radius, and hence even light propagating in the opposite direction than the space will be still dragged by it and unable to escape. As it is well known in general relativity, one can always choose a system of coordinates in such a way that space–time is flat at exactly one point. This would then correspond to the rest frame of some droplet. We have not yet followed this path further, but it seems to be a very fruitful approach towards a deeper understanding of light in a moving dielectric. In our approach, we varied the velocity of the medium but kept the index of refraction constant. The situation where only the index of refraction is varied is described in textbooks. Landau and Lifshitz [5] use a variational approach, whereas the method described by Born and Wolf [15] is similar to ours. A particularly interesting remark can be found in a footnote of the book by Born and Wolf where they mention a 1926 paper by Bortolotti [16]. He showed that the light trajectories in a resting medium with variable index of refraction are geodesics in a three-dimensional space with the metric tensor diag$[n^2, n^2, n^2]$.

5 Slow light and optical black holes

In the previous sections we discussed spectacular effects of light in a moving medium, such as deflection of light rays in the vicinity of a vortex or the optical Aharonov–Bohm effect. For these effects to become strong high values of the index of refraction and high medium velocities are necessary. In order to create an optical black hole, one needs velocities of the medium that are comparable to the velocity of light in the medium, which is incredibly large. It should be clear that in no laboratory can vortices with such velocities be created. Thus the effects of general relativity in a glass of water seem to be nonobservable. Modern interferometric techniques would allow for the observation of the optical Aharonov–Bohm effect, however.

Last year reports were published regarding experiments with light moving at extremely low group velocities [17, 18]. The velocities mentioned in the first publication [17] were as low as 17 m/s. These effects have been seen in alkali Bose–Einstein condensates and in noncondensed atomic vapors. The low group velocities are not due to a very high value of the nondispersive index of refraction. They are a result of the very strongly dispersive character of the medium, i.e. the index of refraction depends strongly on the frequency of the light. Light in a dispersive medium obeys the wave equation

$$k^2 - \frac{\omega^2}{c^2} - \chi(\omega)\frac{\omega^2}{c^2} = 0\,, \tag{63}$$

where $\chi = n^2 - 1$ is the susceptibility. A wave group in a dispersive medium moves with the group velocity

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\,. \tag{64}$$

Using (63), we obtain for the group velocity

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \left(\frac{\sqrt{1+\chi}}{c} + \frac{\omega}{c}\frac{\mathrm{d}\chi}{\mathrm{d}\omega}\frac{1}{2\sqrt{1+\chi}}\right)^{-1}\,. \tag{65}$$

Consequently, a strong dependence of the susceptibility on the frequency leads to a small value of the group velocity. To achieve the strong dispersion an effect called electromagnetically induced transparency (EIT) has been applied.

Electromagnetically induced transparency is an effect of quantum coherence with three atomic levels in an approximate Λ configuration being involved (see Fig. 2).

For the creation of EIT two lasers are needed. One of them – the probe laser – is tuned to the frequency of the transition between the upper level (a) and one of the lower levels (b). It is the propagation of this beam we are actually interested in when performing the experiment. With only the probe laser present, its light would be absorbed by the medium very quickly. The situation changes as soon as the second laser – the drive laser – is used. The drive laser light is strong and it is tuned to the transition between the upper state and the second of the lower states (c), creating a quantum interference that makes the medium effectively transparent to the probe beam. When both lasers are tuned exactly to their respective transition energies, both the real and imaginary parts of the susceptibility vanish. Thus the probe light will propagate through the medium without loss and at the vacuum velocity of light.

In Fig. 3 the dependence of the susceptibility on the detuning is shown. A detailed discussion of the effect can be found in the book by Scully and Zubairy [19]. What is most important here is that the real part shows a strong linear dependence

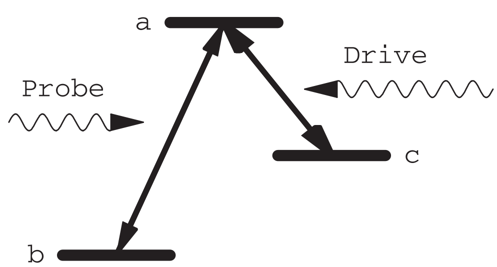

Fig. 2. Three-level system needed for the creation of electromagnetically induced transparency. A strong drive laser field couples the levels a and c, making the medium transparent for the weak probe laser tuned to the transition a↔b

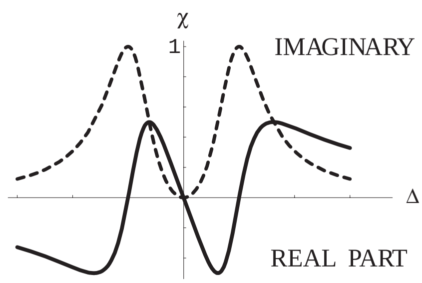

Fig. 3. Susceptibility for the probe laser beam in electromagnetically induced transparency. The plot shows the dependence of the real (*solid lines*) and imaginary (*dashed lines*) parts of the susceptibility χ on the detuning Δ of the probe beam. The drive beam is assumed to be on resonance. Arbitrary units are used

on the detuning of the probe. In [4] we used a simplified model of EIT. We assumed the susceptibility to be of the form

$$\chi(\omega') = \frac{2\alpha}{\omega_0}(\omega' - \omega_0) + O\left((\omega' - \omega_0)^3\right) , \tag{66}$$

with ω_0 being the resonance frequency of the $a \leftrightarrow b$ transition. We neglected the imaginary part of the susceptibility χ and assumed the real part to be linear close to resonance.

