



Emergence of bipartite and tripartite entanglement in a double cavity optomechanical system

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Abstract

In this work, the study focuses on an optomechanical system composed of two Fabry–Pérot cavities each pumped by a coherent laser source, and coupled by photon hopping process, phonon tunneling and by a squeezed light source. We theoretically investigate the impact of increasing the squeezing parameter of squeezed light on the sharing of quantum correlations between two mechanical modes and an optical mode, then between two optical modes and a mechanical mode using logarithmic negativity. Then, we study the effect of different couplings on the emergence and degree of tripartite entanglement by tripartite negativity. As a result, the entanglement shared in each considered three-mode block is monogamous and during the evolution the one-mode versus one-mode entanglement always remains less than one-mode versus two-mode entanglement. Moreover, the combined effect of different couplings can enhance tripartite entanglement in certain conditions as it can degrade it in other ones. Our results indicate that the proposal systems could provide an important platform for the study of macroscopic quantum phenomena.

1 Introduction

In recent years, thanks to its important properties and numerous advantages, quantum entanglement has been widely exploited in quantum systems. It is well-known that entanglement is a purely quantum property which has no equivalent in classical physics [1], manifests itself more particularly in microscopic systems [13–23]. In this respect, it represents a non-local form of quantum correlations between spatially separated systems. Besides entanglement, quantum correlations manifest in many other forms [2], such as Bell non-locality [3] quantum steering [4, 5] and quantum discord [6, 7]. In this respect, they are the main pillar of quantum information processing and computational tasks [8], quantum cryptography [9], quantum teleportation [10], telecloning [11] and quantum metrology [12]. Moreover, entanglement has been widely studied in Fabry–Pérot

cavity-based optomechanical systems [24–45], which are useful and efficient to study the interaction between light and mechanical oscillators [46, 47], and, therefore, allow to share the entanglement between cavity modes and mechanical resonators.

Cavity quantum optomechanics is a recently developed field of research that promises many applications in quantum information; It can be used to perform precision measurements [48, 49] and also generate quantum superpositions [50]. The optomechanical cavity of Fabry–Pérot type consists of two spatially separated mirrors, one of the two mirrors is fixed and semi-transparent, while the other is mobile and perfectly reflective [51]. The device made it possible to highlight the quantum effects of the mutual interaction by radiation pressure, also called optomechanical coupling, between electromagnetic radiation and the mechanical vibrations of the mobile mirror [52]. The optical modes of the cavity are controlled by a laser beam, the part of which transmitted inside the cavity will act on the mobile mirror by the force of the radiation pressure, which causes its movement (mechanical mode). On the other hand, the movement of the mirror causes a modification of the length of the cavity and, therefore, involves the modulation of the phase of the laser light in the cavity. It is the combination of these two processes (movement of the mirror and modification of the phase of the cavity field) which can generate various

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optomechanical phenomena (entanglement, cooling, transparency, etc.). Some works have used two Fabry–Pérot cavities coupled via photon hopping process and by two-mode squeezed light [53], while others have considered two Fabry–Pérot cavities coupled by photon hopping process and phonon tunneling process [54].

Unlike classical correlations, the sharing of correlations in multipartite quantum systems is governed by monogamy law [55, 56], i.e., quantum correlations cannot be freely shared between different parts of the system. Therefore, for example, in a tripartite quantum system, if two parts are maximally entangled, there is no shared quantum correlation with the remaining third part [57–62]. Monogamy plays an important role in quantum information protocols such as quantum cryptography [63] and entanglement distillation [64] as well as in the characterization of multipartite quantum systems [65–73]. In particular, tripartite entanglement has great importance in the area of quantum information, so it seems reasonable to recall tripartite entanglement quantifiers in multimode Gaussian states like tripartite negativity [74], Residual contangle and CKW–monogamy inequality [75].

In the present work, we consider a double-cavity optomechanical system where we explore intracavity quantum correlations in terms of phonon tunneling strength and photon hopping strength to find an appropriate choice of these two coupling strengths to enhance quantum correlations. The two cavities are also coupled by squeezed light; therefore, the generated quantum correlations will also be affected by the squeezed light.

This paper is organized as follows. In Sect. 2, we present the optomechanical model of the system under consideration, we give the effective Hamiltonian for this system, then, we derive the non linear quantum Langevin equations. In Sect. 3, we linearize the quantum Langevin equations describing the dynamics of the quadrature fluctuations, then, we develop the dynamics in the rotating wave approximation. In Sect. 4, we derive the explicit formula for the covariance matrix of the two–three-mode states. We present in Sect. 5 three different quantum measures to explore quantum correlations. In Sect. 6, we discuss the numerical results of different quantum measures. Finally, our concluding remarks are given in Sect. 7.

2 Model

We present in Fig. 1 an illustration of the considered hybrid optomechanical system consisting of two Fabry–Pérot cavities k ($k = 1, 2$) coupled to a two mode squeezed light (SL). Each of both cavities have movable end mirror M_k and pumped by coherent laser source (CLS). Inside each cavity,

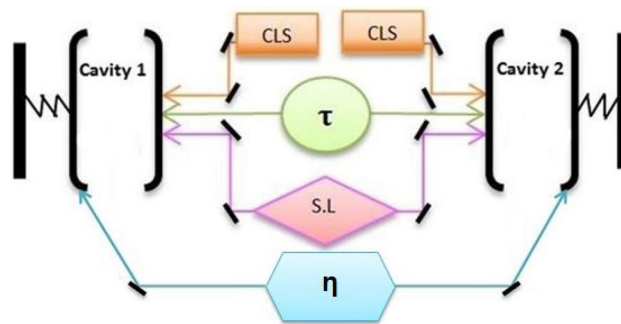


Fig. 1 Schematic of two cavities 1(2). Both cavities are pumped by coherent light source (CLS) and coupled to a two-mode squeezed light (SL). The different couplings strength between photons (τ) and phonons (η)

the mechanical mode b_k characterized by a frequency ω_{b_k} is coupled to the corresponding optical mode a_k by the optomechanical coupling strength, i.e., radiation pressure, at a rate $g_k = \frac{\omega_{a_k}}{L_k} \sqrt{\frac{\hbar}{m_k \omega_{b_k}}}$ [46], where L_k and ω_{a_k} are, respectively, the length and the frequency of k th cavity M_k ; ($k = 1, 2$) and m_k is the mass of the movable mirror. Moreover, we consider two other coupling strengths between the two cavities, namely, the coupling between the optical modes a_k ($k = 1, 2$) by a photon hopping process at rate τ , and the coupling strength between the mechanical modes b_k ($k = 1, 2$) via a phonon tunneling process at rate η [76] assuming that phonon and photon tunneling can be achieved between two neighboring Fabry–Pérot cavities due to the finite overlap between their modes. Thereafter, we denote the operators of annihilation (creation) of the k th optical and mechanical modes, respectively, $a_k(a_k^+)$ and $b_k(b_k^+)$ which satisfy the usual commutation relations: $[a_k, a_k^+] = 1$ and $[b_k, b_k^+] = 1$.

The Hamiltonian governing the system at hand, in a rotating frame at the lasers frequency ω_{L_k} , reads ($\hbar = 1$):

$$H = \sum_{k=1}^2 \left[\omega_{b_k} \left(b_k^+ b_k + \frac{1}{2} \right) - \Delta_k a_k^+ a_k - g_k a_k^+ a_k (b_k^+ + b_k) + (a_k^+ E_k e^{i\varphi_k} + a_k E_k e^{-i\varphi_k}) \right] - \tau (a_1^+ a_2 + a_2^+ a_1) - \eta (b_1^+ b_2 + b_2^+ b_1) \tag{1}$$

where $E_k = \sqrt{\frac{2\mu_k \rho_k}{\hbar \omega_{L_k}}}$ and φ_k ; ($k = 1, 2$) are, respectively, the amplitude and the phase of the input coherent laser field, μ_k is the damping rate of k th cavity and ρ_k is the drive pump power of k th input field.

The terms $\sum_{k=1}^2 \omega_{b_k} \left(b_k^+ b_k + \frac{1}{2} \right)$ and $-\sum_{k=1}^2 \Delta_k a_k^+ a_k$ represent the free Hamiltonians describing, respectively, the mechanical modes and the optical modes, the term $-g_k a_k^+ a_k (b_k^+ + b_k)$ denotes the optomechanical interaction via radiation pressure, the optical driving of the system is described by the term

$\sum_{k=1}^2 (a_k^\dagger E_k e^{i\varphi_k} + a_k E_k e^{-i\varphi_k})$ and the coupling between the two mechanical modes (two optical modes) through a phonon tunneling (photon hopping) reads as $\eta(b_1^\dagger b_2 + b_2^\dagger b_1)$ ($\tau(a_1^\dagger a_2 + a_2^\dagger a_1)$).

