Appl. Phys. B 72, 43–50 (2001) / Digital Object Identifier (DOI) 10.1007/s003400000480

Applied Physics B
Lasers
and Optics

Conditional quantum-state engineering in repeated 2-photon down-conversion

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Received: 12 July 2000/Revised version: 19 September 2000/Published online: 30 November 2000 –

Abstract. The U(1,1) and U(2) transformations realized by three-mode interaction in the respective parametric approximations are studied in conditional measurement, and the corresponding non-unitary transformation operators are derived. As an application, the preparation of single-mode quantum states using an optical feedback loop is discussed, with special emphasis on Fock-state preparation. For that example, the influence of non-perfect detection and feedback is also considered.

PACS: 42.50.Ct; 42.50.Dv; 03.65.Bz

Conditional measurement offers a promising way to manipulate the state of a given quantum system. The basic idea is to entangle the state of the system under consideration with the state of an auxiliary system and to prepare the system in the desired state owing to the state reduction associated with an appropriate measurement on the auxiliary system. In what follows, we restrict our attention to travelling optical fields. The quantum states of two travelling modes can be entangled by mixing them at an appropriately chosen multiport. Possible basic transformations are the U(1,1) transformation as realized by a non-degenerate parametric amplifier and the U(2) transformation as realized by a frequency converter or a beam splitter.

The aim of this paper is to generalize and unify previous work on conditional measurement at U(2) and U(1,1) couplers. This includes the description of the quantum-state transformation in terms of a non-unitary operator [1, 2] as well as possible applications such as the generation of Fock states [3], optical qubits [4], Schrödinger-cat-like states [5], and photon-subtracted or photon-added Jacobi polynomial states [6] or the measurement of specific overlaps [7, 8]. By combining a theoretical concept for preparing single-mode quantum states by alternate coherent displacement and photon-adding [9] with an experimental proposal to employ a 2-photon down-converter inside a feedback loop [10], a way is offered to prepare quantum states without non-classical input. In particular, emphasis is placed on the generation of Fock states, for which possible imperfections of the optical components are analysed.

The paper is organized as follows. In Sect. 1 appropriately factorised representations of the unitary U(1,1) and U(2) transformation operators are introduced. They are used in Sect. 2 in order to derive the non-unitary transformation operators realized by conditional measurement. In Sect. 3 the results are applied to the generation of specific quantum states, with special emphasis on Fock states and simple superpositions of Fock states. Finally, a summary and some concluding remarks are given in Sect. 4.

1 Parametrically approximated three-wave mixing

Let us consider the transformation

$$\hat{\varrho}'_{abc} = \hat{U}\hat{\varrho}_{abc}\hat{U}^{\dagger} \tag{1}$$

of the quantum state $\hat{\varrho}_{abc}$ of three travelling optical modes (denoted by a, b, and c), with $\hat{U} = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}$ being realized by a three-wave mixer,

$$\hat{H} = \hbar\omega_a\hat{a}^{\dagger}\hat{a} + \hbar\omega_b\hat{b}^{\dagger}\hat{b} + \hbar\omega_c\hat{c}^{\dagger}\hat{c} + \hbar\chi^{(2)}(\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{c} + \hat{c}^{\dagger}\hat{b}\hat{a})\,, \tag{2}$$

where $\chi^{(2)}$ corresponds to the second-order non-linear susceptibility. In what follows, we discuss three different parametric approximations (for details on the conditions under which such approximations hold, see [11]).

1.1 U(1,1) mixing of modes a and b: parametric amplification

If we assume that the mode c is prepared in a coherent state, $\hat{\varrho}_{abc} = \hat{\varrho}_{ab} \otimes |\gamma\rangle\langle\gamma|$, then in the limit $\chi^{(2)} \to 0$ and $|\gamma| \to \infty$ with $\chi^{(2)}\gamma = \text{const.}$ the reduced density operator $\hat{\varrho}'_{ab} = \mathrm{Tr}_c\hat{\varrho}'_{abc}$ reads

$$\hat{\varrho}'_{ab} = \hat{U}_{\mathrm{A}}\hat{\varrho}_{ab}\hat{U}^{\dagger}_{\mathrm{A}}, \tag{3}$$

where $\hat{U}_{\mathrm{A}}=\mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{A}}t/\hbar}$ is a U(1,1) transformation realized by a non-degenerate parametric amplifier,

$$\hat{H}_{\mathrm{A}} = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} + \hbar\chi^{(2)}\big(\gamma \hat{a}^\dagger \hat{b}^\dagger + \gamma^* \hat{b}\hat{a}\big). \tag{4}$$

Introducing the quantities

$$\begin{aligned}
\phi_0 &= -(\omega_a - \omega_b)t, & \hat{K}_0 &= \tfrac{1}{2}(\hat{a}\hat{a}^\dagger - \hat{b}^\dagger \hat{b}),\\
\phi_1 &= -(\gamma + \gamma^*)\chi^{(2)} t, & \hat{K}_1 &= \tfrac{1}{2}(\hat{b}^\dagger \hat{a}^\dagger + \hat{a}\hat{b}),\\
\phi_2 &= -\mathrm{i}(\gamma - \gamma^*)\chi^{(2)} t, & \hat{K}_2 &= \tfrac{1}{2\mathrm{i}}(\hat{b}^\dagger \hat{a}^\dagger - \hat{a}\hat{b}),\\
\phi_3 &= -(\omega_a + \omega_b)t, & \hat{K}_3 &= \tfrac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger \hat{b}),
\end{aligned} \tag{5}$$

where the commutation relation

$$[\hat{K}_j, \hat{K}_k] = \mathrm{i} \sum_{l,m=0}^{3} \varepsilon_{0jkl}\, g_{lm} \hat{K}_m \tag{6}$$

is valid ($(g_{lm})=\mathrm{diag}(1, 1, 1, -1)$, ε_{ijkl} is the four-dimensional Levi–Civita symbol), and applying the respective disentanglement theorem [12], we may factorise $\hat{U}_{\mathrm{A}}$ as follows:

$$\begin{aligned}
\hat{U}_{\mathrm{A}} &= \mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{A}}t/\hbar} = \mathrm{e}^{\mathrm{i}\omega_a t}\mathrm{e}^{\mathrm{i}(\phi_0\hat{K}_0+\phi_1\hat{K}_1+\phi_2\hat{K}_2+\phi_3\hat{K}_3)}\\
&= \mathrm{e}^{-\mathrm{i}(\phi_0+\phi_3)/2}\mathrm{e}^{\mathrm{i}\phi_0\hat{K}_0}\mathrm{e}^{\mathrm{i}(\varphi_T+\varphi_R)\hat{K}_3}\mathrm{e}^{2\mathrm{i}\vartheta\hat{K}_2}\mathrm{e}^{\mathrm{i}(\varphi_T-\varphi_R)\hat{K}_3}\\
&= \bar{T}^{*-1}(P\,T\,)^{*-\hat{a}^\dagger\hat{a}}\mathrm{e}^{-P^*R\hat{b}^\dagger\hat{a}^\dagger}\mathrm{e}^{PR^*\hat{a}\hat{b}}(P\,T^*)^{-\hat{b}^\dagger\hat{b}},
\end{aligned} \tag{7}$$

where

$$T = \cosh\vartheta\,\mathrm{e}^{\mathrm{i}\varphi_T} = \cos\frac{\phi}{2} + \mathrm{i}\frac{\phi_3}{\phi}\sin\frac{\phi}{2}\,, \tag{8}$$

$$R = \sinh\vartheta\,\mathrm{e}^{\mathrm{i}\varphi_R} = \frac{\phi_2+\mathrm{i}\phi_1}{\phi}\sin\frac{\phi}{2}\,, \tag{9}$$

$$P = \mathrm{e}^{\mathrm{i}\phi_0/2} \tag{10}$$

and $\bar{T} = T\mathrm{e}^{-\mathrm{i}\phi_3/2}$ ($\phi=\sqrt{\phi_3^2-\phi_2^2-\phi_1^2}$). Note the hyperbolic behaviour of (8) and (9) for $|\chi^{(2)}\gamma| > (\omega_a+\omega_b)/2$. For $\phi_0=0$ we have $P=1$ and $\hat{U}_{\mathrm{A}}$ in (7) reduces to a SU(1,1) transformation operator; compare [13].