In the experiment we discuss in [4], the medium is moving relative to the light source. Consequently the light frequency in the rest frame of the medium is different from the frequency in the frame of the light source. But it is the frequency in the medium frame that fixes the value of the susceptibility, and so the susceptibility will depend on the velocity of the medium. Thus the frequency dependence of the susceptibility comes into play. Although the probe light is assumed to be monochromatic, while transforming the wave equation (63) into the laboratory system we have to keep in mind that the frequency appearing in the argument of the susceptibility is still the frequency in the rest frame of the medium. We obtain for the wave equation

$$k^2 - \frac{\omega_0^2}{c^2} - \chi(\omega')\frac{\omega'^2}{c^2} = 0 , \quad \omega' = \frac{\omega_0 - \boldsymbol{u}\cdot\boldsymbol{k}}{\sqrt{1-u^2/c^2}} , \tag{67}$$

where ω_0 is the frequency of the probe laser. In (67) the relation between frequency and wave vector can become very involved. For our purpose we use the susceptibility in the form (66) and expand the wave equation (67) to the second order in u, and thus we obtain

$$\boldsymbol{k}^2 - 4\left(\boldsymbol{k}\cdot\frac{\boldsymbol{u}}{c}\right)^2\alpha + 2\frac{\boldsymbol{u}}{c}\cdot\boldsymbol{k}\alpha\frac{\omega}{c} - \left(1 + \frac{u}{c}^2\alpha\right)\frac{\omega^2}{c^2} = 0 . \tag{68}$$

Since this equation is quadratic in k and ω we can rewrite it in the form

$$g^{\mu\nu}k_\mu k_\nu = 0 , \tag{69}$$

with the definition

$$k_\nu = \left(\frac{\omega}{c}, -\boldsymbol{k}\right) \tag{70}$$

and the metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1+\alpha\frac{u^2}{c^2} & \alpha\frac{\boldsymbol{u}}{c} \\ \alpha\frac{\boldsymbol{u}}{c} & -\mathbf{1} + 4\alpha\frac{\boldsymbol{u}\otimes\boldsymbol{u}}{c^2} \end{pmatrix} . \tag{71}$$

Note that the precise result for the wave equation (68) strongly depends on the approximations made. Using higher orders in the expansion of the wave equation (67) or in the expansion of the susceptibility would create higher terms in k and ω. In that case, no relativistic model would be valid anymore. Note also that the susceptibility (66) is only valid within a narrow frequency range. If the frequency detuning due to the Doppler effect of the moving medium exceeds this range our analysis is not applicable anymore. However, one may compensate for the overall Doppler detuning in the particular region of the flow where one is interested in by adjusting the frequency of the probe light. As long as we accept our model and the approximations made, all the techniques used in the case of the nondispersive medium can be used once again.

The particular example for a flow is the same one we used in the case of the nondispersive medium. Again we consider the velocity profile in the form of a vortex (35). As discussed in the previous section light rays are null geodesics of the metric (71), and the vortex acts as a black hole that can swallow light rays coming too close to it. Note a small difference of the black hole when compared to the nondispersive case. In the nondispersive case only light rays propagating in the direction opposite to the flow can fall into the black hole. They do so when they come to close to the core of the vortex. Light rays propagating with the flow could come arbitrarily close to the vortex's core and still escape. In the dispersive medium this is not the case. Here two kinds of critical radii are possible. Whereas the soft critical radius corresponds to that encountered in the nondispersive case, the hard critical radius is a new feature. All light rays coming to the core closer than the radius fall into it. But even this radius does not constitute a real event horizon [9]. There are still solutions with light coming out of the vortex and escaping to infinity, as has been discussed in the preceding chapter. Numerical examples show that hard critical radii are far beyond feasibility. But the soft critical radius might become realizable with the methods of today's experimental techniques. As relativistic effects in the moving medium become visible when the velocity of the medium becomes comparable to the velocity of light in the medium, it is clear that the strongly reduced group velocity will make the observability of optical black holes much more likely. Thus the ideas of optical black holes in highly dispersive media provide us with an interesting prospect of observing general relativistic effects in the laboratory.

6 Summary

A moving medium appears to light as a change in the metric of space and time, i.e. as an effective gravitational field. When the medium moves at moderate velocities the flow acts similar to a vector potential, and so the medium appears as an effective magnetic field. A vortex flow will generate the optical analogue of the Aharonov–Bohm effect. Spectacular relativistic effects are expected when the medium moves faster than the local speed of light. The advent of extremely slow light shows that this is not an entirely unrealistic regime and thus opens the prospect of making artificial astronomical objects in an earthly laboratory.

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