The non-linear quantum Langevin equations of the mechanical and optical modes can be written as

$$\frac{\partial b_1}{\partial t} = -\left(i\omega_{b_1} + \frac{\lambda_1}{2}\right)b_1 + ig_1 a_1^\dagger a_1 + i\eta b_2 + \sqrt{\lambda_1} b_1^{in} \tag{2}$$

$$\frac{\partial b_2}{\partial t} = -\left(i\omega_{b_2} + \frac{\lambda_2}{2}\right)b_2 + ig_2 a_2^\dagger a_2 + i\eta b_1 + \sqrt{\lambda_2} b_2^{in} \tag{3}$$

$$\frac{\partial a_1}{\partial t} = \left(i\Delta_1 - \frac{\mu_1}{2}\right)a_1 + ig_1 a_1 (b_1^\dagger + b_1) + i\tau a_2 - iE_1 e^{i\varphi_1} + \sqrt{\mu_1} a_1^{in} \tag{4}$$

$$\frac{\partial a_2}{\partial t} = \left(i\Delta_2 - \frac{\mu_2}{2}\right)a_2 + ig_2 a_2 (b_2^\dagger + b_2) + i\tau a_1 - iE_2 e^{i\varphi_2} + \sqrt{\mu_2} a_2^{in} \tag{5}$$

with b_k^{in} being the k th noise operator representing the coupling between the mechanical mode and its own environment and a_k^{in} is the k th squeezed vacuum operator. λ_k and $\Delta_k = \omega_{L_k} - \omega_{a_k}$ are, respectively, the k th mechanical damping rate and the k th laser detuning. Assuming that $\omega_{b_k} \gg \lambda_k$ and $\Delta_k \gg \mu_k$, we can apply the rotating wave approximation. In the Markovian approximation, the operator b_k^{in} is characterized by the following correlation functions [77, 78]:

$$\langle b_k^{in}(t)b_k^{in\dagger}(t') \rangle = (\hat{n}_k + 1)\delta(t - t') \tag{6}$$

$$\langle b_k^{in\dagger}(t)b_k^{in}(t') \rangle = \hat{n}_k \delta(t - t') \tag{7}$$

where $\hat{n}_k = \left[\exp\left(\frac{\hbar\omega_{b_k}}{K_B T_k}\right) - 1\right]^{-1}$ is the mean number of thermal phonons at frequency ω_{b_k} of the k th mechanical mode, K_B and T_k are, respectively, the Boltzmann constant and the k th thermal bath temperature.

The squeezed vacuum operators a_k^{in} and $a_k^{in\dagger}$ satisfy the following correlation functions [79]:

$$\langle a_k^{in}(t)a_k^{in\dagger}(t') \rangle = (N + 1)\delta(t - t') \tag{8}$$

$$\langle a_k^{in\dagger}(t)a_k^{in}(t') \rangle = N\delta(t - t') \tag{9}$$

$$\langle a_k^{in}(t)a_k^{in}(t') \rangle = M e^{-i\omega_b(t+t')} \delta(t - t') \quad k \neq k' \tag{10}$$

$$\langle a_k^{in\dagger}(t)a_k^{in\dagger}(t') \rangle = M e^{i\omega_b(t+t')} \delta(t - t') \quad k \neq k' \tag{11}$$

we have assumed that $\omega_{b_1} = \omega_{b_2} = \omega_b$, N and M depend on the squeezing parameter characterizing the squeezed light r , with $N = \sinh^2 r$ and $M = \sinh r \cosh r$.

3 Linearization of quantum Langevin equations

Since the non-linearity of the Langevin quantum equations (2)–(5) makes their analytical resolution very difficult, we proceed to the linearization scheme consisting in decomposing each operator as follows [75]:

$$b_k = \langle b_k \rangle + \delta b_k \quad a_k = \langle a_k \rangle + \delta a_k \quad (k = 1, 2)$$

where δb_k and δa_k are the fluctuations operators. The steady-state mean values denoted $\langle b_k \rangle$ and $\langle a_k \rangle$ can be read as

$$\langle b_1 \rangle = \frac{ig_1(i\omega_{b_2} + \frac{\lambda_2}{2})|\langle a_1 \rangle|^2 - \eta g_2 |\langle a_2 \rangle|^2}{(i\omega_{b_1} + \frac{\lambda_1}{2})(i\omega_{b_2} + \frac{\lambda_2}{2}) + \eta^2} \tag{12}$$

$$\langle b_2 \rangle = \frac{ig_2(i\omega_{b_1} + \frac{\lambda_1}{2})|\langle a_2 \rangle|^2 - \eta g_1 |\langle a_1 \rangle|^2}{(i\omega_{b_2} + \frac{\lambda_2}{2})(i\omega_{b_1} + \frac{\lambda_1}{2}) + \eta^2} \tag{13}$$

$$\langle a_1 \rangle = \frac{i\beta_2 E_1 e^{i\varphi_1} + \tau E_2 e^{i\varphi_2}}{\beta_1 \beta_2 + \tau^2} \tag{14}$$

$$\langle a_2 \rangle = \frac{i\beta_1 E_2 e^{i\varphi_2} + \tau E_1 e^{i\varphi_1}}{\beta_2 \beta_1 + \tau^2}; \tag{15}$$

with $\beta_k = -\frac{\mu}{2} + i\Delta'_k$ and Δ'_k is the effective cavity detuning $\{\Delta'_k = \Delta_k + g_k(\langle b_k \rangle^* + \langle b_k \rangle)\}$. Now, we deal with identical cavities ($m_1 = m_2 = m; g_1 = g_2 = g; \omega_{b_1} = \omega_{b_2} = \omega_b; \omega_{a_1} = \omega_{a_2} = \omega_a; \omega_{L_1} = \omega_{L_2} = \omega_L; \lambda_1 = \lambda_2 = \lambda; \mu_1 = \mu_2 = \mu$), equal temperature ($T_1 = T_2 = T$); for $E_1 = E_2 = E$, $\beta_1 = \beta_2 = \beta$, $\varphi_1 = \varphi_2 = \varphi$, one has $\langle a \rangle = -i \frac{E e^{i\varphi}}{\frac{\mu}{2} - i(\Delta' + \tau)}$. We take $\varphi = -\arctan[2(\Delta' + \tau)/\mu]$ and $\langle a \rangle = -i|\langle a \rangle|$. In this case, the many-photon optomechanical coupling inside each cavity is [46]:

$$\Lambda = g|\langle a \rangle| = \frac{\omega_a}{L} \sqrt{\frac{2\mu\varrho}{m\omega_b\omega_L[(\Delta' + \tau)^2 + \mu^2/4]}} \tag{16}$$

The linearized QLES describing the dynamics of the quadrature fluctuations are given by

$$\delta\dot{b}_1 = -\left(i\omega_b + \frac{\lambda}{2}\right)\delta b_1 + \Lambda(\delta a_1^+ - \delta a_1) + i\eta\delta b_2 + \sqrt{\lambda}b_1^{in} \tag{17}$$

$$\delta\dot{b}_2 = -\left(i\omega_b + \frac{\lambda}{2}\right)\delta b_2 + \Lambda(\delta a_2^+ - \delta a_2) + i\eta\delta b_1 + \sqrt{\lambda}b_2^{in} \tag{18}$$

$$\delta\dot{a}_1 = \beta\delta a_1 + \Lambda(\delta b_1^+ + \delta b_1) + i\tau\delta a_2 + \sqrt{\mu}a_1^{in} \tag{19}$$

$$\delta\dot{a}_2 = \beta\delta a_2 + \Lambda(\delta b_2^+ + \delta b_2) + i\tau\delta a_1 + \sqrt{\mu}a_2^{in} \tag{20}$$

We introduce the operators: $\delta b_k(t) = \delta \tilde{b}_k(t)e^{-i\omega_b t}$, $\delta a_k(t) = \delta \tilde{b}_k(t)e^{i\Delta' t}$, $\tilde{a}_k^{in} = e^{-i\Delta' t}a_k^{in}$, $\tilde{b}_k^{in} = e^{i\omega_b t}b_k^{in}$ ($k = 1, 2$) and by assuming that $\omega_b \gg \mu$, and $\Delta' = -\omega_b$, we can neglect the rotating terms at $\pm\omega_b$. This allows us rewrite Eqs. (17)–(20) in a frame rotating wave approximation (RWA) [46, 80] as

$$\delta\dot{\tilde{b}}_1(t) = -\frac{\lambda}{2}\delta \tilde{b}_1(t) - \Lambda\delta \tilde{a}_1(t) + i\eta\delta \tilde{b}_2(t) + \sqrt{\lambda}\delta \tilde{b}_1^{in} \tag{21}$$

$$\delta\dot{\tilde{b}}_2(t) = -\frac{\lambda}{2}\delta \tilde{b}_2(t) - \Lambda\delta \tilde{a}_2(t) + i\eta\delta \tilde{b}_1(t) + \sqrt{\lambda}\delta \tilde{b}_2^{in} \tag{22}$$

$$\delta\dot{\tilde{a}}_1(t) = -\frac{\mu}{2}\delta \tilde{a}_1(t) + \Lambda\delta \tilde{b}_1(t) + i\tau\delta \tilde{a}_2(t) + \sqrt{\mu}\delta \tilde{a}_1^{in} \tag{23}$$

$$\delta\dot{\tilde{a}}_2(t) = -\frac{\mu}{2}\delta \tilde{a}_2(t) + \Lambda\delta \tilde{b}_2(t) + i\tau\delta \tilde{a}_1(t) + \sqrt{\mu}\delta \tilde{a}_2^{in} \tag{24}$$

4 Covariance matrix

To study the quantum entanglement between different tripartite subsystems in the suggested scheme, we consider the following steps to get the steady state of the correlation matrix of quantum fluctuations.