1.2 U(2) mixing of modes a and c: frequency conversion

If, alternatively, we assume that the mode b is prepared in a coherent state, $\hat{\varrho}_{abc}=\hat{\varrho}_{ac}\otimes|\beta\rangle\langle\beta|$, then in the limit $\chi^{(2)}\to 0$ and $|\beta|\to\infty$ with $\chi^{(2)}\beta=$const. the reduced density operator $\hat{\varrho}'_{ac} = \mathrm{Tr}_b\hat{\varrho}'_{abc}$ reads

$$\hat{\varrho}'_{ac} = \hat{U}_{\mathrm{C}}\hat{\varrho}_{ac}\hat{U}_{\mathrm{C}}^\dagger, \tag{11}$$

where $\hat{U}_{\mathrm{C}}=\mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{C}}t/\hbar}$ is a U(2) transformation realized by a frequency converter,

$$\hat{H}_{\mathrm{C}} = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_c \hat{c}^\dagger \hat{c} + \hbar\chi^{(2)}\big(\beta \hat{c}^\dagger \hat{a} + \beta^* \hat{a}^\dagger \hat{c}\big). \tag{12}$$

In order to factorise $\hat{U}_{\mathrm{C}}$, we introduce the quantities

$$\begin{aligned}
\varphi_0 &= -(\omega_a + \omega_c)t, & \hat{L}_0 &= \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{c}^\dagger \hat{c}),\\
\varphi_1 &= -(\beta^* + \beta)\chi^{(2)} t, & \hat{L}_1 &= \tfrac{1}{2}(\hat{a}^\dagger \hat{c} + \hat{c}^\dagger\hat{a}),\\
\varphi_2 &= -\mathrm{i}(\beta^* - \beta)\chi^{(2)} t, & \hat{L}_2 &= \tfrac{1}{2\mathrm{i}}(\hat{a}^\dagger \hat{c} - \hat{c}^\dagger\hat{a}),\\
\varphi_3 &= -(\omega_a - \omega_c)t, & \hat{L}_3 &= \tfrac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{c}^\dagger \hat{c}),
\end{aligned} \tag{13}$$

where now

$$[\hat{L}_j, \hat{L}_k] = \mathrm{i}\sum_{l=0}^{3}\varepsilon_{0jkl}\hat{L}_l. \tag{14}$$

We apply the respective disentanglement theorem [12] to obtain

$$\begin{aligned}
\hat{U}_{\mathrm{C}} &= \mathrm{e}^{-\mathrm{i}\hat{H}_{\mathrm{C}}t/\hbar} = \mathrm{e}^{\mathrm{i}(\varphi_0\hat{L}_0+\varphi_1\hat{L}_1+\varphi_2\hat{L}_2+\varphi_3\hat{L}_3)}\\
&= \mathrm{e}^{\mathrm{i}\varphi_0\hat{L}_0}\mathrm{e}^{\mathrm{i}(\varphi_T+\varphi_R)\hat{L}_3}\mathrm{e}^{2\mathrm{i}\vartheta\hat{L}_2}\mathrm{e}^{\mathrm{i}(\varphi_T-\varphi_R)\hat{L}_3}\\
&= (\mathcal{P}\mathcal{T})^{\hat{a}^\dagger\hat{a}}\mathrm{e}^{-\mathcal{P}\mathcal{R}^*\hat{c}^\dagger\hat{a}}\mathrm{e}^{\mathcal{P}^*\mathcal{R}\hat{a}^\dagger\hat{c}}(\mathcal{P}^*\mathcal{T})^{-\hat{c}^\dagger\hat{c}},
\end{aligned} \tag{15}$$

where

$$\mathcal{T} = \cos\vartheta\,\mathrm{e}^{\mathrm{i}\varphi_T} = \cos\frac{\varphi}{2} + \mathrm{i}\frac{\varphi_3}{\varphi}\sin\frac{\varphi}{2}\,, \tag{16}$$

$$\mathcal{R} = \sin\vartheta\,\mathrm{e}^{\mathrm{i}\varphi_R} = \frac{\varphi_2+\mathrm{i}\varphi_1}{\varphi}\sin\frac{\varphi}{2}\,, \tag{17}$$

$$\mathcal{P} = \mathrm{e}^{\mathrm{i}\varphi_0/2} \tag{18}$$

($\varphi=\sqrt{\varphi_3^2+\varphi_2^2+\varphi_1^2}$). For $\varphi_0=0$ we have $\mathcal{P}=1$ and $\hat{U}_{\mathrm{C}}$ in (15) reduces to a SU(2) transformation operator; compare [14].

1.3 Transformation properties of U(1,1) and U(2)

From (7) and (15) the transformation matrices for the respective mode operators are deduced to be

$$\hat{U}_{\mathrm{A}}^\dagger \begin{pmatrix}\hat{a}\\ \hat{b}^\dagger\end{pmatrix}\hat{U}_{\mathrm{A}} = P\begin{pmatrix}T & -R\\ -R^* & T^*\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{b}^\dagger\end{pmatrix}\,, \tag{19}$$

$$\hat{U}_{\mathrm{C}}^\dagger \begin{pmatrix}\hat{a}\\ \hat{c}\end{pmatrix}\hat{U}_{\mathrm{C}} = \mathcal{P}\begin{pmatrix}\mathcal{T} & \mathcal{R}\\ -\mathcal{R}^* & \mathcal{T}^*\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{c}\end{pmatrix}. \tag{20}$$

In turn, (19) and (20) can themselves be used to define the four-parametric action of a parametric amplifier and a frequency converter. In this case we consider T, R, P and $\mathcal{T}$, $\mathcal{R}$, $\mathcal{P}$ as six complex numbers that satisfy the four conditions

$$|T|^2 - |R|^2 = |P|^2 = \ 1\ = |\mathcal{P}|^2 = |\mathcal{T}|^2 + |\mathcal{R}|^2 \tag{21}$$

and are otherwise arbitrary. Using (19) and (20), it is not difficult to verify that

$$[\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}, \hat{U}_{\mathrm{A}}] = \ 0\ = [\hat{a}^\dagger\hat{a} + \hat{c}^\dagger\hat{c}, \hat{U}_{\mathrm{C}}]. \tag{22}$$