We define the quadrature operators for the two mechanical and optical modes as

$$\begin{aligned} \delta\tilde{q}_{b_k} &= \frac{\delta\tilde{b}_k^+ + \delta\tilde{b}_k}{\sqrt{2}}; \\ \delta\tilde{Y}_{b_k} &= \frac{\delta\tilde{b}_k - \delta\tilde{b}_k^+}{i\sqrt{2}}; \\ \delta\tilde{q}_{a_k} &= \frac{\delta\tilde{a}_k^+ + \delta\tilde{a}_k}{\sqrt{2}}; \\ \delta\tilde{Y}_{a_k} &= \frac{\delta\tilde{a}_k - \delta\tilde{a}_k^+}{i\sqrt{2}}; (k = 1, 2) \end{aligned} \tag{25}$$

Therefore, we get:

$$\delta\dot{\tilde{q}}_{b_1} = \frac{-\lambda}{2}\delta\tilde{q}_{b_1} - \Lambda\delta\tilde{q}_{a_1} - \eta\delta\tilde{Y}_{b_2} + \sqrt{\lambda}\tilde{q}_{b_1}^{in} \tag{26}$$

$$\delta\dot{\tilde{q}}_{b_2} = \frac{-\lambda}{2}\delta\tilde{q}_{b_2} - \Lambda\delta\tilde{q}_{a_2} - \eta\delta\tilde{Y}_{b_1} + \sqrt{\lambda}\tilde{q}_{b_2}^{in} \tag{27}$$

$$\delta\dot{\tilde{Y}}_{b_1} = \frac{-\lambda}{2}\delta\tilde{Y}_{b_1} - \Lambda\delta\tilde{Y}_{a_1} + \eta\delta\tilde{q}_{b_2} + \sqrt{\lambda}\tilde{Y}_{b_1}^{in} \tag{28}$$

$$\delta\dot{\tilde{Y}}_{b_2} = \frac{-\lambda}{2}\delta\tilde{Y}_{b_2} - \Lambda\delta\tilde{Y}_{a_2} + \eta\delta\tilde{q}_{b_1} + \sqrt{\lambda}\tilde{Y}_{b_2}^{in} \tag{29}$$

$$\delta\dot{\tilde{q}}_{a_1} = \frac{-\mu}{2}\delta\tilde{q}_{a_1} + \Lambda\delta\tilde{q}_{b_1} - \tau\delta\tilde{Y}_{a_2} + \sqrt{\mu}\tilde{q}_{a_1}^{in} \tag{30}$$

$$\delta\dot{\tilde{q}}_{a_2} = \frac{-\mu}{2}\delta\tilde{q}_{a_2} + \Lambda\delta\tilde{q}_{b_2} - \tau\delta\tilde{Y}_{a_1} + \sqrt{\mu}\tilde{q}_{a_2}^{in} \tag{31}$$

$$\delta\dot{\tilde{Y}}_{a_1} = \frac{-\mu}{2}\delta\tilde{Y}_{a_1} + \Lambda\delta\tilde{Y}_{b_1} + \tau\delta\tilde{q}_{a_2} + \sqrt{\mu}\tilde{Y}_{a_1}^{in} \tag{32}$$

$$\delta\dot{\tilde{Y}}_{a_2} = \frac{-\mu}{2}\delta\tilde{Y}_{a_2} + \Lambda\delta\tilde{Y}_{b_2} + \tau\delta\tilde{q}_{a_1} + \sqrt{\mu}\tilde{Y}_{a_2}^{in} \tag{33}$$

We can rewrite Eqs. (26)–(34) in the following compact matrix form [81]:

$$\dot{\theta}(t) = \sigma\theta(t) + \nu(t) \tag{34}$$

where $\theta^T(t) = (\delta\tilde{q}_{b_1}, \delta\tilde{Y}_{b_1}, \delta\tilde{q}_{b_2}, \delta\tilde{Y}_{b_2}, \delta\tilde{q}_{a_1}, \delta\tilde{Y}_{a_1}, \delta\tilde{q}_{a_2}, \delta\tilde{Y}_{a_2})$ is the quadrature vector,

$v^T(t) = (\tilde{q}_{b_1}^{in}, \tilde{Y}_{b_1}^{in}, \tilde{q}_{b_2}^{in}, \tilde{Y}_{b_2}^{in}, \tilde{q}_{a_1}^{in}, \tilde{Y}_{a_1}^{in}, \tilde{q}_{a_2}^{in}, \tilde{Y}_{a_2}^{in})$. The matrix σ takes the form:

$$\sigma = \begin{bmatrix} -\frac{\lambda}{2} & 0 & 0 & -\eta & -\Lambda & 0 & 0 & 0 \\ 0 & -\frac{\lambda}{2} & \eta & 0 & 0 & -\Lambda & 0 & 0 \\ 0 & -\eta & -\frac{\lambda}{2} & 0 & 0 & 0 & -\Lambda & 0 \\ \eta & 0 & 0 & -\frac{\lambda}{2} & 0 & 0 & 0 & -\Lambda \\ \Lambda & 0 & 0 & 0 & -\frac{\mu}{2} & 0 & 0 & -\tau \\ 0 & \Lambda & 0 & 0 & 0 & -\frac{\mu}{2} & \tau & 0 \\ 0 & 0 & \Lambda & 0 & 0 & -\tau & -\frac{\mu}{2} & 0 \\ 0 & 0 & 0 & \Lambda & \tau & 0 & 0 & -\frac{\mu}{2} \end{bmatrix} \quad (35)$$

The steady state covariance matrix can be derived directly by solving the Lyapunov equation [82, 83]:

$$\sigma \Pi + \Pi \sigma^T + \lambda = 0 \quad (36)$$

where λ is the stationary noise matrix defined by: $\lambda_{pq} \delta(t-t') = \frac{1}{2}(\langle v_p(t)v_q(t') + v_q(t')v_p(t) \rangle)$, which is written as

$$\lambda = \begin{bmatrix} \lambda' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu' & 0 & M\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu' & 0 & -M\mu & 0 \\ 0 & 0 & 0 & 0 & M\mu & 0 & \mu' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -M\mu & 0 & \mu' & 0 \end{bmatrix} \quad (37)$$

where $\lambda' = \lambda(n + \frac{1}{2})$ and $\mu' = \mu(N + \frac{1}{2})$. The covariance matrix describing the two mechanical modes and the two optical modes can be written in $(\delta\tilde{q}_{b_1}, \delta\tilde{Y}_{b_1}, \delta\tilde{q}_{b_2}, \delta\tilde{Y}_{b_2}, \delta\tilde{q}_{a_1}, \delta\tilde{Y}_{a_1}, \delta\tilde{q}_{a_2}, \delta\tilde{Y}_{a_2})$ basis as

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & 0 & \Pi_4 & \Pi_5 & \Pi_6 & \Pi_7 \\ \Pi_2 & \Pi_1 & 0 & -\Pi_3 & \Pi_5 & \Pi_4 & -\Pi_7 & -\Pi_6 \\ \Pi_3 & 0 & \Pi_1 & \Pi_2 & \Pi_6 & \Pi_7 & \Pi_4 & \Pi_5 \\ 0 & -\Pi_3 & \Pi_2 & \Pi_1 & -\Pi_7 & -\Pi_6 & \Pi_5 & \Pi_4 \\ \Pi_4 & \Pi_5 & \Pi_6 & -\Pi_7 & \Pi_{10} & \Pi_8 & \Pi_9 & 0 \\ \Pi_5 & \Pi_4 & \Pi_7 & -\Pi_6 & \Pi_8 & \Pi_{10} & 0 & -\Pi_9 \\ \Pi_6 & -\Pi_7 & \Pi_4 & \Pi_5 & \Pi_9 & 0 & \Pi_{10} & \Pi_8 \\ \Pi_7 & -\Pi_6 & \Pi_5 & \Pi_4 & 0 & -\Pi_9 & \Pi_8 & \Pi_{10} \end{bmatrix} \quad (38)$$

where the different elements of this matrix are

$$\Pi_1 = \frac{(1 + 2n)(\mu^2 + \mu(2 + C)\lambda + (1 + C)\lambda^2 + 4(\Gamma\mu - \chi\mu^2)) + \mu C(\mu + \lambda) \cosh(2r)}{2(\mu^2(1 + C) + 2\mu(1 + C)\lambda + (1 + C)\lambda^2 + 4(\Gamma\mu - \chi\mu^2))} \quad (39)$$

$$\Pi_2 = \frac{\mu^2 C \lambda (\mu^3 \Gamma + \mu(2 + C)\lambda(\Gamma\mu + \chi\mu) + \chi\mu(\lambda^2 - 4\Gamma\mu(\Gamma\mu + \chi\mu))) \sinh(2r)}{(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma\mu)^2) - 8C\lambda\Gamma\chi\mu^3 + 4(\lambda^2 + 4(\Gamma\mu)^2)(\chi\mu^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma\mu + \chi\mu)^2))} \quad (40)$$

$$\Pi_3 = \frac{\mu^2 C \lambda (\mu^2(1 + C)\lambda + \mu(1 + C)\lambda^2 - 4\lambda\chi\mu(2\Gamma\mu + \chi\mu) - 4\Gamma\mu^2(\Gamma\mu + 2\chi\mu)) \sinh(2r)}{2(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma\mu)^2) - 8C\lambda\Gamma\chi\mu^3 + 4(\lambda^2 + 4(\Gamma\mu)^2)(\chi\mu^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma\mu + \chi\mu)^2))} \quad (41)$$

$$\Pi_4 = -\frac{\sqrt{\mu C \lambda}(\mu + \lambda)(-1 - 2n + \cosh(2r))}{2(\mu^2(1 + C) + 2\mu(1 + C)\lambda + (1 + C)\lambda^2 + 4(\Gamma\mu - \chi\mu^2))} \quad (42)$$

$$\Pi_5 = \frac{\mu\sqrt{\mu C \lambda}(C\lambda(\Gamma\mu^3 - \lambda\chi\mu^2) - (\lambda^2 + 4(\Gamma\mu)^2)(\lambda\chi\mu + \mu^2(\Gamma + 2\chi))) \sinh(2r)}{(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma\mu)^2) - 8C\lambda\Gamma\chi\mu^3 + 4(\lambda^2 + 4(\Gamma\mu)^2)(\chi\mu^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma\mu + \chi\mu)^2))} \quad (43)$$

$$\Pi_6 = -\frac{\mu\sqrt{\mu C \lambda}(\mu C \lambda(\mu\lambda + \lambda^2 + 4\Gamma\mu^2(\Gamma + \chi)) + (\lambda^2 + 4(\Gamma\mu)^2)(\mu^2 + \mu\lambda - 4\chi\mu^2(\Gamma + \chi))) \sinh(2r)}{2(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma\mu)^2) - 8C\lambda\Gamma\chi\mu^3 + 4(\lambda^2 + 4(\Gamma\mu)^2)(\chi\mu^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma\mu + \chi\mu)^2))} \quad (44)$$

$$\Pi_7 = \frac{\sqrt{\mu C \lambda}(\Gamma \mu - \chi \mu)(1 + 2n - \cosh(2r))}{(\mu^2 + 2\mu\lambda + \lambda^2)(1 + C) + 4(\Gamma \mu - \chi \mu)^2} \tag{45}$$

$$\begin{aligned} \Pi_8 &= \frac{\mu(\mu^2 C^2 \lambda^2 (\Gamma \mu + \chi \mu) + (\lambda^2 + 4(\Gamma \mu)^2) \chi \mu (\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma \mu + \chi \mu)^2)) \sinh(2r)}{(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma \mu)^2) - 8C\lambda\Gamma \chi \mu^3 + 4(\lambda^2 + 4(\Gamma \mu)^2)(\chi \mu)^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma \mu + \chi \mu)^2)} \\ &- \frac{\mu(\mu C \lambda (\lambda^2 (\Gamma \mu - \chi \mu) - 2\lambda \chi \mu^2 + 4\Gamma \mu ((\Gamma \mu)^2 + 3\Gamma \chi \mu^2 + 2(\chi \mu)^2))) \sinh(2r)}{(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma \mu)^2) - 8C\lambda\Gamma \chi \mu^3 + 4(\lambda^2 + 4(\Gamma \mu)^2)(\chi \mu)^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma \mu + \chi \mu)^2)} \end{aligned} \tag{46}$$

$$\begin{aligned} \Pi_9 &= \frac{\mu^2(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma \mu)^2) + (\lambda^2 + 4(\Gamma \mu)^2)((1 + C)\lambda^2 + 4(\Gamma \mu + \chi \mu)^2)) \cosh(r) \sinh(r)}{(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma \mu)^2) - 8C\lambda\Gamma \chi \mu^3 + 4(\lambda^2 + 4(\Gamma \mu)^2)(\chi \mu)^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma \mu + \chi \mu)^2)} \\ &+ \frac{\mu^2(\mu((2 + 3C + C^2)\lambda^3 - 4\lambda\Gamma \mu((-2 + C)\Gamma \mu + 2C\chi \mu))) \cosh(r) \sinh(r)}{(\mu^2((1 + C)^2 \lambda^2 + 4(\Gamma \mu)^2) - 8C\lambda\Gamma \chi \mu^3 + 4(\lambda^2 + 4(\Gamma \mu)^2)(\chi \mu)^2)(\mu^2 + 2\mu\lambda + \lambda^2 + 4(\Gamma \mu + \chi \mu)^2)} \end{aligned} \tag{47}$$

$$\Pi_{10} = \frac{(1 + 2n)C\lambda(\mu + \lambda) + (\mu^2(1 + C) + \mu(2 + C)\lambda + \lambda^2 + 4(\Gamma \mu - \chi \mu)^2) \cosh(2r)}{2((\mu^2 + 2\mu\lambda + \lambda^2)(1 + C) + 4(\Gamma \mu - \chi \mu)^2)} \tag{48}$$

The optomechanical cooperativity C is defined as [46, 53]:

$$C = \frac{4\Lambda^2}{\lambda\mu} = \frac{8\omega_a^2}{m\lambda\omega_b\omega_L L^2} \frac{\rho}{\mu^2[\frac{1}{4} + (\chi - \omega_b/\mu)^2]} \tag{49}$$

we have set $\chi = \frac{\tau}{\mu}$ and $\Gamma = \frac{\eta}{\mu}$, where χ is the photon hopping strength and Γ the phonon tunneling strength.

5 Bipartite and tripartite entanglement

In this paragraph, we examine bipartite and tripartite entanglement in two–three-mode blocks (b_1, b_2, a_1) and (b_2, a_1, a_2) whose corresponding reduced covariance matrices are, respectively, $\Pi_{(b_1, b_2, a_1)}$ and $\Pi_{(b_2, a_1, a_2)}$:

$$\begin{aligned} \Pi_{(b_1, b_2, a_1)} &= \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & 0 & \Pi_4 & \Pi_5 \\ \Pi_2 & \Pi_1 & 0 & -\Pi_3 & \Pi_5 & \Pi_4 \\ \Pi_3 & 0 & \Pi_1 & \Pi_2 & \Pi_6 & \Pi_7 \\ 0 & -\Pi_3 & \Pi_2 & \Pi_1 & -\Pi_7 & -\Pi_6 \\ \Pi_4 & \Pi_5 & \Pi_6 & -\Pi_7 & \Pi_{10} & \Pi_8 \\ \Pi_5 & \Pi_4 & \Pi_7 & -\Pi_6 & \Pi_8 & \Pi_{10} \end{bmatrix}; \\ \Pi_{(b_2, a_1, a_2)} &= \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_6 & \Pi_7 & \Pi_4 & \Pi_5 \\ \Pi_2 & \Pi_1 & -\Pi_7 & -\Pi_6 & \Pi_5 & \Pi_4 \\ \Pi_6 & -\Pi_7 & \Pi_{10} & \Pi_8 & \Pi_9 & 0 \\ \Pi_7 & -\Pi_6 & \Pi_8 & \Pi_{10} & 0 & -\Pi_9 \\ \Pi_4 & \Pi_5 & \Pi_9 & 0 & \Pi_{10} & \Pi_8 \\ \Pi_5 & \Pi_4 & 0 & -\Pi_9 & \Pi_8 & \Pi_{10} \end{bmatrix} \end{aligned} \tag{50}$$

Two-mode bipartite entanglement can be quantified by means of one-mode (l) versus one-mode (s) logarithmic

negativity $\epsilon_{l|s}$. For a two-mode Gaussian state, the logarithmic negativity is given by [84, 85]

$$\epsilon_{l|s} = \max[0, -\ln(2\alpha_{l|s}^-)] \tag{51}$$

$\alpha_{l|s}^-$ is the smallest symplectic eigenvalue of partial transposed covariance matrix $\Pi_{(l,s)}$

$$\text{where } \Pi_{(l,s)} = \begin{bmatrix} X(t) & Z(t) \\ Z^T(t) & Y(t) \end{bmatrix} \Delta_{l|s} = \det X(t) + \det Y(t) - 2\det Z(t)$$

$$\alpha_{l|s}^- = \sqrt{\frac{\Delta - \sqrt{\Delta^2 - 4\det \Pi_{(l,s)}}}{2}}; (l \neq s)$$

To quantify three-mode bipartite entanglement, we use the one-mode (l) versus two-mode (sp) logarithmic negativity $\epsilon_{l|sp}$, defined as [74, 87]

$$\epsilon_{l|sp} = \max[0, -\ln(2\alpha_{l|sp}^-)] \tag{52}$$

where, $\alpha_{l|sp}^- = \min \text{eig} |i\Omega_3(\Sigma_{l|sp} \Pi_{(l,sp)} \Sigma_{l|sp})|$, with, $\Omega_3 = \bigoplus_{k=1}^3 i\sigma_y$ (σ_y is the y-Pauli matrix), $\Sigma_{l|sp}$ is the matrix that inverts the sign of momentum of mode l ($l \neq s \neq p$).