The inverse transformations are obtained by replacing ϕ_j with $-\phi_j$ and φ_j with $-\varphi_j$ in (8)–(10) and (16)–(18), respectively, i.e.,

$$\hat{U}_{\mathrm{A}}^{-1}(T, R, P\,) = \hat{U}_{\mathrm{A}}(T^*, -R, P^*), \tag{23}$$

$$\hat{U}_{\mathrm{C}}^{-1}(\mathcal{T}, \mathcal{R}, \mathcal{P}) = \hat{U}_{\mathrm{C}}(\mathcal{T}^*, -\mathcal{R}, \mathcal{P}^*), \tag{24}$$

and interchanging signal and idler modes leads to

$$\hat{U}_{\mathrm{A}}(\hat{b}, \hat{a};\, T, R, P\,) = \hat{U}_{\mathrm{A}}(\hat{a}, \hat{b};\, T, R, P^*), \tag{25}$$

$$\hat{U}_{\mathrm{C}}(\hat{c}, \hat{a};\, \mathcal{T}, \mathcal{R}, \mathcal{P}) = \hat{U}_{\mathrm{C}}(\hat{a}, \hat{c};\, \mathcal{T}^*, -\mathcal{R}^*, \mathcal{P}). \tag{26}$$

1.4 Coherent displacement of mode a

If we assume that modes b and c are simultaneously prepared in coherent states, $\hat{\varrho}_{abc} = \hat{\varrho}_a \otimes |\beta\rangle\langle\beta| \otimes |\gamma\rangle\langle\gamma|$, then in the limit $\chi^{(2)} \to 0$, $|\beta| \to \infty$, $|\gamma| \to \infty$ with $\chi^{(2)}\beta\gamma^* =$ const. the reduced density operator $\hat{\varrho}'_a = \mathrm{Tr}_{bc}\hat{\varrho}'_{abc}$ reads $\hat{\varrho}'_a = \hat{U}_\mathrm{D}\hat{\varrho}_a\hat{U}^\dagger_\mathrm{D}$, where

$$\hat{U}_\mathrm{D} = \mathrm{e}^{-\mathrm{i}[\omega_a \hat{a}^\dagger \hat{a} + \chi^{(2)}(\beta^*\gamma\hat{a}^\dagger + \beta\gamma^*\hat{a})]t} = \mathrm{e}^{-\mathrm{i}\omega_a t \hat{a}^\dagger \hat{a}} \hat{D}(\alpha)\mathrm{e}^{\mathrm{i}\zeta} \tag{27}$$

can be written as the product of a U(1) transformation $\mathrm{e}^{-\mathrm{i}\omega_a t\hat{a}^\dagger\hat{a}}$ and a coherent displacement $\hat{D}(\alpha = \mathrm{i}F^*) = \mathrm{e}^{\mathrm{i}\hat{F}}$, where $\hat{F} = F\hat{a} + F^*\hat{a}^\dagger$ and $F = \mathrm{i}\chi^{(2)}\beta\gamma^*\omega_a^{-1}(1-\mathrm{e}^{-\mathrm{i}\omega_a t})$ ($\mathrm{e}^{\mathrm{i}\zeta}$ is an irrelevant phase factor).

1.5 U(p,q) mixing of $p+q$ modes

Linear coupling of more than two modes can be reduced to a successive application of two-mode couplers. For instance, as a generalization of (19) and (20), let us consider a U(p,q)-coupling of two sets of modes $a_1, \cdots, a_p$ and $a_{p+1}, \cdots, a_N$ ($N = p+q$),

$$\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{a}}^\dagger H\hat{\boldsymbol{a}}}\hat{\boldsymbol{a}}\mathrm{e}^{\mathrm{i}\hat{\boldsymbol{a}}^\dagger H\hat{\boldsymbol{a}}} = \mathrm{e}^{\mathrm{i}GH}\hat{\boldsymbol{a}}, \tag{28}$$

where $\hat{\boldsymbol{a}}$ is a column vector whose elements are $\hat{a}_1, \cdots, \hat{a}_p$, $\hat{a}^\dagger_{p+1}, \cdots, \hat{a}^\dagger_N$ and $\hat{\boldsymbol{a}}^\dagger$ is a row vector with elements $\hat{a}^\dagger_1, \cdots$, $\hat{a}^\dagger_p, \hat{a}_{p+1}, \cdots, \hat{a}_N$. H is a Hermitian $N\times N$ matrix, and G is $N\times N$ diagonal matrix whose upper p (lower q) diagonal elements are equal to 1 (-1) (see Appendix A) Corresponding to a factorisation of $\mathrm{e}^{\mathrm{i}\hat{\boldsymbol{a}}^\dagger H\hat{\boldsymbol{a}}}$, (28) can be implemented by successive application of $\binom{N}{2}$ two-mode couplers, each connecting two of the N modes by either a U(2) transformation (if the two modes belong to the same set) or a U(1,1) transformation (if the two modes belong to different sets). The special case of implementing $\mathrm{U}(N) = \mathrm{U}(N,0)$ by means of beam splitters is discussed in [15]. Note that $\mathrm{e}^{\mathrm{i}\hat{\boldsymbol{a}}^\dagger H\hat{\boldsymbol{a}}}$ can alternatively be factorised into U(2) transformations and single-mode squeezing operations [16].

2 Conditional measurement at U(1,1) and U(2) couplers

Let us consider the scheme in Fig. 1. The signal mode a (index 1) prepared in a state $\hat{\varrho}_1$ and the idler mode (b or c, index 2) prepared in a state $|F\rangle$ are mixed at a U(1,1) parametric amplifier or a U(2) frequency converter, and a device D performs some measurement on the output idler mode. (The pump mode c or b, respectively, prepared in the strong coherent state is not shown in the figure.) Under the condition that D has detected a state $|G\rangle$, the reduced state of the output-signal mode becomes

$$\hat{\varrho}'_1 = \frac{1}{p}\mathrm{Tr}_2(\hat{U}\hat{\varrho}_1 \otimes |F\rangle\langle F|\hat{U}^\dagger\hat{\Pi}), \tag{29}$$

where $\hat{\Pi} = |G\rangle\langle G|$. The norm $p = \mathrm{Tr}_1\mathrm{Tr}_2(\ldots)$ is the probability of measuring the state $|G\rangle$ and thus the probability of

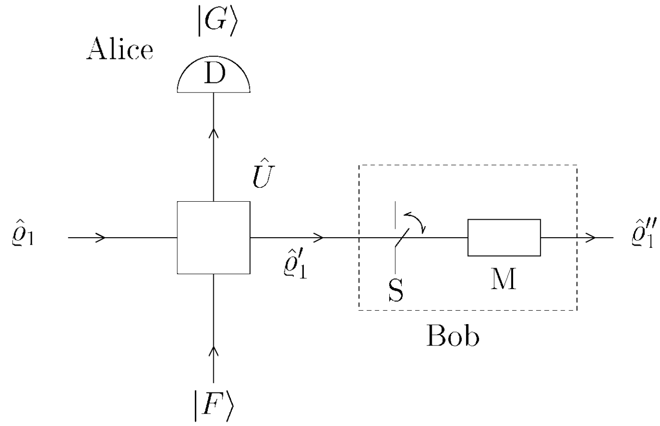

Fig. 1. Scheme of controlled quantum-state engineering by conditional measurement at a U(1,1) or U(2) coupler realizing the transformation $\hat{U} = \hat{U}_\mathrm{A}$ or $\hat{U} = \hat{U}_\mathrm{C}$, respectively. If Alice's measurement device has detected a desired state $|G\rangle$, she informs Bob who opens the aperture S and 'stores' the pulse in the medium M until needed [17]

generating the state $\hat{\varrho}'_1$. Introducing the non-unitary (conditional) operator

$$\hat{Y} = \langle G|\hat{U}|F\rangle \tag{30}$$

acting in the signal-mode Hilbert space, we can rewrite (29) as

$$\hat{\varrho}'_1 = \frac{1}{p}\hat{Y}\hat{\varrho}_1\hat{Y}^\dagger, \tag{31}$$

where the probability now reads $p = \mathrm{Tr}_1(\hat{Y}\hat{\varrho}_1\hat{Y}^\dagger)$. In practice, synchronized sequences of light pulses could be fed into the input ports of the two-mode coupler, each pulse being prepared in the respective state, and the modes are thus non-monochromatic ones. In what follows we assume that the U(1, 1) and U(2) transformations do not vary with frequency within the spectral bandwidth of the pulses, so that the formulas given in Sect. 1 directly apply.