To study the genuine tripartite entanglement, we introduce the tripartite negativity defined as [74, 87]

$$\epsilon_{(l,s,p)} = (\epsilon_{l|sp} \epsilon_{s|pl} \epsilon_{p|ls})^{\frac{1}{3}} \tag{53}$$

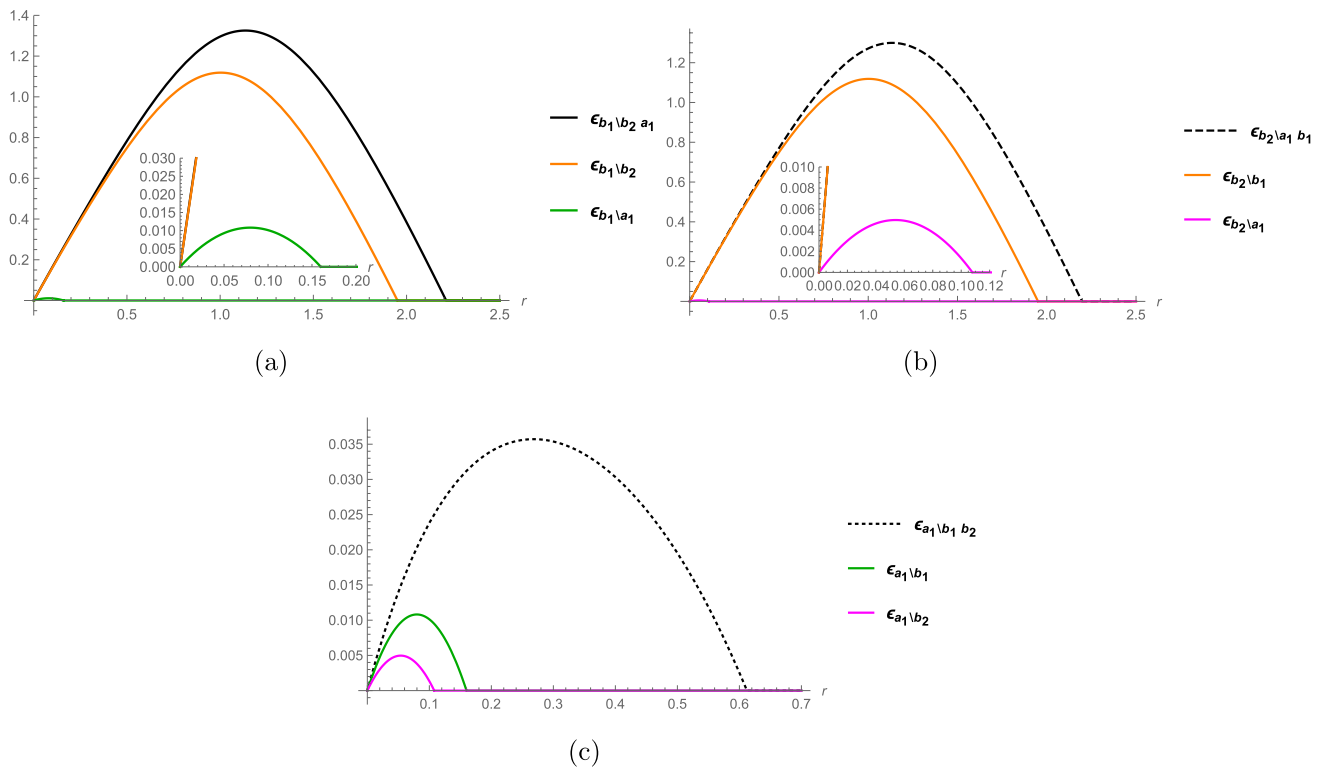


Fig. 2 Plots of logarithmic negativity ($\epsilon_{i|s, \epsilon_{i|sp}}$) as a function of squeezing parameter r , with $\chi = 0.05$; $\Gamma = 0.05$; $T = 0.1mk$; $C = 32.35$ in the three-mode block (b_1, b_2, a_1)

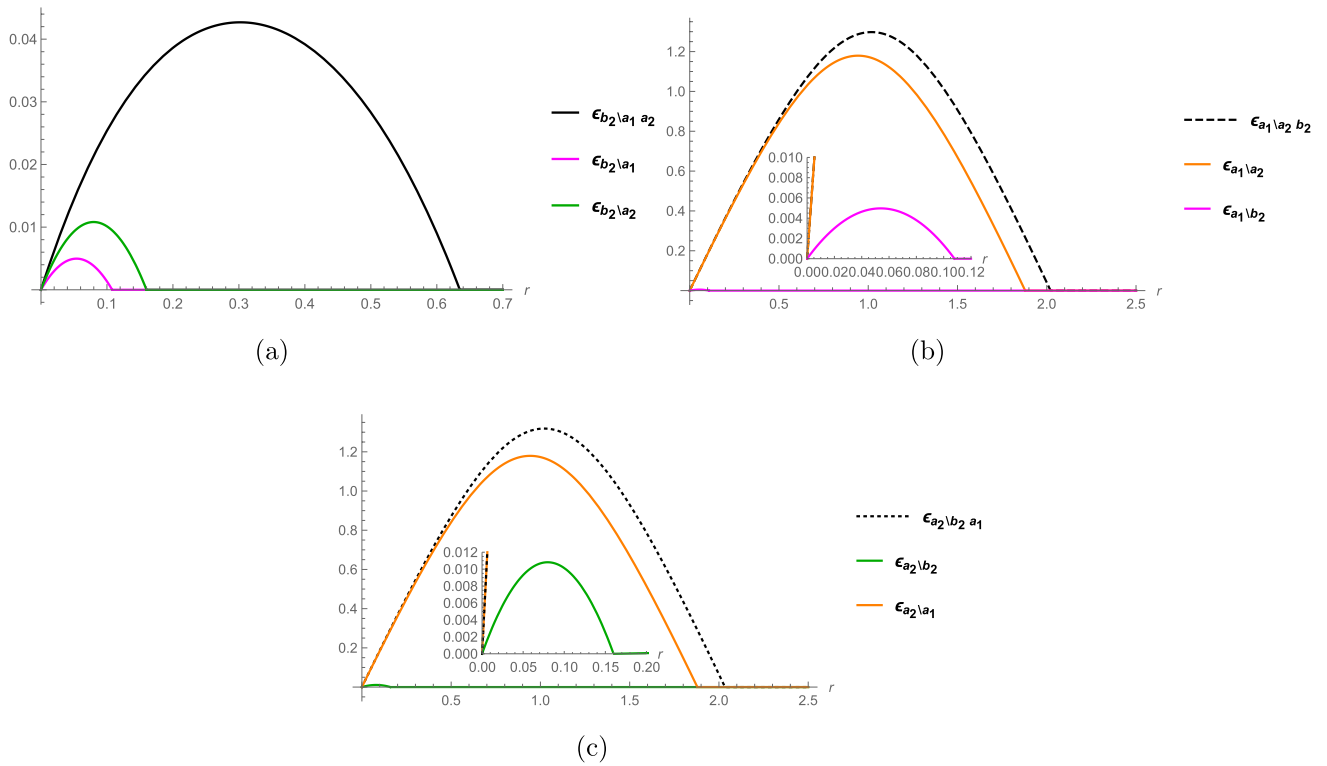


Fig. 3 Plots of logarithmic negativity ($\epsilon_{i|s, \epsilon_{i|sp}}$) as a function of squeezing parameter r , with $\chi = 0.05$; $\Gamma = 0.05$; $T = 0.1mk$; $C = 32.35$ in the three-mode block (b_2, a_1, a_2)