In order to write $\hat{Y}$ as function of the signal-mode operators, we first represent the states $|F\rangle$ and $|G\rangle$ in the form of

$$|F\rangle = \hat{F}(\hat{b}^\dagger)|0\rangle = \sum_{m=0}^{\infty} F_m \hat{b}^{\dagger m}|0\rangle, \tag{32}$$

$$|G\rangle = \hat{G}(\hat{b}^\dagger)|0\rangle = \sum_{n=0}^{\infty} G_n \hat{b}^{\dagger n}|0\rangle \tag{33}$$

(or with $\hat{b}^\dagger$ being replaced by $\hat{c}^\dagger$) and substitute these expressions into (30). Using (7) and (15), applying the s-ordering rule [18]

$$\{\hat{a}^{\dagger m}\hat{a}^n\}_s = \sum_{k=0}^{\min[m,n]} k!\binom{m}{k}\binom{n}{k}\left(\frac{t-s}{2}\right)^k \times \{\hat{a}^{\dagger m-k}\hat{a}^{n-k}\}_t, \tag{34}$$

and introducing $\hat{n} = \hat{a}^\dagger \hat{a}$, we obtain (see Appendix B)

$$\begin{aligned}\hat{Y}_{\mathrm{A}} &= \langle 0|\hat{G}^\dagger(\hat{b}^\dagger)\hat{U}_{\mathrm{A}}\hat{F}(\hat{b}^\dagger)|0\rangle \\ &= \left\{\hat{G}^\dagger\left(-R^*T^{-1}\hat{a}\right)\hat{F}(P^*R^*\hat{a})\right\}_{s_{\mathrm{A}}}\bar{T}^{*-1}(P\,T\,)^{*-\hat{n}}, \qquad (35)\end{aligned}$$

$$\begin{aligned}\hat{Y}_{\mathrm{C}} &= \langle 0|\hat{G}^\dagger(\hat{c}^\dagger)\hat{U}_{\mathrm{C}}\hat{F}(\hat{c}^\dagger)|0\rangle \\ &= \left\{\hat{G}^\dagger\left(-\mathcal{R}\mathcal{T}^{*-1}\hat{a}^\dagger\right)\hat{F}(\mathcal{P}\mathcal{R}\hat{a}^\dagger)\right\}_{s_{\mathrm{C}}}(\mathcal{P}\mathcal{T})^{\hat{n}}, \qquad (36)\end{aligned}$$

where the respective ordering parameters are given by

$$s_{\mathrm{A}} = (|T|^2+1)\,|R|^{-2}, \qquad (37)$$

$$s_{\mathrm{C}} = (|\mathcal{T}|^2+1)\,|\mathcal{R}|^{-2}. \qquad (38)$$

A consequence of (21)) is $|\mathcal{T}|^2 \le 1 \le |T|^2$, from which we see that $s_{\mathrm{A}}, s_{\mathrm{C}} \ge 1$. Note that the arguments of $\hat{F}$ and $\hat{G}$ in (36) are the adjoints of the corresponding arguments in (35).

Coherent displacements can be separated from the ordering procedure. To see this, we derive from (19) and (20) together with (23) and (24) the transformation formulas for the displacement operators of the signal and idler modes,

$$\begin{aligned}&\hat{U}_{\mathrm{A}}\;\hat{D}_1(\alpha)\hat{D}_2(\beta)\;\hat{U}_{\mathrm{A}}^\dagger \\ &\quad = \hat{D}_1[P(T\alpha - R\beta^*)]\hat{D}_2[P^*(-R\alpha^* + T\beta)], \qquad (39)\end{aligned}$$

$$\begin{aligned}&\hat{U}_{\mathrm{C}}\;\hat{D}_1(\alpha)\hat{D}_2(\beta)\;\hat{U}_{\mathrm{C}}^\dagger \\ &\quad = \hat{D}_1[\mathcal{P}(\mathcal{T}\alpha + \mathcal{R}\beta)]\hat{D}_2[\mathcal{P}(-\mathcal{R}^*\alpha + \mathcal{T}^*\beta)]. \qquad (40)\end{aligned}$$

Combining (39) and (40) with (35) and (36), respectively, yields

$$\begin{aligned}\hat{Y}'_{\mathrm{A}} &= \langle 0|\hat{G}^\dagger(\hat{b}^\dagger)\hat{D}_2^\dagger(\beta)\hat{U}_{\mathrm{A}}\hat{D}_2(\alpha)\hat{F}(\hat{b}^\dagger)|0\rangle \\ &= \hat{D}_1\!\left(\frac{P\alpha^* - T\beta^*}{R^*}\right)\hat{Y}_{\mathrm{A}}\hat{D}_1\!\left(\frac{P^*\beta^* - T^*\alpha^*}{R^*}\right), \qquad (41)\end{aligned}$$

$$\begin{aligned}\hat{Y}'_{\mathrm{C}} &= \langle 0|\hat{G}^\dagger(\hat{c}^\dagger)\hat{D}_2^\dagger(\beta)\hat{U}_{\mathrm{C}}\hat{D}_2(\alpha)\hat{F}(\hat{c}^\dagger)|0\rangle \\ &= \hat{D}_1\!\left(\frac{\mathcal{P}\alpha - \mathcal{T}\beta}{\mathcal{R}^*}\right)\hat{Y}_{\mathrm{C}}\hat{D}_1\!\left(\frac{\mathcal{P}^*\beta - \mathcal{T}^*\alpha}{\mathcal{R}^*}\right), \qquad (42)\end{aligned}$$

i.e., a coherent displacement of the idler mode is equivalent to a corresponding coherent displacement of the signal mode.

Since each trial in Fig. 1 yields a desired measurement outcome only with some probability, a lockable aperture S is needed in order to extract the properly transformed outgoing signal modes and dump the others. The desired output states are thus available at random times. It may however be demanded to provide them at certain times. In this case, a quantum-state memory M has to be used into which the pulses can be fed and released when desired. One possibility to realize M is offered by electromagnetically induced transparency [17]. If, in particular, a pulse train of a certain repetition frequency is required, one may apply, e.g., an array of delays with variable optical path lengths which step by step adjust the waiting periods between the pulses to each other. Besides electromagnetically induced transparency, the cross-Kerr effect should also offer a way to realize a variable optical delay. In the latter case the outgoing signal mode in the state $\hat{\varrho}'_1$ is mixed with a reference mode (index r) prepared in a coherent state $|\alpha\rangle$ at a cross-Kerr medium $\hat{U} = \mathrm{e}^{\mathrm{i}\kappa\hat{n}_1\hat{n}_r}$. For $\kappa \to 0$ and $|\alpha| \to \infty$ with $\kappa|\alpha|^2 = \mathrm{const.}$, the reduced signal state becomes $\hat{\varrho}''_1 = \mathrm{Tr}_r(\hat{U}\hat{\varrho}'_1|\alpha\rangle\langle\alpha|\hat{U}^\dagger) = \mathrm{e}^{\mathrm{i}\kappa|\alpha|^2\hat{n}_1}\hat{\varrho}'_1\mathrm{e}^{-\mathrm{i}\kappa|\alpha|^2\hat{n}_1}$. By choosing $|\alpha|^2$ we can therefore control the refractive index and with it the optical length and time delay caused by the cross-Kerr medium.