6 Analyse and discussion

By exploiting experimental values reported in Ref [86] $\{\frac{\omega_b}{2\pi} = 947 \times 10^3 \text{Hz}$, the mechanical damping rate $\frac{\lambda}{2\pi} = 140 \text{Hz}$ and the mass of the movable mirror $m = 145 \text{ng}$, the cavity length and frequency are, respectively, $L = 25 \text{mm}$ and $\frac{\omega_a}{2\pi} = 5.26 \times 10^{14} \text{Hz}$, the laser frequency $\frac{\omega_L}{2\pi} = 2.82 \times 10^{14} \text{Hz}$ and the drive laser power $\rho = 11 \text{mW}$ }, we plot one-mode versus one-mode logarithmic negativity $\epsilon_{l|s}$; ($l \neq s$) and one-mode versus two-mode logarithmic negativity $\epsilon_{l|sp}$; ($l \neq s \neq p$) as a function of the squeezing parameter r for fixed values of the photon hopping strength ($\chi = \frac{\tau}{\mu}$) and the phonon tunneling strength ($\Gamma = \frac{\eta}{\mu}$). The numerical results are shown in Figs. 2 and 3. In Fig. 2a which presents the behavior of the entanglement in the block with three modes (b_1, b_2, a_1), i.e., the entanglement shared by the mode b_1 is either individually with mode b_2 and mode a_1 ($\epsilon_{b_1|b_2}, \epsilon_{b_1|a_1}$), or jointly with the two-mode ($b_2 - a_1$) ($\epsilon_{b_1|b_2 a_1}$). Note that for small values of r , $\epsilon_{b_1|a_1}$ rapidly vanishes, while $\epsilon_{b_1|b_2}$ and $\epsilon_{b_1|b_2 a_1}$ remain non-zero. This means that mode b_1 can simultaneously share entanglement with the two-mode ($b_2 - a_1$) and also individually with mode b_2 but it cannot share entanglement with single mode a_1 . $\epsilon_{b_1|b_2}$ remains robust against the rise of r , but eventually vanishes for $r \geq 1.96$. For $1.96 \leq r \leq 2.2$, there is no entanglement sharing individually between the mode b_1 and each single remaining modes (b_2, a_1), while, there is entanglement sharing by mode b_1 to the two-mode ($b_2 - a_1$), i.e., $\{\epsilon_{b_1|b_2} = 0, \epsilon_{b_1|a_1} = 0, \epsilon_{b_1|b_2 a_1} \neq 0\}$. Now, we investigate the behavior of entanglement in the configurations $\epsilon_{b_2|a_1}, \epsilon_{b_2|b_1}$ and $\epsilon_{b_2|a_1 b_1}$ (Fig. 2b). We remark that all terms $\epsilon_{b_2|a_1}, \epsilon_{b_2|b_1}$ and $\epsilon_{b_2|a_1 b_1}$ are non-zero, meaning that the mode b_2 is correlated with single modes b_1 and a_1 and with the two-mode ($b_1 - a_1$) in a small range of very lower values of r ($0.00 \leq r \lesssim 0.11$). Obviously, we observe that with an increase of the squeezing parameter, $\epsilon_{b_2|a_1}$ becomes zero, while the quantities $\epsilon_{b_2|b_1}$ and $\epsilon_{b_2|a_1 b_1}$ are non-zero. This means that the mode b_2 is not able to share entanglement individually with mode a_1 but it can remarkably share it individually with mode b_1 as well as with two modes ($a_1 - b_1$). Unlike $\epsilon_{b_2|b_1}$ and $\epsilon_{b_2|a_1}$, when r becomes larger ($1.96 \leq r \leq 2.2$), $\epsilon_{b_2|a_1 b_1}$ remains non-zero. Thus, although b_2 is unable to share the entanglement with the individual modes a_1 and b_1 , it can nevertheless share it simultaneously with the two-mode ($a_1 - b_1$). In the configurations $\epsilon_{a_1|b_1}, \epsilon_{a_1|b_2}$ and $\epsilon_{a_1|b_1 b_2}$ (Fig. 2c), the entanglement shared by a_1 conjointly with b_1 and b_2 is the most dominated especially for a higher values of r ($0.16 \leq r \lesssim 0.62$), where $\epsilon_{a_1|b_1}$ and $\epsilon_{a_1|b_2}$ are totally vanished. In this case a_1 can share entanglement with modes b_1 and b_2 only in a collective way. We notice also that for the smallest values of r ($r \leq 0.11$), a_1 can share entanglement individually and collectively with b_1 and b_2 , i.e., $\{\epsilon_{a_1|b_1} \neq 0, \epsilon_{a_1|b_2} \neq 0, \epsilon_{a_1|b_1 b_2} \neq 0\}$. In addition

we remark that for a small range of r ($0.11 \leq r \leq 0.16$), $\epsilon_{a_1|b_2}$ is zero, while $\epsilon_{a_1|b_1}$ and $\epsilon_{a_1|b_1 b_2}$ are non-zero, which means that a_1 is correlated individually only to mode b_1 , and that entanglement sharing occurs with the two-mode ($b_1 - b_2$). From this part of our study, we can conclude that sharing entanglement between modes b_1, b_2 and a_1 is monogamous.

Subsequently, we will inspect the property of monogamy in a three-mode block (b_2, a_1, a_2). Figure 3a shows that entanglement cannot be shared freely between modes. Indeed, in the configurations $\epsilon_{b_2|a_2}, \epsilon_{b_2|a_1}$ and $\epsilon_{b_2|a_1 a_2}$, the simultaneous emergence of individual and collective sharing of entanglement between mode b_2 and modes a_1 and a_2 ($\epsilon_{b_2|a_2} \neq 0, \epsilon_{b_2|a_1} \neq 0, \epsilon_{b_2|a_1 a_2} \neq 0$) occurs just in a very restricted range of small values of r ($0.00 \leq r \leq 0.11$). By increasing the values of r , we find that $\epsilon_{b_2|a_1} = 0$, while $\epsilon_{b_2|a_2}$ and $\epsilon_{b_2|a_1 a_2}$ are non-zero, which means that the mode b_2 can share the entanglement with the mode a_2 and with the two-mode ($a_1 - a_2$), but cannot be individually entangled with mode a_1 . If we further increase the values of r , then $\epsilon_{b_2|a_2}$ vanishes, while $\epsilon_{b_2|a_1 a_2}$ is non-zero. In this range of large values of r ($0.16 \leq r \lesssim 0.64$), the mode b_2 can share entanglement with the two-mode ($a_1 - a_2$), but it is unable to share entanglement individually with each of them.

In the following, we will examine the sharing of entanglement in the three-mode block (a_1, a_2, b_2), discussing in more detail the bipartite and tripartite logarithmic negativities $\epsilon_{a_1|b_2}, \epsilon_{a_1|a_2}$ and $\epsilon_{a_1|a_2 b_2}$ (Fig. 3b). For a wide range of values of r ($0.11 \lesssim r \lesssim 1.9$), we remark that $\epsilon_{a_1|b_2} = 0, \epsilon_{a_1|a_2}$ and $\epsilon_{a_1|a_2 b_2}$ are non-zero, meaning that the mode a_1 cannot be entangled individually with b_2 but it is able to share entanglement individually with mode a_2 in addition to sharing it with the two-mode ($a_2 - b_2$). For very small values of r ($0.00 \leq r \lesssim 0.11$), the mode a_1 can share entanglement individually with b_2 and a_2 in addition to sharing it with the two-mode ($a_2 - b_2$), i.e., $\epsilon_{a_1|b_2} \neq 0, \epsilon_{a_1|a_2} \neq 0$ and $\epsilon_{a_1|a_2 b_2} \neq 0$. When $1.89 \leq r \leq 2.0$, we find that $\epsilon_{a_1|b_2} = 0, \epsilon_{a_1|a_2} = 0$ and $\epsilon_{a_1|a_2 b_2} \neq 0$, meaning that mode a_1 cannot be entangled individually neither with b_2 nor with a_2 , but it is able to share entanglement with the two-mode ($a_2 - b_2$).