3 Preparation of single-mode quantum states

3.1 Displaced photon-adding for generating qubits

In order to illustrate the general results derived in Sect. 2, let us first study the generation of a single qubit. When a parametric amplifier as described in Sect. 1.1 is fed with an idler pulse prepared in a single-mode coherent state $|F\rangle = |\alpha_1\rangle$ and a single photon is detected, $|G\rangle = |1\rangle$, then (35) and (41) yield the non-unitary (conditional) operator

$$\begin{aligned}\hat{Y}_{\mathrm{A}}^{(k)} &= -\,RP^*\bar{T}^{*-1}\hat{D}\big(PR^{*-1}\alpha_k^*\big) \\ &\quad \times\,(P\,T\,)^{*-\hat{n}}\hat{a}^\dagger\hat{D}\big(-T^*R^{*-1}\alpha_k^*\big) \qquad (43)\end{aligned}$$

(the index $k=1$ is introduced for later purposes). Let us further assume that the signal input channel is unused, $\hat{\varrho} = |0\rangle\langle 0|$ (for notational convenience we omit the mode index). The outgoing signal pulse is then prepared in a state

$$\hat{\varrho}' = \frac{1}{p}\hat{Y}_{\mathrm{A}}^{(1)}|0\rangle\langle 0|\hat{Y}_{\mathrm{A}}^{(1)\dagger} = |\Psi\rangle\langle\Psi|, \qquad (44)$$

where

$$|\Psi\rangle = \frac{|0\rangle + q|1\rangle}{\sqrt{1+|q|^2}} \qquad (45)$$

is a superposition of the vacuum and a single-photon Fock state. The parameter $q = -PR/\alpha_1$ can be controlled by varying R or α_1. It is however convenient to choose $|R|$ and $|\alpha_1|$ such that for a desired q the probability

$$p = \|\hat{Y}_{\mathrm{A}}^{(1)}|0\rangle\|^2 = (|R|^2 + |\alpha_1|^2)|T|^{-4}\mathrm{e}^{-|\alpha_1|^2} \qquad (46)$$

$\big(\||\Phi\rangle\| = \sqrt{\langle\Phi|\Phi\rangle}\big)$ of generating the qubit (45) attains a maximum. This is the case for

$$|R|^2 = \frac{\sqrt{(|q|^{-2}+1)^2 + 4|q|^{-2}} - (|q|^{-2}+1)}{2|q|^{-2}}. \qquad (47)$$

The maximum values of p together with the corresponding values of $|R|$ and $|\alpha_1|$ are shown in Fig. 2.

3.2 Repeated displaced photon-adding for generating arbitrary superpositions of Fock states

The scheme can be extended to the generation of an arbitrary superposition of a finite number of Fock states

$$|\Psi\rangle = \sum_{n=0}^{N} |n\rangle\langle n|\Psi\rangle. \qquad (48)$$

Since the states of that type are completely determined by the N zeros of the Q-function, i.e., the N solutions $\beta_1, \cdots, \beta_N$

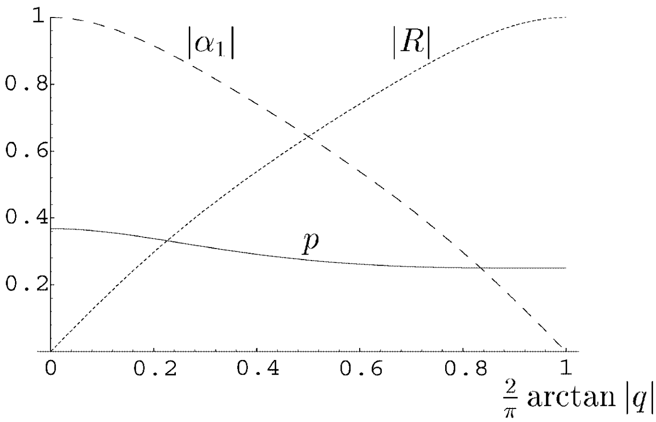

Fig. 2. The maximized probability (46) of generating a qubit (45) and the corresponding values of the parameters $|R|$ and $|\alpha_1|$ as functions of $|q|$

of the equation $\langle\Psi|\beta\rangle = 0$, they can be generated from the vacuum by alternate application of the coherent displacement operator and the creation operator,

$$\begin{aligned}|\Psi\rangle &= \frac{\langle N|\Psi\rangle}{\sqrt{N!}} \prod_{k=1}^{N} (\hat{a}^\dagger - \beta_k^*)|0\rangle \\ &= \frac{\langle N|\Psi\rangle}{\sqrt{N!}} \prod_{k=1}^{N} \left[\hat{D}(\beta_k)\hat{a}^\dagger \hat{D}^\dagger(\beta_k)\right]|0\rangle, \end{aligned} \tag{49}$$

which may be realized repeating the procedure in Sect. 3.1 according to Fig. 3. The pulse prepared in the state (45) is sent back through a ring resonator to the amplifier and used as a signal input. Simultaneously, an idler pulse prepared in a coherent state $|\alpha_2\rangle$ is fed into the second input port of the amplifier. If the detector again registers a single photon, then the outgoing signal pulse is prepared in a state $\sim \hat{Y}_\mathrm{A}^{(2)}\hat{Y}_\mathrm{A}^{(1)}|0\rangle$. Under the condition that in each round trip a single photon is detected, after N round trips the signal pulse is prepared in a state

$$|\Psi\rangle \sim \hat{Y}_\mathrm{A}|0\rangle = \hat{Y}_\mathrm{A}^{(N)} \cdots \hat{Y}_\mathrm{A}^{(2)}\hat{Y}_\mathrm{A}^{(1)}|0\rangle. \tag{50}$$

If the preparation of the desired state has been successful, the mirror M_1 in Fig. 3 can be removed in order to open the cavity and release the pulse. Otherwise the pulse is dumped in order to start the next trial from the very beginning, with the signal-input port of the amplifier being unused.

Inserting (43) and rearranging the operator order such that the photon-creation operators are on the left of the exponential operators, we derive

$$\begin{aligned}\hat{Y}_\mathrm{A}|0\rangle &= \mathrm{e}^{\mathrm{i}\xi}|R|^N|T|^{-N(N+3)/2} \exp\left(-\frac{1}{2}\sum_{k=1}^{N}|\alpha_k|^2\right) \\ &\times \prod_{k=1}^{N}\left[\hat{a}^\dagger - \frac{(P\,T\,)^{*N}}{R}\sum_{l=k}^{N}\frac{P^*\alpha_l - T\alpha_{l+1}}{(P\,T\,)^{*l}}\right]|0\rangle \end{aligned} \tag{51}$$

($\alpha_{N+1} = 0$; $\mathrm{e}^{\mathrm{i}\xi}$ is an irrelevant phase factor). In the derivation of (51) we have used the relations $\hat{D}(\alpha)\, f(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha) = f(\hat{a}-\alpha,\hat{a}^\dagger-\alpha^*)$ and $\alpha^{\hat{n}}\, f(\hat{a},\hat{a}^\dagger)\alpha^{-\hat{n}} = f(\alpha^{-1}\hat{a},\alpha\hat{a}^\dagger)$. Comparing (51) with (49), we see that for

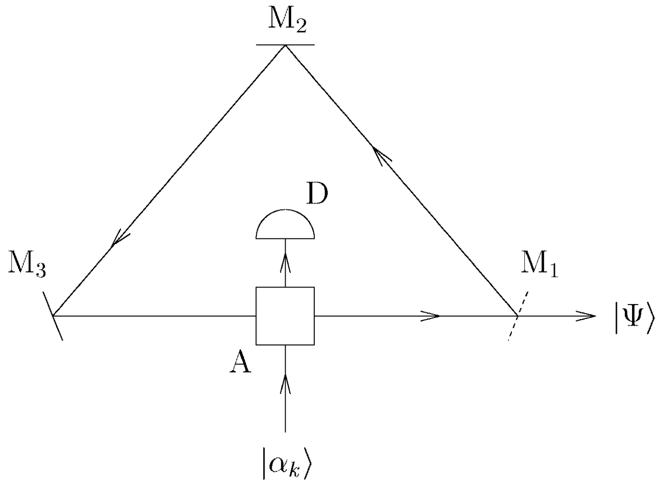

Fig. 3. Scheme for preparing a travelling optical field in a quantum state (48). A parametric amplifier A is fed with a sequence of idler pulses prepared in appropriately chosen coherent states $|\alpha_k\rangle$. The idler pulses arrive at A simultaneously with the produced signal pulse circulating in the ring resonator consisting of mirrors M_2, M_3 and a (removable) mirror M_1. (The pumping of A by a synchronized sequence of pump pulses is not shown in the figure.) The desired quantum state is generated if in each round trip of the signal field the detector D registers a single photon

$$\beta_k = \frac{(P\,T\,)^N}{R^*}\sum_{l=k}^{N}\frac{P\alpha_l^* - T^*\alpha_{l+1}^*}{(P\,T\,)^l} \tag{52}$$

or equivalently

$$\alpha_k = \frac{PR}{(P\,T\,)^k(P\,T\,)^{*N}}\sum_{l=k}^{N}|T|^{2l}(\beta_l^* - \beta_{l+1}^*) \tag{53}$$

($\beta_{N+1} = 0$) the desired state $|\Psi\rangle$ is just realized. The probability of generating the state is given by

$$\begin{aligned} p &= \|\hat{Y}_\mathrm{A}|0\rangle\|^2 \\ &= \frac{N!}{|\langle N|\Psi\rangle|^2}\frac{|R|^{2N}}{|T|^{N(N+3)}}\exp\left(-\sum_{k=1}^{N}|\alpha_k|^2\right), \end{aligned} \tag{54}$$

as is seen by comparing the norms of both sides of (51) and inserting (49). It decreases rapidly with increasing N in general, so that the applicability of this method is effectively restricted to low numbers of round trips.

3.3 Possibility of generating Fock states independently of coincidence measurements

In order to generate a Fock state $|n\rangle$, no feeding with idler modes is necessary, since all the β_k in (49) are zero. According to (54), the probability of detecting one photon at each of the n round trips is

$$p = n!|R|^{2n}|T|^{-n(n+3)} \tag{55}$$

and becomes maximal for fixed n if $|R|^2 = 2/(n+1)$. Substituting this expression into (55) and applying Stirling's formula yields the asymptotic behaviour $p_{\max} \approx a\mathrm{e}^{-bn}$, where

$a=\sqrt{2\pi}/\mathrm{e}$ and $b=2-\ln 2$, i.e., the probability of preparing an n-photon Fock state decreases exponentially with increasing n. Note that exactly the same asymptotic behaviour is observed when the state is generated by conditional measurement at a beam-splitter array [9].

However, the scheme considered here offers the possibility of circumventing the problem of low preparation probability. Since the idler field remains in the vacuum state, $|\alpha_k\rangle = |0\rangle$, the circulating signal pulse is always in a Fock state whose number is simply the sum of detected idler photons. This means that every trial sooner or later results in the desired state $|n\rangle$, provided that the respective n is not skipped, and thus there is no need to wait for a sequence of n consecutive single-photon clicks. The idler detector is simply used for photon book-keeping and the cavity is opened in that moment when the sum of all detected idler photons has reached the desired value n.

3.4 Influence of non-perfect photodetection and non-perfect cavity feedback on Fock-state generation

If the idler detector and the feedback mirrors are not perfect, the situation becomes more complicated. Let us assume that the state of the signal pulse after the jth round trip is a statistical mixture of Fock states described by a density operator $\hat{\varrho}^{(j)}$. This pulse now enters the signal-input port of the parametric amplifier whose idler-input port is unused, $|F\rangle=|0\rangle$. When k outgoing idler photons are detected with efficiency η_{D}, then the state $\hat{\varrho}^{(j)\prime}$ of the outgoing signal field is given by (29), where now

$$\hat{\Pi}(k) = :\frac{\left(\eta_{\mathrm{D}}\hat{n}\right)^k}{k!}\,\mathrm{e}^{-\eta_{\mathrm{D}}\hat{n}}: = \hat{b}_{k\hat{n}}(\eta_{\mathrm{D}}). \tag{56}$$

Here, the symbol : : introduces normal ordering, and

$$b_{kn}(z) = \binom{n}{k} z^k (1-z)^{n-k}. \tag{57}$$

Inserting (56) into (29), we find that the prepared state is a mixture of Fock states with

$$\varrho_{mm}^{(j)\prime} = \frac{1}{|T|^2 p(k)} \sum_l b_{k,m-l}(\eta_{\mathrm{D}}) b_{lm}(|T|^{-2}) \varrho_{ll}^{(j)}, \tag{58}$$

where

$$p(k) = |T|^{-2} \sum_{l,m} b_{k,m-l}(\eta_{\mathrm{D}}) b_{lm}(|T|^{-2}) \varrho_{ll}^{(j)} \tag{59}$$

is the probability of detecting k photons. Next, the pulse prepared in the state $\hat{\varrho}^{(j)\prime}$ is fed back by the mirrors $\mathrm{M}_1, \cdots, \mathrm{M}_3$ into the signal-input port of the amplifier. If the reflectances of the mirrors $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$ are given by $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, the state of the pulse after the $(j+1)$th round trip is still a mixture of Fock states with

$$\varrho_{mm}^{(j+1)} = \sum_l b_{ml}(\eta_{\mathrm{F}}) \varrho_{ll}^{(j)\prime}, \tag{60}$$

where $\eta_{\mathrm{F}} = |\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3|^2$ determines the feedback efficiency. Combining (60) and (58), we obtain the recursion relation

$$\varrho_{mm}^{(j+1)} = \frac{1}{|T|^2 p(k_{j+1})} \sum_{l,n} b_{ml}(\eta_{\mathrm{F}}) \times b_{k_{j+1},l-n}(\eta_{\mathrm{D}}) b_{nl}(|T|^{-2}) \varrho_{nn}^{(j)}. \tag{61}$$

If we start from the vacuum state, $\varrho_{nn}^{(0)} = \delta_{n0}$, and measure the numbers k_j of (outgoing) idler photons detected at the jth round trip, the circulating pulse therefore evolves into a mixture of Fock states, which depends according to (61) on the respective sequence $\{k_1, k_2, \cdots\}$. For this reason, some arbitrariness has to be introduced if the state evolution needs to be simulated.