Finally, we investigate entanglement behavior in the three-mode block (a_2, a_1, b_2), i.e., $\epsilon_{a_2|b_2}, \epsilon_{a_2|a_1}$ and $\epsilon_{a_2|b_2 a_1}$ (Fig. 3c). We notice as Fig. 3 shows that during the evolution of quantities $\epsilon_{a_2|b_2}, \epsilon_{a_2|a_1}$ and $\epsilon_{a_2|b_2 a_1}$ versus r , three intervals of r can be distinguished. In the first one ($0.00 \leq r \leq 0.16$) we remark that $\{\epsilon_{a_2|b_2} \neq 0, \epsilon_{a_2|a_1} \neq 0, \epsilon_{a_2|b_2 a_1} \neq 0\}$, which shows that mode a_2 is able to share entanglement individually with mode b_2 and mode a_1 and also collectively with the two-mode ($b_2 - a_1$). In the second range ($0.16 \leq r \leq 1.88$) which is the largest one, the mode a_2 is individually entangled only with mode a_1 and with the two-mode ($b_2 - a_1$). Regarding the third range ($1.88 \leq r \leq 2.03$), there is no individual

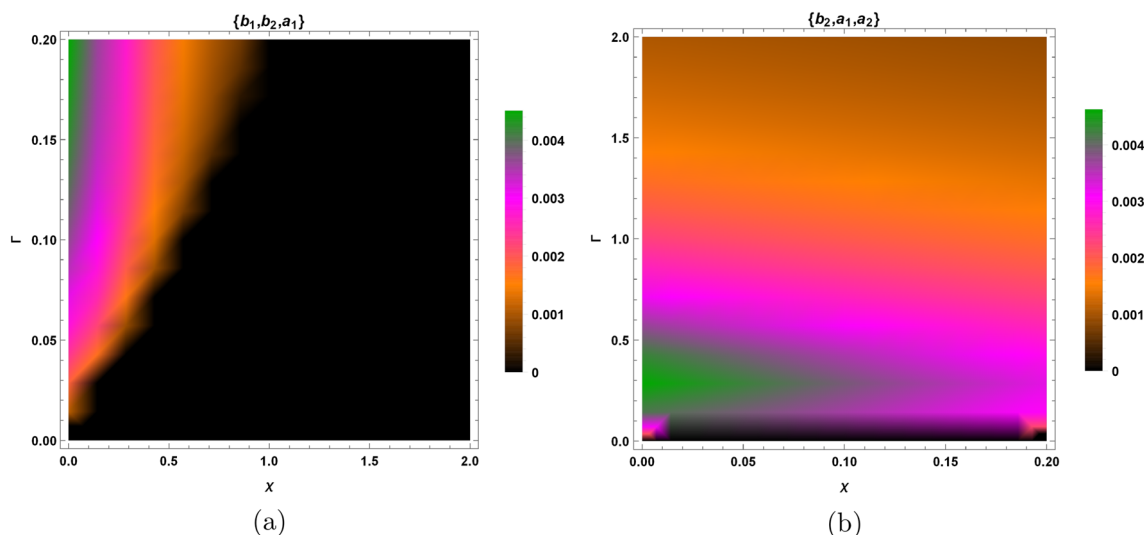


Fig. 4 Effect of phonon tunneling strength Γ and photon hopping strength χ on tripartite entanglement with $r = 0.1$ and $T = 0.1\text{mK}$

entanglement shared between mode a_2 and the other remaining modes, i.e., $\{\epsilon_{a_2|b_2} = 0, \epsilon_{a_2|a_1} = 0\}$, whereas there exist collective entanglement sharing between mode a_2 and the two-mode $(b_2 - a_1)$, i.e., $\epsilon_{a_2|b_2a_1} \neq 0$.

In general, we notice that all the quantities $(\epsilon_{l|s}; (l \neq s))$ and $(\epsilon_{l|sp}; (l \neq s \neq p))$ in the two three-mode blocks (b_1, b_2, a_1) and (b_2, a_1, a_2) have the same evolution with r : the entanglement begins by increasing, reaches a maximum then it decreases and ends up canceling out. As shown in Fig. 2(3), the degree of entanglement shared by $a_1(b_2)$ jointly or individually with modes b_1 and b_2 (a_1 and a_2) remains very low compared to that of modes $b_1(a_1)$ and that of mode $b_2(a_2)$. In addition, we find that one-mode versus two-mode entanglement is robust against increasing of r values than one-mode versus one-mode entanglement. Moreover, the former has a very large degree compared to the degree of the latter one, as expected by CKW-monogamy [75].

In Fig. 4, we plot the tripartite negativity as a function of the phonon tunneling strength Γ and the photon jumping strength χ in the two three-mode blocks (b_1, b_2, a_1) (Fig. 4a) and (b_2, a_1, a_2) (Fig. 4b). Based on the results shown in Fig. 4a, it is clearly indicated that for a fixed value of Γ , the tripartite entanglement degrades with increasing χ and eventually vanishes for: $0.9 \leq \chi \leq 2.0$ and $0.0 \leq \Gamma \leq 0.2$, signifying the disappearance of tripartite entanglement when χ has reached 0.9. The emergence of tripartite entanglement requires a minimum value of Γ ($\Gamma = 0.01$), then it increases with the increasing of Γ values. Remarkably, the area for which the entanglement exists widens with the rise of both Γ and χ . For instance, for $\Gamma = 0.05$, tripartite entanglement is observed when $0.0 \leq \chi \leq 0.3$, while for $\Gamma = 0.20$ it is observed when $0.0 \leq \chi \leq 0.9$. As

an example of the optimal situation for strong entanglement, the area selected is one in which the value of Γ can be chosen between 0.18 and 0.20, while χ can be taken between 0.0 and 0.1. In Fig. 4b, we examine the effects of the parameters χ and Γ on tripartite entanglement strength in the three-mode block (b_2, a_1, a_2) . Note that, in the range of values of Γ between 0.0 and 0.8, the maximum of shared entanglement is obtained for $\Gamma = 0.3$ whatever the value of χ . For a fixed value of χ , the tripartite entanglement increases with Γ , then it decreases when $\Gamma = 0.3$. For a fixed value of Γ , varying the values of χ has a negligible effect on the amount of entanglement. Besides, Fig. 4b shows a strong decrease in the range of entanglement with increasing values of χ . For example, for $\chi = 0.00$, tripartite entanglement is strong when $0.2 \leq \Gamma \leq 0.4$, whereas for $\chi = 0.03$, tripartite entanglement is strong when $0.25 \leq \Gamma \leq 0.35$. Contrary to the first region, the tripartite entanglement degrades and remains low for higher values of Γ compared to values of χ , i.e., $0.00 \leq \chi \leq 0.2$ and $0.9 \leq \Gamma \leq 2.0$.

7 Conclusion

This work focused on the study of entanglement monogamy as well as the behavior of tripartite entanglement, in two blocks with three modes (b_1, b_2, a_1) and (b_2, a_1, a_2) , depending on the phonon tunneling force Γ and photon jump force χ . It follows that, in the case of the two blocks, the sharing of the entanglement obeys the law of monogamy, this means that each mode cannot freely share the entanglement with the other modes. We also analyzed the effects of the combined actions of squeezing light, photon

hopping process and phonon tunneling process on entanglement sharing.

For our purpose, we have fixed the value of r and we have varied both the values of χ and the values of Γ . Combining the effects of χ and Γ can produce strong tripartite entanglement. Moreover, the tripartite entanglement degrades and sometimes vanishes due to a very large difference between the values of χ and the values of Γ . Finally, the main result of this article is that quantum entanglement cannot be freely shared between the three modes constituting each of the two blocks studied (b_1, b_2, a_1) and (b_2, a_1, a_2). The analysis of the combined action of the parameters governing the system on the emergence and the degree of entanglement are of great importance to use the considered system with a greater precision to bring a great performance.

Author Contributions All of the authors have contributed substantially to the manuscript.

Data availability No data associated in the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