Let us therefore first consider the evolution of the *mean* photon number of the circulating pulse, which can be obtained by considering the case when no measurement is performed in the idler-output channel. Equation (61) together with $\eta_{\mathrm{D}}=0$ and $k_{j+1}=0$ (and the initial condition $\varrho_{nn}^{(0)}=\delta_{n0}$) then yields a thermal state

$$\hat{\varrho}^{(N)} = \frac{1}{\langle\hat{n}\rangle^{(N)}+1}\, \hat{b}_{0\hat{n}}\!\left(\frac{1}{\langle\hat{n}\rangle^{(N)}+1}\right), \tag{62}$$

where the mean photon number

$$\langle\hat{n}\rangle^{(N)} = \eta_{\mathrm{F}}|R|^2 \frac{\left(\eta_{\mathrm{F}}|T|^2\right)^N - 1}{\eta_{\mathrm{F}}|T|^2 - 1} \tag{63}$$

can be deduced from the (general, compare (19) and (20)) recursion relation

$$\langle\hat{n}\rangle^{(j+1)} = \eta_{\mathrm{F}}\left[|T|^2\langle\hat{n}\rangle^{(j)} + |R|^2\right]. \tag{64}$$

If $\eta_{\mathrm{F}}|T|^2 > 1$, then $\langle\hat{n}\rangle^{(N)}$ increases exponentially with N, while for $\eta_{\mathrm{F}}|T|^2 < 1$ a stationary value is observed, $\lim_{N\to\infty}\langle\hat{n}\rangle^{(N)} = \eta_{\mathrm{F}}|R|^2/(1-\eta_{\mathrm{F}}|T|^2)$. The critical value $\eta_{\mathrm{F}}|T|^2=1$ leads to a linear increase, $\langle\hat{n}\rangle^{(N)}=|R/T|^2 N$.

To give an example, let us now consider the generation of the state $|n=4\rangle$. Since the amplification is typically weak ($|R|^2 \ll 1$), a large number of round trips is likely and a high-quality cavity is demanded. Note that $|R|^2 = {}_s\langle\mathrm{vac}|\hat{a}^\dagger\hat{a}|\mathrm{vac}\rangle_s = {}_s\langle\mathrm{vac}|\hat{b}^\dagger\hat{b}|\mathrm{vac}\rangle_s$ characterizes the mean photon numbers of a two-mode squeezed vacuum $|\mathrm{vac}\rangle_s = \hat{U}_{\mathrm{A}}|0,0\rangle$ that is generated by means of a parametric amplifier. In order to provide a rough estimate of the required efficiency η_{F}, we approximate the number N of cycles needed on average by identifying $\langle\hat{n}\rangle^{(N)}$ in (63) with n and assume a linear increase $\langle\hat{n}\rangle^{(N)} = |R/T|^2 N$ (we have $|T|^2 \approx 1$ and further assume $\eta_{\mathrm{F}} \approx 1$; therefore $\eta_{\mathrm{F}}|T|^2 \approx 1$), so that $N \approx |R|^{-2}n$. We now estimate η_{F} from the requirement that after that number of round trips in an empty resonator (i.e., without the amplifier) an initially present photon can still be found with probability $\frac{1}{2}$; thus $\eta_{\mathrm{F}}=2^{-|R|^2/n}$. For $n=4$ and $|R|^2=3\times10^{-3}$ this yields $\eta_{\mathrm{F}} \approx 0.999$. We insert these quantities into (63) and (arbitrarily) consider the case when the first idler photon is detected if $\eta_{\mathrm{F}}\langle\hat{n}\rangle$ exceeds 1, the second if $\eta_{\mathrm{F}}\langle\hat{n}\rangle$ exceeds 2 and so on, until eventually the cavity is opened after detecting the 4th idler photon. The resulting density matrix is then obtained from (61) together with $\varrho_{nn}^{(0)} = \delta_{n0}$. Examples are shown in Fig. 4. The plots confirm the sensitivity

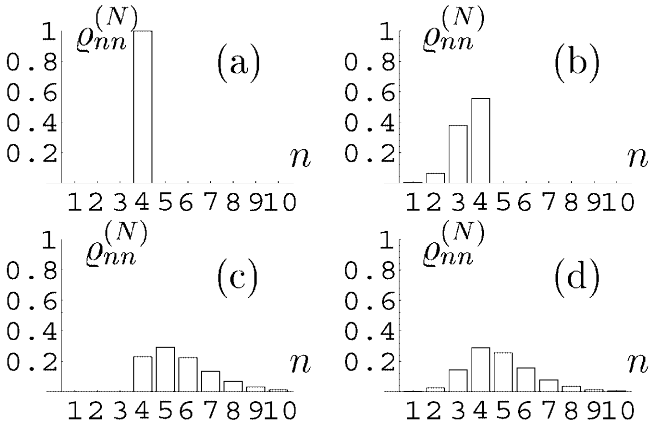

Fig. 4a–d. Computer simulation of the preparation of a desired Fock state $|n=4\rangle$. The density-matrix elements $\varrho_{nn}^{(N)}$ obtained according to (61) are shown for $|R|^2=3\times10^{-3}$ and different feedback efficiencies η_{F} and detection efficiencies η_{D}. The cavity is assumed to be opened after detecting the 4th idler photon, which determines the number of round trips N. **a** $\eta_{\mathrm{F}}=1$ and $\eta_{\mathrm{D}}=1$ ($N=538$), **b** $\eta_{\mathrm{F}}=0.999$ and $\eta_{\mathrm{D}}=1$ ($N=652$), **c** $\eta_{\mathrm{F}}=1$ and $\eta_{\mathrm{D}}=0.7$ ($N=636$), **d** $\eta_{\mathrm{F}}=0.999$ and $\eta_{\mathrm{D}}=0.7$ ($N=788$)

to cavity loss for small $|R|^2$. Note that the Mandel parameter $Q=\langle(\Delta\hat n)^2\rangle/\langle\hat n\rangle-1$ of the mixture plotted in (d) is $Q=-0.527$.

4 Conclusion

In this paper we have studied conditional quantum-state engineering at parametric amplifiers and frequency converters, regarding each apparatus as being effectively a two-port device, whose action in conditional measurement can be described by a non-unitary operator $\hat Y$ defined in the Hilbert space of the signal mode. We have presented $\hat Y$ for arbitrary quantum states of the incoming idler mode and arbitrary detected quantum states of the outgoing idler mode as s-ordered products of the operators that generate the quantum states from the vacuum, s being entirely determined by the device parameters.

To illustrate the results, we have proposed a scheme allowing the generation of arbitrary finite single-mode quantum states of travelling waves by a parametric amplifier equipped with a ring resonator as an optical feedback loop. We have applied the method to the problem of Fock-state preparation, for which we have also addressed the influence of non-perfect photodetection and non-perfect cavity feedback.

Acknowledgements. This work was supported by the Deutsche Forschungsgemeinschaft.