References

- M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000)
- R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009)
- J.S. Bell, *Physics* **1**, 195 (1964)
- H.M. Wiseman, S.J. Jones, A.C. Doherty, *Phys. Rev. Lett.* **98**, 140402 (2007)
- S.J. Jones, H.M. Wiseman, A.C. Doherty, *Phys. Rev. A* **76**, 052116 (2007)
- G. Adesso, A. Datta, *Phys. Rev. Lett.* **105**, 030501 (2010)
- P. Giorda, M.G.A. Paris, *Phys. Rev. Lett.* **105**, 020503 (2010)
- M.A. Nielson, I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000)
- S. Pirandola, U.L. Andersen, L. Banchi, M. Berta, D. Bunandar, R. Colbeck, D. Englund, T. Gehring, C. Lupo, C. Ottaviani, J.L. Pereira, M. Razavi, J. Shamsul Shaari, M. Tomamichel, V.C. Usenko, G. Vallone, P. Villoresi, P. Wallden, *Adv. Opt. Photonics* **12**, 1012 (2020)
- C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W.K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993)
- V. Giovannetti, S. Lloyd, L. Maccone, *Nat. Photonics* **5**, 222 (2011)
- S. Pirandola, J. Eisert, C. Weedbrook, A. Furusawa, S.L. Braunstein, *Nat. Photonics* **9**, 641 (2015)
- L. Zhou, Y. Han, J. Jing, W. Zhang, *Phys. Rev. A* **83**, 052117 (2011)
- A. Nunnenkamp, K. Børkje, S.M. Girvin, *Phys. Rev. Lett.* **107**, 063602 (2011)
- T. Purdy, *Science* **339**, 801 (2013)
- C.-H. Bai, D.-Y. Wang, H.-F. Wang, A.-D. Zhu, S. Zhang, *Sci. Rep.* **6**, 33404 (2016). <https://doi.org/10.1038/srep33404>
- S. Bougouffa, Z. Ficek, *Phys. Rev. A* **93**, 063848 (2016)
- X. Liang, Q. Guo, W. Yuan, *Int. J. Theor. Phys.* **58**, 58 (2019)
- W. Ge, M. Al-Amri, H. Nha, M.S. Zubairy, *Phys. Rev. A* **88**, 052301 (2013)
- W. Ge, M.S. Zubairy, *Phys. Rev. A* **91**, 013842 (2015)
- W. Ge, M.S. Zubairy, *Phys. Scr.* **90**, 074015 (2015)
- L.-G. Si, H. Xiong, M.S. Zubairy, Y. Wu, *Phys. Rev. A* **95**, 033803 (2017)
- S. Asiri, Z. Liao, M.S. Zubairy, *Phys. Scr.* **93**, 124002 (2018)
- M. Aspelmeyer, K. Schwab, *New J. Phys.* **10**, 095001 (2008)
- M. Blencowe, *Phys. Rep.* **395**, 159 (2004)
- C. Genes, A. Mari, D. Vitali, P. Tombesi, *Adv. Atom Mol. Opt. Phys.* **57**, 33 (2009)
- M. Aspelmeyer, S. Gröblacher, K. Hammerer, N. Kiesel, *JOSA B* **27**, A189 (2010)
- A.A. Clerk, F. Marquardt, *Basic theory of cavity optomechanics, in Cavity Optomechanics: Nano- and Micromechanical Resonators Interacting with Light.* (Springer, New York, 2014), pp.5–23
- S. Pirandola, S. Mancini, D. Vitali, P. Tombesi, *Phys. Rev. A* **68**, 062317 (2003)
- M. Amazioug, M. Nassik, N. Habiballah, *Chin. J. Phys.* **58**, 1 (2019)
- M. Bhattacharya, P. Meystre, *Phys. Rev. Lett.* **99**(7), 073601 (2007)
- M. Asjad, P. Tombesi, D. Vitali, *Opt. Express* **23**(6), 7786–7794 (2015)
- J. Teufel, T. Donner, D. Li, J. Harlow, M. Allman, K. Cicak, A. Sirois, J. Whittaker, K. Lehnert, R. Simmonds, *Nature* **475**, 359 (2011)
- M. Amazioug, M. Nassik, N. Habiballah, *Int. J. Quantum Inf.* **16**, 1850043 (2018)
- M. Asjad, S. Zippilli, D. Vitali, *Phys. Rev. A* **93**, 062307 (2016)
- M. Asjad, P. Tombesi, D. Vitali, *Phys. Rev. A* **94**, 052312 (2016)
- M. Amazioug, M. Nassik, N. Habiballah, *Eur. Phys. J. D* **72**, 171 (2018)
- M. Asjad, N.E. Abari, S. Zippilli, D. Vitali, *Opt. Express* **27**, 32427 (2019)
- M. Amazioug, M. Nassik, *Int. J. Quantum Inf.* **17**(05), 1950045 (2019)
- S.K. Singh, C.H. Raymond Ooi, *J. Opt. Soc. Am. B* **31**, 2390 (2014)
- S.K. Singh, J.-X. Peng, M. Asjad, M. Mazaheri, *J. Phys. B Atom. Mol. Opt. Phys.* **54**, 215502 (2021)
- M. Amazioug, M. Nassik, N. Habiballah, *Optik Int. J. Light Electron Opt.* **158**, 1186 (2018)
- E.A. Sete, H. Eleuch, *Phys. Rev. A* **85**, 043824 (2012)
- A. Kronwald, F. Marquardt, A.A. Clerk, *New J. Phys.* **16**, 063058 (2014)
- T. Yousif, W. Zhou, L. Zhou, *J. Mod. Opt.* **61**, 1180 (2014)
- M. Aspelmeyer, T.J. Kippenberg, F. Marquardt, *Rev. Mod. Phys.* **86**, 1391 (2014)
- J.Q. Liao, C.K. Law, *Phys. Rev. A* **84**, 053838 (2011)
- L. LSC, *The Laser Interferometer Gravitational-Wave Observatory*, *Tech. Rep.* (LIGO-P070082-01, 2007)
- T.P. Purdy, R.W. Peterson, C. Regal, *Science* **339**, 801 (2013)
- J.Q. Liao, L. Tian, *Phys. Rev. Lett.* **116**, 163602 (2016)
- A. Cleland, M. Roukes, *Appl. Phys. Lett.* **69**, 2653 (1996)
- M. Aspelmeyer, T.J. Kippenberg, F. Marquardt, *Rev. Mod. Phys.* **86**, 1391 (2014)
- M. Amazioug, B. Maroufi, M. Daoud, *Eur. Phys. J. D* **74**, 54 (2020)
- S. Bougouffa, M. Al-Hmoud, J.W. Hakami. Probing quantum correlations in a hybrid optomechanical system. *arXiv preprint arXiv:2204.07753* (2022)

55. V. Coffman, J. Kundu, W.K. Wootters, *Phys. Rev. A* **61**, 052306 (2000)
56. C. Lancien, S. Di Martino, M. Huber, M. Piani, G. Adesso, A. Winter, *Phys. Rev. Lett.* **117**, 060501 (2016)
57. B.M. Terhal, *IBM J. Res. Dev.* **48**, 71 (2004)
58. C. Lancien, S. Di Martino, M. Huber, M. Piani, G. Adesso, A. Winter, *Phys. Rev. Lett.* **117**, 060501 (2016)
59. A.K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991)
60. C.H. Bennett, H.J. Bernstein, S. Popescu, B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996)
61. V. Coffman, J. Kundu, W.K. Wootters, *Phys. Rev. A* **61**, 052306 (2000)
62. J.S. Kim, G. Gour, B.C. Sanders, *Contemp. Phys.* **53**, 417 (2012)
63. N. Gisin, G. Ribordy, W. Tittel, H. Zbinden, *Rev. Mod. Phys.* **74**, 145 (2002)
64. C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, *Phys. Rev. A* **54**, 3824 (1996)
65. A. Chandran, D. Kaszlikowski, A. Sen(De), U. Sen, V. Vedral, *Phys. Rev. Lett.* **99**, 170502 (2007)
66. H.S. Dhar, A. Sen(De), *J. Phys. A Math. Theor.* **44**, 465302 (2011)
67. S. Singha Roy, H.S. Dhar, D. Rakshit, A. Sen(De), U. Sen. [arXiv: 1607.05195](https://arxiv.org/abs/1607.05195) (2016)
68. D. Sadhukhan, S. Singha Roy, D. Rakshit, R. Prabhu, A. Sen(De), U. Sen, *Phys. Rev. E* **93**, 012131 (2016)
69. M. Allegra, P. Giorda, A. Montorsi, *Phys. Rev. B* **84**, 245133 (2011)
70. X.-K. Song, T. Wu, L. Ye, *Quantum Inf. Process.* **12**, 3305 (2013)
71. L. Qiu, G. Tang, X.-Q. Yang, A.-M. Wang, *Europhys. Lett.* **105**, 30005 (2014)
72. M. Qin, Z.-Z. Ren, X. Zhang, *Quantum Inf. Process.* **15**, 255 (2016)
73. K.R.K. Rao, H. Katiyar, T.S. Mahesh, A. Sen(De), U. Sen, A. Kumar, *Phys. Rev. A* **88**, 022312 (2013)
74. J. Zhang, T. Zhang, A. Xuereb, D. Vitali, J. Li, *Ann. Phys. (Berlin)* **527**, 147 (2015)
75. G. Adesso, A. Serafini, F. Illuminati, *Phys. Rev. A* **73**, 032345 (2006)
76. M. Ludwig, F. Marquardt, *Phys. Rev. Lett.* **111**, 073603 (2013)
77. V. Giovannetti, D. Vitali, *Phys. Rev. A* **63**, 023812 (2001)
78. C.W. Gardiner, P. Zoller, *Quantum Noise* (Springer, Berlin, 2000), p.71
79. C.W. Gardiner, *Phys. Rev. Lett.* **56**, 1917 (1986)
80. Y.D. Wang, S. Chesi, A.A. Clerk, *Phys. Rev. A* **91**, 013807 (2015)
81. A. Mari, J. Eisert, *Phys. Rev. Lett.* **103**, 213603 (2009)
82. D. Vitali, S. Gigan, A. Ferreira, H.R. Bohm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, M. Aspelmeyer, *Phys. Rev. Lett.* **98**, 030405 (2007)
83. P.C. Parks, V. Hahn, *Stability Theory* (Prentice Hall, New York, 1993)
84. G. Vidal, R.F. Werner, *Phys. Rev. A* **65**, 032314 (2002)
85. G. Adesso, A. Serafini, F. Illuminati, *Phys. Rev. A* **70**, 022318 (2004)
86. S. Gröblacher, K. Hammerer, M.R. Vanner, M. Aspelmeyer, *Nature (London)* **460**, 724 (2009)
87. A. Lakhfif, A. Hidki, J. El Qars, M. Nassik, *Phys. Lett. A* **445**, 128247 (2022)

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