Appendices

Appendix A Derivation of equation (28)

The commutation relation between $\hat{\boldsymbol a}$ and $\hat{\boldsymbol a}^\dagger$ is invariant under a U(p,q) transformation, i.e., for $U^{-1}=G^\dagger U^\dagger G$ we have

$$[(U\hat{\boldsymbol a})_\lambda,(U\hat{\boldsymbol a})^\dagger_\mu]=[\hat{\boldsymbol a}_\lambda,\hat{\boldsymbol a}^\dagger_\mu]=G_{\lambda\mu}. \tag{A.1}$$

After writing

$$\begin{aligned}\hat{\boldsymbol a}\mathrm{e}^{\mathrm{i}\hat{\boldsymbol a}^\dagger H\hat{\boldsymbol a}}&=U^{-1}U\hat{\boldsymbol a}\mathrm{e}^{\mathrm{i}\hat{\boldsymbol a}^\dagger U^\dagger U^{\dagger-1}HU^{-1}U\hat{\boldsymbol a}}\\&=U^{-1}(U\hat{\boldsymbol a})\mathrm{e}^{\mathrm{i}(U\hat{\boldsymbol a})^\dagger(U^{-1\dagger}HU^{-1})(U\hat{\boldsymbol a})},\end{aligned}\tag{A.2}$$

we choose U such that $U^{-1\dagger}HU^{-1}$ becomes diagonal, i.e., $(U^{-1\dagger}HU^{-1})_{\lambda\mu}=(U^{-1\dagger}HU^{-1})_{\lambda\lambda}\delta_{\lambda\mu}$. Now making use of $\alpha^{\hat n}f(\hat a,\hat a^\dagger)\alpha^{-\hat n}=f(\alpha^{-1}\hat a,\alpha\hat a^\dagger)$, we see that the relation $\hat{\boldsymbol a}_\mu\mathrm{e}^{\mathrm{i}\alpha\hat{\boldsymbol a}^\dagger_\lambda\hat{\boldsymbol a}_\lambda}=\mathrm{e}^{\mathrm{i}\alpha\hat{\boldsymbol a}^\dagger_\lambda\hat{\boldsymbol a}_\lambda}\mathrm{e}^{\mathrm{i}G_{\lambda\mu}\alpha}\hat{\boldsymbol a}_\mu$ holds. Inserting it into (A.2) with $\alpha=(U^{-1\dagger}HU^{-1})_{\lambda\lambda}$, we obtain together with (A.1)

$$\begin{aligned}\hat{\boldsymbol a}\mathrm{e}^{\mathrm{i}\hat{\boldsymbol a}^\dagger H\hat{\boldsymbol a}}&=U^{-1}(U\hat{\boldsymbol a})\prod_\lambda\mathrm{e}^{\mathrm{i}(U^{-1\dagger}HU^{-1})_{\lambda\lambda}(U\hat{\boldsymbol a})^\dagger_\lambda(U\hat{\boldsymbol a})_\lambda}\\&=U^{-1}\mathrm{e}^{\mathrm{i}\hat{\boldsymbol a}^\dagger H\hat{\boldsymbol a}}\mathrm{e}^{\mathrm{i}G(U^{-1\dagger}HU^{-1})}(U\hat{\boldsymbol a})\\&=\mathrm{e}^{\mathrm{i}\hat{\boldsymbol a}^\dagger H\hat{\boldsymbol a}}U^{-1}\mathrm{e}^{\mathrm{i}GU^{-1\dagger}HU^{-1}}U\hat{\boldsymbol a}\\&=\mathrm{e}^{\mathrm{i}\hat{\boldsymbol a}^\dagger H\hat{\boldsymbol a}}\mathrm{e}^{\mathrm{i}GH}\hat{\boldsymbol a}.\end{aligned}\tag{A.3}$$

Note that $\mathrm{e}^{\mathrm{i}GH}$ itself is just a U(p,q) matrix, because $(\mathrm{e}^{\mathrm{i}GH})^{-1}=G^\dagger(\mathrm{e}^{\mathrm{i}GH})^\dagger G$.

Appendix B Derivation of equation (35)

Applying (19), we first write

$$\hat b^n\hat U_{\mathrm{A}}=\hat U_{\mathrm{A}}(\hat U_{\mathrm{A}}^\dagger\hat b^\dagger\hat U_{\mathrm{A}})^{\dagger n}=\hat U_{\mathrm{A}}(-P^*R\hat a^\dagger+P^*T\hat b)^n. \tag{B.1}$$

Using $\alpha^{\hat n}f(\hat a,\hat a^\dagger)\alpha^{-\hat n}=f(\alpha^{-1}\hat a,\alpha\hat a^\dagger)$, we see from (7) that for $|0\rangle\equiv|0\rangle_2$

$$\langle0|\hat U_{\mathrm{A}}=\bar T^{*-1}(P\,T)^{*-\hat a^\dagger\hat a}\langle0|\mathrm{e}^{\frac{R^*}{T^*}\hat a\hat b}. \tag{B.2}$$

With the help of (B.1) and (B.2) we now get

$$\begin{aligned}\hat Y_{\mathrm{A}}&=\sum_{m,n}F_mG_n^*\langle0|\hat b^n\hat U_{\mathrm{A}}\hat b^{\dagger m}|0\rangle\\&=\sum_{m,n,k}F_mG_n^*\bar T^{*-1}(P\,T)^{*-\hat a^\dagger\hat a}\langle0|\mathrm{e}^{\frac{R^*}{T^*}\hat a\hat b}\\&\quad\times\binom{n}{k}(P^*T\hat b)^k(-P^*R\hat a^\dagger)^{n-k}\hat b^{\dagger m}|0\rangle\\&=\sum_{m,n,k}F_mG_n^*\bar T^{*-1}(P\,T)^{*-\hat a^\dagger\hat a}k!\binom{m}{k}\binom{n}{k}(P^*T)^k\\&\quad\times(R^*T^{*-1}\hat a)^{m-k}(-P^*R\hat a^\dagger)^{n-k}\\&=\sum_{m,n,k}F_mG_n^*\bar T^{*-1}k!\binom{m}{k}\binom{n}{k}(P^*T)^k\\&\quad\times(P^*R^*\hat a)^{m-k}(-RT^{*-1}\hat a^\dagger)^{n-k}(P\,T)^{*-\hat n}\\&=\sum_{m,n}F_mG_n^*(P^*R^*)^m(-RT^{*-1})^n\\&\quad\times\sum_kk!\binom{m}{k}\binom{n}{k}\left(-\left|\frac{T}{R}\right|^2\right)^k\hat a^{m-k}\hat a^{\dagger n-k}\\&\quad\times\bar T^{*-1}(P\,T)^{*-\hat n}.\end{aligned}\tag{B.3}$$

Applying (34) with $t=-1$ (denoting antinormal order) and $s=s_{\mathrm{A}}$ gives

$$
\begin{aligned}
\hat{Y}_{\mathrm{A}} &= \sum_{m,n} F_m G_n^* (P^* R^*)^m (-RT^{*-1})^n \\
&\quad \times \{\hat{a}^{\dagger n}\hat{a}^m\}_{s_{\mathrm{A}}} \bar{T}^{*-1} (P\ T\ )^{*-\hat{n}} \\
&= \left\{ \sum_{m,n} G_n^* (-R^* T^{-1}\hat{a})^{\dagger n} F_m (P^* R^* \hat{a})^m \right\}_{s_{\mathrm{A}}} \\
&\quad \times \bar{T}^{*-1} (P\ T\ )^{*-\hat{n}}, \qquad \text{(B.4)}
\end{aligned}
$$

from which (35) directly follows. Equation (36) can be derived analogously.

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